ON THE FLATS FOR SPECHT MATROIDS CORRESPONDING TO PARTITIONS [N-2,1,1]

by

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Abstract. There exists a novel conjecture that characterize the orbits of the Specht matroid's flats. We initially describe the fundamental topics to understand how to find the irreducible representations of Sn and then construct the Specht matroid itself. Following, we expand on Wiltshire-Gordon, Woo, and Zajaczkowska, [12] by writing code which takes a hook and gives us the corresponding Specht Matroid in order compute the hyperplanes, flats, lattice of flats, and S_n-orbits of these flats. Given the maximal flats of hyperplanes of a Specht matroid generated by a hook partition $[n-2,1,1]$ for some integer n, we conjecture that every flat is contained in a "special flat", coming from a larger Specht matroid.

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1. Introduction

Representation theory is the algebraic study of group elements as linear transformations on a vector space. In standard abstract algebra courses, students of mathematics will encounter group theory; often one of the first few they encounter is the symmetric group S_n. The symmetric group is an important construction in algebra for the sake of understanding the combinatorial perspective on all other algebraic subjects. S_n is always described acting on a finite set, S, of size n, where the group elements are bijective functions from S to itself, called "permutations". [9] We understand the permutations as an abstraction for how we can rearrange objects in a line. As a result we can imagine how S_n acts upon any other group by its underlying set. (Or we can imagine complex problems with a lot of different results depending on how we choose indeterminates; we can abstract these problems by making those choices elements of S_n.) [3]. If the elements of S_n are linear transformations on $\mathbb{C}$ then we can expand upon this using the fact that linear transformations can be represented as matrices with a choice for basis to computationally model all kinds of relations in abstract algebra.

The theory on representations of S_n originated from Alfred Young. He introduced, and is the namesake for, the *Young subgroups*, *Young diagrams* and *Young tableaux*. Young's work [13] was the foundation for the work of Wilhelm Specht [10] who introduced the complete set of irreducible representations of S_n, aptly named the *Specht modules*. The combination of the two mathematicians' work provided the result that there is exactly one Specht module of S_n for each partition of the integer n. The necessary definitions and theorems to reach this conclusion are described in detail and in the language of *modules* in Bruce E. Sagan's *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*.[9] Following Young and Specht, Henri Garnir would contribute a way to produce the basis for a Specht module using Young tableaux and various developments on the original ideas.[5] Another supporting work is *Specht Polytopes and Specht Matroids* [12] which describes the construction of the Specht modules in a different way than Sagan. The method that the authors (Wiltshire-Gordon, Woo, Zajaczkowska) use to the construct the modules is the motivation for this thesis and the foundation of computational work during my research. The authors of [11] offers the same as [12] but using more detail and technical language. This article mentions more of the properties of Specht modules that we interested in investigating.

The goal is to contribute to the study of irreducible representations of S_n or Specht modules. And since there is exactly 1 Specht module for each partition of n, we choose to investigate a generalization for a partition that depends on n in hopes that we can make some conclusion about many Specht modules at once. The key idea from Young's work is that every partition has a corresponding diagram. Let $n = 3$, then we can take the partition $\lambda = (2, 1)$. The corresponding diagram to λ looks like

$$\ydiagram{2,1}\ .$$

By the result mentioned, there exists a Specht module of S_3 corresponding to this partition. This shape of diagram is referred to as a *hook* and hook Specht modules have more structure than general ones.[11] Currently, hooks of particular shapes such as the diagrams generated by a partition $[n-1,1]$ or $[2,1,\overset{n-2}{\dots},1]$ have been the subject of some research.

Wilteshire-Gordon, Woo, and Zajaczkowska provide the procedure and SageMath code to computationally research hook shapes. [12] Given a set of vectors that span an irreducible representation of S_n that describes a Specht module by conveying a partition of n as two words of the same length. Pairing two words of the same length vertically showed there existed only one instance where the two words could be rearranged such that no columns of letters are repeated. The two words are called complementary and can generate a Specht module which we describe in more detail later. Simply put, the two complementary words can be used to draw a diagram of the multiplicities of each letter and that diagram matches the Young diagram of a given partition of n.

With a *Specht matrix* we can construct a Specht module. The Specht matrix records when the letters of the two words can be rearrange in a compatible way. The rearrangements of word 1 index the rows and rearrangements of word 2 index the columns such that the entries of the matrix is the *Young's character* of the two rearranged words. Indeed, the column vectors of the Specht matrix span the vector space that is the Specht module. Now we can finally describe how we explore the properties of a Specht module via *matroid theory*. *Matroid* is the name given to a system which generalizes the dependence relations inside a vector space. Matroids differ in how they are defined from study to study but in our case we work with linear matroids which are responsible ways to deal with a large spanning set of vectors. [7] We define our *Specht matroid* as the matroid with a ground set consisting of the column vectors of the Specht matrix and all the independent subsets of the column vectors.

At this point, we are able to computationally construct a Specht matroid using SageMath [2] and ask for any of its components: independent sets, bases, rank functions, hyperplanes, flats, lattice of flats. SageMath's software is equipped with the tools to run these computations; however, we are restricted in what we might ask by computing power. Recall that we are interested in the representation corresponding to a hook. The hooks of partitions $[n-1,1]$ and $[2,1,\overset{n-2}{\dots},1]$ generate matroids named matroids: the *braid matroid* and the *uniform matroid* [8][11] respectively. The general form for the hook shape that we are investigating in comes from the partition $[n-2,1,1]$. Since the computations in SageMath can be computationally difficult, we investigate the hook for the smallest n, i.e. $n=5$ so our partition is $[3,1,1]$ using code from [12]. Then we calculate the Specht matrix and construct the Specht matroid. We've written code to take the

Specht matroid and produce the ”big flats” which are the flats with nearly maximal rank in order to compute flats for matroids of larger values of n than 5.

From here, we are tasked with modeling the properties of the Specht matroid. We encountered that our Specht matrices consisted of some redundancies including duplicate columns and negative duplicate columns which is expected by the definition of the Specht matrix. Removing those columns reveals a row-reducible matrix which after row-reducing is a matrix whose matroid is the same as the one we started with but is easier for Sage to work with. Then, we computed the big flats of the matroid. We are interested in a particular conjecture which describes the structure of certain Specht matroids of hook shape partitions. We use the code we wrote to test this conjecture because it is a reasonable test to compute and the consequence if a matroid were to pass relates to other theory, specifically of algebraic geometry. This research is a single part of a general process to writing new theorems and making new conclusions on the subject of representation stability. More on the structure of these matroids will be found in paper by Eric Ramos [4] which is currently in preparation.

2. Vector Spaces

2.1. Basic Defintions.

DEFINITION 1. *A* vector space *is a set V with operations of addition and scalar multiplication on V such that the following properties hold: commutativity, associativity, additive identity, additive inverse, multiplicative identity, distributive properties. [1]*

The most common example of a vector space is $\mathbb{R}^n$ or $\mathbb{C}^n$ and each vector space has a ground field, F, and in these two examples those fields are $\mathbb{R}$ and $\mathbb{C}$ respectively. We will soon define dimension of vector spaces 7 but we should note that we only need to consider finite-dimensional vector spaces.

DEFINITION 2. *The elements of a vector space V are* vectors. *[1]*

DEFINITION 3. *A* linear combination *of a list $v_1, ..., v_m$ of vectors is*

$$a_1v_1 + ... + a_mv_m$$

where $a_1, ..., a_m \in F$. [1]

DEFINITION 4. *The set of all linear combinations of $v_1, ..., v_m$ in V is the* span *of $v_1, ..., v_m$ written as* $\operatorname{span}(v_1, ..., v_m)$.

DEFINITION 5. *A set of vectors, $v_1, ..., v_m$ are* linearly independent *if the following linear combination,*

$$a_1v_1 + ... + a_mv_m = 0,$$

has only the choice for the scalars $a_1, ..., a_m \in F$ that $a_1 = ... = a_m = 0$.

DEFINITION 6. *A* basis *is a set of vectors, $v_1, ..., v_m$ in the vector space, V, such that every element of V can be uniquely written as a linear combination of the basis vectors. For $v_1, ..., v_m$ to be a basis it must be true that the vectors span V and are linearly independent.*

DEFINITION 7. *A vector space, V, has a* dimension. *V is either* finite-dimensional *or* infinite-dimensional. *The function* $\dim(V)$ *is written to indicate the dimension of V. If V is finite-dimensional that its dimension is the size of a basis for V.*

An example in $\mathbb{R}^3$ would be the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ which $v \in \mathbb{R}^3$ and we say $\dim(\mathbb{R}^3) = 3$.

