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Exploring Regularization Techniques for Small Data Sets

1 Introduction

In this paper we focus on regularization techniques that improve linear regression modeling on small data sets. Ordinary Least Squares (OLS) is a popular solution for the weights of a linear model, however the method becomes numerically unstable as the number of samples decrease, and has no solution when the number of parameters exceed the sample size. Statistical regularization techniques stabilize our solution and reduce the variance which becomes increasingly important as the number of samples in our data set shrink. Unforunatley, statistical regularization introduces bias into our model so we must choose an appropriate balance. The models we consider in this paper control the degree of regularization through one or two tuning parameters that are completely suppressed at zero, and have more influence on the solution as they increase towards infinity. We can estimate the ideal tuning parameter values by using cross-validation to estimate the MSE for a small universe of tuning parameters, and select the parameters that minimize MSE. Each regularization technique that we consider is formulated as an optimization problem that minimizes an objective function J of the form:

$$\hat{\theta} = \arg\min_{\theta} J(\theta) = \arg\min_{\theta} \sum_{i=1}^{N} L(\theta) + \sum_{i=1}^{P} R(\theta, \lambda)$$

Where $L(\theta)$ is the loss function and $R(\theta, \lambda)$ is the regularization rule that depends on the tuning parameter λ, and sometimes an additional tuning parameter α. We often refer to the regularization rule as the *penalty*. We only consider squared loss in this paper, i.e. $L(\theta) = (Y_i - X_i'\theta)^2$, however we consider and discuss several options for choice of the regularization rule. The regularization techniques that we consider are:

- L1 penalty (LASSO)
- L2 penalty (Ridge)
- $L\frac{1}{2}$
- Elastic Net
- L2 with Custom Soft-Thresholding

Many of the regularization techniques we discuss do not have a closed analytical solution, so we will consider two algorithms for numerically arriving at a solution: Batch Gradient Descent and Pathwise Coordinate Descent. The following sections detail how to use each regularization rule with each algorithm.

For the remainder of this paper we assume that all data X and responses Y are standardized.

2 Gradient Descent

Gradient Descent algorithms work by moving in the direction that the gradient of θ points, with distance scaled by a learning rate parameter α. The algorithm continuously updates it value for the parameter vector θ until convergence. Gradients point towards local optima, so we clearly need to use convex objective functions to arrive at a gobally optimal solution in Gradient Descent. Fortunately, computing the gradient of objective functions listed above is straightforward which makes Gradient Descent fairly simple to implement. Notice that Gradient Descent works by computing the gradient; it does not itself solve an optimization problem per se; since the algorithm is only dependent on the gradient of the objective function, we can parameterize it based on the gradient. Therefore we end up with a core algorithm the form: gradient_descent($\nabla\theta, X, Y$)

3 Pathwise Coordinate Descent

While Gradient Descent operates on the entire parameter vector at once, Pathwise Coordinate Descent (PCD) concentrates on each parameter individually. Pathwise Coordinate Descent considers each parameter θ_j as its own optimization problem[1]:

$$\hat{\theta}_j = \arg\min_{\theta_j} J(\theta)$$

In each iteration of the PCD algorithm, we update each θ_j to be the solution to the individual optimization problem.

3.1 Parameterizing the PCD Algorithm

Pathwise Coordinate Descent is parameterized by the objective function J that it solves for each θ_j. However, for the universe of objective functions we consider in this paper, we can parameterize the PCD algorithm by just two nonnegative parameters we call A and B. For each objective function we consider below, we will show that the solution to optimizing each θ_j takes the form:

$$\theta_j = S\left(\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k), B\right)/A$$

Where we define[1] $S(t, B) = \text{ sign}(t)(|t| - B)_+$, and $()_+$ is the soft-thresholding operator. It follows that each algorithm reduces to the following parameterized form PATHWISE_CD(A, B). In the following sections we will show how each of the objective functions we consider fit into this parameterization.

3.2 LASSO

LASSO uses the L1 norm for a penalty:

$$\hat{\theta}_j = \arg\min_{\theta_j} \frac{1}{2}\sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{p} x_{ik}\theta_k\right)^2 + \lambda\sum_{k=1}^{p}|\theta_k|$$

Where $\lambda \geq 0$ is a tuning parameter. Let J be the objective function we want to optimize above. We can solve for θ_j by finding the stationary points of J:

$$\begin{aligned}\frac{\partial J}{\partial \theta_j} &= -\sum_{i=1}^{N} X_{ij}(Y_i - X_{ij}\theta_j - \sum_{k\neq j}^{p} X_{ik}\theta_k) + \lambda \text{ sign}(\theta_j) \\ &= -\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k) + \sum_{i=1}^{N} X_{ij}^2\theta_j + \lambda \text{ sign}(\theta_j) \\ &= -\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k) + (N-1)\theta_j + \lambda \text{ sign}(\theta_j)\end{aligned} \tag{1}$$

The last step follows from the fact that our data is standardized, and thus $\sum_{i=1}^{N} X_{ij}^2 = N - 1$.

Set $\frac{\partial J}{\partial \theta_j} = 0$ and, for convenience, let $Q = \sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k)$ and and we get:

$$(N-1)\theta_j + \lambda \text{ sign}(\theta_j) = Q$$

To solve for θ_j notice that when $\theta_j \geq 0$, $\theta_j = \frac{Q-\lambda}{N-1}$ so $Q \geq \lambda$. However when $\theta_j \leq 0$, $\theta_j = \frac{Q+\lambda}{N-1}$ so $Q \leq -\lambda$. Notice the following two properties:

(1) $|Q| \geq \lambda$

(2) $\theta_j \geq 0$ iff $Q \geq 0$ and $\theta_j \leq 0$ iff $Q \leq 0$

Therefore, we can write the solution for θ_j as:

$$\theta_j = \frac{\text{sign}(Q)(|Q| - \lambda)_+}{N-1}$$

Where the soft-thresholding operator guarantees property (1), while multiplying by $\text{sign}(Q)$ makes use of (and guarantees) property (2). Recall from Section 2 that $S(t, B) = \text{sign}(t)(|t| - B)_+$, then our final solution is:

$$\theta_j = S\left(\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k), \lambda\right)/(N-1)$$

Notice this yields a solution nearly identical to Elements of Statistical Learning[1]; the difference is that we divide by $N-1$ and we believe this is likely a typo in the book.

Notice that the soft-thresholding parameter is λ; this explains why larger λ values causes LASSO to drive parameter values to zero. It is now clear that LASSO performs subset selection due to the choice of regularization penalty *and* algorithm.

Finally, it is clear that the parameterization for LASSO is PATHWISE_CD$(N-1, \lambda)$.

3.3 Elastic Net

The Elastic Net uses the tuning parameter $\alpha \in [0, 1]$ to compromise between Ridge and Lasso penalization. Ridge penalizes the L2 norm, so Elastic Net solves the following optimization problem:

$$\hat{\theta}_j = \arg\min_{\theta_j} \frac{1}{2}\sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{p} x_{ik}\theta_k\right)^2 + \lambda\sum_{k=1}^{p}\left(\alpha\theta_k^2 + (1-\alpha)|\theta_k|\right)$$

This math is very similar to LASSO above so we have left the details to the appendix. Our final solution is:

$$\theta_j = S\left(\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k), B\right)/A$$

Where $A = (N-1) + 2\lambda\alpha$, and $B = \lambda(1-\alpha)$. This equation is already in our desired form, and hence the parameterization is PATHWISE_CD$(N - 1 + 2\lambda\alpha, \lambda(1-\alpha))$.

3.4 Ridge

Ridge Regularization has a closed form analytical solution so it does not require an algorithm. Nevertheless, we will discuss Ridge in the context of PCD to realize why the L2 penalty does not perform subset selection while the L1 does. We can simply use our solution for the Elastic Net with $\alpha = 1$ for the Ridge solution. This implies that $A = (N-1) + 2\lambda$ and $B = 0$, yielding:

$$\theta_j = S\left(\sum_{i=1}^{N} X_{ij}(Y_i - \sum_{k\neq j}^{p} X_{ik}\theta_k), 0\right)/(N - 1 + 2\lambda)$$

With parameterization PATHWISE_CD$(N - 1 + 2\lambda, 0)$. Notice that the soft-thresholding parameter is zero; this implies that *no* soft-thresholding will be applied for Ridge, regardless of the parameter λ. This explains why Ridge tends to shrink parameter values but doesn't drive them towards zero.

