M-N Without Permutations

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March 30, 2022

1 Introduction

In this note, we consider the verification problem of a transition system $T = (I, \Delta)$ where I is the initial constraint, Δ is the transition relation, and the system is parameterized by a single sort $E = \{e_1, ...\}$ of indistinguishable elements.

To begin, we will introduce notation for the template and finite instances of a transition system. We adopt the convention of [1] where $T(E)$ is the template of T and $T(|E|)$ is a finite instance. We can also refer to the template or a finite instance of a quantified formula F and the sort E. For example, suppose F is in Prenex Normal Form (PNF) and univerally quantifies over j variables, i.e. F can be written as:

$$F := \forall x_1, ..., x_j \in E, \phi(x_1, ..., x_j)$$

where ϕ is a non-quantified statement whose only free variables are $x_1, ..., x_j$. Then $F(k)$ is identical to the formula F, except E is replaced by $E(k) \subseteq E$, where $E(k) = \{e_1, ..., e_k\}$, that is, k distinct arbitrary elements of E. Thus we see:

$$F(k) = \forall x_1, ..., x_j \in E(k), \phi(x_1, ..., x_j)$$

In this note, we are concerned with the specific scenario in which we are given a candidate inductive invariant Φ, and the finite instances $\Phi(1), ..., \Phi(k)$ have been proved to be inductive invariants for $T(1), ..., T(k)$; we want to know whether Φ is an inductive invariant for T. We are specifically concerned with the case in which both Δ and Φ are written in PNF and Φ is restricted to universal quantification.

Throughout this note, we will build several lemmas that lead to an interesting result: let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over; if we suppose that $\Phi(m+n)$ is an inductive invariant for $T(m+n)$, then $\Phi(k)$ is also an inductive invariant for $T(k)$ for all $k > m+n$. We will refer to this as the M-N Theorem in this note. This result is useful because it reduces the verification problem on T to model checking a finite number of instances $T(1), ..., T(m+n)$. Essentially, $m+n$ is a cutoff instance size for proving that our inductive invariant holds.

Note: I think it is likely that if $\Phi(m+n)$ is an inductive invariant, then it is *also* the case for $\Phi(k)$ for all $k < m+n$, but I left this out of this note for the time being to focus on the $k > m+n$ case.

2 Notation

Definition 1 (*E(k)*)**.** We let $E(k) := \{e_1, ..., e_k\} \subseteq E$, where each e_i is distinct and arbitrarily chosen from E. In particular, it is always the case that $|E(k)| = k$.

Definition 2 (*F(k)*)**.** Let F be a quantified formula of the form $Q_1 x_1, ..., Q_m x_m \in E, f(x_1, ..., x_m)$, where each $Q_i \in \{\forall, \exists\}$. Then for any $k > 0$:

$$F(k) := Q_1 x_1, ..., Q_m x_m \in E(k), f(x_1, ..., x_m)$$

Definition 3 (Finite Instances)**.** Let F be a quantified formula of the form $Q_1 x_1, ..., Q_m x_m \in E, f(x_1, ..., x_m)$, where each $Q_i \in \{\forall, \exists\}$. Then for any $k > 0$:

$$\text{FinInstances}(F, k) := \{Q_1 x_1, ..., Q_m x_m \in H, f(x_1, ..., x_m) \mid H \subseteq E \wedge |H| = k\}$$

The sort E is assumed to be unbound, and hence $\text{FinInstances}(F, k)$ is an infinite set.

3 Lemmas

Lemma 1. Let $k \in \mathbb{N}$ such that $s \in \text{States}(k)$ and F is a universally quantified formula. Then:

$$(s \models F(k)) \leftrightarrow (\forall f \in \text{Gr}(F, k), s \models f)$$

Proof. Suppose that $s \models F(k)$. For an arbitrary formula $f \in \text{Gr}(F, k)$, $F(k) \models f$ and hence we see that $s \to F(k) \wedge F(k) \to f$. It follows that $s \models f$.

Now suppose that $\forall f \in \text{Gr}(F, k), s \models f$. Suppose, for the sake of contradiction, that $s \not\models F(k)$. Then it must be the case that $s \wedge \neg F(k)$. We know that F is unversally quantified, so let $F(k) := \forall x_1, ..., x_m \in P, \phi(x_1, ..., x_m)$ where $m \geq 1$. Then, because $\neg F(k)$ holds, it must be the case that $\exists x_1, ..., x_m \in P, \neg\phi(x_1, ..., x_m)$. However, $\phi(x_1, ..., x_m) \in \text{Gr}(F, k)$ which, by our original assumption, implies $\neg s$. Hence we have both s and $\neg s$ and we have reached a contradiction. □

Lemma 2. Let F be a quantified formula and $k > 0$ be given, then:

$$F(k) \leftrightarrow \forall f \in \text{FinInstances}(F, k), f$$

Proof. Let F be a quantified formula of the form $Q_1 x_1, ..., Q_m x_m \in E, f(x_1, ..., x_m)$.