DEFINITION 8. *A subset of V is a* subspace *of V if and only if it has an additive identity, is closed under addition, and is closed under scalar multiplication. [1]*

DEFINITION 9. *A* coordinate vector *is defined with a choice of basis* $B = \{b_i\}$ *and a vector* $v = c_1 b_1 + ... + c_n b_n$ *such that* $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Next we want to define an important operation for linear algebra, the *dot product*.

DEFINITION 10. *Given two vectors in* $u, v \in V$ *such that if* V *has dimension* n *then* $u = (u_1, ..., u_n)$ *and* $v = (v_1, ..., v_n)$ *then the* dot product *of* u *and* v *is*

$$u \cdot v = u_1 v_1 + ... + u_n v_n.$$

The dot product lets us define orthogonality, length, angle, etc. in high dimensional vector spaces. To be specific, the length of a vector, $|u|$ *is defined by*

$$|u| = \sqrt{u \cdot u}.$$

And the angle, θ*, between two vectors can be found from*

$$u \cdot v = |u||v| \cos(\theta).$$

And if it happens that $u \cdot v = 0$ *then we say that* u *is orthogonal to* v*.*

2.2. Hyperplanes and Arrangements. The geometric surfaces we will be considering are *hyperplanes*. That is the name for planes in higher dimension but the definition is more particular.

DEFINITION 11. *Given a vector space* V *and a vector* $v \in V$ *we define a* hyperplane H *as the subspace*

$$H = \{x : x \cdot v = 0\}.$$

That is, a hyperplane is the set of all vectors x *that are orthogonal to* v*.*

If the dimension of V is n then H will have dimension $n-1$ since we can get a basis for V by adding v to any basis for H. As an example, $\mathbb{R}^3$ is the usual 3 dimensional vector space. The hyperplanes of $\mathbb{R}^3$ will have dimension 2 and 2 dimensional hyperplanes are just your usual planes in a Euclidean space. Hyperplanes in $\mathbb{R}^2$ are 1 dimensional hyperplanes which would simply be lines in the vector space.

Given a set of vectors $v_{i_1}, ..., v_{i_k}$ we find a corresponding set of hyperplanes $A = \{H_{i_1}, ..., H_{i_k}\}$ where A is called an *arrangement.* We want to look at the intersection of some of these planes in the vector space so we look at

$$S = H_{i_1} \cap ... \cap H_{i_k}.$$

We find out what a handful of hyperplanes have in common and then we look to the rest to see if there are any existing dependency relations. This is what defines the flats we are interested in.

That is, the subset defined by

$$F = \{v_i : S \subseteq H_i\}.$$

The elements of the flat are the vectors $v_i \in V$ such that the corresponding hyperplane to that vector H_i contains the intersection of some of the hyperplanes. The dimension of our flat is the codimension of the hyperplanes. That is, if $\dim(V) = n$ then H_i have dimension $n-1$ for all i. Therefore, the intersection S will have dimension $n-k$. So if we take the maximal rank flats of our vector space then $\dim(S) = 1$. If S is 1 dimensional then that means we're looking at a intersection of nearly all hyperplanes in our vector space.

2.3. Matroids.

DEFINITION 12. *A* Matroid, $(E, \mathcal{I})$ *consists of a finite set E and a$\mathcal{I}$, the collection of independent subsets of E. A matroid must hold under these three properties:*

(1) $\emptyset \in \mathcal{I}$
(2) *If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.*
(3) *If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cap e \in I$.*

[7]

It is important to note that different matroids can be equivalently defined so, rather than using explicit definitions, matroids are always defined by their independent sets, bases, *circuits*, *rank function*, flats, and more.

An example of a matroid that we will refer to later is the *uniform matroid*. Take some ground set E with n elements and fix some natural number m such that $m \leq n$. Then we find that taking all the m element subsets of E gives you the set of bases for a matroid; in particular these are the bases for the uniform matroid $U_{m,n}$.[7]

Another example which we refer to later is the *braid matroid*. The braid matroid comes from the n-dimensional collection of hyperplanes called the *braid arrangement*. The braid arrangement contains all the hyperplanes $H_{i,j}$ consisting of points $x_1, ..., x_n \in R^n$ when $x_i = x_j$ for $1 \leq i < j \leq n$. [6]

DEFINITION 13. *A flat in a matroid, M, is a closed subset X of the ground set E of M.*

DEFINITION 14. *Given a matroid M, the rank function $r(X)$ is a map from the power set of E to the non-negative integers. The rank function is the cardinality of a basis B of $M|X$ where B is a basis of X. The rank function satisfies the following two properties:*

(1) *If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.*
(2) *If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.*

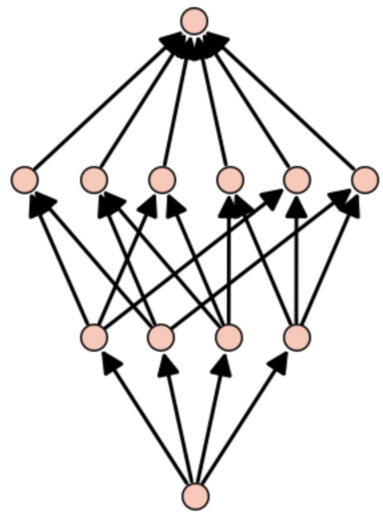


FIGURE 1. An example of a Hasse Diagram corresponding to the lattice of flats of a matroid made from the partition [2,1,1]

A hyperplane in a matroid 2.2 is a flat with rank of $n-1$. And a flat of rank r is a maximal rank-r subset F of E and will be important for the definition of a *lattice of flats*.

DEFINITION 15. *A* partially ordered set *or* poset *is a set with a reflexive, antisymmetric, and transitive binary relation.*

In combinatorics, given a poset we are able to make a *Hasse diagram*. A Hasse diagram is a graph which has vertices for each of the elements of a poset and then lines connecting vertices determined by the order relations of each element.

In the case that for every pair of elements in the poset has a least upper bound and a greatest lower bound then we call the poset a *lattice*. A lattice is visually depicted by a Hasse diagram.

Using the fact that a flat of rank r is a maximal rank-r subset F of E we can define the following:

LEMMA 1. *For any given matroid M, the poset of all flats of M is a lattice.*

PROOF. If F_1, F_2 are flats of a linear matroid M cut out vector spaces V_1, V_2 respectively then $\sup(F_1, F_2) \Leftrightarrow V_1 \cap V_2$ and $\inf(F_1, F_2) \Leftrightarrow V_1 + V_2$ where $\sup$ is the *supremum* operator and $\inf$ is the *infimum* operator. $\square$

REMARK. *The lattice is "upside-down": the vertices of the Hasse diagram are ordered from top to bottom with the vertex as the top having the highest rank and the vertex as the bottom having the lowest rank. In actuality the vertex at the bottom represents the ambient space with the maximal dimension and the highest vertex represents the smallest subspace of the ambient space with a very small dimension.*

3. The Symmetric Group

3.0.1. *Group Theory.* The notation and results in this section of the thesis follow Sagan [9]. First we will introduce the idea of group theory and some of its consequences. Then we will expand into the *the symmetric group* and more.

DEFINITION 16. *A group is a non-empty set G, paired with a binary operation "$\cdot$", that satisfies these three axioms:*

- *Associativity: for any a,b,c $\in G:$ $\quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$*
- *Identity: $\exists\, e \in G : \forall\, a \in G : e \cdot a = a$ and $a \cdot e = a$*
- *Inverses: $\forall\, a \in G$, $\exists\, b \in G : a \cdot b = e$ and $b \cdot a = e$, such that $b = a^{-1}$*

Formally, the correct notation of a group is $(G, \cdot)$ and some authors choose to use this. However, most mathematicians will slightly abuse this notation by referring to a group just by its underlying set.

A few examples of groups are

- $\mathbb{Z},$ the set of all integers with the operation of addition: $''+''$,
- $\mathbb{R}^*,$ the set of all real, nonzero numbers with the operation of multiplication: $''\cdot''$,
- $\mathbb{Z}_n,$ the set of all integers modulo n with operation of addition: $''+''$.
- $S_n,$ the Symmetric group over n elements, with the operation of composition (see below).

