# A Cutoff Rule For Parameterized Distributed Protocols in Prenex Normal Form

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## 1 Introduction

In this note, we consider the verification problem of a transition system $T = (I, \Delta)$ parameterized by a single sort $P$ of identical elements. We assume that a candidate inductive invariant $\Phi$ (which implies our key safety property) is given, and that both $\Delta$ and $\Phi$ are in Prenex Normal Form (PNF). We adopt the convention of [1] where $T(P)$ is the template of $T$, and $T(|P|)$ is a finite instantiation. We also will consider the prime (’) symbol to be an operator that can be recursively applied to a formula, only affecting (sticking to) state variables.

In this note, we will build several lemmas that lead to an interesting result: $\Phi(P)$ is an inductive invariant for $T(P)$ iff $\Phi(m+n)$ is an inductive invariant for $T(m+n)$, where $m$ is the number of variables that $\Phi$ quantifies over and $n$ is the number of variables that $\Delta$ quantifies over.

## 2 Finitely Instantiated Properties (FIPs)

In this section we introduce the FIP, a key tool for proving the M-N Theorem.

### 2.1 FIP Basics

**Definition 1** (FIP)**.** Let $M_\Phi$ and $M_\Delta$ be the respective matrices of $\Phi$ and $\Delta$. A Finitely Instantiated Property (FIP) of $[\Phi \wedge \Delta](|P|)$ is a formula $(M_\Phi \wedge M_\Delta)[v_i \mapsto e_i]$, where each free variable $v_i$ has been substituted for a concrete element $e_i \in P$.

**Example 1.** Let $\Phi = \forall p, q \in P, M_\Phi(p, q)$ and $\Delta = \exists p \in P, M_\Delta(p)$. Then $M_\Phi(1, 3) \wedge M_\Delta(2)$ and $M_\Phi(1, 1) \wedge M_\Delta(1)$ are both FIPs of $[\Phi \wedge \Delta](3)$ (i.e. for the case when $P = \{1, 2, 3\}$). □

*Remark* 1. We will often write a FIPs of $[\Phi \wedge \Delta](|P|)$ in the abstract as $\phi \wedge \delta$. We specifically choose “$\phi \wedge \delta$” to show the syntactic correspondence of $\phi$ to $\Phi$ and $\delta$ to $\Delta$. In other words, if $M_\Phi$ is the matrix of $\Phi$ and $M_\Delta$ is the matrix of $\Delta$, then it is always the case that $\phi = M_\Phi[v_i \mapsto e_i]$ and $\delta = M_\Delta[v_i \mapsto e_i]$.

**Example 2.** Let $\phi \wedge \delta = M_\Phi(1, 3) \wedge M_\Delta(2)$ be a FIP of $[\Phi \wedge \Delta](3)$. By Remark 1, $\phi = M_\Phi(1, 3)$ and $\delta = M_\Delta(2)$. □

### 2.2 Protocol Assumptions

Now that we have formally defined FIPs, we can describe our requirement that “the elements of sort $P$ are identical” in precise terms.

**Assumption 1** (Sort Elements Are Identical)**.** Let $\phi \wedge \delta$ be an arbitrary FIP of $[\Phi \wedge \Delta](|P|)$ and $g : P \to P$ be a bijective function. Then we assume:

$$\phi \leftrightarrow \phi[P \mapsto g(P)]$$

and

$$\phi \wedge \delta \leftrightarrow (\phi \wedge \delta)[P \mapsto g(P)]$$

Note that $(\phi \wedge \delta)[P \mapsto g(P)]$ is said to be a *permutation* of $\phi \wedge \delta$.

**Example 3.** Let $\Phi = \forall p \in P, M_\Phi(p)$ and $\Delta = \exists p, q \in P, M_\Delta(p, q)$. Consider $M_\Phi(2) \wedge M_\Delta(1, 3)$ and $M_\Phi(3) \wedge M_\Delta(1, 2)$ which are both FIPs of $[\Phi \wedge \Delta](3)$. Let $g$ be the permutation:

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Then these two FIPs are permutations of each other because

$$g(M_\Phi(2) \wedge M_\Delta(1, 3)) = M_\Phi(3) \wedge M_\Delta(1, 2)$$

and

$$M_\Phi(2) \wedge M_\Delta(1, 3) = g(M_\Phi(3) \wedge M_\Delta(1, 2))$$

□

Because each sort element is identical, we can make the following assumption without loss of generality:

**Assumption 2.** For any sort $P$, $P = \{1, ..., |P|\}$.

This leads to the trivial result:

**Lemma 1.** Let $P$ and $Q$ be sorts, then $|P| \leq |Q| \leftrightarrow P \subseteq Q$.

*Proof.* Suppose $|P| \leq |Q|$. Then by Assumption 2:

$$P = \{1, ..., |P|\} \subseteq \{1, ..., |Q|\} = Q$$

Now suppose that $P \subseteq Q$. Then by Assumption 2:

$$\{1, ..., |P|\} = P \subseteq Q = \{1, ..., |Q|\}$$

Hence it follows that $|P| \leq |Q|$. □

## 2.3 FIP Equivalency

In this section we introduce the notion of *equivalency* for FIPs and prove some useful lemmas.

