Some random notes, remarks and observations

- 'Most' graphs on n vertices have about $\frac{1}{2}\binom{n}{2} \approx \frac{n^2}{4}$ edges.
- If edges are chosen with probability p then we will average $p\binom{n}{p}$ edges, with a variance of $p(1-p)\binom{n}{2}$. The standard deviation is therefore smaller than $\sqrt{\frac{n^2}{8}} \approx 0.35n$. When n gets at all large different values of p will give measures of effectively disjoint support.
- the 'typical' mgr seems to be closely related to the number of edges. At the endpoints we get

  * $edges = n-1$, gives $mgr > 1$.
  * $edges = \binom{n}{2}$ gives $mgr = \infty$.

  In the middle the typical mgrs seem to cluster into a smaller and smaller range, whose mean is going to zero. This is presumably because a large graph with a middling number of edges will almost surely have a certain bipartite graph inside it.
- Experimentally, the observed mgrs from random graphs ($p = 0.5$) are completely disjoint from the graphs with very small or a very large number of edge
- You can get a large mgr by joining together two smaller graphs with large mgr's with a single edge (or at a single point). In particular you can join together to complete graphs K_m. For these graphs you can often explicitly write down an expression for $\det(D_p)$ and $\langle D_p^{-1}1, 1\rangle$. As $n \to \infty$ these expressions become even simpler and you can calculate the limiting value. For example, if you take an even value of n and let G be two $K_{n/2}$ graphs joined by a single edge, then the mgr is given by the sum of entries test, which boils down to solving (details omitted!)

  $$(n-2)(n-4)\cdot 2^p - (\frac{n}{2}-1)(n-2)\cdot 3^p - (\frac{n}{2}-1)(2n-5)\cdot 4^p + (\frac{n}{2}-1)(n-2)\cdot 8^p \\ + (\frac{n}{2}-1)^2\cdot 9^p - (\frac{n}{2}-1)^2\cdot 12^p + n - 3 = 0.$$

  If you write $X = 2^p$ and $Y = 3^p$ then this simplifies to solving

  $$((X - \frac{Y}{2})n - 2X + Y + 1)((X^2 - 2X - Y + 2)n - 2X^2 + 4X + 2Y - 6) = 0.$$

  If you normalize by dividing by n^2 and then let $n \to \infty$ you are left with

  $$(2X - Y)(X^2 - 2X - Y + 2) = 0.$$

  Replacing X and Y and solving this tells you that as n gets large you should get that the mgr should approach

  $$\frac{1}{\log_2 3 - 1} \approx 1.709511$$

  and this is indeed what you see when you calculate the mgrs (by $n = 20$ it is correct to this number of digits.

Bounds

The proof in my talk shows that for all $\epsilon > 0$, $\delta \in (0,1)$ there exists N such that if $n \geq N$ then $P\big(mgr(X) < \epsilon : X \in G(n,p)\big) > 1-\delta$. The proof is 'effective' in that you can calculate a value of $N(\epsilon,\delta)$, but the proof is so sloppy that the bound is insane. (Doing a few rough estimates I got something like[1] $N(\epsilon,\delta) \lessapprox \frac{6}{\epsilon}\log\left(\frac{1}{\delta}\right)10^{5/\epsilon^2}$.)

The main point is that if $m > \frac{3}{\epsilon}$ then $\log_2(1+\frac{1}{m-1}) < \epsilon$ and so if $K_{m,m}$ sits inside X, then $mgr(X) < \epsilon$. So the real issue is

Question. How big does n need to be so that $X \in G(n, 0.5)$ contains $K_{m,m}$ 'with high probability'.

The enumerative graph theorists should have an answer to this, but I decided to try it empirically. Testing for a copy of $K_{m,m}$ inside X quickly becomes very hard. For $m = 2, 3, 4$ I tried 1000 random graphs of each size until I was finding a copy of $K_{m,m}$ inside every single one. This occurred around $n = 12$ for $K_{n,n}$, so with probability $> 0.99?$, $mgr(X) \leq 1$ for $X \in G(12, 0.5)$. This agrees with the calculated mgrs: in 1000 elements of $G(n, 0.5)$, the smallest n where the maximum mgr seen was smaller than one was 12.

For $m = 3$, every random graph of size at least 25 produced had a copy of $K_{3,3}$, so with probability $> 0.99?$, $mgr(X) \leq 0.55$ for $X \in G(25, 0.5)$. The actual mgrs from a random sample of 25 vertex graphs had a maximum of 0.408. At this point I was needing to check 10^7 different subgraphs of each random graph — not easy!

(The non-rough calculations via the sloppy proof show that if $\delta = \frac{1}{1000}$, then $P(K_{m,m} \subseteq X) > 0.999$ if $n > 589$ for $m = 2$, $n > 135,811$ for $m = 3$ and $n > 423,836,900$ for $m = 4$.)

Past this, the computation was even more difficult. However, of the 12 random 100 vertex graphs I checked, I found a copy of $K_{4,4}$ in each of them.

