

Math 207C Homework #1

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Holmes Ex #1.1, 1.3, 1.5, 1.7, 1.12, 1.18 (a)(e)(h)(p), 1.19, 1.20, 1.21

Problem 1 (1.1)

(a) What values of α , if any, yield $f = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$ for each of the following functions?

(i) $f = \sqrt{1 + \varepsilon^2}$

(v) $f = \varepsilon \ln(\varepsilon)$

(ii) $f = \varepsilon \sin(\varepsilon)$

(vi) $f = \sin(1/\varepsilon)$

(iii) $f = (1 - e^\varepsilon)^{-1}$

(vii) $f = \sqrt{x + \varepsilon}$, where $0 \leq x \leq 1$

(iv) $f = \ln(1 + \varepsilon)$

(b) For the functions listed in (a), what values of α , if any, yield $f = o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$?

Solution

(i) We can expand by plugging x^2 into the Taylor series for $\sqrt{1 + x}$:

$$f = \sqrt{1 + \varepsilon^2} = 1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \dots$$

Now, taking the ratio of limits we get,

$$\lim_{\varepsilon \downarrow 0} \frac{f}{\varepsilon^\alpha} = \lim_{\varepsilon \downarrow 0} \left(\varepsilon^{-\alpha} + \frac{1}{2}\varepsilon^{2-\alpha} - \frac{1}{8}\varepsilon^{4-\alpha} + \dots \right) = \begin{cases} \infty, & \alpha > 0 \\ 1, & \alpha = 0 \\ 0, & \alpha < 0 \end{cases}$$

Therefore, $f = O(\varepsilon^\alpha)$ for $\alpha \leq 0$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 0$

(ii) Now, we can expand sine into its Taylor series

$$f = \varepsilon \sin(\varepsilon) = \varepsilon \left(\varepsilon - \frac{\varepsilon^3}{3!} + \dots \right) = \varepsilon^2 - \frac{\varepsilon^4}{3!} + \dots$$

For similar reasons as above, we have that $f = O(\varepsilon^\alpha)$ for $\alpha \leq 2$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 2$.

(iii) For this, we can use another Taylor series expansion $e^x = 1 + x + \frac{x^2}{2} + \dots$. The following then holds

$$\begin{aligned} f &= \frac{1}{1 - e^\varepsilon} \\ &= \frac{1}{-\varepsilon - \frac{\varepsilon^2}{2} - \dots} \\ &= -\frac{1}{\varepsilon} \cdot \frac{1}{1 + \left(\frac{\varepsilon}{2} + \dots\right)} \\ &= -\frac{1}{\varepsilon} - \frac{1}{2} + \dots \end{aligned}$$

It follows that $f = O(\varepsilon^\alpha)$ for $\alpha \leq -1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < -1$.

(iv) Use the Taylor series expansion for $\ln(1+x)$:

$$f = \ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \dots$$

Therefore, $f = O(\varepsilon^\alpha)$ for $\alpha \leq 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

(v) For $f = \varepsilon \ln(\varepsilon)$ Start by setting up the limit and then apply L'Hospital's rule:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \ln(\varepsilon)}{\varepsilon^\alpha} &= \lim_{\varepsilon \downarrow 0} \frac{\ln(\varepsilon)}{\varepsilon^{\alpha-1}} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\frac{1}{\varepsilon}}{(\alpha-1)\varepsilon^{\alpha-2}} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{(\alpha-1)\varepsilon^{\alpha-1}}. \end{aligned}$$

So we have that $f = O(\varepsilon^\alpha)$ for $\alpha < 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

(vi) Set up the limit

$$\lim_{\varepsilon \downarrow 0} \left| \frac{\sin(1/\varepsilon)}{\varepsilon^\alpha} \right| \leq \lim_{\varepsilon \downarrow 0} \varepsilon^{-\alpha}.$$

which is finite and thus $f = O(\varepsilon^\alpha)$ for $\alpha \leq 0$. It is zero when $\alpha < 0$, so $f = o(\varepsilon^\alpha)$ there.

(vii) If $x = 0$ then $f = \varepsilon^{1/2}$ and is $O(\varepsilon^\alpha)$ for $\alpha \leq 1/2$. Similarly, $f = o(\varepsilon^\alpha)$ for $\alpha < 1/2$.

For $x \neq 0$ then $\sqrt{x+\varepsilon} = \sqrt{x} + \frac{1}{2\sqrt{x}}\varepsilon + \dots$ and we find that $f = O(\varepsilon^\alpha)$ for $\alpha \leq 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

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Problem 2 (1.3)

This problem establishes some of the basic properties of the order symbols, some of which are used extensively in this book. The limit assumed here is $\varepsilon \downarrow 0$.