3.5 Custom Shrinkage Methods

We have found above that LASSO uses λ exclusively for linear shrinkage and soft-thresholding, while Ridge uses λ exclusively for inverse shrinkage. The Elastic Net implicitly combines both shrinkage techniques using two tuning parameters; here we propose three "custom" ideas for combining the shrinkage types using a single tuning parameter. Our hope is that using a single tuning parameter makes it easier to find the optimal bias-variance tradeoff using cross-validation. These custom ideas, inspired by Nearest Shrunken Centroids, apply soft-thresholding to the Ridge solution in the form: PATHWISE_CD($N - 1 + 2\lambda, STP(\lambda)$), where STP is the soft-thresholding penalty function. The three custom ideas we will explore are:

1. Ridge with Soft-Thresholding: PATHWISE_CD($N - 1 + 2\lambda, \lambda$)
2. Ridge with Quadratic Soft-Thresholding: PATHWISE_CD($N - 1 + 2\lambda, \lambda^2$)
3. Ridge with Exponential Soft-Thresholding: PATHWISE_CD($N - 1 + 2\lambda, e^\lambda - 1$)

4 Experimental Design

In order to compare each regularization technique, we generated two data sets on which we trained and tested our suite of algorithms. Our models include 27 individual parameters, 26 of which are continuous and one of which is categorical (month). The first model includes four orthogonal design variables:

$$m_{ortho}(X) = 2X_1 + 3X_7^2 + 4X_8 + 2X_{37} + \epsilon_{ortho}$$

Where $\epsilon_{ortho} \sim \mathcal{N}(0, 4)$. Our second model includes two design variables that are each correlated with two parameters:

$$m_{corr}(X) = 2X_1^3 I_{Dec}(X_{26}) + 3X_{23}X_{24} + \epsilon_{corr}$$

Where $\epsilon_{corr} \sim \mathcal{N}(0, 1)$, and the indicator I_{Dec} is one if the given argument is "Dec" and zero otherwise. For both data sets we generated four sets of samples:

1. "$n < p$": a training data set where the number of samples is smaller than the number of parameters. We chose 20 samples.
2. "$n = p$": a training data set where the number of samples is approximately equal to the number of parameters We chose 40 samples even though we have 38 parameters. We use 5-fold cross-valdiation in which we require the number of samples to be divisble by 5; hence the choice of 40 samples.
3. "$n > p$": a training data set where the number of samples exceeds the number of parameters. We chose 80 samples; we purposely chose a number of samples that exceeds the number of parameters, but is still relatively small.
4. "Test": a testing data set with a large number of samples for estimating the Mean Squared Error (MSE) of the fitted model. We chose 10,000 samples.

For each data set, we trained each algorithm using a design matrix that is linear in each of the 26 continuous variables, and included 12 extra indicator variables for the categorical month variable. We chose optimal tuning parameters for each algorithm using 5-fold cross-validation and then computed the test MSE on the "Test" data set. Our results are included in the following section.

5 Results

We have summarized the results of our experiment in the figures below. In addition to the Test MSE and parameters that each model selected, we have included the number of correct and incorrect parameter selections, as well as a comparison of each algorithm to the sklearn library implementation for LASSO. The "MSE Diff" column subtracts each algorithm's MSE from the library MSE; the "# Correct Diff" column subtracts the number of correct paramters for each algorithm from the number of correct parameters that the library predicts; the "# Incorrect Diff" column subtracts the number of incorrect paramters for each algorithm from the

number of incorrect parameters that the library predicts. Judging by lowest MSE, there is no clear algorithm that emerges as the best. All algorithms perform very similarly in the correlated data set, however Ridge with Soft-Thresholding does perform best for "$n < p$", the library performs best for "$n = p$", and Ridge with Soft-Thresholding is tied with Ridge with Quadratic Soft-Thresholding for best for "$n > p$". Judging by subset selection, LASSO with Batch Gradient Descent clearly performs the worst in each case; this algorithm does not inherently perform subset selection so this is expected. Disregarding Gradient Descent, our implementations mostly perform better at subset selection than the library. Ridge with Exponential Soft-Thresholding performs the best overall, only misclassifying a parameter in one trial and always choosing a better parameter subset than the library.

N<p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	7.22	1, 4, 8	2	1	0.93	0	3
Ridge w/ST	7.08	1, 4, 8, 11	2	2	1.07	0	2
Ridge w/Quad ST	8.8	1, 8	2	0	-0.65	0	4
Ridge w/Exp ST	9.64	8	1	0	-1.49	1	4
Elastic Net	9.5	8	1	0	-1.35	1	4
LASSO w/BGD	10.36	All	4	34	-2.21	-2	-30
sklearn.linear_model	8.15	1, 4, 5, 8, 9, 11	2	4	0	0	0

N=p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	8.2	8	1	0	-1.27	1	3
Ridge w/ST	7.84	1, 8	2	0	-0.91	0	3
Ridge w/Quad ST	7.85	1, 8	2	0	-0.92	0	3
Ridge w/Exp ST	7.99	1, 8	2	0	-1.06	0	3
Elastic Net	7.48	1, 8	2	0	-0.55	0	3
LASSO w/BGD	7.76	All	4	34	-0.83	-2	-31
sklearn.linear_model	6.93	1, 2, 6, 8, 14	2	3	0	0	0

N>p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	7.35	1, 7, 8	3	0	0.2	0	3
Ridge w/ST	7.18	1, 7, 8	3	0	0.37	0	3
Ridge w/Quad ST	7.18	1, 7, 8	3	0	0.37	0	3
Ridge w/Exp ST	7.44	1, 7, 8	3	0	0.11	0	3
Elastic Net	7.78	1, 4, 7, 8	3	1	-0.23	0	2
LASSO w/BGD	7.95	All	4	34	-0.4	-1	-31
sklearn.linear_model	7.55	1, 4, 7, 8, 13, 20	3	3	0	0	0

Figure 1: Orthogonal Dataset Results

N<p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	1.42	None	0	0	0	0	0
Ridge w/ST	1.42	None	0	0	0	0	0
Ridge w/Quad ST	1.42	None	0	0	0	0	0
Ridge w/Exp ST	1.42	None	0	0	0	0	0
Elastic Net	1.42	None	0	0	0	0	0
LASSO w/BGD	1.42	All	5	33	0	-5	-33
sklearn.linear_model	1.42	None	0	0	0	0	0

N=p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	1.53	14, 18, 21, 25	0	4	-0.11	0	-3
Ridge w/ST	1.5	14, 18, 21	0	3	-0.08	0	-2
Ridge w/Quad ST	1.43	14	0	1	-0.01	0	0
Ridge w/Exp ST	1.44	14	0	1	-0.02	0	0
Elastic Net	1.64	2, 7, 14, 18, 21, 24, 25	2	5	-0.22	-2	-4
LASSO w/BGD	1.42	All	5	33	0	-5	-32
sklearn.linear_model	1.42	14	0	1	0	0	0

N>p

Algorithm	Test MSE	Parameters Selected	# Correct	# Incorrect	MSE Diff	# Correct Diff	# Incorrect Diff
LASSO	1.41	3	0	1	0.01	0	-1
Ridge w/ST	1.43	3, 14, 24	1	2	-0.01	-1	-2
Ridge w/Quad ST	1.47	3, 4, 8, 14, 24	1	4	-0.05	-1	-4
Ridge w/Exp ST	1.42	None	0	0	0	0	0
Elastic Net	1.42	3, 24	1	1	0	-1	-1
LASSO w/BGD	1.42	All	5	33	0	-5	-33
sklearn.linear_model	1.42	None	0	0	0	0	0

Figure 2: Correlated Dataset Results

6 Discussion

First and foremost, it is clear that regularization is important when modeling data sets that have a small number of samples compared to parameters. We noticed our algorithms performed far better as we applied regularization to stabilize our solutions; this is made clear by the range lambda values in the charts in the appendix; we simply did not consider lamba values of zero because numerical overflow happens in nearly every case. Second, we found that implementing Pathwise Coordinate Descent is far tougher than we initially thought; Batch Gradient Descent and Cross-Validation were not as tough, but still required several hours fine tuning and debugging. Despite the challenges we faced, our results show that our implementations perform similarly to the sklearn library implementation of LASSO, with our implementation outperforming the library in some cases. Finally, it's tough to judge whether our "custom" Ridge shrinkage methods worked well. Our results here do look moderately promising, yet our data sets our small and more investigation is needed. Furthermore, our data sets are sparse which tend to favor LASSO[2], and likely other soft-thresholding methods that we have experimented with in this paper; our analysis would benefit from more diverse models, including models that make sure of a higher percentage of the available parameters.

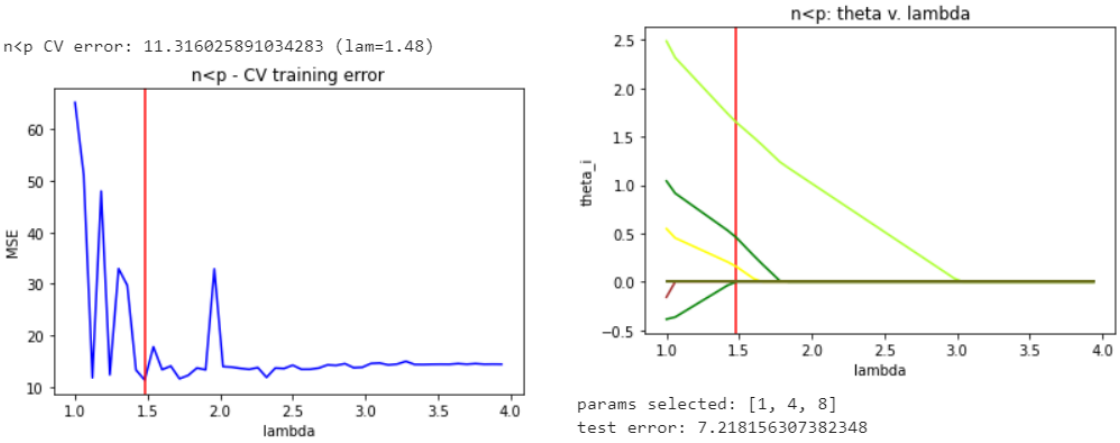
References

[1] Trevor Hastie, Robert Tibshirani, and Jerome Friedman, The Elements of Statistical Learning 2nd Edition

