A Cutoff Rule For A Special Class Of Parameterized Distributed Protocols

Ian Dardik

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1 Introduction

In this note, we consider the verification problem of a transition system $T = (I, \Delta)$ where I is the initial constraint, Δ is the transition relation, and the system is parameterized by a single sort P of indistinguishable elements (We make the notion of "indistinguishable" precise in Assumption 1 below).

To begin, we will introduce notation for the template and finite instances of a transition system. We adopt the convention of [2] where $T(P)$ is the template of T and $T(|P|)$ is a finite instance. We can also refer to the template or a finite instance of a quantified formula F. For example, suppose F is in Prenex Normal Form (PNF) and univerally quantifies over j variables, i.e. F can be written as:

$$F := \forall x_1, ..., x_j \in P, \phi(x_1, ..., x_j)$$

where ϕ is a non-quantified statement whose only free variables are $x_1, ..., x_j$. Then $F(k)$ is identical to the formula F, except P is replaced by a finite subset of k elements from P; without loss of generality, we will let this finite set be $\{1, ..., k\} \subseteq P$. Thus we see:

$$F(k) = \forall x_1, ..., x_j \in \{1, .., k\}, \phi(x_1, ..., x_j)$$

In this note, we are concerned with the specific scenario in which we are given a candidate inductive invariant Φ, and the finite instances $\Phi(1), ..., \Phi(k)$ have been proved to be inductive invariants for $T(1), ..., T(k)$; we want to know whether Φ is an inductive invariant for T. We are specifically concerned with the case in which Φ is restricted to PNF with only universal quantifiers, and Δ is restricted to PNF with only existential quantifiers.

Throughout this note, we will build several lemmas that lead to an interesting result: let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over; if we suppose that $\Phi(m + n)$ is an inductive invariant for $T(m + n)$, then $\Phi(k)$ is also an inductive invariant for $T(k)$ for all $k > m + n$. We will refer to this as the M-N Theorem in this note. This result is useful because it reduces the verification problem on T to model checking a finite number of instances $T(1), ..., T(m + n)$. Essentially, $m + n$ is a cutoff instance size for proving that our inductive invariant holds.

Note: I think it is likely that if $\Phi(m + n)$ is an inductive invariant, then it is *also* the case for $\Phi(k)$ for all $k < m + n$, but I left this out of this note for the time being to focus on the $k > m + n$ case.

2 Preliminaries

In this section we introduce several assumptions, definitions, and notation that we will use to prove the M-N Theorem.

2.1 Inductive Invariant Of T

Definition 1. A formula F is an inductive invariant for T iff $\forall k \in \mathbb{N}$, $F(k)$ is an inductive invariant for $T(k)$.

2.2 States And Ground Formulas

Definition 2 (States)**.** Let $k \in \mathbb{N}$, then:

$$\text{States}(k) := \{s \mid s \text{ is a state of } T(k)\}$$

In this note we consider a "state" $s \in \text{States}(k)$ to be a formula. More specifically, s is a non-quantified conjunction of constraints that describe a single state in $T(k)$.

Definition 3 (Satisfaction)**.** Let f and g be formulas in First Order Logic (FOL). Then we write $f \models g$ iff $f \rightarrow g$. Alternatively, f satisfies g iff f is stronger than g.

Definition 4 (Ground Formula)**.** A *ground formula* is a non-quantified FOL sentence (has no free variables).

Definition 5 (Ground Formula *of F(k)*)**.** Let F be a quantified formula and $k \in \mathbb{N}$. We say that f is a ground formula *of* $F(k)$ iff f is a ground formula that is identical in structure to F without quantifiers, and with all free variables replaced by members of $\{1, ..., k\}$.

Example 1. Consider the transition system T with two state variables, $x \in (P \rightarrow \mathbb{N})$ and $y \in \mathbb{Z}$. Let $s := (x[1] = 6 \wedge x[2] = 0 \wedge y = -22)$ be a state in the transition system. Let $F := \forall p, q \in P, x[p] \neq x[q]$ and $f := (x[1] \neq x[2])$.

Then $F(2) = \forall p, q \in \{1, 2\}, x[p] \neq x[q]$. Furthermore, f is a ground formula of $F(2)$, $F(2) \models f$, $s \models F(2)$, and $s \models f$.

Definition 6 (Gr)**.** Let F be a quantified formula and $k \in \mathbb{N}$. Then:

$$\text{Gr}(F, k) := \{f \mid (f \text{ is a ground formula of } F(k)) \wedge (F(k) \models f)\}$$

Example 2. $\text{Gr}([\forall p, q \in P, p = q], 2) = \{(1 = 1), (1 = 2), (2 = 1), (2 = 2)\}$
Note: we sometimes use square braces to wrap formulas when it looks better than parentheses.
Notice that $\text{Gr}([\forall p, q \in P, p = q], 2)$ contains elements that are false. This indicates that the statement $[\forall p, q \in P, p = q](2)$ is not valid.