In any case, what is apparent from the data is that the mgr's are not strictly determined by the largest $K_{m,m}$ subgraph inside the big graph. In fact the graphs of the max and min mgrs for different numbers of vertices are smoother than I might expect if that were the case.

Upper and lower envelopes

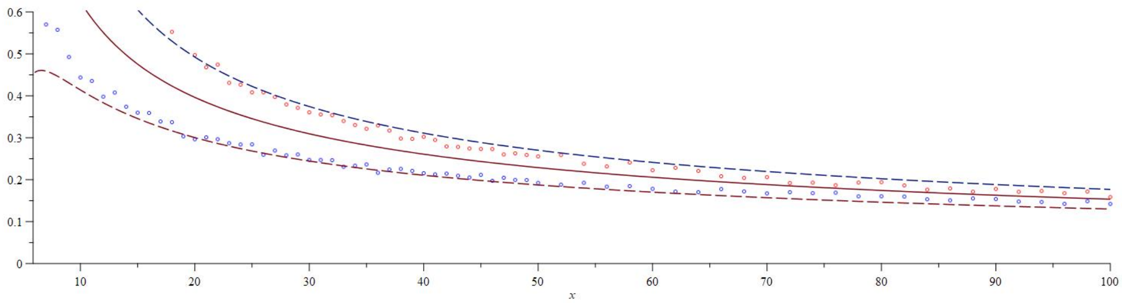
You would expect that as n gets big, what you actually observe from even quite large random samples is that

$$c_n < m_n \leq mgr(X) \leq M_n \ll 2$$

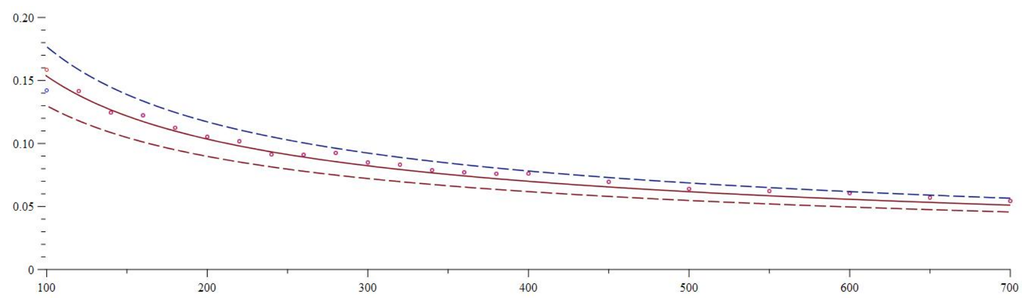
for some rather ill-defined sequences m_n and M_n. Certainly you expect that $M_n - m_n$ will go to zero as $n \to \infty$.

[1] The $10^{5/\epsilon^2}$ term is really $2^{-\binom{2m}{2}}\binom{2m-1}{m}$ which VERY roughly appears to be of order $10^{-m^2/2}$.

Recall that $c_n \approx \frac{2}{n-2}$. Again looking at samples of size 1000, the curve $\frac{2}{(n-2)^{0.56}}$ seems to sit nicely between the max and min observed mgrs for $6 \leq n \leq 100$, and so it appears that the 'typical' mgr of a random graph is not going to zero as fast as the extreme ones.



In the graph, the dots are the max and min mgrs for each n, the solid curve is $y = \frac{2}{(n-2)^{0.56}}$, and the dashed lines are $y \pm d$ where $d = \frac{1.4}{(x-2)^{0.56}\log(x-2)}$. I was thinking of d as being roughly 3σ where σ is the standard deviation of the observed mgrs. (So I would expect the extreme values to be close to the dashed lines for $n \leq 50$, and rather less so as n goes past 65, when I looked at far fewer graphs.) Of course all this is just pulling numbers out of the air — I just fitted some curves by eye and I have no idea whether the 0.56 and 1.4 actually mean anything. **Some** degree of confidence is that I then computed a few mgrs of random graphs up to size $n = 700$ and they all fall within the predicted range:



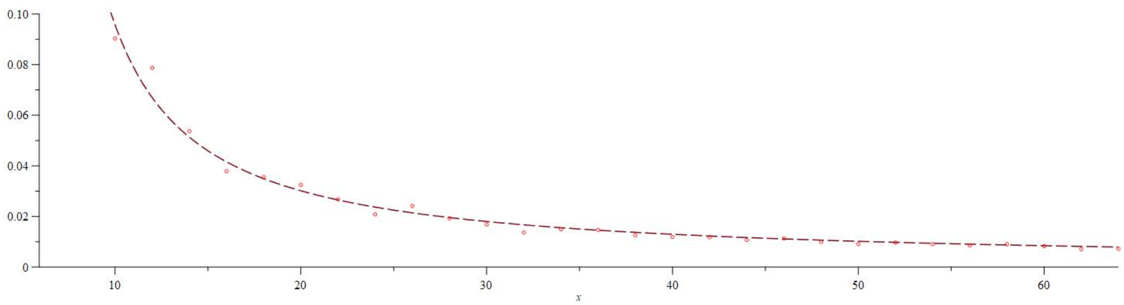
(I took much smaller samples for large n, as the computation time blows up. Odd fact: The computation time grows significantly slower than I would expect for n up to 64 — it is not much worse than linear. There is then a big jump as one goes from 64 to 65. This seems to be something in MAPLE's determinant algorithm, presumably because $64 = 2^6 \ldots$)

All this is in the p-0_5 directory.