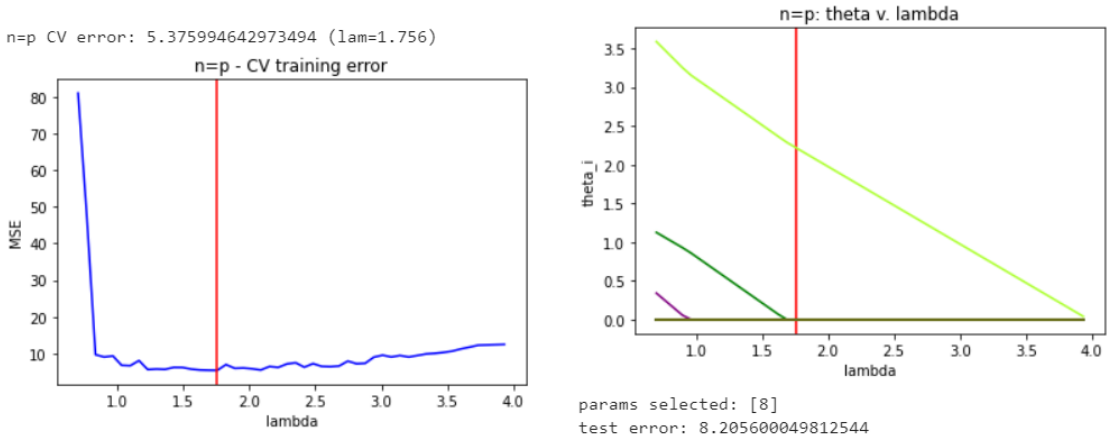
[2] Tibshirani's paper

7 Appendix

7.1 Cross-Validation Training Results

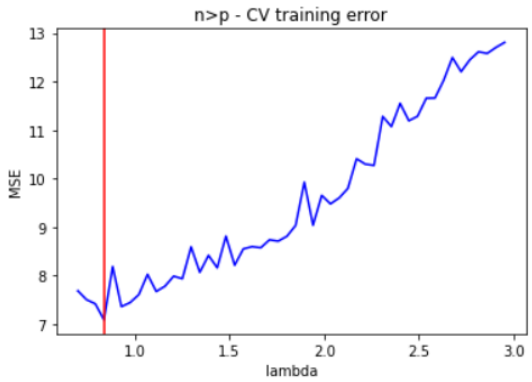


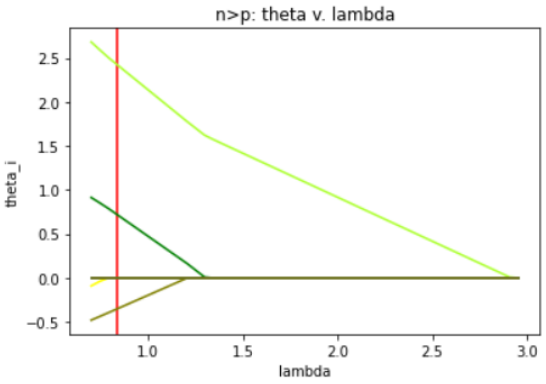
LASSO, $n < p$ orthogonal data set



LASSO, $n = p$ orthogonal data set

```
n>p CV error: 7.09161639683978 (lam=0.838)
```

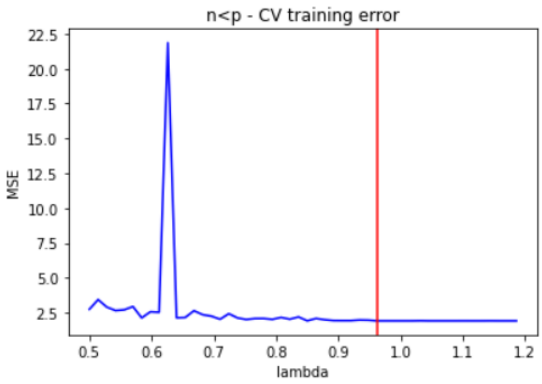


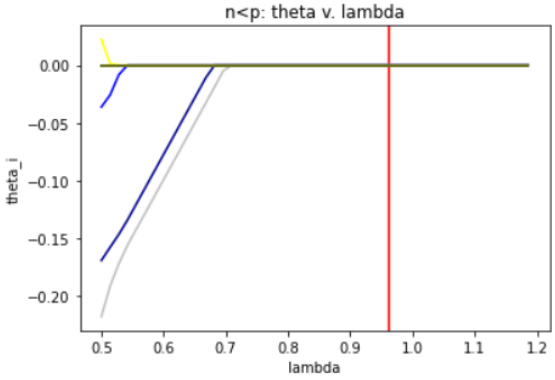



```
params selected: [1, 7, 8]
test error: 7.347932494569225
```

LASSO, $n > p$ orthogonal data set

```
n<p CV error: 1.9266625090183453 (lam=0.962)
```

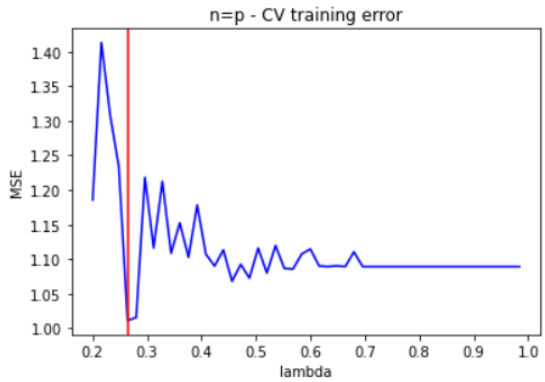


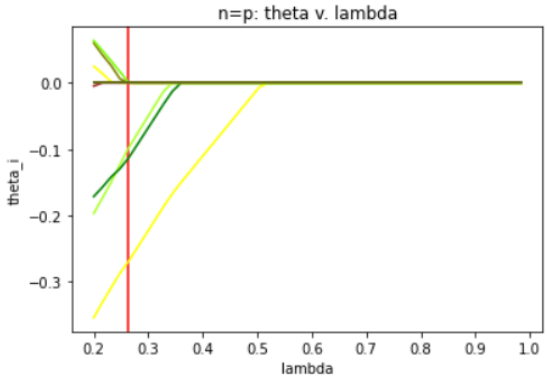



```
params selected: []
test error: 1.4160738183074826
```

LASSO, $n < p$ correlated data set

```
n=p CV error: 1.0112681644334178 (lam=0.264)
```

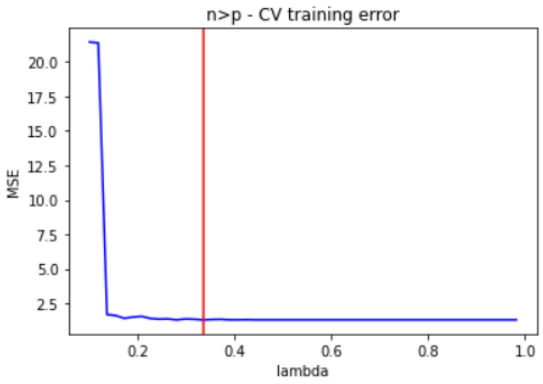


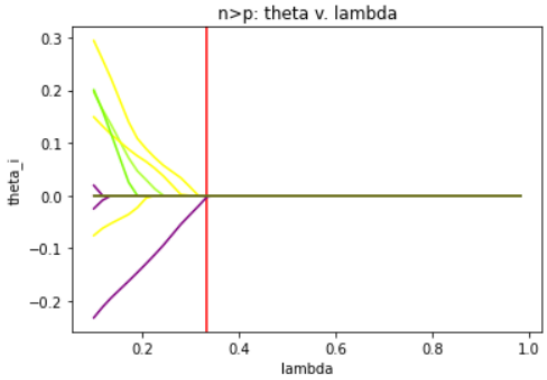



```
params selected: [14, 18, 21, 25]
test error: 1.5321562394350738
```

LASSO, $n = p$ correlated data set

```
n>p CV error: 1.3042428681808445 (lam=0.334)
```

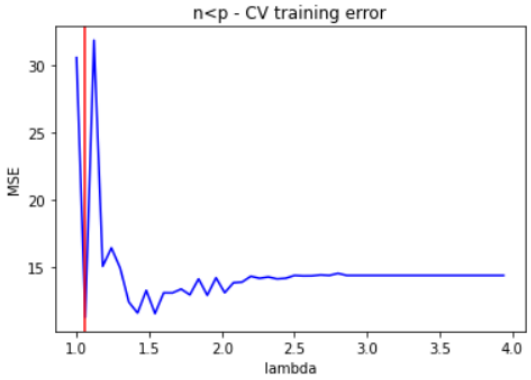


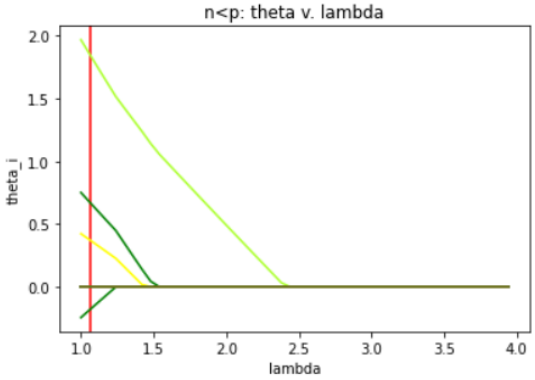



```
params selected: [3]
test error: 1.4160660460764922
```

