Math 207C Homework #1

Ian Gallagher

April 10, 2025

Holmes Ex #1.1, 1.3, 1.5, 1.7, 1.12, 1.18 (a)(e)(h)(p), 1.19, 1.20, 1.21

Problem 1 (1.1)

(a) What values of α, if any, yield $f = O\big(\varepsilon^\alpha\big)$ as $\varepsilon \downarrow 0$ for each of the following functions?

(i) $f = \sqrt{1+\varepsilon^2}$

(ii) $f = \varepsilon \sin(\varepsilon)$

(iii) $f = \big(1 - e^\varepsilon\big)^{-1}$

(iv) $f = \ln(1+\varepsilon)$

(v) $f = \varepsilon \ln(\varepsilon)$

(vi) $f = \sin(1/\varepsilon)$

(vii) $f = \sqrt{x+\varepsilon}$, where $0 \geq x \geq 1$

(b) For the functions listed in (a), what values of α, if any, yield $f = o\big(\varepsilon^\alpha\big)$ as $\varepsilon \downarrow 0$?

Solution

(i) We can expand by plugging x^2 into the Taylor series for $\sqrt{1+x}$:

$$f = \sqrt{1+\varepsilon^2} = 1 + \frac{1}{2}\varepsilon^2 + o(\varepsilon^4).$$

Now, taking the ratio of limits we get,

$$\lim_{\varepsilon \downarrow 0} \frac{f}{\varepsilon^\alpha} = \lim_{\varepsilon \downarrow 0} \left(\varepsilon^{-\alpha} + \frac{1}{2}\varepsilon^{2-\alpha} + o(\varepsilon^{4-\alpha})\right) = \begin{cases} \infty, & \alpha > 0 \\ 1, & \alpha = 0 \\ 0, & \alpha < 0 \end{cases}$$

Therefore, $f = O(\varepsilon^\alpha)$ for $\alpha \leq 0$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 0$

(ii) Now, we can expand sine into its Taylor series

$$f = \varepsilon \sin(\varepsilon) = \varepsilon\left(\varepsilon - \frac{\varepsilon^3}{3!} + o(\varepsilon^5)\right) = \varepsilon^2 - \frac{\varepsilon^4}{3!} + o(\varepsilon^5)$$

For similar reasons as above, we have that $f = O(\varepsilon^\alpha)$ and $f = o(\varepsilon^\alpha)$ for $\alpha \leq 2$.

(iii) For this, we can use another Taylor series truncation $e^x = 1 + x + o(x^2)$

$\blacksquare$

Problem 2 (1.3)
This problem establishes some of the basic properties of the order symbols, some of which are used extensively in this book. The limit assumed here is $\varepsilon \downarrow 0$.

(a) If $f = o(g)$ and $g = O(h)$, or if $f = O(g)$ and $g = o(h)$, then show that $f = o(h)$. Note that this result can be written as $o(O(h)) = O(o(h)) = o(h)$.

(b) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f+g = O(|\phi_1|+|\phi_2|)$. Also, explain why the absolute signs are necessary. Note that this result can be written as $O(f) + O(g) = O(|f| + |g|)$.

(c) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $fg = O(\phi_1\phi_2)$. This result can be written as $O(f)O(g) = O(fg)$.

(d) Show that $O(O(f)) = O(f)$.

(e) Show that $O(f)o(g) = o(f)o(g) = o(fg)$.

Solution

(a) For the first case, the following holds for all δ_1 and some constants $C, \varepsilon_1, \varepsilon_2$

$$|f(\varepsilon)| \le \delta_1 |g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_1$$

and

$$|g(\varepsilon)| \le C|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_2$$

Therefore,

$$|f(\varepsilon)| \le \delta|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$$

where $\delta = C\delta_1$. Since δ_1 is arbitrary, for any $\delta > 0$, we can set $\delta_1 = \delta/C$ and we may conclude that $f = o(h)$.

Similarly, for all δ_1' and some constants $C', \varepsilon_1', \varepsilon_2'$

$$|f(\varepsilon)| \le C'|g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_1'$$

and

$$|g(\varepsilon)| \le \delta_1'|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_2'$$

Therefore,

$$|f(\varepsilon)| \le \delta'|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon_1', \varepsilon_2'\}$$

where $\delta' = C'\delta_1'$ and we may conclude that $f = o(h)$ in the same way.