Suppose that $F(k)$ is true. Consider arbitrary $f \in \text{FinInstances}(F, k)$. f has the form $Q_1 x_1, ..., Q_m x_m \in H, f(x_1, ..., x_m)$ where $H = \{h_1, ..., h_k\}$ is a particular subset of E. Notice that $F(k) \to f$, and hence f must be true.

Now suppose that $\forall f \in \text{FinInstances}(F, k), f$, then the result follows trivially. □

4 The M-N Theorem

In this section, we will establish initiation and consecution in two separate lemmas using similar techniques. The M-N Theorem follows immediately from these two lemmas.

Lemma 3 (M-N Initiation)**.** Suppose that $\Phi(m)$ is an inductive invariant for $T(m)$, then $I(k) \to \Phi(k)$ for all $k > m$.

Lemma 4 (M-N Consecution)**.** Suppose that Φ and Δ are both in PNF, while Φ is restricted to universal quantification. Let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over. Then if $\Phi(m+n)$ is an inductive invariant, $\Phi(k)$ is inductive for any $k > m + n$.

Proof. Assume that $[\Phi \wedge \Delta \to \Phi'](m + n)$ is valid. Let $k > m + n$ be given, we want to show that $[\Phi \wedge \Delta \to \Phi'](k)$ is also valid. We will fix the sort domain as $E(k) = \{e_1, ..., e_k\}$ (need to clean up notation here). Let $s \in \text{States}(k)$ such that $s \models \Phi(k)$ and let $\delta \in \text{Gr}(\Delta, k)$ such that $\delta \models \Delta(k)$. Then $(s \wedge \delta)$ is a formula that describes the states reachable from s in one "δ step", and it suffices to show that $(s \wedge \delta) \models \Phi'(k)$. Furthermore, let $\phi' \in \text{Gr}(\Phi', k)$ be arbitrary, then, by Lemma 1 and the fact that Φ' is in PNF and universally quantified, it suffices to show that $(s \wedge \delta) \models \phi'$.

Let $\alpha_1, ..., \alpha_i$ be the unique elements of $\{e_1, ..., e_k\}$ in $(\phi \wedge \delta)$, then we know that $i \leq m + n$ because $\phi \in \text{Gr}(\Phi, k)$ where Φ quantifies over m variables and $\delta \in \text{Gr}(\Delta, k)$ where Δ quantifies over n variables.

Let $j = m + n - i$, then we can choose $\beta_1, ..., \beta_j$ such that $\{\beta_1, ..., \beta_j\} \subseteq (\{e_1, ..., e_k\} - \{\alpha_1, ..., \alpha_i\})$. It is clear that:

$$[\Phi \wedge \Delta](E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\}) \in \text{FinInstances}(F, m + n)$$

Now, $s \models \Phi(E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\})$ because Φ is in PNF and universally quantified (need lemma). Furthermore, $\delta \models \Delta(E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\})$ because...? (this is clear if Δ is restricted to existential quantification). Thus we see:

$$(s \wedge \delta) \models [\Phi \wedge \Delta](E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\}) \in \text{FinInstances}(F, m + n)$$

By our initial assumption and Lemma 2, $[\Phi \wedge \Delta \rightarrow \Phi'](E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\})$ is a valid formula. Hence:

$$(s \wedge \delta) \models \Phi'(E \mapsto \{\alpha_1, ..., \alpha_i, \beta_1, ..., \beta_j\}) \models \phi'$$

□

Next we present the M-N Theorem:

Theorem 1 (M-N)**.** Suppose that Φ is in PNF with only universal quantifiers, while Δ is in PNF with only existential quantifiers. Let m be the number of variables that Φ quantifies over and n be the number of variables that Δ quantifies over. If $\Phi(m + n)$ is an inductive invariant, then $\Phi(k)$ is also an inductive invariant for any $k > m + n$.

Proof. This follows immediately from the previous two lemmas. □

References

[1] Aman Goel and Karem Sakallah. On Symmetry and Quantification: A New Approach to Verify Distributed Protocols. In *NASA Formal Methods Symposium*, pages 131–150. Springer, 2021.