

# M-N In EPR

Ian Dardik

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Suppose that  $\Phi$  is universally quantified and in PNF while  $\Delta$  is existentially quantified and in PNF. We will assume a single sort  $P$ , and let  $\Phi$  quantify over  $m$  variables and  $\Delta$  quantify over  $n$  variables. We will write  $\Phi = \forall \hat{p}, \phi(\hat{p})$ ,  $\Delta = \exists \hat{q}, \delta(\hat{q})$ , and  $\Phi' = \forall \hat{r}, \phi(\hat{r})'$ , where  $\phi$  and  $\delta$  are non-quantified, uninterpreted, and contain no function symbols. It is the case that  $|\hat{p}| = |\hat{r}| = m$  and  $|\hat{q}| = n$ . We use different quantifier variables for  $\Phi$  and  $\Phi'$  so the variables are already standardized apart.

Then we can prove the M-N Theorem for this special case quite quickly:

$$\begin{aligned} & \Phi \wedge \Delta \rightarrow \Phi' \\ \Leftrightarrow & (\forall \hat{p}, \phi(\hat{p})) \wedge (\exists \hat{q}, \delta(\hat{q})) \rightarrow (\forall \hat{r}, \phi(\hat{r})') \\ \Leftrightarrow & \neg(\forall \hat{p}, \phi(\hat{p})) \vee \neg(\exists \hat{q}, \delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \\ \Leftrightarrow & (\exists \hat{p}, \neg\phi(\hat{p})) \vee (\forall \hat{q}, \neg\delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \\ \Leftrightarrow & (\forall \hat{q}, \neg\delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \vee (\exists \hat{p}, \neg\phi(\hat{p})) \\ \Leftrightarrow & \forall \hat{q}, \forall \hat{r}, \exists \hat{p}, \neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p}) \end{aligned}$$

In other words, we want to know if:

$$\forall \hat{q}, \forall \hat{r}, \exists \hat{p}, \neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p})$$

is valid. We can turn this into a SAT question by negating the formula, which results in:

$$\exists \hat{q}, \exists \hat{r}, \forall \hat{p}, \neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p})$$

This formula is in EPR, and according to Proposition 14.2 (Libkin, Elements of Finite Model Theory), if the formula is satisfiable then it has a model whose universe is size  $|\hat{r}| + |\hat{q}| = m + n$ .