- (a) If $f = o(g)$ and $g = O(h)$, or if $f = O(g)$ and $g = o(h)$, then show that $f = o(h)$. Note that this result can be written as $o(O(h)) = O(o(h)) = o(h)$.
- (b) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f + g = O(|\phi_1| + |\phi_2|)$. Also, explain why the absolute signs are necessary. Note that this result can be written as $O(f) + O(g) = O(|f| + |g|)$.
- (c) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $fg = O(\phi_1\phi_2)$. This result can be written as $O(f)O(g) = O(fg)$.
- (d) Show that $O(O(f)) = O(f)$.
- (e) Show that $O(f)o(g) = o(f)o(g) = o(fg)$.

Solution

- (a) For the first case, the following holds for all δ_1 and some constants $C, \varepsilon_1, \varepsilon_2$

$$|f(\varepsilon)| \leq \delta_1 |g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_1$$

and

$$|g(\varepsilon)| \leq C |h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_2$$

Therefore,

$$|f(\varepsilon)| \leq \delta |h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$$

where $\delta = C\delta_1$. Since δ_1 is arbitrary, for any $\delta > 0$, we can set $\delta_1 = \delta/C$ and we may conclude that $f = o(h)$.

Similarly, for all δ'_1 and some constants $C', \varepsilon'_1, \varepsilon'_2$

$$|f(\varepsilon)| \leq C' |g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon'_1$$

and

$$|g(\varepsilon)| \leq \delta'_1 |h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon'_2$$

Therefore,

$$|f(\varepsilon)| \leq \delta' |h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon'_1, \varepsilon'_2\}$$

where $\delta' = C'\delta'_1$ and we may conclude that $f = o(h)$ in the same way.

- (b) Assuming $f = O(\phi_1), g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| &\leq C_1 |\phi_1(\varepsilon)|, & 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq C_2 |\phi_2(\varepsilon)|, & 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(f + g)(\varepsilon)| &= |f(\varepsilon) + g(\varepsilon)| \\ &\leq |f(\varepsilon)| + |g(\varepsilon)| \\ &\leq C_1 |\phi_1(\varepsilon)| + C_2 |\phi_2(\varepsilon)| \\ &\leq C (|\phi_1(\varepsilon)| + |\phi_2(\varepsilon)|) \end{aligned}$$

holds for $C = \max\{C_1, C_2\}$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $f + g = O(|\phi_1| + |\phi_2|)$. The necessity of the absolute values are evident from the last inequality.

(c) Assuming $f = O(\phi_1)$, $g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| &\leq C_1|\phi_1(\varepsilon)|, & 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq C_2|\phi_2(\varepsilon)|, & 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(fg)(\varepsilon)| &= |f(\varepsilon)g(\varepsilon)| \\ &= |f(\varepsilon)||g(\varepsilon)| \\ &\leq C_1|\phi_1(\varepsilon)| \cdot C_2|\phi_2(\varepsilon)| \\ &= C|\phi_1(\varepsilon)\phi_2(\varepsilon)| \end{aligned}$$

holds for $C = C_1C_2$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $fg = O(\phi_1\phi_2)$.

(d) For this we need to show that

$$|O(f(\varepsilon))| \leq C|O(g(\varepsilon))|$$

for some constant C and ε in the same range as above. Equality holds for $C = 1$ and we are done.

(e) Assuming $f = O(\phi_1)$, $g = o(\phi_2)$, there exists constants $C, \varepsilon_1, \varepsilon_2$ such that, for all $\delta' > 0$,

$$\begin{aligned} |f(\varepsilon)| &\leq C|\phi_1(\varepsilon)|, & 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq \delta'|\phi_2(\varepsilon)|, & 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(fg)(\varepsilon)| &= |f(\varepsilon)g(\varepsilon)| \\ &= |f(\varepsilon)||g(\varepsilon)| \\ &\leq C|\phi_1(\varepsilon)|\delta'|\phi_2(\varepsilon)| \\ &= \delta_1|\phi_1(\varepsilon)|\delta_2|\phi_2(\varepsilon)| \\ &= \delta|\phi_1(\varepsilon)\phi_2(\varepsilon)| \end{aligned}$$

holds for $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ and all possible choices δ_1, δ_2 where we left $\delta' = \frac{\delta_1\delta_2}{C}$ or $\delta' = \frac{\delta}{C}$ in the last line. This suffices to show that $O(f)o(g) = o(f)o(g) = o(fg)$. ■

Problem 3

1.5 Suppose $f = o(\phi)$ for small ε , where f and ϕ are continuous.

- (a) Give an example to show that it is not necessarily true that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon \phi d\varepsilon\right).$$

- (b) Show that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon |\phi| d\varepsilon\right).$$