Of these examples, the groups $\mathbb{Z}, \mathbb{R}$ are groups of infinite size whereas $\mathbb{Z}_n, S_n$ are finite groups. For the remainder of this thesis, the set G will always be finite. Indeed, I will focus on the symmetric group.

DEFINITION 17. *A group action of a group G on a set X is a map from $G \times X$ to X (written as $g \cdot x$, for all $g \in G$ and $x \in X$ satisfying the following properties:*

(1) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ *for all* $g_1, g_2 \in G, a \in A$, *and*
(2) $e \cdot a = a$, *for all* $a \in A$.

[3]

DEFINITION 18. *Given a group G with an underlying set X we define an orbit of an element $x \in X$ to be*

$$\mathcal{O} = \{g \cdot x \in G : g \in G\}$$

Orbits are an important classification for all possible destinations that G can send an element x and they will show up later.

3.1. The Symmetric Group. The *symmetric group* (denoted by S_n) is a finite group. To define it, fix a set of n elements, $[n] = \{1, 2, ..., n\}$. As a set, S_n consists of every bijective function that maps $[n]$ to itself. The elements of the Symmetric group are referred to as *permutations*; we will typically use the Greek characters π or σ for elements of S_n.

The group operation $\cdot : S_n \times S_n \to S_n$ is the composition of functions. Typically, function composition is written $\pi \circ \sigma$, and defined as

$$(\pi \circ \sigma)(x) = \pi(\sigma(x)).$$

In terms of group theory and the symmetric group, we write composition as $\pi \cdot \sigma$ and refer to it as the multiplication of two permutations. By convention, we multiply permutations right to left so on a set of n elements we permute by σ first then permute by π. Before giving an example of this, we will describe the notation for writing a permutation.

3.2. Notations. Permutations in S_n are written in a few different ways and it is necessary to understand what the notation looks like in order to multiply permutations. The first notation is *two-line notation* which in the general case for $\pi \in S_n$ appears as

$$\pi = \begin{pmatrix} 1 & 2 & ... & n \\ \pi(1) & \pi(2) & ... & \pi(n) \end{pmatrix}.$$

An example of this could be $\pi \in S_6$ where $\pi(1) = 3, \pi(2) = 2, \pi(3) = 4, \pi(4) = 1, \pi(5) = 6, \pi(6) = 5$ such that

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}.$$

For all $\pi \in S_n$ observe that the top row is always $1, 2, ..., n$ so we can simplify this notation by just writing the bottom row, referring to the index for the same information. This is called *one-line notation*. Writing our ongoing example in one-line notation we have

$$\pi = \begin{pmatrix} 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}.$$

One-line notation is not to be confused with the last of our permutation notations, *cycle notation.* Using the fact that for any $i \in \{1, 2, ..., n\}$ the values of $i, \pi(i), \pi^2(i), \pi^3(i), ...$ will eventually result in $\pi^m(i) = i$ for some m. Powers greater than m will repeat these values thus giving us a *cycle*

$$(i, \pi(i), \pi^2(i), ..., \pi^{m-1}(i)).$$

An example if we have the cycle $(1, 3, 4)$. We read this as a permutation π which sends 1 to 3 and 3 to 4 then 4 back to 1. Cycles can be of any length less than or equal to n for a partition

S_n. An example of a cycle of length 1 could be (2) which says π sends 2 to 2; and this is referred to as a *fixed point*. An example of a cycle of length n would be the entire permutation $\sigma = (1, 2, 3, ..., n)$. This naively says σ sends 1 to 2 and then 2 to 3 and so on until eventually n is sent back to 1.

In general, a cycle does not describe the whole permutation but just one of its cycles. *Cycle notation* is the result of denoting π by the collection of its cycles. Usually by convention, we leave out fixed points (in the event of multiplication by fixed points, nothing changes).

Take our example π. We identify that $\pi(1) = 3, \pi(3) = 4, \pi(4) = 1$ so this forms the cycle $(1, 3, 4)$ and since $\pi(2) = 2$ we know that (2) is a fixed point and will not be included. Lastly, $\pi(5) = 6$ and $\pi(6) = 5$ so we have the cycle $(5, 6)$. Therefore π in cycle notation is

$$\pi = (1, 3, 4)(5, 6).$$

The arrangement of the cycles within a permutation and the ordering of the cycles themselves can be arbitrary so it is also true that $\pi = (4, 1, 3)(6, 5)$ and $\pi = (5, 6)(1, 3, 4)$.

These examples of cycle notation have all been *disjoint cycles* - that is, there don't exist any two cycles sharing an element. An example of a permutation that is not in *disjoint cycle notation* (or *adjacent*) might be

$$\pi = (1, 3)(1, 4)(5, 6).$$

When necessary, we put this permutation in disjoint cycle notation with permutation multiplication. The two cycles $(1, 3)(1, 4)$ are the only adjacent cycles so we want to multiply them. Permutation multiplication can take the cycle $(1, 4)$ and see that 1 is sent to 4, then take the cycle $(1, 3)$, since 4 is not in this cycle we send 4 back to 1. Now taking $(1, 3)$, observe 1 is sent to 3 and since 3 is sent to 1 which is sent to 4 in $(1, 4)$ we say that their product is $(1, 3, 4)$. Now we are back to

$$\pi = (1, 3, 4)(5, 6).$$

This example of multiplying the cycles within a permutation is the same as multiplying distinct permutations in cycle notation. Ideally, we usually prefer to write the product of permutations as disjoint cycles.

Multiplying permutations in the other notations is the same exact process of composing the cycles of the permutations right to left.

3.3. Inverses. To justify that S_n is a group and more, we will now describe how to invert a permutation. By definition of a group, S_n should have an inverse element for each $\pi \in S_n$. Inverse elements, $\pi^{-1} \in S_n$, are defined such that $\pi \cdot \pi^{-1} = e$ and $\pi^{-1} \cdot \pi = e$ where e is the identity permutation of S_n. The identity permutation in S_6 is

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Inverting permutations in any notation is easy. If we are given

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

and asked to find π^{-1}, we simply flip the top and bottom rows to get

$$\begin{pmatrix} 3 & 2 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

And then rearrange the top row to be in the standard ascending order

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

If you wanted to invert the π in the cycle notation form of $\pi = (1, 3, 4)$, all we need to do is consider where each element is sent and do the opposite.

$$1 \mapsto 3, \quad 3 \mapsto 4, \quad 4 \mapsto 1$$

And doing the opposite gives us

$$3 \mapsto 1, \quad 4 \mapsto 3, \quad 1 \mapsto 4$$

Giving us $\pi^{-1} = (1, 4, 3)$ which is the same as we got in *two-line notation*.

3.4. Conjugacy. Now that we know how to invert, we will describe *conjugacy* and *conjugacy classes* for S_n. In general, a group G has a *conjugate elements* $a, b \in G$ when for some other element $c \in G$ we have

$$a = cbc^{-1}.$$

In S_n conjugacy is the "relabeling" the elements of G. Suppose S_n acts on some arbitrary set X with $\pi, \sigma \in S_n$ and $a, a', b, b' \in X$. If $\pi(a') = b'$, $\sigma(a) = a'$, $\sigma(b') = b$ then

$$\sigma\pi\sigma^{-1}(a) = \sigma\pi(a') = \sigma(b') = b$$

. We can see that conjugating the permutation π gives us some other permutation in S_n say τ such that $\tau = \sigma\pi\sigma^{-1}$ and $\tau(a) = b$ like $\sigma\pi\sigma^{-1}(a) = b$.

Conjugacy is an equivalence relation between elements of G such that there are sets of the elements of G that are all conjugate to each other which is called a conjugacy class.

Since conjugacy and conjugacy classes are a consequence of group theory we know that conjugacy classes are a part of S_n. The conjugacy classes of S_n are important when we get into *representation theory* and *character theory* as they will be part of describing the *irreducible representations* of S_n.

Our last topic on S_n is the *sign* of permutations. There exists a function called the *sign function* $sgn : S_n \to \{-1, 1\}$ such that

$$sgn(\pi) = (-1)^k.$$

The sign function includes the variable k which requires we define another item: *transpositions*.

