Integration of the Stellar Structure Equations
ASTRONOMY 3Y03

The following is drawn almost entirely from *Structure and Evolution of Stars*, Martin Schwarzschild, 1958. This is an old method (see below) but is also the most straight-forward to understand and implement.

We have four equations of stellar structure. Written in terms of radius as the independent variable, and for a radiative region, they are:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$
$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \tag{2}$$
$$\frac{dl}{dr} = 4\pi r^2 \rho \epsilon \tag{3}$$
$$\frac{dT}{dr} = \frac{-3}{4ac} \frac{\kappa\rho}{T^3} \frac{l}{4\pi r^2} \tag{4}$$

We also have the constitutive equations:

$$\kappa = \kappa_0 \rho T^{-3.5} \tag{5}$$
$$\epsilon = \epsilon_0 \rho T^4 \tag{6}$$
$$P = \frac{\Re \rho T}{\mu} \tag{7}$$

and the boundary conditions:

at $r = 0 : m = 0; l = 0$

at $r = R : T = 0; P = KT^{2.5}$ for a convective envelope

In order to solve these equations in a straight-forward way, we need to make two changes of variables. One of these changes is to make our variables dimensionless (reducing the numerical errors in our calculations), and the second is to reduce our boundary conditions at the centre so that we only have one free parameter. The first set is:

$$\begin{aligned} r &= x^* R \\ m &= q^* M \\ l &= f^* L \\ T &= t^* \frac{\mu G M}{\Re R} \\ P &= p^* \frac{GM^2}{4\pi R^4} \end{aligned} \tag{8}$$

and then let us replace the dimensionless variables $(x^*, q^*, f^*, t^*, p^*)$ with a slightly modified set:

$$
\begin{aligned}
x^* &= x_0 x & (9)\\
q^* &= q_0 q & (10)\\
f^* &= f_0 f & (11)\\
t^* &= t_0 t & (12)\\
p^* &= p_0 p & (13)
\end{aligned}
$$

Let us further define some relationships between the second transformation constants:

$$
\begin{aligned}
\frac{q_0}{t_0 x_0} &= 1\\
\frac{p_0 x_0^3}{t_0 q_0} &= 1\\
C\frac{p_0^2 f_0}{t_0^{9.5} x_0} &= 1\\
D\frac{p_0^2 t_0^2 x_0^3}{f_0} &= 1\\
t_0 &= t_c
\end{aligned}
\tag{14}
$$

where t_c is the central temperature in dimensionless units. The constants are

$$
C = \frac{3\kappa_0}{4ac}\frac{1}{(4\pi)^3}\left(\frac{\Re}{\mu}\right)^{7.5}\frac{1}{G^{7.5}}\frac{LR^{0.5}}{M^{5.5}} \tag{15}
$$

and

$$
D = \epsilon_0\left(\frac{\mu}{\Re}\right)^4\frac{G^4}{4\pi}\frac{M^6}{LR^7} \tag{16}
$$

Under all the previous transformations, the equations of stellar structure simplify to become

$$
\begin{aligned}
\frac{dq}{dx} &= \frac{px^2}{t} & (17)\\
\frac{dp}{dx} &= \frac{-pq}{tx^2} & (18)\\
\frac{df}{dx} &= p^2 t^2 x^2 & (19)\\
\frac{dt}{dx} &= \frac{-p^2 f}{t^{8.5} x^2} & (20)
\end{aligned}
$$

The boundary conditions at the centre are formally $q = 0, f = 0, t = t_c$ at $x = 0$, leaving only p_c as a free variable that you choose. However, the formal boundary conditions cannot be used directly, as most of the differential equations diverge at x=0. Therefore, we use a series expansion for the first step, which is valid for x $\sim$ 0.01 or so.

$$\begin{aligned} q &= \frac{1}{3} p_c x^3 && (21) \\ p &= p_c - \frac{1}{6} p_c^2 x^2 && (22) \\ f &= \frac{1}{3} p_c^2 x^3 && (23) \\ t &= 1 - \frac{1}{6} p_c^4 x^4 && (24) \\ & && (25) \end{aligned}$$

With these series expansions as your starting conditions, you can guess a value of p_c and integrate the equations outwards, increasing x as far as you want to go. A good value is something like $x =$ 10-15.

The derivation here has been for an outwards solution of the equations for a radiative core. We need to fit this solution to the solution for a convective envelope. To calculate an envelope, similar transformations and series expansions are used. I will not ask you to do this integration, but instead have provided you with a family of convective envelopes. To fit your interior solutions to the envelopes, you need to use two variables which are invariant to the choices of transformations. These are the *homology invariants* U and V, which don't change under the kinds of transformations we have chosen. This makes them useful for comparing different integrations. They are given by:

$$\begin{aligned} U &= \frac{px^3}{tq} && (26) \\ V &= \frac{q}{tx} && (27) \end{aligned}$$

One definition of a convective zone is one where the temperature gradient is the adiabatic temperature gradient (which has a particular value, 0.4, for an ideal gas). Written in terms of something called the polytropic index, n (another homology invariant), we can say

$$n + 1 = 2.5 = \frac{t^{8.5} q}{p^2 f} \tag{28}$$

Therefore, your interior solution must fit an envelope solution when $n + 1 = 2.5$. This fitting is done in the UV plane as outlined in the assignment sheet.

There is one additional boundary condition which is relevant for this problem. At the surface, the original boundary condition for pressure was $P = KT^{2.5}$. The envelope solutions are calculated using the same dimensionless variables shown in equation 8, and so the boundary condition becomes $p^* = Et^{*2.5}$. The envelope solutions are labeled by their values of E, which is related to K through

$$E = 4\pi \left(\frac{\mu}{\Re}\right)^{2.5} G^{1.5} M^{0.5} R^{1.5} K \tag{29}$$

A historical perspective:

This procedure was developed by Martin Schwarzschild in the early 50's. He was one of the first people to use computers for scientific work, and worked extensively with IBM to develop research computing ('high performance computing' of the day). However, even in 1958, that was not his main mode of working. Here is what he has to say about solving the equations of stellar structure:

"For many problems in the theory of the stellar interior the speed of numerical integrations by hand is entirely sufficient. A person can usually accomplish more than twenty integration steps per day..."

"...the entire numerical work for this fairly typical case can be accomplished by one person in one month. However, if extensive evolutionary model sequences including a variety of physical complications are to be derived, then numerical integrations by hand may become prohibitive and the advantage of large electronic machines will be incontestable." SES, pp 119-120.