The fourth-order Runge-Kutta integration algorithm
Astronomy 3Y03

The theory

In order to solve differential equations numerically, you need to make some approximations. Typically what this means is you approximate a differential quantity (dr in the stellar structure equations) as a small step (Δ r), and then do some clever things to minimize the error this assumption has introduced.

A workhorse method for solving ordinary differential equations is the Runge-Kutta method. References can be found in any good numerical methods book. I particularly like the description in *Numerical Recipes* by Press, Teukolsky, Vettering & Flannery.

Briefly, if we have a series of N differential equations of the form

$$\frac{dy_i(x)}{dx} = f_i(x, y_1, y_2, y_3 ... y_N), i = 1, ...N \tag{1}$$

then an approximation which advances the solution to these equations from x_n to $x_{n+1} = x_n + h$ is

$$y_i(x_{n+1}) = y_i(x_n) + \frac{k_{1,i}}{6} + \frac{k_{2,i}}{3} + \frac{k_{3,i}}{3} + \frac{k_{4,i}}{6} \tag{2}$$

where the y_i's are solutions to the equations, and

$$\begin{aligned} k_{1,i} &= hf_i(x_n, y_i) \\ k_{2,i} &= hf_i(x_n + \frac{h}{2}, y_i + \frac{k_{1,i}}{2}) \\ k_{3,i} &= hf_i(x_n + \frac{h}{2}, y_i + \frac{k_{2,i}}{2}) \\ k_{4,i} &= hf_i(x_n + h, y_i + k_{3,i}) \end{aligned} \tag{3}$$

All the y_i's in these last equations are the solution of the equation at position x_n.

Computationally, you write a routine which calculates the functions f_i for given values of x and y_i. Then, you call it four times, changing the values of x and y_i to calculate the k's. It's a good idea to use temporary variables for these changes, as you want to retain the information from the original x_n and y_i's for the final calculate. Then you add everything up following equation 2, and spit out the answer. Again, I suggest you look in *Numerical Recipes* or Appendix H in Carroll & Ostlie for some examples.

An example

Consider the following system of equations:

$$\begin{aligned} \frac{dy_1}{dx} &= y_2 \\ \frac{dy_2}{dx} &= -by_2 - c^2 y_1 \end{aligned} \tag{4}$$

where b and c are free parameters. The boundary values of this problem are

$$\begin{aligned} y_1(x=0) &= A \\ y_2(x=0) &= 0.0 \end{aligned} \tag{5}$$

Write a Runge-Kutta solver to find the solution to these equations over a range in x. You will need to specify the constants A, b and c. Your program should have three parts: a subroutine which calculates the derivatives $f(x, y_i) = \frac{dy_i}{dx}$; a subroutine which calls the derivatives subroutine four times to calculate the k's and then calculates the new y_i's; and a main routine which sets up the initial conditions and then calls the Runge-Kutta routine for each new value of x.

The system of equations given above can be solved analytically. They are one way of writing the differential equation which governs damped simple harmonic motion. If you set the constant $b = 0$, you should get SHM exactly. The solution to the general equation is

$$y_1 = Ae^{-bx/2}\cos(cx) \tag{6}$$

Check that your numerical method is working by plotting both your solution and the analytic solution on the same plot.