Cycle decomposition refers to how we reduce permutations into their disjoint cycle forms but beyond the disjoint cycles, every permutation of S_n can be written as the product of adjacent transpositions.

$$(a_1a_2a_3...a_k) = (a_1a_k)(a_1a_{k-1})...(a_1a_3)(a_1a_2)$$

and

$$(1n) = (12)(23)...(n-1n)...(23)(12)$$

Cycles of length 2 that share elements with neighboring cycles are referred to as adjacent transpositions. A permutation written as the product of adjacent transpositions is in its furthest reduced form and solving the permutation will result in its disjoint cycle form.

Using the example from earlier of $\pi = (1,3,4)(5,6)$ with (2) as fixed point. We can write π in terms of its transpositions by breaking up π into τ_1 and τ_2:

$$\tau_1 = (1,3) \text{ and } \tau_2 = (1,4) \quad \text{such that} \quad \tau_1 \cdot \tau_2 = (1,3) \cdot (1,4) = (1,3,4).$$

Then we have,

$$\pi = \tau_1 \cdot \tau_2 \cdot (5,6) = (1,3)(1,4)(5,6).$$

The sign function takes the number of distinct transpositions of the given permutation and returns $+1$ if that number is even and -1 if that number is odd. As we can see with the π in the example we have three transpositions so $k = 3$ and thus $sgn(\pi) = (-1)^3 = -1$ thus π is an odd permutation.

[9]

4. Representation Theory

Everything in the following chapter is from chapter 1 of Sagan [9] from sections 1.2 to 1.7 and the proofs for any theorems can be found there.

Representation Theory is the study of how groups act on vector spaces while preserving the vector space structure i.e. how groups act on geometric objects.

For this section, V will refer to an arbitrary finite vector space over $\mathbb{C}$ with $\dim V = d$ and $GL_d(V)$ will refer to the set of all invertible linear transformations from V to itself.

4.1. Group Representation.

DEFINITION 19. *A* group representation *is defined as homomorphism:*

$$\rho : G \to GL_d(V).$$

A vector space V is called a G-module *if there exists such a ρ.*

The following axioms also imply that V carries a representation of G given that for $g, h \in G$, $v, w \in V$, $a, b \in \mathbb{C}$:

- *G acts on V,*
- *The action of G is linear.*

While the term G-module is the proper name, I will refer to V as a *representation* of G even though that is technically the name for ρ.

An example of a representation is such a V that there exists a ρ that

$$\rho : S_3 \to GL_3(V).$$

We describe the defining representation of S_3 where S_3 acts on the set $\{1, 2, 3\}$. Choose a basis $\{b_1, b_2, b_3\}$ for V. Let S_3 act on the indices.

$$\pi(b_i) = b_{\pi(i)}$$

This is extended linearly to an action on V.

We can take a group action on a set S and promote it to the "permutation representation" of the same group (by permutation matrices) on the vector space $\mathbb{C}S$. Take the module $\mathbb{C}S$ which is the the vector space made from the set of vectors, S, over the complex numbers, $\mathbb{C}$. Every set S that G acts on can be a representation of G. Whatever S is, is called "standard" basis of the vector space. Depending on how we choose S we get different representations of G such as the trivial representation, the defining representation, the regular representation, the sign representation, and more.

4.2. Matrix Representation.

DEFINITION 20. *The following homomorphism of groups,*

$$X : G \to Mat_{n\times n},$$

is a called a matrix representation *where for all $g \in G$ we have $X(g) \in Mat_{n\times n}$. The following properties must be true for all $g, h \in G$,*

- $X(e) = I,$ *as well as*
- $X(gh) = X(g)X(h).$

The distinction between the matrix representation and the definition of the group representation is the fact that matrix representations give us matrices from an existing group homomorphism ρ depending on a choice of basis. Note that a matrix representation defines a linear G action on the vector space $\mathbb{R}^n$.

An example of a matrix representation which will come up later is defined as

$$X : S_n \to GL_n(V).$$

We can take $n = 3$ such that S_3 contains the following permutations:

$$\{(123), (132), (213), (231), (312), (321)\}.$$

We demonstrate how X works by mapping these permutations by choosing the standard basis of $\mathbb{R}^3$:

$$X((123)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad X((132)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$X((213)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \ldots$$

This is the *defining representation* of S_3 where $X(\pi)$ for all $\pi \in S_3$ are called *permutation matrices*. We note that the matrix of $X((1, 2, 3))$ is the identity matrix for a dimension of 3. That corresponds to the fact that $(1, 2, 3)$ is the identity permutation of S_3 i.e. $(1, 2, 3) = e$.

4.3. Reducibility. If V is a representation of G we call W a *submodule* or *subrepresentation* of G if W is a subspace of V and it is closed under the action of G. That is,

$$\forall w \in W, g \in G \quad \Rightarrow \quad gw \in W.$$

A relevant example of the reducibility of a representation for S_n would be to take $n = 5$ such that $V = \mathbb{C}^5$ where $e_1, e_2, ..., e_5$ are the standard basis. Now, let W be the subspace spanned by the vector $e_1 + e_2 + e_3 + e_4 + e_5$ that is

$$W = \mathbb{C}\{e_1 + e_2 + e_3 + e_4 + e_5\} = \mathbb{C}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \subseteq V.$$

In verifying that W is in fact a subrepresentation, we observe that for any $\pi \in S_5$,

$$\pi(e_1 + e_2 + e_3 + e_4 + e_5) \in W,$$

$$\begin{aligned}\pi(e_1 + e_2 + e_3 + e_4 + e_5) &= \pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4) + \pi(e_5)\\ &= e_1 + e_2 + e_3 + e_4 + e_5.\end{aligned}$$

We recall that any π will map an element of V back to itself so for all $\pi \in V$, we know that $\pi(e_1 + e_2 + e_3 + e_4 + e_5) = e_1 + e_2 + e_3 + e_4 + e_5$ with only some change in order. Thus, we've defined W to be a trivial representation ($\dim W = 1$) of $\mathbb{C}S$ that is a subrepresentation of V.

4.4. Irreducibility and Maschke's Theorem.

DEFINITION 21. *An* irreducible representation *V is one that does not contain any non-trivial subrepresentations.*

DEFINITION 22. *If V is a vector space with U, W as subspaces of V then we say V is the direct sum of U and W: $V = U \oplus W$, if and only if for every $v \in V$ there uniquely exists $u \in U, w \in W$ such that $v = u + w$.*

In this situation ($V = U \oplus W$), if $u_1, ..., u_k$ is a basis for U and $w_1, ..., w_l$ is a basis for W then $u_1, ..., u_k, w_1, ..., w_l$ is basis for V. The uniqueness of the elements of U, W mean there is linear independence in the combination of their bases.

If U, W are invariant under the action of G, then the linear transformation that comes from multiplying by any $g \in G$ can be written as block matrices by choosing the basis for V that is the combination of bases for U, W ($u_1, ..., u_k, w_1, ..., w_l$). Such a block matrix looks like

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right].$$

Where A is a $k \times k$ matrix and B is a $l \times l$ matrix. Saying U, W are invariant under G on some basis is equivalent to $V = U \oplus W$ as representations of G if all matrices are of this block form.

We can say that U is the span of the first k vectors and W is the span of the last l vectors of the matrix to check that $V = U \oplus W$.

Take the representation, $S_2 \times S_3 \curvearrowright \mathbb{C}^5$. If we have a basis for S_2, v_1, v_2, and we have a basis for S_3, u_1, u_2, u_3, then we can choose the basis for $S_2 \times S_3$ that is v_1, v_2, u_1, u_2, u_3. Then the corresponding matrices for the group action of $S_2 \times S_3$ will look like

$$\left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right], \text{ etc.}$$

THEOREM 1 (Maschke's Theorem). *Given a finite group G and a non-trivial representation of G, V, we can write V as the direct sum of irreducible representations W_i for $i \in \{1, 2, ..., k\}$ such that*

$$V = \bigoplus_{i=1}^{k} W_i = W_1 \oplus W_2 \oplus ... \oplus W_k$$

An example of this theorem is if $S_5 \curvearrowright \mathbb{C}^5$ we found a 1-dimensional invariant subspace $\mathbb{C}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ from 4.3. Maschke's theorem says there's a complement to that subspace W_2. It turns out that $W_2 = \mathrm{span}(e_1 - e_2, e_1 - e_3, e_1 - e_4, e_1 - e_5)$.

