Lasso for L 0.5 or L1/2 norm:-

$$\theta_j = \arg\min_{\Theta} \left\{ \frac{1}{2} \sum_{i=1}^{N} \left(Y_i - \sum_{k=1}^{p} x_{ik}\theta_k \right)^2 + \lambda \sum_{k=1}^{p} \sqrt{|\theta_k|} \right\}$$

$\lambda \geq 0$ —> tuning parameter

For optimisation,we have to take the derivative.

So,

$$\frac{\partial O}{\partial \theta_i} = -\sum_{i=1}^{N} x_{ij} \left(Y_i - x_{ij}\Theta_j - \sum_{k \neq j}^{p} x_{ij}\Theta j \right) + \frac{\lambda}{2} \frac{sign\,(\theta_k)}{\sqrt{|\theta_k|}}$$

$$-\sum_{i=1}^{N} x_{ij} \left(Y_i - \sum_{k \neq j}^{p} x_{ij}\Theta j \right) - \sum_{i=1}^{N} x_{ij}^2 \theta_j + \frac{\lambda}{2} \frac{sign\,(\theta_k)}{\sqrt{|\theta_k|}}$$

$Now;$

$$\sum_{i=1}^{N} x_i j^2 = N - 1$$

$$-\sum_{i=1}^{N} x_{ij} \left(Y_i - \sum_{k \neq j}^{p} x_{ij}\Theta j \right) + (N-1)\,\theta j + \frac{\lambda}{2} \frac{sign\,(\theta_k)}{\sqrt{|\theta_k|}}$$

Now, we set the derivative to 0 i.e. $\dfrac{\partial O}{\partial \theta_j} = 0$

$S_0;$

$$\sum_{i=1}^{N} x_{ij} \left(Y_i - \sum_{k=j}^{p} x_{ij}\Theta j \right) = (N-1)\,\Theta j + \frac{\lambda}{2} \frac{sign\,(\theta_k)}{\sqrt{|\theta_k|}}$$

$When \quad \theta_j \geq 0 \qquad \dfrac{T - \dfrac{\lambda}{2\sqrt{|\theta_k|}}}{N-1} \qquad S_0, \quad T \geq \dfrac{\lambda}{2\sqrt{|\theta_k|}}$

$$And \quad when \quad \theta_j < 0 \; ; \quad \frac{T + \frac{\lambda}{2\sqrt{|\theta_k|}}}{N-1} \quad S_0; \quad T < \frac{\lambda}{2\sqrt{|\theta_k|}}$$

Now, as we can infer, we have 2 properties;

1) $$T \geq \frac{\lambda}{2\sqrt{|\theta_k|}}$$

2) $\theta_j \geq 0 \quad if \quad T \geq 0$ & $\theta_j \leq 0 \quad if \quad T \leq 0$

$$\theta j = \frac{sign\,(\tau)\left(|T| - \frac{\lambda}{2\sqrt{|\theta_k|}}\right)}{N-1}$$

S_0; $$A = N - 1 \quad ; \quad B = \frac{\lambda}{2\sqrt{|\theta_k|}}$$