**Definition 2** (Equivalent)**.** Let $\phi_1 \wedge \delta_1$ and $\phi_2 \wedge \delta_2$ be FIPs, then $\phi_1 \wedge \delta_1 \equiv \phi_2 \wedge \delta_2$ iff $\phi_2 \wedge \delta_2$ is a permutation of $\phi_1 \wedge \delta_1$. In this case, $\phi_1 \wedge \delta_1$ and $\phi_2 \wedge \delta_2$ are said to be *equivalent*.

**Example 4.** Let $F_1 = M_\Phi(1, 2) \wedge M_\Delta(1)$, $F_2 = M_\Phi(2, 3) \wedge M_\Delta(2)$ and $F_3 = M_\Phi(2, 2) \wedge M_\Delta(2)$ be FIPs of $[\Phi \wedge \Delta](3)$. Let $g = (1\ 2\ 3)$ be a permutation (using cycle notation) on $P$. Then $F_1 \equiv F_2$ because $g(F_1) = F_2$, however, $F_3$ is a permutation of neither $F_1$ nor $F_2$ and hence is not equivalent to either. □

The notion of equivalency is important because it partitions a formula $\Phi \wedge \Delta$ into distinct classes of FIPs. In the example above, $F_1$ and $F_2$ describe the same class of properties/actions because each element of $P$ is interchangeable for one another. We now present several basic lemmas about FIPs.

**Lemma 2** (Equivalency Is Commutative)**.** Let $\phi_1 \wedge \delta_1$ and $\phi_2 \wedge \delta_2$ be FIPs for $[\Phi \wedge \Delta](|P|)$, then

$$(\phi_1 \wedge \delta_1 \equiv \phi_2 \wedge \delta_2) \leftrightarrow (\phi_2 \wedge \delta_2 \equiv \phi_1 \wedge \delta_1)$$

*Proof.* Suppose that $\phi_1 \wedge \delta_1 \equiv \phi_2 \wedge \delta_2$. Then there exists a permutation $g$ such that $g(\phi_1 \wedge \delta_1) = \phi_2 \wedge \delta_2$. Permutations are bijective and hence are invertible. Then

$$\phi_1 \wedge \delta_1 = g^{-1}(g(\phi_1 \wedge \delta_1)) = g^{-1}(\phi_2 \wedge \delta_2)$$

But $g^{-1}$ is a permutation itself, and hence $\phi_1 \wedge \delta_1$ is a permutation of $\phi_2 \wedge \delta_2$.

We omit the proof in the other direction because the argument is nearly identical. □

**Lemma 3.** Let $\phi_1 \wedge \delta_1$ and $\phi_2 \wedge \delta_2$ both be FIPs of $[\Phi \wedge \Delta](|P|)$. Then:

$$((\phi_1 \wedge \delta_1) \equiv (\phi_2 \wedge \delta_2)) \rightarrow ((\phi_1 \equiv \phi_2) \wedge (\delta_1 \equiv \delta_2))$$

*Proof.* Suppose $(\phi_1 \wedge \delta_1) \equiv (\phi_2 \wedge \delta_2)$. Then there exists a permutation $g$ such that $g(\phi_1 \wedge \delta_1) = \phi_2 \wedge \delta_2$. But:

$$g(\phi_1) \wedge g(\delta_1) = g(\phi_1 \wedge \delta_1) = \phi_2 \wedge \delta_2$$

From Remark 1, we see that $\phi_1$ and $\phi_2$ are identical up to a finite instantiation, and hence $g(\phi_1) = \phi_2$; a similar argument shows that $g(\delta_1) = \delta_2$. □

**Lemma 4.** Let $\phi_1$ and $\phi_2$ be quantifier-free properties parameterized by the sort $P$. Then:

$$(\phi_1 \equiv \phi_2) \leftrightarrow (\phi_1' \equiv \phi_2')$$

*Proof.* Suppose that $\phi_1 \equiv \phi_2$. Then there exists a permutation $g$ such that $g(\phi_1) = \phi_2$. Recall that the prime operator only affects state variables, and not $P$ or its elements; on the other hand, $g$ can only affect the elements of $P$. Thus it is the case that $g(\phi_1)' = g(\phi_1')$. We now see:

$$\phi_2' = g(\phi_1)' = g(\phi_1')$$

Showing that $\phi_1' \equiv \phi_2'$ by definition. We omit the proof in the other direction since it is nearly identical. □