Measuring the spread

I just re-ran things calculating the actual standard deviation of the samples for n from 10 to 64. It is crazy how close these are to

$$sd(n) = \frac{1.4}{(n-2)^{0.56}\log(n-2)\log\log(n-2)}.$$



I expect a proof of this by the time I return ......

Random Trees

I did roughly the same thing with random trees. (Here that means: "Starting with the empty undirected graph T on n vertices, edges are chosen uniformly at random and inserted into T if they do not create a cycle. This is repeated until T has $n-1$ edges.")

I sampled 1000 random trees for $6 \leq n50$, then 100 trees for $n = 60, 70, \ldots, 190, 200$. Remember that the (presumed) lower bound on $mgr(T)$ for $|T| = n$ is $1 + \log_2\left(1 + \frac{1}{n-2}\right) \sim 1 + \frac{1}{n-2}$. For small n, you are quite likely to get a straight line with $mgr(T) = 2$, but as n gets bigger, this becomes increasingly unlikely.

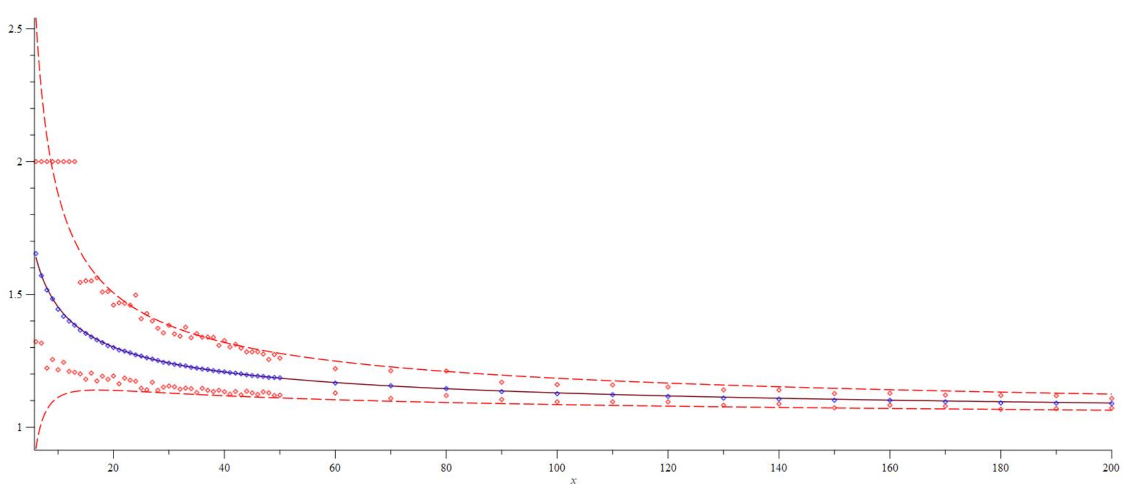
The trial suggests that the mean mgr for trees on n vertices is very close to

$$1 + \frac{1.28}{\sqrt{x-2}}$$

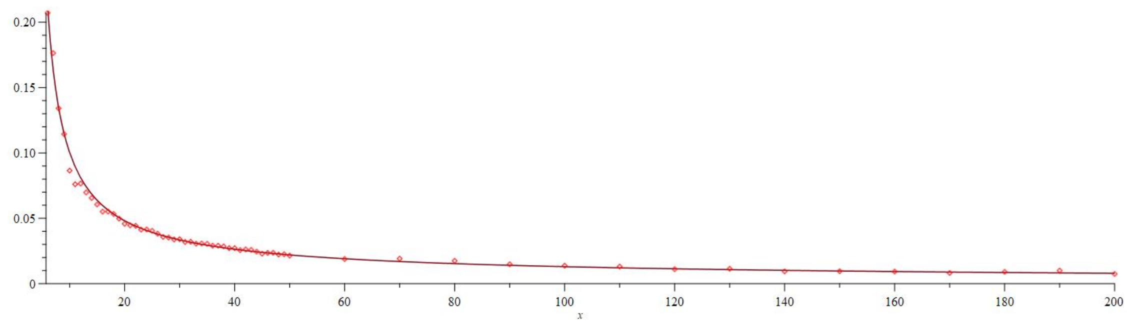
while (similarly to above) the standard deviation is close to

$$\frac{0.59}{\sqrt{n-2}\,\log(n-2)}.$$

The graphs are



which has the max, min, mean, and a rather random pair of bounding curves which need re-doing; and



which gives the actual standard deviation of the samples up to $n = 200$ and the above curve.