LASSO, $n > p$ correlated data set

```
n<p CV error: 11.268002948781955 (lam=1.06)
```

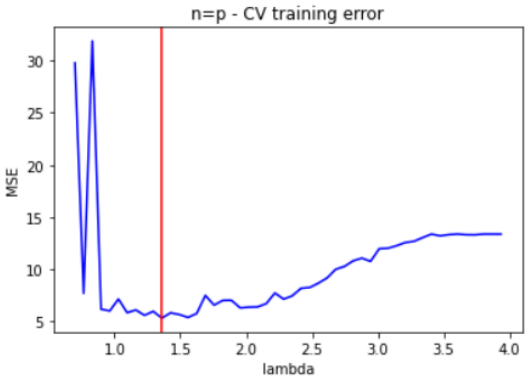


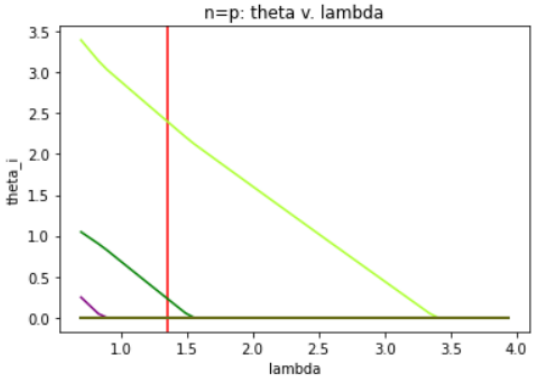



```
params selected: [1, 4, 8, 11]
test error: 7.079880200681896
```

Ridge with Soft-Thresholding, $n < p$ orthogonal data set

```
n=p CV error: 5.338765655876978 (lam=1.3599999999999999)
```

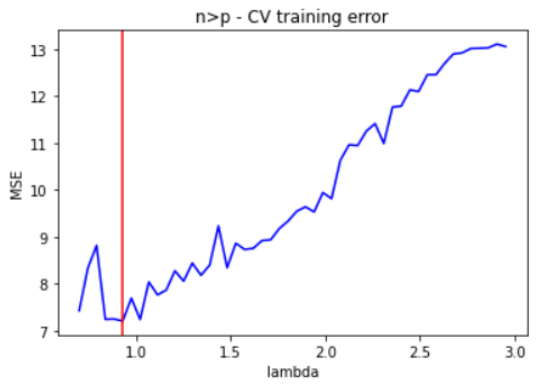


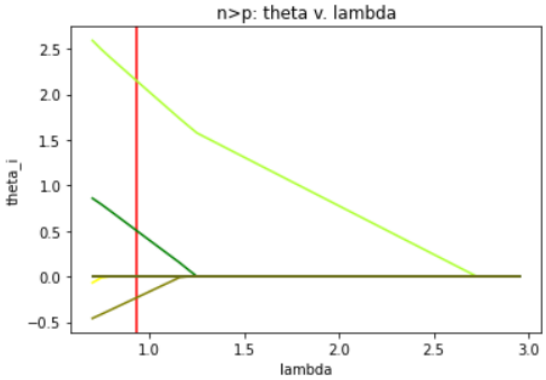



```
params selected: [1, 8]
test error: 7.844885140022174
```

Ridge with Soft-Thresholding, $n = p$ orthogonal data set

```
n>p CV error: 7.208925217062908 (lam=0.9299999999999999)
```

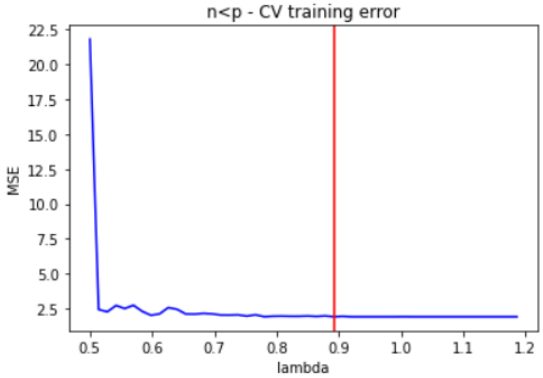


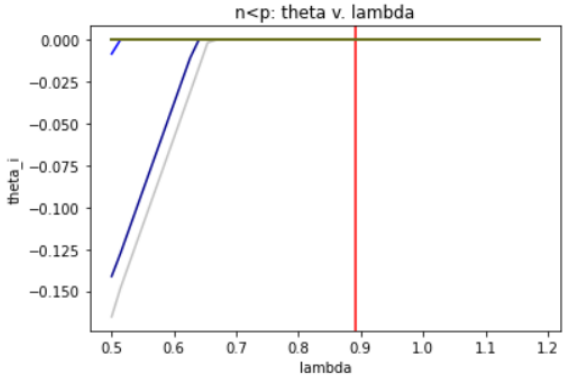



```
params selected: [1, 7, 8]
test error: 7.176750424642003
```

Ridge with Soft-Thresholding, $n > p$ orthogonal data set

```
n<p CV error: 1.9266625090183453 (lam=0.8919999999999999)
```

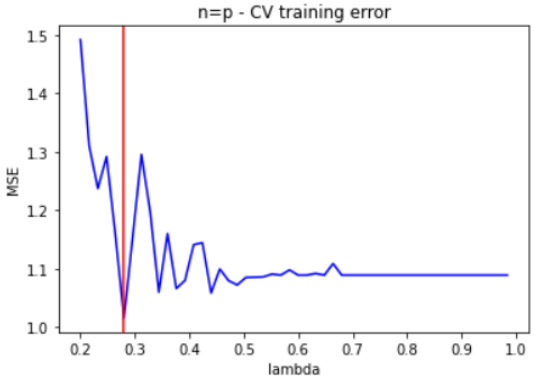


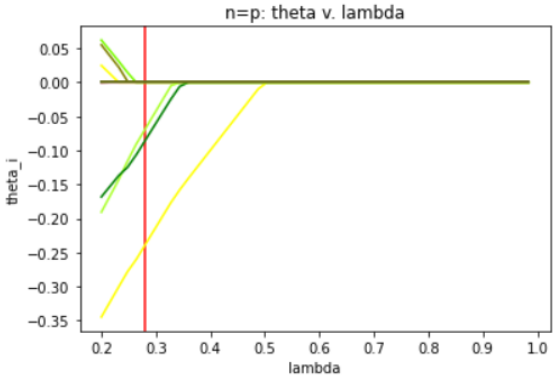



```
params selected: []
test error: 1.4160738183074826
```

Ridge with Soft-Thresholding, $n < p$ correlated data set

```
n=p CV error: 1.0158229666883483 (lam=0.28)
```

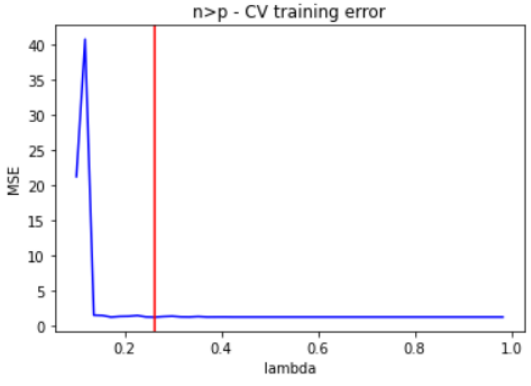


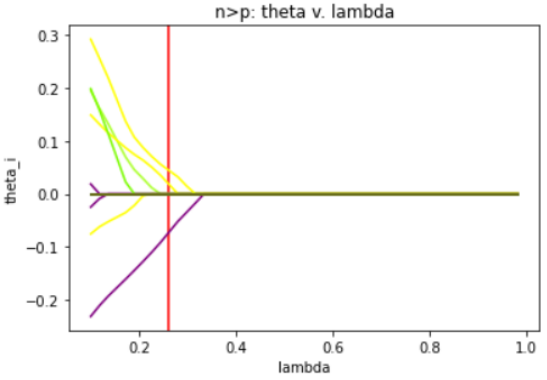



```
params selected: [14, 18, 21]
test error: 1.4999553543772113
```

Ridge with Soft-Thresholding, $n = p$ correlated data set

```
n>p CV error: 1.2908435870120456 (lam=0.262)
```

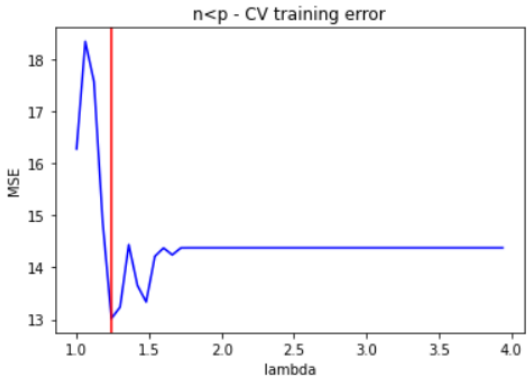


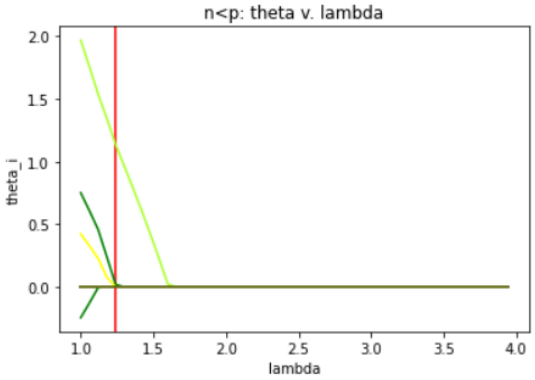



```
params selected: [3, 14, 24]
test error: 1.428218090701521
```