**Lemma 5.** Let $\phi_1$ and $\phi_2$ be quantifier-free properties parameterized by the sort $P$. Suppose that $\phi_1 \equiv \phi_2$. Then:

$$\phi_1 \leftrightarrow \phi_2$$

*Proof.* Because $\phi_1 \equiv \phi_2$, there exists a permutation $g$ such that $g(\phi_1) = \phi_2$. However $g$ is bijective, and hence by Assumption 1:

$$\phi_1 \leftrightarrow (\phi_1)[P \mapsto g(P)] = \phi_2$$

□

**Lemma 6.** Let $\phi_1 \wedge \delta_1$ and $\phi_2 \wedge \delta_2$ both be FIPs for $[\Phi \wedge \Delta](|P|)$. Suppose that $\phi_1 \wedge \delta_1 \equiv \phi_2 \wedge \delta_2$. Then:

$$(\phi_1 \wedge \delta_1) \leftrightarrow (\phi_2 \wedge \delta_2)$$

*Proof.* Because $\phi_1 \wedge \delta_1 \equiv \phi_2 \wedge \delta_2$, there exists a permutation $g$ such that $g(\phi_1 \wedge \delta_1) = \phi_2 \wedge \delta_2$. However $g$ is bijective, and hence by Assumption 1:

$$\phi_1 \wedge \delta_1 \leftrightarrow (\phi_1 \wedge \delta_1)[P \mapsto g(P)] = \phi_2 \wedge \delta_2$$

□

### 2.4 The FIPSOperator

In this section we introduce the FIPS operator:

**Definition 3.** Let $\Phi$ and $\Delta$ be PNF properties with respective matrices $\phi$ and $\delta$. Suppose that $\Phi$ quantifies over $m \in \mathbb{N}$ variables while $\Delta$ quantifies over $n \in \mathbb{N}$ variables. Then:

$$\text{FIPS}(\Phi \wedge \Delta, |P|) := \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in P\}$$

The FIPS operator simply contains every possible FIP for a given formula $[\Phi \wedge \Delta](|P|)$. Next, we prove an intuitive result:

**Lemma 7.** $\text{FIPS}(\Phi \wedge \Delta, |P|) \subseteq \text{FIPS}(\Phi \wedge \Delta, |Q|) \leftrightarrow |P| \leq |Q|$

*Proof.* We begin by showing that $|P| \leq |Q| \rightarrow \text{FIPS}(\Phi \wedge \Delta, |P|) \subseteq \text{FIPS}(\Phi \wedge \Delta, |Q|)$. Suppose $|P| \leq |Q|$, and then it follows that $P \subseteq Q$ by Lemma 1. Then:

$$\begin{aligned}
\text{FIPS}(\Phi \wedge \Delta, |P|) =& \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in P\} \\
\subseteq& \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in P\} \cup \\
& \quad \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in Q \setminus P\} \\
=& \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in Q\} \\
=& \text{FIPS}(\Phi \wedge \Delta, |Q|)
\end{aligned}$$

Next we show that $\text{FIPS}(\Phi \wedge \Delta, |P|) \subseteq \text{FIPS}(\Phi \wedge \Delta, |Q|) \rightarrow |P| \leq |Q|$. Suppose that $\text{FIPS}(\Phi \wedge \Delta, |P|) \subseteq \text{FIPS}(\Phi \wedge \Delta, |Q|)$, then:

$$\begin{aligned}
& \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in P\} \\
\subseteq& \{\phi(v_1, ..., v_m) \wedge \delta(w_1, ..., w_n) | v_1, ..., v_m, w_1, ..., w_n \in Q\}
\end{aligned}$$

Which implies that $P \subseteq Q$, which in turn implies that $|P| \leq |Q|$ by Lemma 1. □

## 3 Intuition

We will build intuition by proving the M-N Theorem for small examples. Coming soon.

## 4 M-N Theorem

**Lemma 8** (FIP Saturation)**.** Suppose that $\Phi$ quantifies over $m \in \mathbb{N}$ variables and $\Delta$ quantifies over $n \in \mathbb{N}$ variables. Then every FIP of $[\Delta \wedge \Phi](k)$, for $k > m + n$, has an equivalent FIP in $[\Delta \wedge \Phi](m + n)$.

