M-N In EPR

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Suppose that Φ is universally quantified and in PNF while Δ is existentially quantified and in PNF. We will assume a single sort E, and let Φ quantify over m variables and Δ quantify over n variables. We will write $\Phi = \forall \hat{p}, \phi(\hat{p})$, $\Delta = \exists \hat{q}, \delta(\hat{q})$, and $\Phi' = \forall \hat{r}, \phi(\hat{r})'$, where ϕ and δ are non-quantified, uninterpreted, and contain no function symbols. It is the case that $|\hat{p}| = |\hat{r}| = m$ and $|\hat{q}| = n$. We use different quantifer variables for Φ and Φ' so the variables are already standardized apart.

Then we can prove the M-N Theorem for this special case quite quickly:

$$
\begin{aligned}
& \Phi \wedge \Delta \rightarrow \Phi' \\
\leftrightarrow\ & (\forall \hat{p}, \phi(\hat{p})) \wedge (\exists \hat{q}, \delta(\hat{q})) \rightarrow (\forall \hat{r}, \phi(\hat{r})') \\
\leftrightarrow\ & \neg(\forall \hat{p}, \phi(\hat{p})) \vee \neg(\exists \hat{q}, \delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \\
\leftrightarrow\ & (\exists \hat{p}, \neg\phi(\hat{p})) \vee (\forall \hat{q}, \neg\delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \\
\leftrightarrow\ & (\forall \hat{q}, \neg\delta(\hat{q})) \vee (\forall \hat{r}, \phi(\hat{r})') \vee (\exists \hat{p}, \neg\phi(\hat{p})) \\
\leftrightarrow\ & \forall \hat{q}, \forall \hat{r}, \exists \hat{p}, \neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p})
\end{aligned}
$$

In other words, we want to know if:

$$\forall \hat{q}, \forall \hat{r}, \exists \hat{p}, \neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p})$$

is valid. We can turn this into a SAT question by negating the formula, which results in:

$$\exists \hat{q}, \exists \hat{r}, \forall \hat{p}, \neg(\neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p}))$$

This formula is in EPR, and according to Proposition 14.2 (Libkin, Elements of Finite Model Theory), if the formula is satisfiable then it has a model whose universe is size $|\hat{r}| + |\hat{q}| = m + n$. However, we assume that $[\Phi \wedge \Delta \rightarrow \Phi'](m+n)$ is valid, and hence:

$$\exists \hat{q}, \exists \hat{r}, \forall \hat{p} \in E(m+n), \neg(\neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p}))$$

is unsatisfiable. Therefore it follows that, for $k > m + n$:

$$\exists \hat{q}, \exists \hat{r}, \forall \hat{p} \in E(k), \neg(\neg\delta(\hat{q}) \vee \phi(\hat{r})' \vee \neg\phi(\hat{p}))$$

is unsatisfiable, and hence $[\Phi \wedge \Delta \rightarrow \Phi'](k)$ is valid.