Ridge with Soft-Thresholding, $n > p$ correlated data set

```
n<p CV error: 13.018172016236113 (lam=1.24)
```

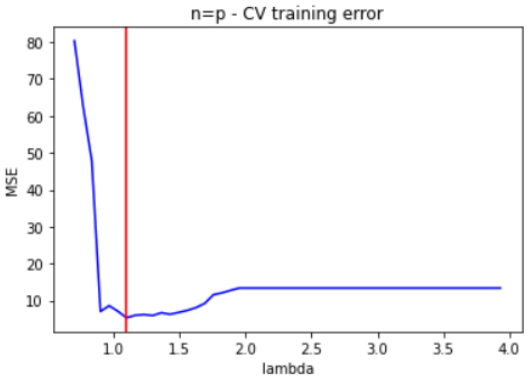


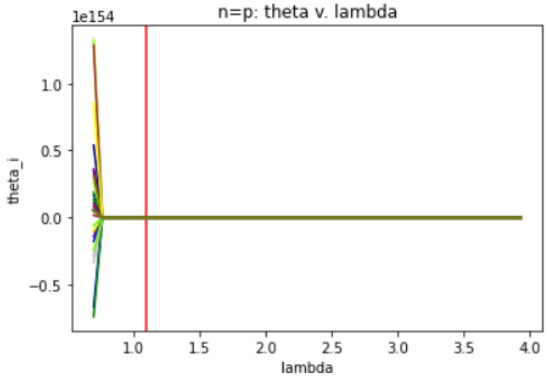



```
params selected: [1, 8]
test error: 8.802726933142381
```

Ridge with Quadratic Soft-Thresholding, $n < p$ orthogonal data set

```
n=p CV error: 5.322434291901843 (lam=1.0959999999999999)
```

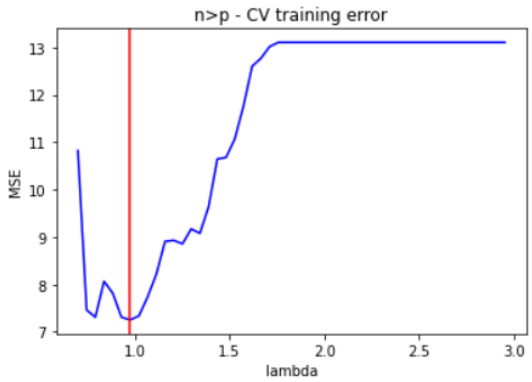


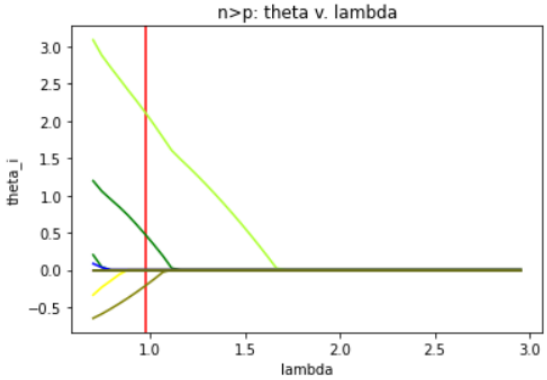



```
params selected: [1, 8]
test error: 7.847848970557611
```

Ridge with Quadratic Soft-Thresholding, $n = p$ orthogonal data set

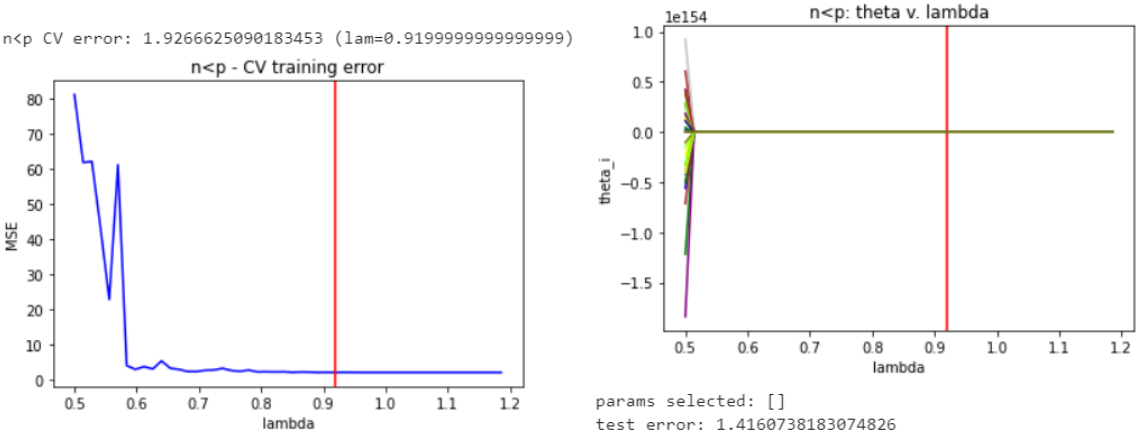
```
n>p CV error: 7.25262454726182 (lam=0.976)
```



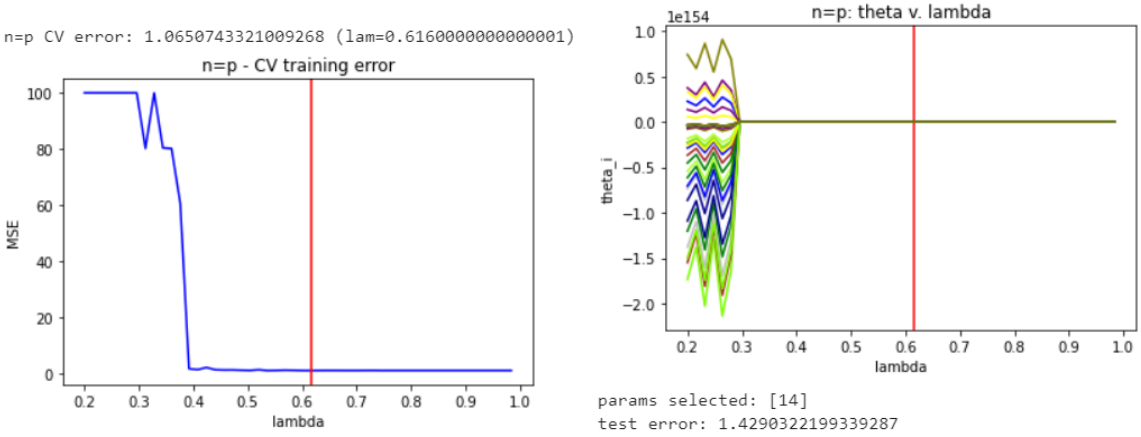



```
params selected: [1, 7, 8]
test error: 7.184188030187201
```

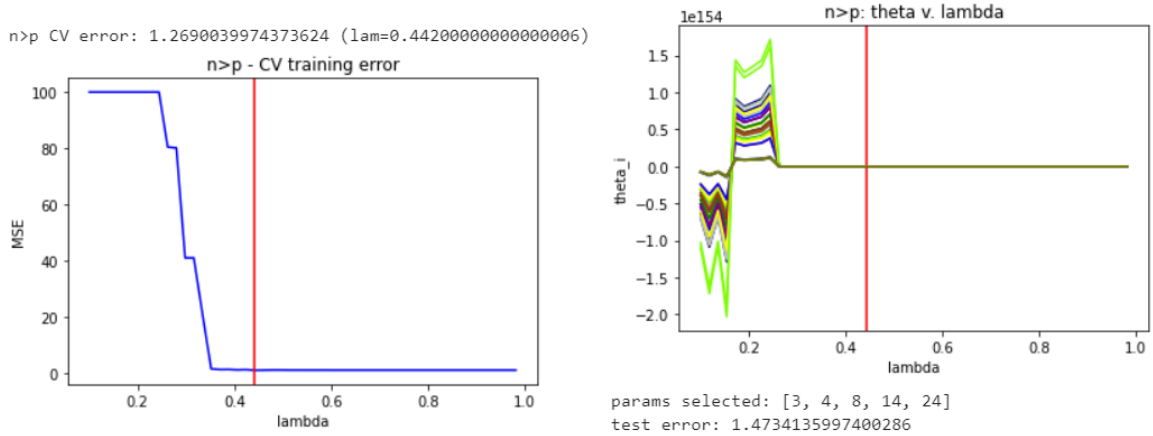
Ridge with Quadratic Soft-Thresholding, $n > p$ orthogonal data set