*Proof.* Let $k = |P| = m + n + z$ where $z \in \mathbb{Z}_{>0}$, and then $P = \{1, ..., k\}$ by Assumption 2. Let $\phi \wedge \delta$ be an arbitrary FIP of $[\Phi \wedge \Delta](k)$. Then, because $[\Phi \wedge \Delta](k)$ quantifies over exactly $m + n$ variables, there must be at least $z$ elements in $P$ that do not appear in $\phi \wedge \delta$. Let $u \leq m + n$ be the *number* of elements of $P$ that are used in $\phi \wedge \delta$, and let $\{e_1, ..., e_u\} \subseteq P$ be the *set* of elements that are used. Consider the following permutation:

$$g = \begin{pmatrix} e_1 \dots e_u \\ 1 \dots u \end{pmatrix}$$

Notice that $g(\phi \wedge \delta)$ contains only the elements $1...u$. Since $u \leq m + n$, it must be the case that $g(\phi \wedge \delta)$ is a FIP of $[\Phi \wedge \Delta](m + n)$. □

**Theorem 1** (M-N)**.** Suppose that $\Phi$ quantifies over $m \in \mathbb{N}$ variables and $\Delta$ quantifies over $n \in \mathbb{N}$ variables. Then $\Phi(P)$ is an inductive invariant for $T(P)$ iff it is an inductive invariant for the finite instantiation $T(m+n)$.

*Proof.* It is clear that if $\Phi(P)$ is an inductive invariant, then it must be an inductive invariant for $T(m+n)$. We prove the opposite direction in the remainder of the proof.

TODO revisit case when the finite instantiation is less than $m+n$.

Suppose that $[\Phi \wedge \Delta](m+n) \to \Phi(m+n)'$. Let $k > m+n$, then we must show that $[\Phi \wedge \Delta](k) \to \Phi(k)'$. Consider an arbitrary FIP $\phi \wedge \delta$ of $[\Phi \wedge \Delta](k)$. By Lemma 8, there exists a FIP $\phi_2 \wedge \delta_2$ of $[\Phi \wedge \Delta](m+n)$ such that $\phi_2 \wedge \delta_2 \equiv \phi \wedge \delta$. By Lemma 6, we see that $\phi_2 \wedge \delta_2$ holds because $\phi \wedge \delta$ holds. However, because $\phi_2 \wedge \delta_2$ holds, and we know that $[\Phi \wedge \Delta](m+n) \to \Phi(m+n)'$, it follows that $\phi_2'$ holds. Now Lemma 3 implies that $\phi \equiv \phi_2$, and Lemma 4 implies that $\phi' \equiv \phi_2'$, so we have $\phi_2'$ and $\phi' \equiv \phi_2'$. Thus by Lemma 5 and Lemma 2, we can conclude that $\phi'$ holds. □

# 5 Case Studies

In this section we visit several (more coming soon) distributed protocols that are parameterized by a single sort and adhere to the property of Assumption 1.

## 5.1 Peterson's Mutex Protocol

Peterson's Mutex Protocol can be encoded with a transition function $\Delta$ in PNF that quantifies over two variables. A sample inductive invariant candidate is given in [2] that quantifies of two variables and works for $|P| = 2$:

```
Phi == \A p,q \in ProcSet :
   /\ pc[p] \in {"a3","a4","cs"} => flag[p]
   /\ (p#q /\ pc[p] = "cs" /\ pc[q] = "a4") => turn = p
   /\ (p # q) => ~(pc[p] = "cs" /\ pc[q] = "cs")
```

However, by the M-N Theorem, we must show that $\Phi$ is an inductive invariant for the case when $|P| = 4$. In fact, we easily see that $\Phi$ fails to be inductive in the case:

```
/\ turn = 1              /\ turn = 2
/\ pc[1] = "cs"          /\ pc[1] = "cs"
/\ pc[2] = "a4"    ->    /\ pc[2] = "a4"
/\ pc[3] = "a3"          /\ pc[3] = "a4"
/\ a3(3,2)
```

In this case, we can see the counter example by analyzing the following two FIPs from $T(4)$:

$$M_\Phi(3,2) \wedge M_\Delta(3,2) \text{ and } M_\Phi(1,2) \wedge M_\Delta(3,2)$$

These two FIPs are true, and they are precisely how we achieve the counterexample. However these two FIPs can only be described by a minimum of three elements in $P$, which is why we do not detect the counter example when $|P| = 2$.

# References

[1] Aman Goel and Karem Sakallah. On Symmetry and Quantification: A New Approach to Verify Distributed Protocols. In *NASA Formal Methods Symposium*, pages 131–150. Springer, 2021.

[2] Parametric Peterson's Mutex Protocol, 2022.