Example 3. Let sv be a state variable, then:

$$\text{Gr}((\forall p, q \in P, p \neq q \rightarrow \text{sv[p]} \neq \text{sv[q]}), 3) = \{(1 \neq 1 \rightarrow \text{sv}[1] \neq \text{sv}[1]), (1 \neq 2 \rightarrow \text{sv}[1] \neq \text{sv}[2]), ...\}$$

2.3 Permutation Transformations

Definition 7 (Permutation Transformation)**.** Let $Q := \{1, ..., k\}$ be a finite instance of the sort P, i.e. $\{1, ..., k\} \subseteq P$. Then let $\pi : Q \rightarrow Q$ be a permutation on Q, and let G be the set of all possible ground formulas. Then $M_\pi : G \rightarrow G$ is the *permutation transformation* on π, a syntactic transformation that replaces each element from Q in a ground formula with its permuted value.

Example 4. Let π be the following permutation:

$$\pi := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Let sv be a state variable, then:

$$M_\pi(3 \neq 1 \rightarrow \text{sv}[3] \neq \text{sv}[1]) = (1 \neq 2 \rightarrow \text{sv}[1] \neq \text{sv}[2])$$

2.4 Indistinguishable Elements

We have loosely stipulated that T must have "indistinguishable" elements. In this section, we make this assumption precise.

Assumption 1 (P Has Indistinguishable Elements)**.** Let $j, k \in \mathbb{N}$ such that $j \geq k$ and F be a quantified sentence in PNF. Let $s \in \text{States}(j)$ such that $s \models F(k)$. If π is a permutation then it is also the case that $M_\pi(s) \models F(k)$.

3 Helper Lemmas

Lemma 1. Let $j, k \in \mathbb{N}$ such that $j \geq k$, $s \in \text{States}(j)$, and F be a universally quantified formula. Then:

$$(s \models F(k)) \leftrightarrow (\forall f \in \text{Gr}(F, k), s \models f)$$

Proof. Suppose that $s \models F(k)$. For an arbitrary formula $f \in \text{Gr}(F, k)$, $F(k) \models f$ and hence we see that $s \to F(k) \wedge F(k) \to f$. It follows that $s \models f$.

Now suppose that $\forall f \in \text{Gr}(F, k), s \models f$. Suppose, for the sake of contradiction, that $s \not\models F(k)$. Then it must be the case that $s \wedge \neg F(k)$. We know that F is unversally quantified, so let $F(k) := \forall x_1, ..., x_m \in P, \phi(x_1, ..., x_m)$ where $m \geq 1$. Then, because $\neg F(k)$ holds, it must be the case that $\exists x_1, ..., x_m \in P, \neg\phi(x_1, ..., x_m)$. However, $\phi(x_1, ..., x_m) \in \text{Gr}(F, k)$ which, by our original assumption, implies $\neg s$. Hence we have both s and $\neg s$ and we have reached a contradiction. □

Lemma 2. Let $k \in \mathbb{N}$, and $s \in \text{States}(k)$ such that $s \models \Phi(k)$. Then for $j \leq k$, it is also the case that $s \models \Phi(j)$.

Proof. Let k and $j \leq k$ be given and suppose that $s \models \Phi(k)$. By Lemma 1, $\forall f \in \text{Gr}(F, k), s \models \Phi(k)$. Now observe that $\text{Gr}(\Phi, j) \subseteq \text{Gr}(\Phi, k)$ due to the fact that Φ is a universally quantified PNF formula. Thus it is also the case that $\forall f \in \text{Gr}(F, j), s \models \Phi(j)$, and then the result follows from Lemma 1. □

4 The M-N Theorem

In this section, we will establish initiation and consecution in two separate lemmas using similar techniques. The M-N Theorem follows immediately from these two lemmas.

Lemma 3 (M-N Initiation)**.** Suppose that $\Phi(m)$ is an inductive invariant for $T(m)$, then $I(k) \to \Phi(k)$ for all $k > m$.

Proof. Because $\Phi(m)$ is an inductive invariant for $T(m)$, it is the case that $I(m) \to \Phi(m)$. Now consider $I(k)$ where $k > m$. Let $s \in \text{States}(k)$ be arbitrary with the condition $s \models I(k)$, and now it suffices to prove that $s \models \Phi(k)$.

Next, we will construct a permutation as follows: let $x_1, ..., x_j$ be the distinct elements of $\{1, ..., k\}$ used in $\Phi(k)$. We know that $j \leq m$ because Φ quantifies over m variables. Let:

$$\pi := \begin{pmatrix} x_1 & x_2 & ... & x_j \\ 1 & 2 & ... & j \end{pmatrix}$$

Notice that $M_\pi(s)$ only contains the elements $1, ..., j$.

Next, by Lemma 2 (needs to be updated) and the fact that $s \models I(k)$, we know that $s \models I(m)$. By Assumption 1, we also know that $M_\pi(s) \models I(m)$. Hence $M_\pi(s) \models \Phi(m)$. By Assumption 1–noting the fact that π^{-1} is also a permutation of $\{1, ..., k\}$–we also see that $s \models \Phi(m)$.