(b) Assuming $f = O(\phi_1), g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| &\le C_1|\phi_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\le C_2|\phi_2(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(f+g)(\varepsilon)| &= |f(\varepsilon) + g(\varepsilon)| \\ &\le |f(\varepsilon)| + |g(\varepsilon)| \\ &\le C_1|\phi_1(\varepsilon)| + C_2|\phi_2(\varepsilon)| \\ &\le C\left(|\phi_1(\varepsilon)| + |\phi_2(\varepsilon)|\right) \end{aligned}$$

holds for $C = \max\{C_1, C_2\}$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $f+g = O(|\phi_1|+|\phi_2|)$. The necessity of the absolute values are evident from the last inequality.

(c) Assuming $f = O(\phi_1), g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| \le C_1|\phi_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| \le C_2|\phi_2(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(fg)(\varepsilon)| &= |f(\varepsilon)g(\varepsilon)| \\ &= |f(\varepsilon)||g(\varepsilon)| \\ &\le C_1|\phi_1(\varepsilon)| \cdot C_2|\phi_2(\varepsilon)| \\ &= C|\phi_1(\varepsilon)\phi_2(\varepsilon)| \end{aligned}$$

holds for $C = C_1C_2$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $fg = O(\phi_1\phi_2)$.

(d) Assuming $f = O(\phi)$, there exists constants C, ε_1 such that

$$|f(\varepsilon)| \le C|\phi(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1$$

Therefore,

$$\begin{aligned} |O(f)(\varepsilon)| &= |O(\phi(\varepsilon))| \\ &= O(|\phi(\varepsilon)|) \\ &= O(f(\varepsilon)) \end{aligned}$$

holds for $0 < \varepsilon < \varepsilon_1$.

$\blacksquare$

Problem 3

1.5 Suppose $f = o(\phi)$ for small ε, where f and ϕ are continuous.

(a) Give an example to show that it is not necessarily true that

$$\int_0^\varepsilon f \mathrm{d}\varepsilon = o\left(\int_0^\varepsilon \phi \mathrm{d}\varepsilon\right).$$

(b) Show that

$$\int_0^\varepsilon f \mathrm{d}\varepsilon = o\left(\int_0^\varepsilon |\phi| \mathrm{d}\varepsilon\right).$$

Problem 4 (1.7)
Assuming $f \sim a_1\varepsilon^\alpha + a_2\varepsilon^\beta + \cdots$, find α, β (with $\alpha < \beta$) and nonzero a_1, a_2 for the following functions:

(a) $f = \dfrac{1}{1 - e^\varepsilon}$.

(b) $f = \left[1 + \dfrac{1}{\cos(\varepsilon)}\right]^{\frac{3}{2}}$.

(c) $f = 1 + \varepsilon - 2\ln(1+\varepsilon) - \dfrac{1}{1+\varepsilon}$.

(d) $f = \sinh\left(\sqrt{1+\varepsilon x}\right)$, for $0 < x < \infty$.

(e) $f = (1+\varepsilon x)^{\frac{1}{\varepsilon}}$, for $0 < x < \infty$.

(f) $f = \displaystyle\int_0^\varepsilon \sin\left(x + \varepsilon x^2\right) dx$.

(g) $f = \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{\varepsilon}{n}\right)$.

(h) $\displaystyle\int_0^\varepsilon \sin(x + \varepsilon x^2)\mathrm{d}x$.

(i) $f = \displaystyle\int_0^1 \frac{\mathrm{d}x}{\varepsilon + x(x-1)}$.

Problem 5 (1.18)

Find a two-term asymptotic expansion, for small ε, of each solution x of the following equations:

(a) $x^2 + x - \varepsilon = 0$

(e) $\varepsilon x^3 - x + \varepsilon = 0$

(h) $x^2 + \varepsilon\sqrt{2 + x} = \cos(\varepsilon)$

(p) $xe^{-x} = \varepsilon$

Problem 6 (1.19)
This problem considers the equation $1 + \sqrt{x^2 + \varepsilon} = e^x$.

(a) Explain why there is one real root for small ε.

(b) Find a two-term expansion of the root.

Problem 7 (1.20)
In this problem you should sketch the functions in each equation and then use this to determine the number and approximate location of the real-valued solutions. With this, find a three-term asymptotic expansion, for small ε, of the nonzero solutions.

(a) $x = \tanh\left(\frac{x}{\varepsilon}\right)$,

(b) $x = \tan\left(\frac{x}{\varepsilon}\right)$.

Problem 8 (1.21)
To determine the natural frequencies of an elastic string, one is faced with solving the equation $\tan(\lambda) = \lambda$.

(a) After sketching the two functions in this equation on the same graph explain why there is an infinite number of solutions.

(b) To find an asymptotic expansion of the large solutions of the equation, assume that $\lambda \sim \varepsilon^{-\alpha}(\lambda_0 + \varepsilon^{\beta}\lambda_1$. Find $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$ (note that λ_0 and λ_1 are nonzero and $\beta > 0$).