Solution

- (a) This could be accomplished with functions that are oscillatory as ε approaches 0 and affect the decay rate after integrating. I thought about this for some time and think something with $\sin(1/\varepsilon)$ could work but couldn't figure it out.
- (b) If $f = o(\phi)$ then $|f(\varepsilon)| \leq \delta|\phi(\varepsilon)|$ for all $\delta > 0$ and $0 < \varepsilon < \varepsilon_1$ for some ε_1 . The following then holds

$$\begin{aligned} \left|\int_0^\varepsilon f d\varepsilon\right| &\leq \int_0^\varepsilon |f| d\varepsilon \\ &\leq \int_0^\varepsilon |\delta\phi| d\varepsilon \\ &\leq \delta \int_0^\varepsilon |\phi| d\varepsilon \end{aligned}$$

and we are done since this implies $f = o(\phi)$.

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Problem 4 (1.7)

Assuming $f \sim a_1\varepsilon^\alpha + a_2\varepsilon^\beta + \dots$, find α, β (with $\alpha < \beta$) and nonzero a_1, a_2 for the following functions:

(a) $f = \frac{1}{1 - e^\varepsilon}.$

(b) $f = \left[1 + \frac{1}{\cos(\varepsilon)}\right]^{\frac{3}{2}}.$

(c) $f = 1 + \varepsilon - 2 \ln(1 + \varepsilon) - \frac{1}{1 + \varepsilon}.$

(d) $f = \sinh(\sqrt{1 + \varepsilon x}),$ for $0 < x < \infty.$

(e) $f = (1 + \varepsilon x)^{\frac{1}{\varepsilon}},$ for $0 < x < \infty.$

(f) $f = \int_0^\varepsilon \sin(x + \varepsilon x^2) dx.$

(g) $f = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{\varepsilon}{n}\right).$

(h) $f = \prod_{k=0}^n (1 + \varepsilon k)$ where n is a positive integer.

(i) $f = \int_0^\pi \frac{\sin(x)}{\sqrt{1 + \varepsilon x}} dx.$

Solution

(a) First we substitute the Taylor expansion for e^ε into the denominator:

$$\begin{aligned} f &\sim \frac{1}{1 - \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \dots\right)} \\ &\sim \frac{1}{-\varepsilon - \frac{\varepsilon^2}{2} - \dots} \\ &\sim -\frac{1}{\varepsilon} \cdot \frac{1}{1 + \left(\frac{\varepsilon}{2} + \dots\right)} \\ &\sim -\frac{1}{\varepsilon} \cdot \left(1 - \frac{\varepsilon}{2} + \dots\right), \text{ by Taylor expansion of } \frac{1}{1+x} \\ &\sim -\frac{1}{\varepsilon} + \frac{1}{2} + \dots \end{aligned}$$

Therefore, $\alpha = -1, a_1 = -1, \beta = 0, a_2 = 1/2.$

(b) Apply the Taylor expansion of $\cos(\varepsilon)$:

$$\begin{aligned}
 f &\sim \left(1 + \frac{1}{1 - \frac{\varepsilon^2}{2} + \dots}\right)^{\frac{3}{2}} \\
 &\sim \left(2 + \frac{\varepsilon^2}{2} + \dots\right)^{\frac{3}{2}} \\
 &\sim 2^{\frac{3}{2}} \left(1 + \frac{\varepsilon^2}{4} + \dots\right)^{\frac{3}{2}} \\
 &\sim 2^{\frac{3}{2}} \left(1 + \frac{3}{2} \frac{\varepsilon^2}{4} + \dots\right) \\
 &\sim 2^{\frac{3}{2}} \left(1 + \frac{3\varepsilon^2}{8} + \dots\right)
 \end{aligned}$$

Therefore, $\alpha = 0, a_1 = 2^{\frac{3}{2}}, \beta = 2, a_2 = 2^{-\frac{3}{2}}3$.

(c) Apply the Taylor expansion of $\ln(1+x)$ and $\frac{1}{1+x}$:

$$\begin{aligned}
 f &\sim 1 + \varepsilon - 2 \left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \dots \right) \\
 &\quad - \left(1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \varepsilon^4 - \dots \right) \\
 &\sim \frac{1}{3}\varepsilon^3 - \frac{1}{2}\varepsilon^4 + \dots
 \end{aligned}$$

Therefore, $\alpha = 3, a_1 = \frac{1}{3}, \beta = 4, a_2 = -\frac{1}{2}$.