The way to finish the proof is to show that this argument holds for $2, ..., j+1$, $3, ..., j+2$, ... until $k, ..., j+k$ as well as each combination $(1, 3, ..., j+1$, etc.). We will probably need a helper lemma that shows that it suffices to prove this. We will also need to change the lemmas/theorems/wlog assumption so that $F(i)$ refers to the case where $|P| = i$, not necessarily that $P = \{1, ..., i\}$.

$\square$

Lemma 4 (M-N Consecution)**.** Suppose that Φ is in PNF with only universal quantifiers, while Δ is in PNF with only existential quantifiers. Let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over. If $\Phi(m+n)$ is an inductive invariant, then $\Phi(k)$ is inductive for any $k > m+n$.

Proof. Assume that $[\Phi \wedge \Delta \rightarrow \Phi'](m+n)$ is valid. Let $k > m+n$ be given and $s \in \text{States}(k)$ such that $s \models \Phi(k)$. Let $\delta \in \text{Gr}(\Delta, k)$, i.e. δ is a ground "transition". Let $t \in \text{States}(k)$ such that $t' \models (s \wedge \delta)$, that is, t' is an arbitrary "next" state of s. Finally, let $f' \in \text{Gr}(\Phi', k)$ be arbitrary, then, by Lemma 1 and the fact that Φ' is in PNF and universally quantified, it suffices to show that $t' \models f'$.

Next, we will construct a permutation π as follows: let $x_1, ..., x_j$ be the distinct elements of $\{1, ..., k\}$ used in δ and f'. We know that $j \leq m+n$ because Δ quantifies over n variables while Φ quantifies over m variables. Let:

$$\pi := \begin{pmatrix} x_1 & x_2 & ... & x_j \\ 1 & 2 & ... & j \end{pmatrix}$$

Notice that the formulas $M_\pi(\delta)$ and $M_\pi(f')$ now only contain the elements $1, ..., j$. In particular, $M_\pi(\delta) \in \text{Gr}(\Delta, m+n)$ and $M_\pi(f') \in \text{Gr}(\Phi', m+n)$ and hence $M_\pi(\delta) \models \Delta(m+n)$ and $M_\pi(f') \models \Phi'(m+n)$ (this whole sentence needs further explanation). Now because $s \models \Phi(k)$, we see that $s \models \Phi(m+n)$ by Lemma 2, and furthermore $M_\pi(s) \models \Phi(m+n)$ by Assumption 1. Now:

$$M_\pi(t') \models M_\pi(s \wedge \delta) = M_\pi(s) \wedge M_\pi(\delta) \models [\Phi(m+n) \wedge \Delta(m+n)] = [\Phi \wedge \Delta](m+n)$$

Thus $M_\pi(t') \models [\Phi \wedge \Delta](m+n)$, which in turn implies $M_\pi(t') \models \Phi'(m+n)$ by our initial assumption. In particular, by Lemma 1 and the fact that $M_\pi(f') \in \text{Gr}(\Phi', m+n)$, it is the case that $M_\pi(t') \models M_\pi(f')$.

$\square$

Next we present the M-N Theorem:

Theorem 1 (M-N)**.** Suppose that Φ is in PNF with only universal quantifiers, while Δ is in PNF with only existential quantifiers. Let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over. If $\Phi(m+n)$ is an inductive invariant, then $\Phi(k)$ is also an inductive invariant for any $k > m+n$.

5 Case Studies

In this section we visit several (more coming soon) distributed protocols that are parameterized by a single sort and satisfy Assumption 1.

5.1 Peterson's Mutex Protocol

Peterson's Mutex Protocol can be encoded with a transition relation Δ in PNF that quantifies over two variables. A sample inductive invariant candidate is given in [1] that quantifies of two variables and works for $|P| = 2$:

```
Phi == \A p,q \in ProcSet :
   /\ pc[p] \in {"a3","a4","cs"} => flag[p]
   /\ (p#q /\ pc[p] = "cs" /\ pc[q] = "a4") => turn = p
   /\ (p # q) => ~(pc[p] = "cs" /\ pc[q] = "cs")
```

However, by the M-N Theorem, we must show that Φ is an inductive invariant for the cases when $|P| = 1, ..., 4$. In fact, we easily see that $\Phi(3)$ fails to be inductive in the following counter example:

```
/\ turn = 1               /\ turn = 2
/\ pc[1] = "cs"           /\ pc[1] = "cs"
/\ pc[2] = "a4"    ->     /\ pc[2] = "a4"
/\ pc[3] = "a3"           /\ pc[3] = "a4"
/\ a3(3,2)
```

References

[1] Parametric Peterson's Mutex Protocol. `https://github.com/iandardik/iinf/blob/master/ii_cutoff/mn_thm/PetersonParametric.tla`, 2022.

[2] Aman Goel and Karem Sakallah. On Symmetry and Quantification: A New Approach to Verify Distributed Protocols. In *NASA Formal Methods Symposium*, pages 131–150. Springer, 2021.