4.5. Homomorphisms and Schur's Lemma.

DEFINITION 23. *A* homomorphism *is a linear transformation that preserves the group action. If G and V and W are representations of G then a homomorphism defined as*

$$\phi : V \to W$$

exists such that

$$\phi(gv) = g\phi(v)$$

for every $g \in G$ and $v \in V$.

DEFINITION 24. *A* isomorphism *is a bijective homomorphism. That is given the bijective homomorphism $\phi : V \to W$, ϕ is invertible, as well as V, W are* isomorphic *or* equivalent *representations of G ($V \cong W$).*

THEOREM 2 (Schur's lemma). *Let G be a group, let V, W be vector spaces, and let there be two irreducible representations of G ρ_V, ρ_W. If there exists a homomorphism $\phi : V \to W$ then either ϕ is an isomorphism or ϕ is a zero map.*

We want to prove Schur's lemma to show that the decomposition into irreducible representations is unique up to an isomorphism.

PROOF. Given $\phi : V \to W$ is a homomorphism between representations of G then $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ are invariant under G. Therefore we have two cases, the first is $\mathrm{Ker}(\phi) = V$ then $\mathrm{Im}(\phi) = 0$ so ϕ is the zero map. The second is $\mathrm{Ker}(\phi) = 0$ then $\mathrm{Im}(\phi) = W$ so ϕ is bijective and a homomorphism so ϕ is also an isomorphism. $\square$

COROLLARY 1. *Given two irreducible matrix representations of G, X and Y, if there exists a matrix such that for all $g \in G$ we have $TX(g) = Y(g)T$, then T is either invertible, or T is the zero matrix.*

THEOREM 3. *Given a representation of G that is the direct sum of irreducible representations, i.e.*

$$V = m_1V_1 + m_2V_2 + ... + m_kV_k,$$

where $m_1, m_2, ..., m_k$ are the multiplicities of the irreducible representations, $V_1, V_2, ..., V_k$. Then, if we let $\dim V_i = d_i$ then we have the result that

$$\dim V = m_1d_1 + m_2d_2 + ... + m_kd_k.$$

The same result occurs when working with matrix representations.

5. Character Theory

Once again, everything from this chapter is from chapter 1 of [9] from sections 1.8 to 1.10.

5.1. Trace. To define *character* we need to define *trace*.

The *trace* is a linear function from a vector space V to the ground field F, defined as

$$\mathrm{tr} : V \to F$$

such that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}.$$

For a $n \times n$ matrix A. The trace of a matrix is understood as the sum of the elements along the matrix's diagonal. An example for a 3×3 matrix might be

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 2 \\ 6 & 1 & 2 \end{bmatrix} \qquad \mathrm{tr}(A) = a_{11} + a_{22} + a_{33} = 1 + 4 + 2 = 7.$$

The trace of a matrix is invariant under conjugation, which is why we want to use it in representation theory. An example of its invariance would be for matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and invertible matrix $B = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ with inverse $B^{-1} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$. We want to show that $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$. So first take

$$\mathrm{tr}(A) = 1 + 4 = 5.$$

Then find the matrix $B^{-1}AB$:

$$B^{-1}AB = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 19 & 12 \\ -22 & -14 \end{bmatrix}$$

Then take the trace of that matrix:

$$\mathrm{tr}(B^{-1}AB) = 19 + -14 = 5.$$

We can see, $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$. We can show this is true by $\mathrm{tr}(BC) = \mathrm{tr}(CB) = \sum_{i,j} c_{ij} bji$. Let $C = B^{-1}A$ to see that

$$\mathrm{tr}(CB) = \mathrm{tr}(B^{-1}AB).$$

And since $\mathrm{tr}(CB) = \mathrm{tr}(BC) = \mathrm{tr}(BB^{-1}A) = \sum_{i,j} b_{ij} b_{ij}^{-1} a_{ji} = \sum_{i,j} a_{ji} = \mathrm{tr}(A)$. Hence $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$.

To demonstrate, keep A and B from the previous example and take

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 29 & 18 \end{bmatrix}, \text{ and}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 14 \\ 14 & 22 \end{bmatrix}.$$

We can clearly see that $AB \neq BA$; however, when we take the trace:

$$\mathrm{tr}(AB) = 13 + 18 = 31, \text{ and } \mathrm{tr}(BA) = 9 + 22 = 31.$$

Hence $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ so we can justify the trace's invariance under conjugation by $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(B^{-1}BA) = \mathrm{tr}(IA) = \mathrm{tr}(A)$ where I is the identity matrix.

5.2. Character.

DEFINITION 25. *Given a matrix representation X for a chosen basis β of the group G then the* character *of the matrix representation X is the function:*

$$\chi : G \xrightarrow{\mathrm{tr}\, X} \mathbb{C},$$

such that

$$\chi(g) = \mathrm{tr}\, X(g).$$

We verify that the character is well-defined, considering there can be multiple representations for the same element of the group depending on the choice of basis. That is if we have two bases β, β' we want to check that for a change of basis p that we have $\mathrm{tr}(\rho(g)) = \mathrm{tr}(p^{-1}\rho(g)p)$. As we showed trace to be invariant under conjugation, we know this is equality is true. Therefore character is invariant under a choice of basis, so it is well defined.

An example, let X be a matrix representation of S_3 with chosen basis of the standard basis of $\mathbb{R}^3$ and we have a corresponding character χ such that $\chi : S_3 \to \mathbb{C}$ then we are taking the trace of partition matrices. Let $\pi = (1, 3)$ then we get

$$X(\pi) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{so} \qquad \chi(\pi) = \mathrm{tr}\, X(\pi) = 0 + 1 + 0 = 1$$

Notice from what we discussed of representation theory, we have the result that the reducibility of a character relates to the reducibility of the matrix representation. That is, if X is an irreducible matrix representation, then its character χ is also irreducible.

PROPOSITION. *Given X is a matrix representation of G with character χ and $\deg G = d$, we notice the following for all $g \in G$:*

$$\chi(\epsilon) = d.$$

For elements of a conjugacy class of G, $g, h \in K$,

$$\chi(g) = \chi(h).$$

For matrix representation of G, Y and character ψ,

$$X \cong Y \Rightarrow \chi(g) = \psi(g).$$

DEFINITION 26. *Given a group G, we define a* class function *to be a function $f : G \to \mathbb{C}$ such that $f(g) = f(h)$ when elements g and h belong to the same conjugacy class.*

We also define $R(G)$ to be the set of all class functions of G. $R(G)$ can be viewed as a vector space over $\mathbb{C}$. That is $R(G)$ is equipped with addition and scalar multiplication such that for any two class functions $\varphi, \psi \in R(G)$ and any $g \in G$ the following are satisfied:

$$(\varphi + \psi)(g) = \varphi(g) + \varphi(g),$$

and for any $\lambda \in \mathbb{C}$,

$$(\lambda\varphi)(g) = \lambda(\varphi(g)).$$

This allows us to take $\dim R(G)$ to find the number of conjugacy classes of G.

DEFINITION 27. *Given G, we construct the* character table *of G with rows corresponding to unique irreducible characters of G and columns correspond to conjugacy classes. Each entry is the colliding irreducible character of the conjugacy class. We typically make the trivial character the first row and the conjugacy class of the identity $\epsilon \in G$ the first column.*

5.3. Inner products.

DEFINITION 28. *Given two characters of the group G, χ and ψ, (for respective matrix representations, X and Y) we can take the* inner product *of two characters as defined*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

Where $|G| = \deg G$ and we choose to normalize the inner product of two characters by dividing by $|G|$. Additionally, $\overline{\psi(g)}$ refers to the complex conjugate.

THEOREM 4 (Character Relations of the First Kind). *If two characters of G, χ and ψ, are irreducible, then their inner product is equal to the Kronecker delta $\delta_{\chi,\psi}$. That is,*

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi}$$

So $\langle \chi, \psi \rangle$ is equal to 1 when $\chi = \psi$ and it is equal to 0 when $\chi \neq \psi$.

The proof can be found in [9].