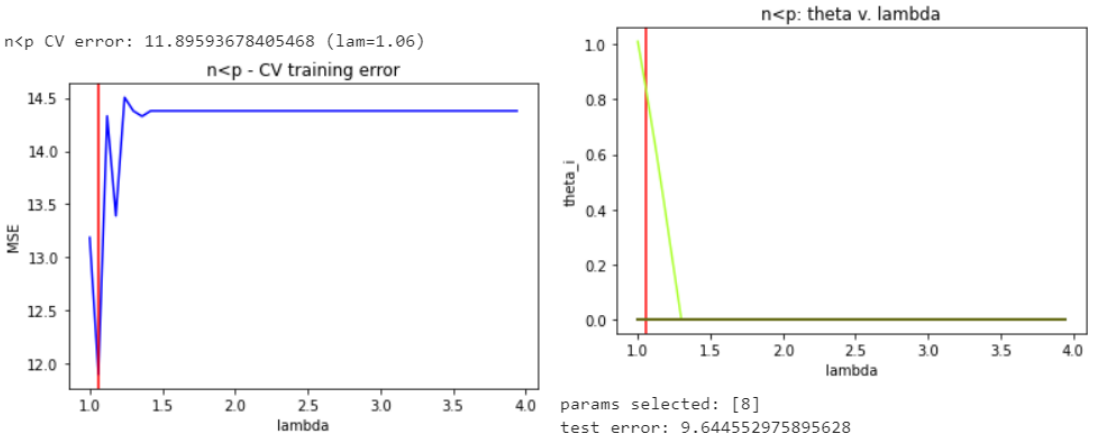
Ridge with Quadratic Soft-Thresholding, $n < p$ correlated data set



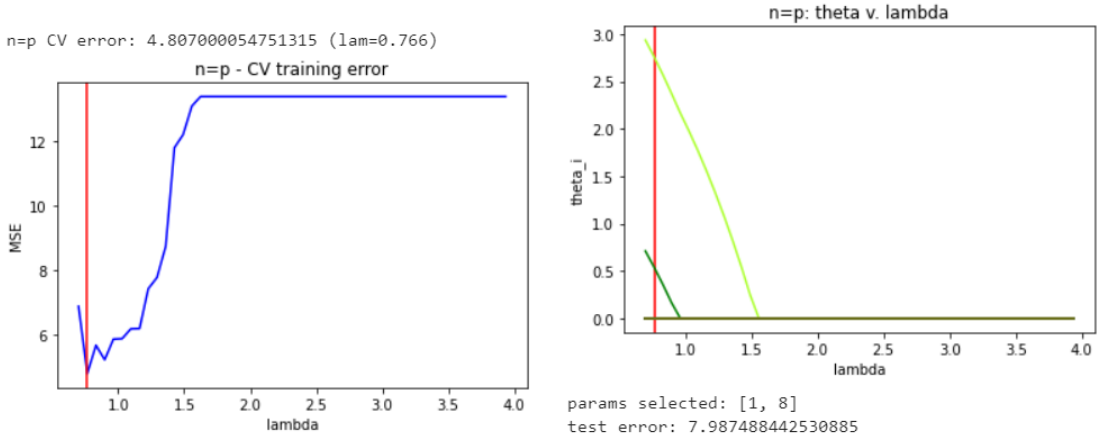
Ridge with Quadratic Soft-Thresholding, $n = p$ correlated data set



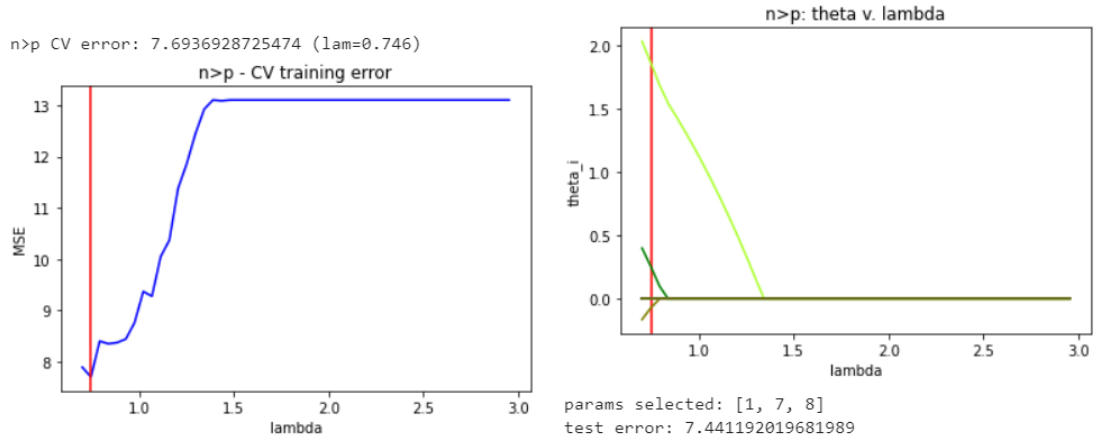
Ridge with Quadratic Soft-Thresholding, $n > p$ correlated data set



Ridge with Exponential Soft-Thresholding, $n < p$ orthogonal data set

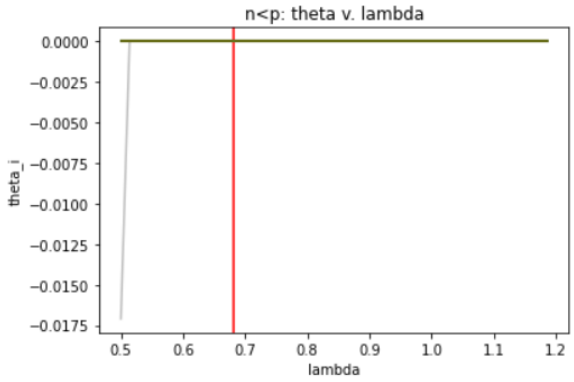


Ridge with Exponential Soft-Thresholding, $n = p$ orthogonal data set



Ridge with Exponential Soft-Thresholding, $n > p$ orthogonal data set

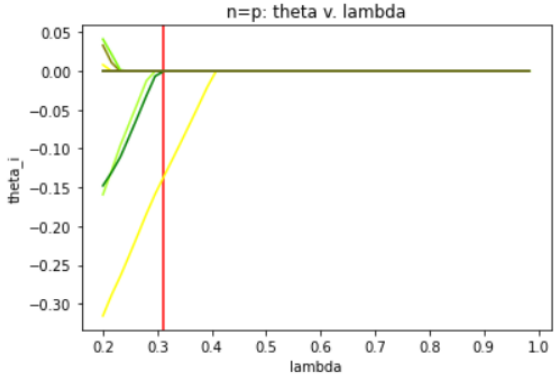
```
n<p CV error: 1.9266625090183453 (lam=0.6819999999999999)
```




```
params selected: []
test error: 1.4160738183074826
```

Ridge with Exponential Soft-Thresholding, $n < p$ correlated data set

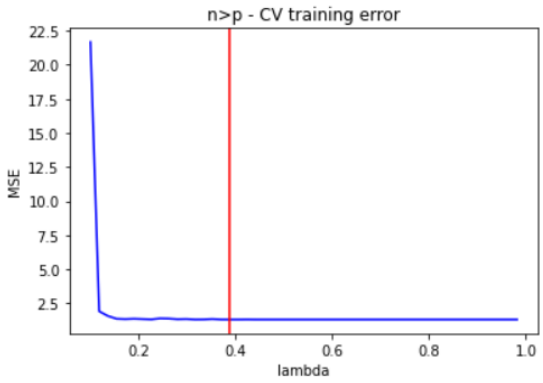
```
n=p CV error: 1.0275730126808031 (lam=0.31200000000000006)
```

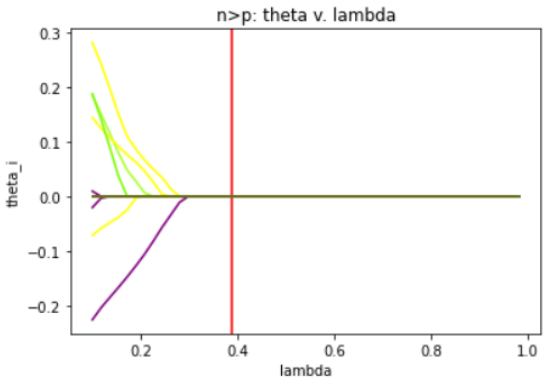



```
params selected: [14]
test error: 1.4354326964192399
```

Ridge with Exponential Soft-Thresholding, $n = p$ correlated data set

```
n>p CV error: 1.3107237739219442 (lam=0.388)
```

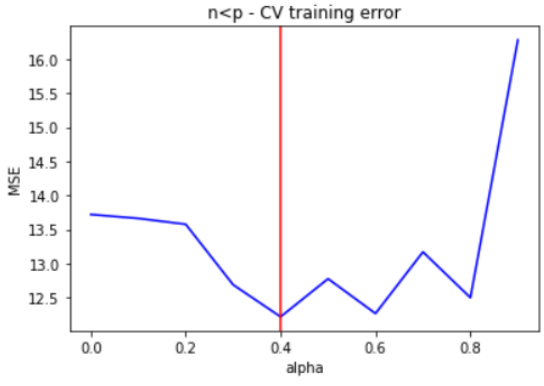


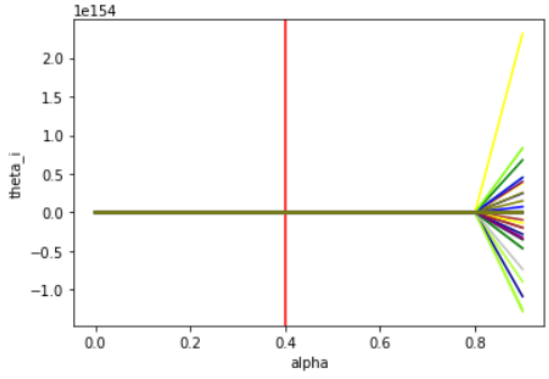



```
params selected: []
test error: 1.4160738183074826
```

