Math 207C Homework #3

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(1) Holmes # 1.32(a)(b), 1.34(b), 1.35, 1.36, 1.38

(2) Holmes # 2.1, 2.2(a)

Problem 1 (1.32)
Find a two-term asymptotic expansion, for small ε, of the solution of the following problems:

(a) $y'' + \varepsilon y' - y = 1$, where $y(0) = 0$ and $y(1) = 1$.

(b) $y'' + f(\varepsilon y) = 0$, where $y(0) = y(1) = 0$ and $f(s)$ is a smooth positive function.

Problem 2 (1.34)
This problem concerns aspects of the projectile problem.

(a) Assuming $\tau_h \sim \tau_0 + \varepsilon\tau_1$, find τ_0 and τ_1 from (1.46). Give a physical reason why τ_1 is positive.

(b) Find a two-term expansion for the time at which the projectile reaches its maximum height. How much higher does the projectile get in the nonuniform gravitational field? (You should find a first-term approximation for this.)

Problem 3 (1.35)
In the projectile problem, to account for air resistance, one obtains the equation

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} - \frac{k}{R+x}\frac{dx}{dt},$$

where k is a nonnegative constant. Assume here that $x(0) = 0$ and $x'(0) = v_0$.

(a) What is the nondimensional version of this problem if one uses the same scaling as in Sect. 1.1?

(b) Find a two-term asymptotic expansion of the solution for small ε. Assume in doing this that $\alpha = kv_0/(gR)$ is independent of ε.

(c) Does the addition of air resistance increase or decrease the flight time?

Problem 4 (1.36)
The eigenvalue problem for the vertical displacement, $y(x)$, of an elastic string with variable density is

$$y'' + \lambda^2\rho(x,\varepsilon)y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. For small ε assume $\rho \sim 1 + \varepsilon\mu(x)$, where $\mu(x)$ is positive and continuous. In this case the solution $y(x)$ and eigenvalue λ depend on ε, and the appropriate expansions are

$$y \sim y_0(x) + \varepsilon y_1(x) \quad \text{and} \quad \lambda \sim \lambda_0 + \varepsilon\lambda_1$$

(better expansions will be discussed in Sect. 3.6).

(a) Find y_0 and λ_0.

(b) Find y_1 and λ_1.

Problem 5 (1.38)
The equation for the displacement, $u(x)$, of a weakly nonlinear string, on an elastic foundation and with a uniform forcing, is

$$\frac{d}{dx}\left(\frac{u_x}{\sqrt{1+\varepsilon(u_x)^2}}\right) - k^2 u = 1, \quad \text{for } 0 < x < 1,$$

where $u(0) = u(1) = 0$ and k is a positive constant. Find a two-term expansion of the solution for small ε. You should find, but do not need to solve, the problem for the second term.

Problem 6 (2.1)
The Friedrichs model problem for a boundary layer in a viscous fluid is (Friedrichs, 1941)

$$\varepsilon y'' = a - y', \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and a is a given positive constant with $a \neq 1$.

(a) After finding the first term of the inner and outer expansions, derive a composite expansion of the solution of this problem.

(b) Derive a two-term composite expansion of the solution of this problem.

Problem 7 (2.2)
Find a composite expansion of the solution of the following problems:

(a) $\varepsilon y'' + 2y' + y^3 = 0$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = \frac{1}{2}$.