MAT 207C: HOMEWORK 4

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Holmes 2.2: Matched Asymptotics Introduction

Problem 2.2. Find a composite expansion of the solution of the following problems:

(c) $\varepsilon y'' + y(y' + 3) = 0$ for $0 < x < 1$, with $y(0) = 1$ and $y(1) = 1$.
(d) $\varepsilon y'' = f(x) - y'$ for $0 < x < 1$, with $y(0) = 0$ and $y(1) = 1$. Also, $f(x)$ is continuous.

Problem 2.3. Consider the problem

$$\varepsilon^2 y'' + ay' = x^2 \quad \text{for } 0 < x < 1,$$

with $y'(0) = \lambda$, $y(1) = 2$, where a and λ are positive constants.

(a) Find a first-term composite expansion for the solution. Explain why the approximation does not depend on λ.

Problem 2.4. A small parameter multiplying the highest derivative does not guarantee that the solution will have a boundary layer for small values of ε. As demonstrated in this problem, this can be due to the form of the differential equation or the particular boundary conditions used in the problem.

(b) Consider the equation $\varepsilon^2 y'' - xy' = 0$ for $0 < x < 1$. From the exact solution, show that there is no boundary layer if the boundary conditions are $y(0) = y(1) = 2$, while there is a boundary layer if the boundary conditions are $y(0) = 1$ and $y(1) = 2$.

Problem 2.9. Consider the problem

$$\varepsilon y'' + p(x)y' + q(x)y = f(x), \quad 0 < x < 1,$$

with $y(0) = \alpha$ and $y(1) = \beta$. Assume $p(x), q(x)$, and $f(x)$ are continuous and $p(x) > 0$ for $0 \le x \le 1$.

(a) In the case where $f = 0$, show that

$$y \sim \beta \exp\left(\int_x^1 \frac{q(s)}{p(s)}\, ds\right) + \left[\alpha - \beta \exp\left(\int_0^1 \frac{q(s)}{p(s)}\, ds\right)\right] h(x),$$

where $h(x) = e^{-p(0)x/\varepsilon}$.

Problem 2.10. Consider the problem

$$\varepsilon y'' + 6\sqrt{x}y' - 3y = -3, \quad 0 < x < 1,$$

with $y(0) = 0$ and $y(1) = 3$.

(a) Find a composite expansion of the problem.

Holmes 2.3: Boundary Layers

Problem 2.15. Find a composite expansion of the solution of the following problems and sketch the solution:

(b) $\varepsilon y'' - y' + y^2 = 1$ for $0 < x < 1$, with $y(0) = 1/3$, $y(1) = 1$.

(e) $\varepsilon y'' - y(y' + 1) = 0$ for $0 < x < 1$, with $y(0) = 3$, $y(1) = 3$.

Holmes 2.5: Interior Layers

Problem 2.32. Find a first-term expansion of the solution of each of the following problems. It should not be unexpected that for the nonlinear problems the solutions are defined implicitly or that the transition layer contains an undetermined constant.

(a) $\varepsilon y'' = -\left(x^2 - \frac{1}{4}\right) y'$ for $0 < x < 1$, with $y(0) = 1$ and $y(1) = -1$.

(f) $\varepsilon y'' + y(y' + 3) = 0$ for $0 < x < 1$, with $y(0) = -1$ and $y(1) = 2$.

Problem 2.33. Consider the problem

$$\varepsilon y'' = yy' \quad \text{for } 0 < x < 1,$$

with $y(0) = a$ and $y(1) = -a$, where $a > 0$.

(b) Find a composite expansion of the solution.