Ridge with Exponential Soft-Thresholding, $n > p$ correlated data set

```
elastic net CV error: 12.217233273558842 (lam=3.0,alpha=0.4)
```

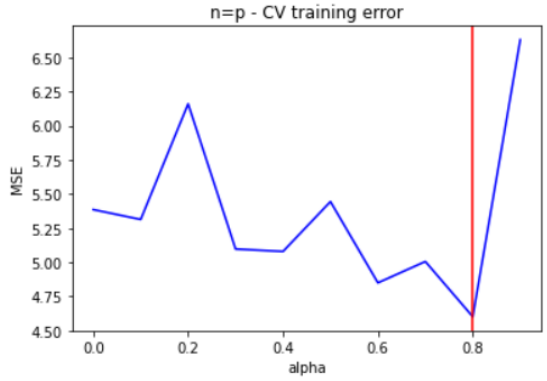


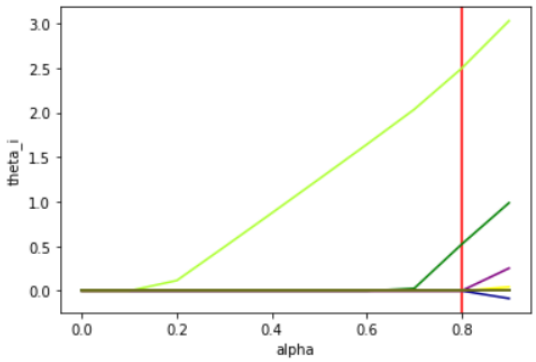



```
params selected: [8]
test error: 9.495145341090655
```

Elastic Net, $n < p$ orthogonal data set

```
elastic net CV error: 4.601332710351868 (lam=4.6,alpha=0.8)
```

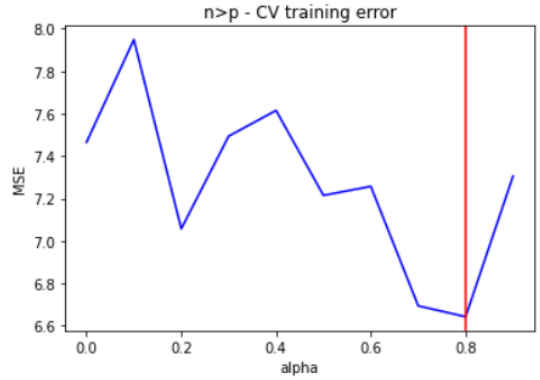


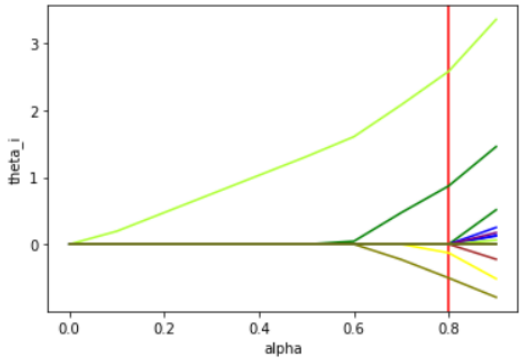



```
params selected: [1, 8]
test error: 7.4789582644839525
```

Elastic Net, $n = p$ orthogonal data set

```
elastic net CV error: 6.642628494436083 (lam=3.0,alpha=0.8)
```

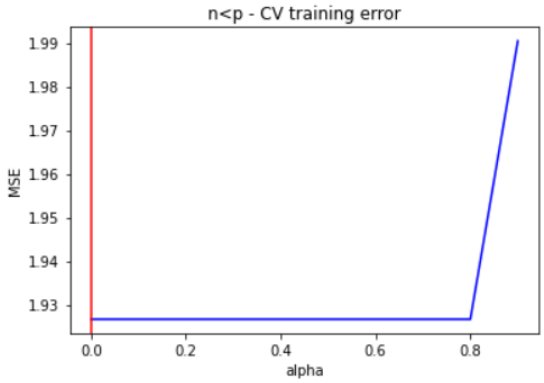


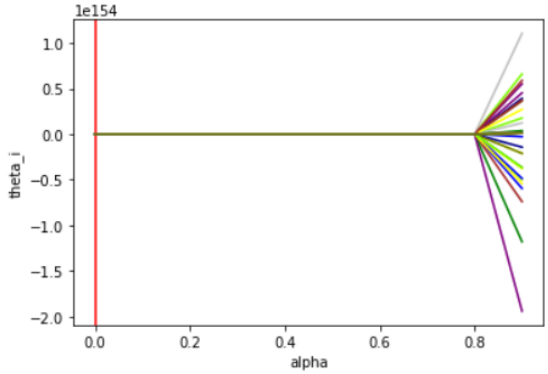



```
params selected: [1, 4, 7, 8]
test error: 7.781175508486528
```

Elastic Net, $n > p$ orthogonal data set

```
elastic net CV error: 1.9266625090183453 (lam=1.4,alpha=0.0)
```

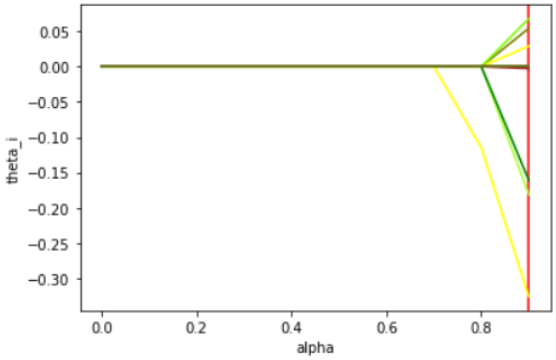





```
params selected: []
test error: 1.4160738183074826
```

Elastic Net, $n < p$ correlated data set

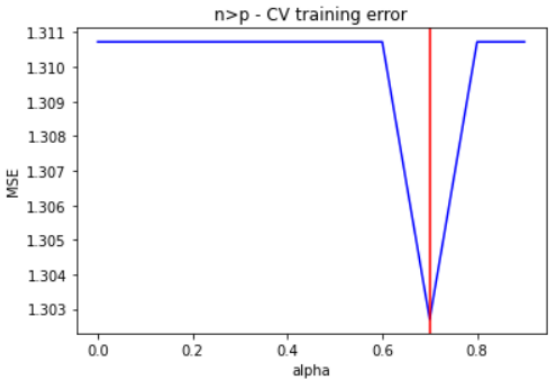
```
elastic net CV error: 1.0234769819032052 (lam=1.8,alpha=0.9)
```

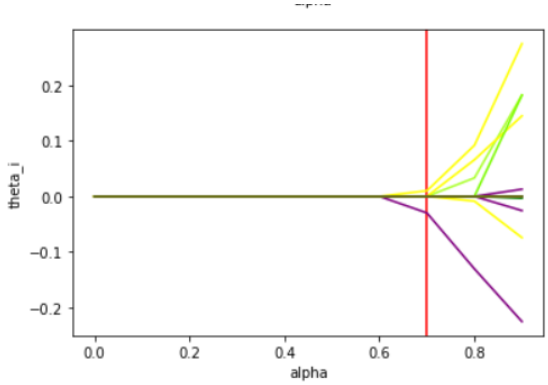



```
params selected: [2, 7, 14, 18, 21, 24, 25]
test error: 1.6433790438502822
```

Elastic Net, $n = p$ correlated data set

```
elastic net CV error: 1.3027029115677597 (lam=1.0,alpha=0.7)
```

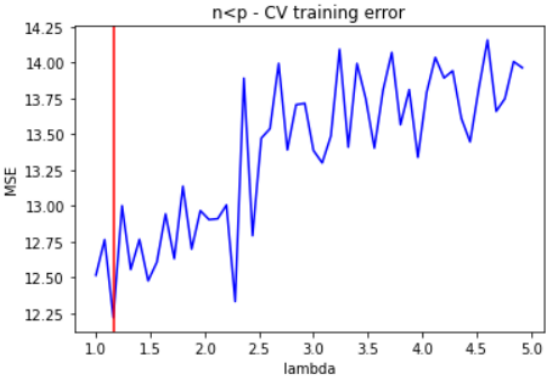


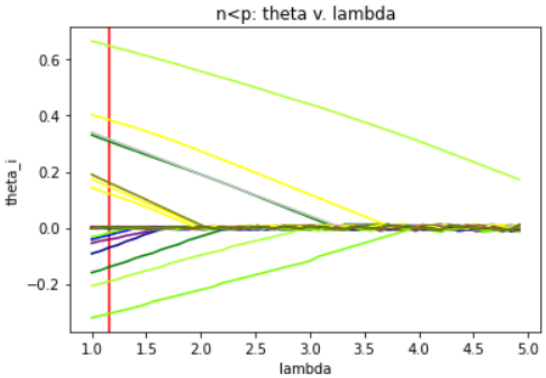



```
params selected: [3, 24]
test error: 1.4175092033116798
```