COROLLARY 2. *Given a matrix representation of G, X and its character χ. We use Maschke's theorem (1) to write*

$$X \cong m_1 X_1 \oplus m_2 X_2 \oplus ... \oplus m_k X_k$$

such that X_i for $i \in 1, 2, ..., k$ are inequivalent and irreducible subrepresentations of G with corresponding characters $\chi_1, ..., \chi_k$. The following properties hold as a result:

(1) $\chi = m_1 \chi_1 + ... m_k \chi_k$,
(2) $\langle \chi, \chi_j \rangle = m_j$,
(3) $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + ... + m_k^2$,
(4) *X is irreducible if and only if $\langle \chi, \chi \rangle = 1$,*
(5) *Given another representation Y with character ψ then $X \cong Y$ is true if and only if $\chi(g) = \psi(g)$ for every $g \in G$. Where m_i for $i \in 1, 2, ..., k$ is the multiplicity of $X_1, ..., X_k$.*

5.4. Unique Factorization Theorem. We claim that any character can be written uniquely as a sum of irreducibles. That is, given a representation of G called, $V = \mathbb{C}[G]$, which we can decompose by Maschke's theorem 1 into

$$V = m_1 V_1 \oplus m_2 V_2 + ... + m_k V_k,$$

then I am claiming that k is equal to the number of conjugacy classes of G. The proof for this claim involves the endomorphic algebra and defining the center of $\mathbb{C}[G]$ which we omit but can be found in [9]. This shows us there is a irreducible representation of G for each conjugacy class of G.

6. Specht Modules and Specht Matroids

We will now introduce two ways to depict the irreducible representations of the Symmetric group, S_n. The first is the *Young tableaux* and the other is the complementary rearrangements of two words. Either one will allow us to construct and then work with the irreducible representations of S_n, called the *Specht Modules*.

6.1. Young Tableaux. The following subsection is from Sagan, [9] chapter 2 sections 2.1 and 2.3.

DEFINITION 29. *A partition of a a natural number n is described as the vector $\lambda = [\lambda_1, \lambda_2, ..., \lambda_\ell]$ where λ_i are decreasing for $i = 1, 2, ..., \ell$ such that $\lambda_1 + \lambda_2 + ... + \lambda_\ell = n$.*

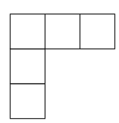
An example of a partition that we are interested in is $\lambda = [3, 1, 1]$.

We call λ as the shape of our diagrams, and indeed sometimes use λ interchangeably with the diagram of that shape.

DEFINITION 30. *A Ferrers diagram or just diagram is a finite subset of $\mathbb{N}^2$ which is depicted in the form of an array of n boxes. Given $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ the diagram has boxes (i, j) in matrix coordinate form (row, col) such that $1 \leq i \leq \lambda_i$ and $1 \leq j \leq n$.*

REMARK. *Different mathematicians have different customs to depict a Ferrers diagram but to be particular, I will be using left-justified rows with boxes as shown in the following example.*

An example of a Ferrers diagram for $\lambda = [3, 1, 1]$ is



In the context of mathematician Alfred Young's work we might refer to a Ferrer's diagram as a Young diagram to distinguish it between a Young tableau and a tabloid (all of which look very similar).

DEFINITION 31. *Given $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$, we call a Young subgroup of S_n to be*

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times ... \times S_{\lambda_l}.$$

An example of a Young subgroup of S_5 would be the one that comes from $\lambda = [3, 1, 1]$ which gives us

$$S_{[3,1,1]} \cong S_3 \times S_1 \times S_1.$$

DEFINITION 32. *A* Young tableau *is a Ferrers diagram, again with shape parameter λ, where the cells of the diagram are replaced with the positive integers less than n.*

The running example for $n = 5$ with a shape of $\lambda = [3,1,1]$ gives us $5! = 120$ possible combinations for this tableaux of this shape. Take 5 tableau t to look like:

$$t_1 = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ 5 & & \end{array}, \quad t_2 = \begin{array}{ccc} 5 & 1 & 2 \\ 3 & & \\ 4 & & \end{array}, \quad t_3 = \begin{array}{ccc} 4 & 5 & 1 \\ 2 & & \\ 3 & & \end{array}, \quad t_4 = \begin{array}{ccc} 3 & 4 & 5 \\ 1 & & \\ 2 & & \end{array},$$

$$t_5 = \begin{array}{ccc} 2 & 3 & 4 \\ 5 & & \\ 1 & & \end{array}.$$

We now define an equivalence relation between two tableaux of the same n and λ. If t_i and t_j have all the same elements in the same rows, regardless of order, then we call them *row equivalent*. Then we can make the class of tableaux that are row equivalent called a $tabloid$. Building off the last example, take

$$t_i = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ 5 & & \end{array} \quad \text{and} \quad t_j = \begin{array}{ccc} 3 & 2 & 1 \\ 4 & & \\ 5 & & \end{array}.$$

Since these two tableaux are row equivalent we know they belong to the same tabloid $\{t\}$ along with 8 other tableaux in that same tabloid (the size is determined to from $5 \operatorname{choose} 3 = 10$).

The notation for the tableau is

$$\{t\} = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array}$$

We will also use $\{t^\lambda\}$ to indicate the complete list of all λ-tabloids.

DEFINITION 33. *Given a permutation λ, let*

$$M^\lambda = \mathbb{C}\{\{t^\lambda\}\},$$

such that M^λ is called the permutation module for the given λ. [9]

THEOREM 5. *Given λ is a partition n. Then for the Young subgroup S_λ and tabloid $\{t^\lambda\}$ we have that the corresponding representations $\mathbb{C}S_\lambda$ and $\mathbb{C}\{t^\lambda\}$ are isomorphic representations.*

DEFINITION 34. *If a tableau t has rows $R_1, ..., R_l$ and columns $C_1, ..., C_k$, then*

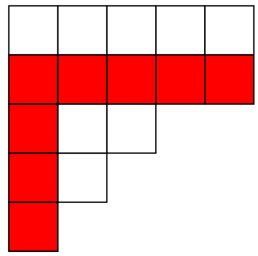
$$R_t = S_{R_1} \times ... \times S_{R_l}$$

and

$$C_t = S_{C_1} \times ... \times S_{C_k}$$

are the row-stabilizer and column-stabilizer of t, respectively. [9]

FIGURE 2. The hook of the cell (1,2) in the Young diagram of the partition [5,5,3,2,1]. Here, $H(r, c)$ = 8.



We also define k_t to be the sign of the column-stabilizer.

$$\kappa_t := C_t^- = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi.$$

DEFINITION 35. *The polytabloid associated with the tabloid t is*

$$e_t = \kappa_t\{t\}.$$

[9]

DEFINITION 36. *Given any λ, the Specht module, S^λ, is the submodule of M^λ the tabloid of shape λ.*

THEOREM 6. *If $V(\lambda)$ is a representation of S_λ for some partition λ then the* Specht modules *are the complete set of all irreducible representations of S_n such that any $V(\lambda)$ is either isomorphic to the Specht modules or it is the sum of Specht modules depending on if $V(\lambda)$ is irreducible or not.*

The proof for this is found in [9] in section 2.4.

We are particularly interested in the Specht modules generated by partitions whose diagrams are *hooks*.

DEFINITION 37. *Given the Ferrers diagram for a partition λ, label its cells in matrix coordinates. The* arm *of a cell (r, c) is the collection of cells*

$$\textit{arm}(r, c) = \{(r, c') \in \lambda \ : \ (c' > c)\}.$$

The leg *of (r, c) is the collection of cells*

$$\textit{leg}(r, c) = \{(r', c) \in \lambda \ : \ (r' > r)\}.$$

The hook *of (r, c) is $\{(r, c) \cup \textit{arm}(r, c) \cup \textit{leg}(r, c).\}$ See Figure 2. We also define $H(n, k) = |\textit{arm}(r, c)| + |\textit{leg}(r, c)| + 1$ to be the number of cells in the hook of (r, c).*

FIGURE 3. A diagram highlighting the corner boxes which are removed or added to during induction and restriction

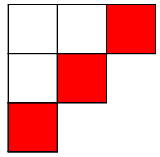
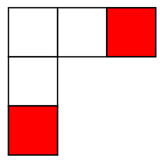


FIGURE 4. A hook diagram highlighting the corner boxes which are removed or added to during induction and restriction



Another theorem which is important to have when working with Specht modules is the*branching rule*.

