Math 207C Homework #2

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Problem 1
Holmes # 1.8, 1.9, 1.16

(1.8) Find the first two terms in the expansion for the following functions:

(a) $f = \displaystyle\int_0^{\pi/4} \frac{dx}{\varepsilon^2 + \sin^2 x}$.

(b) $f = \displaystyle\int_0^{1} \frac{\cos(\varepsilon x)}{\varepsilon + x}\, dx$.

(c) $f = \displaystyle\int_0^{1} \frac{dx}{\varepsilon + x(x-1)}$.

(1.9) This problem derives an asymptotic approximation for the Stieltjes function, defined as

$$S(\varepsilon) = \int_0^{\infty} \frac{e^{-t}}{1+\varepsilon t}\, dt.$$

(a) Find the first three terms in the expansion of the integrand for small ε and explain why this requires that $t \ll 1/\varepsilon$.

(b) Split the integral into the sum of an integral over $0 < t < \delta$ and one over $\delta < t < \infty$, where $1 \ll \delta \ll 1/\varepsilon$. Explain why the second integral is bounded by $e^{-\delta}$, and use your expansion in part (a) to find an approximation for the first integral. From this derive the following approximation:

$$S(\varepsilon) \sim 1 - \varepsilon + 2\varepsilon^2 + \cdots .$$

(1.16) This problem derives asymptotic approximations for the complete elliptic integral, defined as

$$K(x) = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - x\sin^2 s}}.$$

It is assumed that $0 < x < 1$.

(a) Show that, for x close to zero,

$$K \sim \frac{\pi}{2}\left(1 + \frac{1}{4}x\right).$$

(b) Show that, for x close to one,

$$K \sim -\frac{1}{2}\ln(1-x).$$

(c) Show that, for x close to one,

$$K \sim -\frac{1}{2}\left[1 + \frac{1}{4}(1-x)\right]\ln(1-x).$$

Problem 2
Use integration by parts to find asymptotic expansion for large x of the following integrals:

(i) $\displaystyle\int_1^\infty e^{-xt}/t^n\,dt, \quad n \in \mathbb{N}.$

(ii) $\displaystyle\int_0^1 e^{ixt}t^{-1/2}\,dt.$

Problem 3
This exercise examines Laplace's method carefully as applied to

$$I_n(x) = \int_0^\pi e^{x\cos t}\cos(nt)\,dt, \quad x \to \infty.$$

(i) Show that

$$I(x) \sim \int_0^\epsilon e^{x\cos t}\cos(nt)\,dt$$

for any fixed $\epsilon > 0$.

(ii) Prove that

$$\int_0^\epsilon e^{x\cos t}\cos(nt)\,dt \sim \int_0^\epsilon e^{x(1-t^2)}\,dt, \quad \epsilon > 0$$

by breaking up the range of integration $[0,\epsilon)$ into $[0,x^{-\alpha}]$ and $[x^{-\alpha},\epsilon]$ with $\alpha \in (1/4,1/2)$ and showing

$$\cos(nt)e^{x\cos t} \sim e^{x(1-t^2/2)} \quad \text{uniformly for all } t \in [0,x^{-\alpha}],$$

as $x \to \infty$ (Hint: $1 - t^2/2 \le \cos t \le 1 - t^2/2 + t^4/4!$) and the integration range $[x^{-\alpha},\epsilon]$ has an exponentially smaller contribution to the integral.

(iii) What's wrong with the result if you approximate $\cos t$ by 1 and $\cos(nt)$ by 1 for $t \in [0,\epsilon]$?

Problem 4
Find the leading asymptotic of the following integrals as $x \to \infty$

(i)

$$\int_0^\infty \cos[x(t + t^3/3)]\,dt,$$

(ii)

$$\int_{-\infty}^\infty e^{ix(-t+t^3/3)}\,dt.$$