Elastic Net, $n > p$ correlated data set

```
n<p CV error: 12.22141402920315 (lam=1.16)
```

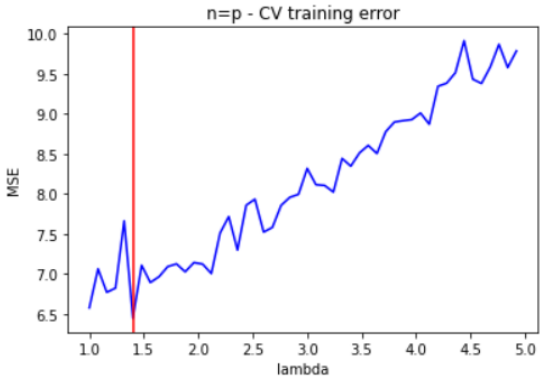


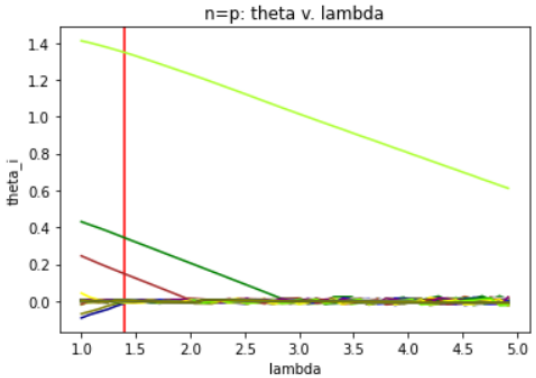



```
params selected: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1
3, 34, 35, 36, 37]
test error: 10.357946358945773
```

LASSO with Batch Gradient Descent, $n < p$ orthogonal data set

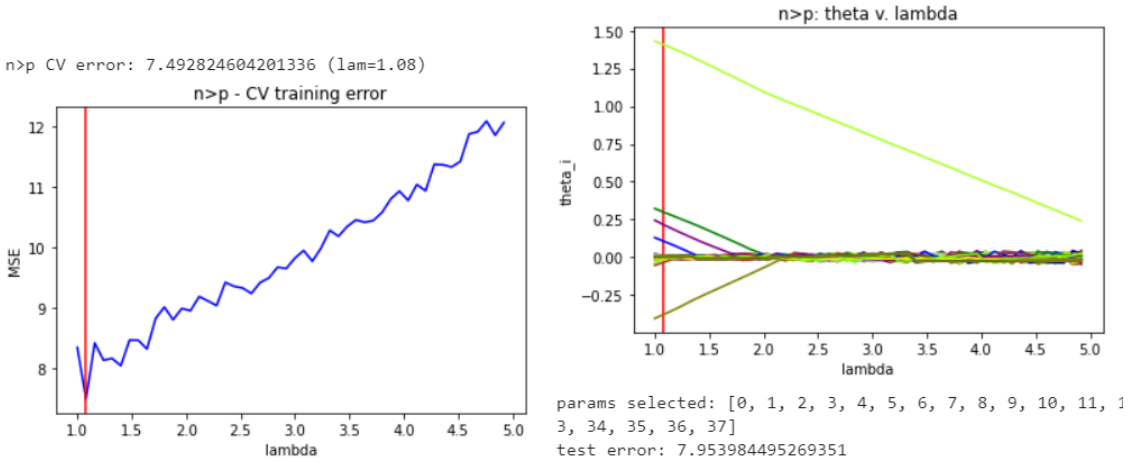
```
n=p CV error: 6.443097458137994 (lam=1.4)
```



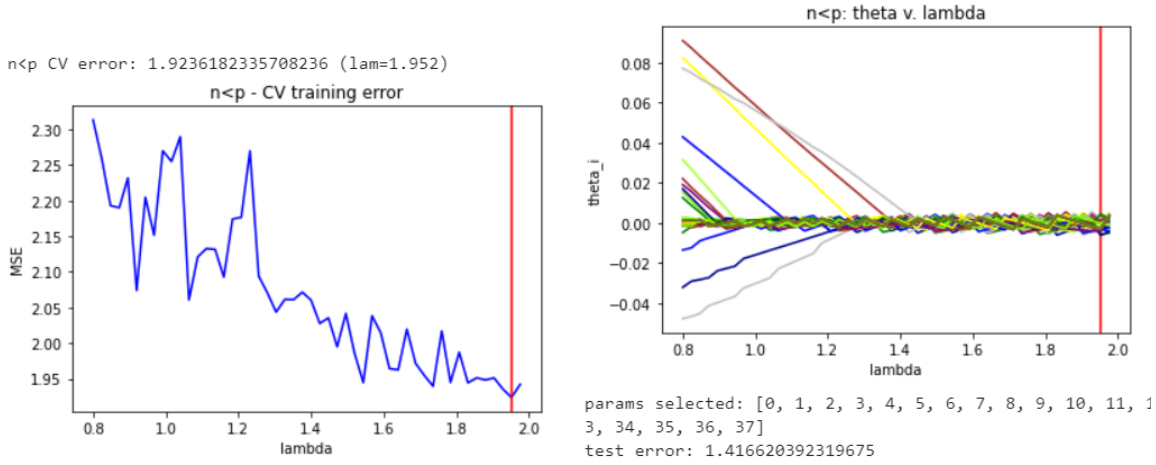



```
params selected: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
3, 34, 35, 36, 37]
test error: 7.757728526653304
```

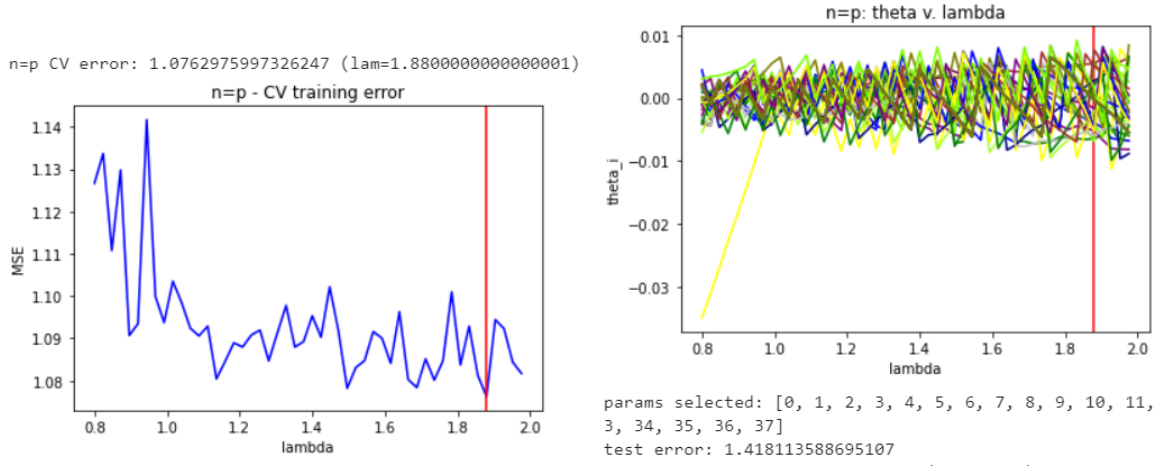
LASSO with Batch Gradient Descent, $n = p$ orthogonal data set



LASSO with Batch Gradient Descent, $n > p$ orthogonal data set



LASSO with Batch Gradient Descent, $n < p$ correlated data set



LASSO with Batch Gradient Descent, $n = p$ correlated data set

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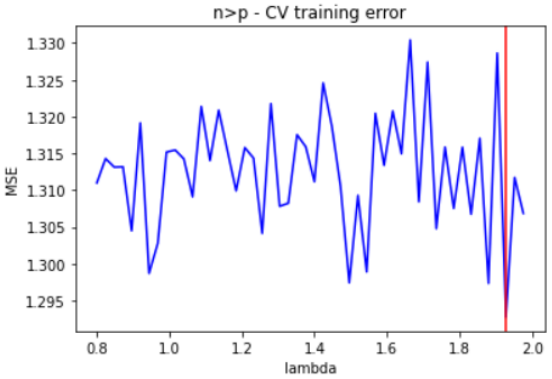
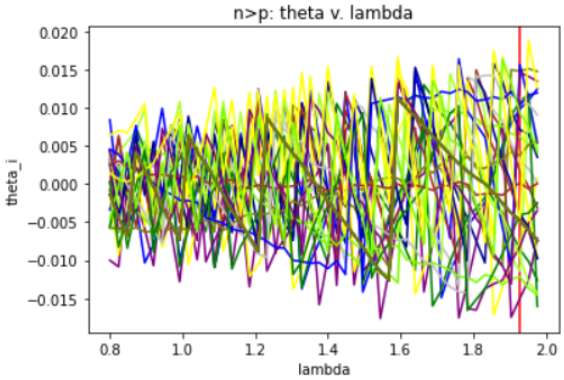
n>p CV error: 1.292746498278167 (lam=1.928)
```



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params selected: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
3, 34, 35, 36, 37]
test error: 1.4194471855500905
```

LASSO with Batch Gradient Descent, $n > p$ correlated data set