THEOREM 7 (Branching Rule). *For some partition λ of n, we can induce or restrict the representation $V(\lambda)$ of S_n. I.e. reducing $V(\lambda)$ to $V(\lambda)\downarrow_{S_{n-1}}$ and inducing $V(\lambda)$ to $V(\lambda)\uparrow^{S_{n+1}}$ such that*

$$V(\lambda)\downarrow_{S_{n-1}}\cong \bigoplus_{\lambda^-} V(\lambda^-), \quad \textit{and} \quad V(\lambda)\uparrow^{S_{n+1}}\cong \bigoplus_{\lambda^+} V(\lambda^+).$$

This induction/restriction concept is applied directly to the corresponding diagram for λ. Given λ is $(3, 2, 1)$ then restricting or inducing $V(\lambda)$ looks like adding to or removing any of the following highlighted cells: 3

This theorem is really important because we will discuss later on that when applying it to hooks it provides us with the hope that we can learn anything but the lattice of flats 1 which otherwise would be very difficult. In the case that you are inducing or restricting a representation that comes from a hook say $\lambda = [3, 1, 1]$ then you are adding to or removing from the following highlighted cells: 4

[9]

6.2. Rearrangements of Complementary Words. We will arrive at this same definition if we consider the alternative way of describing a representation. The following is from *Specht Polytopes and Specht Matroids* by Wiltshire-Gordon, Woo, and Zajaczkowska [12].

We can think of the ways to rearrange a word by the choices of letters we have for each position in a word. For example, the word $\texttt{ONION}$. The number of combinations of the letters in $\texttt{ONION}$ is equal to $5!$. But the number of ways to rearrange the word $\texttt{ONION}$ without trivial repetitions is equal to

$$\#\{\text{rearrangements of } \texttt{ONION}\} = \frac{5!}{2! \cdot 2! \cdot 1!} = 30.$$

Note that any problem in combinatorics can instead be thought of in terms of S_n. In this instance, we are looking at how S_5 acts on the word $\texttt{ONION}$. We include the stabilizer subgroup of S_5 to avoid trivial rearrangements i.e. $S_2 \times S_2 \times S_1$. Thus, the rearrangements of $\texttt{ONION}$ is the same as

$$\{\text{rearrangements of } \texttt{ONION}\} = \frac{S_5}{S_2 \times S_2 \times S_1}$$

Now if we take another word of the same length as ONION such as MAMMA we can consider the two words in an array with one word above the other

$$\begin{pmatrix} \texttt{MAMMA} \\ \texttt{ONION} \end{pmatrix}.$$

This is significant to the orbits of the S_n. Notice the columns $\binom{M}{O}$ and $\binom{A}{N}$ are repeated. We understand that if S_5 acts on this object then we will have an made an orbit for every possible rearrangement such that there is always two $\binom{M}{O}$, $\binom{A}{N}$ columns. This can be the case for any rearrangement such as

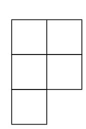
$$\begin{pmatrix} \texttt{MMMAA} \\ \texttt{OONNI} \end{pmatrix}$$

which is another orbit where every element is a $\binom{M}{O}$, $\binom{M}{N}$, $\binom{A}{N}$, or a $\binom{A}{I}$. This example is not in the same orbit as the previous example or else the columns would be the same.

What we want to find are the *free orbits* which will end up giving us the Specht module. The free orbits are those with all distinct columns. That is the rearrangement of

$$\begin{pmatrix} \texttt{MAMAM} \\ \texttt{OONNI} \end{pmatrix}.$$

This free orbit gives us one of the irreducible representations of S_5 and this works for S_n for any words of length n. We take the free orbit and make a diagram from the number of times a letter appears in each word, with the top word being our horizontal axis and the bottom word being our vertical axis and the columns labelling the cells we have:



As you might notice, this diagram is generated by a partition of 5. As we've already defined how the Specht modules are constructed by a given partition, it follows from here that these rearrangements of words can give us a Specht module we'll call $V(w_1, w_2)$ where w_1, w_2 are the first and second word. As a result, $V(w_1, w_2)$ is isomorphic to $V(\lambda)$ for a corresponding λ which gives the shape to the diagram of the letter multiplicities for w_1 and w_2. In this example, the Specht module given by $w_1 = \texttt{MAMAM}$ and $w_2 = \texttt{OONNI}$ is isomorphic to the Specht module given by $\lambda = (2, 2, 1)$ i.e.

$$V(2,2,1) \cong V(\texttt{MAMAM}, \texttt{OONNI})$$

The advantage to constructing the Specht modules via the method described in [12] is that the two words give more to work with. [12] shows how to define a *Specht Matrix* and how the column vectors of this matrix span the corresponding Specht module. To construct the Specht matrix we need to define *Young's character*.

DEFINITION 38. *Given two complementary words of equal length, w_1 and w_2, then let r_1, r_2 be rearrangements of w_1, w_2 such that for any $\pi \in S_n$ that $\pi \cdot w_1 = r_1$ and $\pi \cdot w_2 = r_2$. Then the Young's character is the function specific to w_1, w_2 such that*

$$Y_{w_1,w_2}(r_1, r_2) = \sum_{\pi} \operatorname{sgn}(\pi)$$

THEOREM 8. *The output of the Young's character will be 0 if there is a repeated column when r_1 and r_2 is stacked and -1 or 1 otherwise.*

The proof for this theorem is found in [12]

Now we can construct the Specht matrix.

DEFINITION 39. *Given two complementary words of equal length, w_1 and w_2 then the Specht matrix is the matrix whose columns are indexed by all possible rearrangements of w_1 and rows are indexed by all possible rearrangements of w_2. The entry of the matrix at r_1, r_2 is the Young's character $Y_{w_1,w_2}(r_1, r_2)$.*

For two words $w_1 = 112$ and $w_2 = 121$, the Specht matrix looks something like

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

REMARK. *Specht matrices vary by a change of sign depending on the two words we choose but the corresponding matroid is the same regardless of this.*

The result is that the column vectors of the Specht matrix span the corresponding Specht module. So in this example, since the Specht module $V(112, 121)$ (the same as the Specht module $V(2,1)$ 5) is spanned by

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

.

It is important to note that this is not the only way to get vectors from a Specht module. This method is from [12] which they implemented into SageMath code that we build off of.

6.3. Specht Matroids. Now that we've described the two ways of getting the Specht modules, we want to define the *Specht matroid*.

Beginning with the definition of matroid from earlier 12. We will give an example of a matroid that we will build into the Specht matroid.

Given a ground set of M, $E = \{1, 2, ..., k\}$ such that $\{v_1, v_2, ..., v_k\}$ are vectors spanning $\mathbb{C}^n$. Let $\beta = \{v_1, v_2, ..., v_n\}$ be a basis of $M(v_1, ..., v_k)$ since β is a basis of $\mathbb{C}^n$. The rank function 14 of M is the function $r(X)$ which takes a subset $S \subseteq E$ and gives the dimension of S spanned by the basis $\{v_i | i \in S\}$. The flats 13 of M, as defined are maximal for their rank. If $F \subseteq E$ is a flat then it is the subspace spanned by $\{v_i | i \in F\}$.[12]

In our circumstance, a *Specht matroid* is a matroid $M(\lambda)$ that has a ground set E which consists of the column vectors of the Specht matrix of λ. In all the research, we refer to the basis, independence, rank, and flats of the matroid corresponding to these vectors.

The Specht matroid, $M(\lambda)$, comes from the Specht matrix of λ via the proposition from [7] which says the column vectors and the collection of subsets the column vectors that are linearly independent vectors in $\mathbb{C}^n$ form a matroid. As a result, we get the benefit of all the consequences of a matroid such as the flats. It is expected to be extremely difficult to say anything about the lattice of flats of a Specht matroid unless the Specht matroid is unexpectedly nice for some other reason. In particular, the named Specht matroids: the braid matroid and the uniform matroid for $[n-1, 1]$ and $[2, 1, \overset{n-2}{\ldots}, 1]$ respectively. 2.3 2.3 When a partition is a single hook, it is expected there is a little additional structure. More on why this is the case can be found in [11].

7. Research

7.1. Motivation of Research. In this section, we describe the research component of this project, and the outcome of that research.

The question which we address here is whether flats of the specht matroid $[n-2,1,1]$ "come from contractions"; we define this notion below. Indeed, this is still a highly artificial question and not obviously of intrinsic interest. Indeed, this question comes from a deeper question about the structure of the equivariant Kazhdan-Lusztig (KL) polynomials of matroids [8]. The Kazhdan-Lusztig polynomial of a matroid is an interesting invariant of a matroid; when the matroid's lattice of flats comes with a group action, its coefficients can be promoted to be representations of the same group (S_n, in the case of the Specht matroids), hence the word "equivariant", which for our purposes loosely means "taking a group action into account".

There are several nice families of matroids (in particular the uniform and braid matroids, both of which are Specht matroids) in which this polynomial, viewed itself as a representation of S_n, exhibits representation stability [4]. It is well beyond the scope of this thesis to define these terms - suffice it to say that the linear coefficient of the KL polynomial of a matroid can be readily expressed in terms of the matroid's maximal-rank flats, and this description still holds in the equivariant setting. If a family of matroids exhibits representation stability, then a simple consequence of this fact is that eventually all maximal-rank flats come from contractions. It is this hypothesis that we test here, in the case of the hook Specht matroids $[n-2,1,1]$.

Somewhat to our surprise, the outcome of our tests is negative - the maximal flats in question do *not* appear to be coming from contractions for any n that we can actually compute. This, in turn, is evidence *against* the hypothesis that the equivariant KL polynomials of the matroids $[n-2,1,1]$ exhibit representation stability, unlike the uniform and braid matroids!

The tests which we perform, though straightforward to describe to readers that are well-versed in the previous chapters, required a substantial amount of computational power; as such, we include our sage code in the next chapter.

7.2. 3-element sets. We establish a system to label the vectors of the Specht module that is utilized in the research. This system begins with the the partition associated with the given Specht module and drawing the Young diagram for this partition. We consider the corresponding "words" that would yield the same diagram for their multiplicities as described in subsection 6.2. Instead of using the alphabet for our words we write the cells of the Young diagram using the matrix coordinate convention as shown: 5

This gives us the words $w_1 = 0123400$ and $w_2 = 0000012$. Then we make a set of the indices of the 0's in the first word, w_1. In this example we have $\{0, 5, 6\}$. This 3-element set is a label for all the tabloids on a vector for the Specht module with the first column of the tabloid fixed. And that vector of the Specht module corresponding to the 3-element set can also be considered the

FIGURE 5. A Young diagram with cells labeled by its coordinates

$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$
$(1,0)$				
$(2,0)$				

.

normal vector to a hyperplane in the intrinsic arrangement of the Specht module [11]. Lastly, the vector corresponding to that 3-element set also corresponds to one of the vectors of the respective Specht matrix.

What we see in the computational work is a flat of hyperplanes looking like a set of 3-element sets i.e.

$$\{\{0,5,6\},\{0,4,6\}\{1,2,5\},...\}.$$

This system of labeling the vectors is necessary to describe the result of the research.

7.3. Conjecture of flats. Let λ be a partition of n such that $\lambda = [n-2,1,1]$. Take the Specht matroid $M(\lambda)$ and let the vectors of $M(\lambda)$'s ground set, E, be written in the 3-element set form i.e. (i,j,k) where $0 \leq i,j,k \leq n$. 7.2 We are going to think of the vectors of E as orthogonal vectors to hyperplanes so the vectors correspond to hyperplanes.

Now, we will say that a flat of $M(\lambda)$ "comes from the contraction (i,j)" if, for all k in the ground set other than i,j, all the hyperplanes labelled (i,j,k) are part of the flat.

CONJECTURE 1. *For some Specht matroid corresponding to a partition* $[n-2,1,1]$*, all flats come from some contraction* (i,j) *if* n *is large enough.*

For $n = 5,6,7$, We found that some of the flats in the matroid *do* come from contractions *but* not every flat. It is possible that n is *not* large enough. We initially used the SageMath code from [12] to test $n = 5$ and concluded it was not large enough. Larger values of n than 5 were more intense to compute so we wrote the function `max_rank_flats(n)` to look at a few of the flats for Specht matroids for values of $n > 5$. This function optimizes the Specht matrix by row reducing the matrix and removing redundancies. Then the function takes some set of hyperplanes and checks they are a flat in the matroid. The function repeats this with adding/removing hyperplanes to construct a few flats. This function allowed us to discover the partial results continued for $n = 6$ and $n = 7$ when they did not entirely conform to the conjecture.

We are inclined to believe the conjecture is false. If the conjecture were true and n is simply not large enough then we should expect to stop seeing new flats for larger values of n. The way contractions are encountered in the braid 2.3 and uniform 2.3 matroids demonstrates that the conjecture is true in these cases. The Specht matroid for $[n-2,1,1]$ does not appear like these two other matroids so we are inclined that the structure is not as we believed.

8. SageMath Code

The following is from [12] with addition to functions we wrote including *hook_shape*, *hook_words*, *hook_specht_matrix*, *hook_specht_matroid_slow*.

```
def distinctColumns(w1, w2):
    if len(w1) != len(w2):
        return False
    seen = set()
    for i in range(len(w1)):
        t = (w1[i], w2[i])
        if t in seen:
            return False
        seen.add(t)
    return True


def YoungCharacter(w1, w2):
    assert distinctColumns(w1, w2)
    wp = [(w1[i], w2[i]) for i in range(len(w1))]
    def ycfunc(r1, r2):
        if not distinctColumns(r1, r2):
            return 0
        rp = [(r1[i], r2[i]) for i in range(len(w1))]
        po = [wp.index(rx) + 1 for rx in rp]
        return Permutation(po).sign()
    return ycfunc


def SpechtMatrix(w1, w2):
    Y = YoungCharacter(w1, w2)
    rows = [[Y(u, v) for u in Permutations(w1)] for v in Permutations(w2)]

    return matrix(QQ, rows).transpose()


def hook_shape(n, k):
    return Partition([n-k] + [1]*k)
```

```
def hook_words(n,k):
    w1, w2 = zip(*(hook_shape(n,k).cells()))
    return w1, w2

def hook_matrix(n,k):
    return SpechtMatrix(*hook_words(n,k))

def hook_specht_matroid_slow(n,k):
    M = SpechtMatrix(*hook_words(n,k))
    return Matroid(M)
```

The rest of the code was written by us.

```
def idx_to_threeset(i):
    w1,w2 = hook_words(5,2)
    L = list(Permutations(w2))
    word = L[i]
    threeset = []
    for j,entry in enumerate(word, 1):
        if entry==0:
            threeset.append(j)
    return Set(threeset)


def NarrowerSpechtMatrix(w1, w2):
    Y = YoungCharacter(w1, w2)
    cols = []
    for v in Permutations(w2):
        new_col = vector([Y(u, v) for u in Permutations(w1)])
        if new_col not in cols and -new_col not in cols:
            cols.append(new_col)
    return matrix(QQ, cols).transpose()
```

```
def hook_specht_matrix(n,k):
    M = NarrowerSpechtMatrix(*hook_words(n,k))
    rref = M.rref()
    rowcount = binomial(n-1, k)
    return Matrix(rowcount, M.dimensions()[1], lambda i,j:rref[i,j])


def hook_specht_matroid(n,k):
    return Matroid(hook_specht_matrix(n,k))



def max_rank_flats(n):
    target_rank = binomial(n-1,2)-1
    M = hook_specht_matroid(n,2)
    ground = Set(M.groundset())
    groundsize = len(ground)
    work_stack = []
    bigflats = []
    for S in ground.subsets(groundsize-1):
        assert M.rank(S) == target_rank + 1
        work_stack.append(S)
    step = 0
    while len(work_stack) > 0:
        step += 1
        if step % 100000 == 0:
            print("step", step, " - Stack size now:", len(work_stack), \
                    "flats found:", len(bigflats))
        work = work_stack.pop()
        biggest_absent_thing = max(ground-work)
        for t in range(biggest_absent_thing+1, groundsize):
            assert t in work
            S = work.difference(Set([t]))
            if M.rank(S) == target_rank:
                put_it_back = ground.difference(S)
```

```
                if all(M.rank(S.union(Set([t]))) == \
                        target_rank+1 for t in put_it_back):
                    bigflats.append(S)
            elif M.rank(S) == target_rank+1:
                work_stack.append(S)
            else:
                print("My flat got too small too fast!")
    return bigflats


def act_by(pi, some_set):
    one_bigger = [u+1 for u in some_set]
    return Set([pi(u)-1 for u in one_bigger])


def act_on_index_by(pi, n):
    assert 0 <= n < 20, "that's not a hyperplane"
    H = threesets[n]
    new_H = act_by(pi, H)
    return number[new_H]


def orbit_of(flat):
    orbit = []
    for pi in Permutations(6):
        new_flat = Set(act_on_index_by(pi, i) for i in flat)
        orbit.append(new_flat)
    return Set(orbit)
```

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