(d) Here it is easier to compute the Taylor coefficients directly:

$$\begin{aligned}
 f(x, 0) &= \sinh(1) \\
 \frac{d}{d\varepsilon} f(x, \varepsilon) &= \cosh(\sqrt{1 + \varepsilon x}) \cdot \frac{x}{2\sqrt{1 + \varepsilon x}} \\
 \frac{d}{d\varepsilon} f(x, 0) &= \frac{\cosh(1)x}{2}
 \end{aligned}$$

Therefore, $\alpha = 0, a_1 = \sinh(1), \beta = 1, a_2 = \frac{\cosh(1)x}{2}$.

(e) We can make the change of variables $y = \frac{1}{\varepsilon}$ to get

$$\begin{aligned}
 a_1 &= \lim_{\varepsilon \downarrow 0} (1 + \varepsilon x)^{\frac{1}{\varepsilon}} \\
 &= \lim_{y \rightarrow \infty} \left(1 + \frac{x}{y}\right)^y \\
 &= e^x
 \end{aligned}$$

Then we can find the next asymptotic term:

$$\begin{aligned}
 a_2 &= \lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon) - a_1}{\varepsilon} \\
 &= \lim_{\varepsilon \downarrow 0} \frac{(1 + \varepsilon x)^{\frac{1}{\varepsilon}} - e^x}{\varepsilon} \\
 &= \lim_{y \rightarrow \infty} y \left[\left(1 + \frac{x}{y}\right)^y - e^x \right] \\
 &= \lim_{y \rightarrow \infty} y \left[\exp\left(y \ln\left(1 + \frac{x}{y}\right)\right) - e^x \right] \\
 &= \lim_{y \rightarrow \infty} y \left[\exp\left(y \left(\frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \dots\right)\right) - e^x \right] \\
 &= \lim_{y \rightarrow \infty} y \left[e^x \cdot \exp\left(-\frac{x^2}{2y} + \frac{x^3}{3y^2} - \dots\right) - e^x \right] \\
 &= \lim_{y \rightarrow \infty} y \left[e^x \left(1 - \frac{x^2}{2y} + \frac{x^3}{3y^2} - \dots\right) - e^x \right] \\
 &= \lim_{y \rightarrow \infty} e^x \left(-\frac{x^2}{2} + \frac{x^3}{3y} - \dots\right) \\
 &= -\frac{1}{2}e^x x^2
 \end{aligned}$$

Therefore, $\alpha = 0, a_1 = e^x, \beta = 1, a_2 = -\frac{1}{2}e^x x^2$.

- (f) Since $\sin(x + \varepsilon x^2)$ is integrable on $[0, \varepsilon]$, we can integrate its asymptotic terms to get the desired equation. Computing the first two terms from the Taylor formulas shows $b_1 = f(x, 0) = \sin(x)$ and the second term is computed as follows

$$\begin{aligned}
 \frac{d}{d\varepsilon} f(x, \varepsilon) &= x^2 \cos(x + \varepsilon x^2) \\
 b_2 &= x^2 \cos(x).
 \end{aligned}$$

So we have $\sin(x + \varepsilon x^2) \sim \sin(x) + \varepsilon x^2 \cos(x)$. Integrating term by term yields

$$\begin{aligned}
 f &\sim -\cos(x)|_0^\varepsilon + \varepsilon \left[x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) \right] \Big|_0^\varepsilon + \dots \\
 &\sim 1 - \cos(\varepsilon) + \varepsilon^3 \sin(\varepsilon) + 2\varepsilon^2 \cos(\varepsilon) - 2\varepsilon \sin(\varepsilon) + \dots \\
 &\sim 1 - \left(1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{24} \dots\right) + \varepsilon^3 \left(\varepsilon - \frac{\varepsilon^3}{6} + \dots\right) + 2\varepsilon^2 \left(1 - \frac{\varepsilon^2}{2} + \dots\right) - 2\varepsilon \left(\varepsilon - \frac{\varepsilon^3}{6} + \dots\right) + \dots \\
 &\sim \frac{1}{2}\varepsilon^2 + \frac{7}{24}\varepsilon^4 + \dots
 \end{aligned}$$

Therefore, $\alpha = 2, a_1 = \frac{1}{2}, \beta = 4, a_2 = \frac{7}{24}$.

- (g) Starting by plugging in the Taylor series for $\sin\left(\frac{\varepsilon}{n}\right)$ to get

$$\begin{aligned}
 f &\sim \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\varepsilon}{n} - \frac{\varepsilon^3}{6n^3} + \dots\right) \\
 &\sim \ln(2)\varepsilon - \frac{\varepsilon^3}{6} \sum_{n=1}^{\infty} \frac{1}{2^n n^3} + \dots
 \end{aligned}$$

Therefore, $\alpha = 1, a_1 = \ln(2), \beta = 3, a_2 = -\frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{2^n n^3}$.

- (h) The expansion will be finite and exact in this case and we can just multiply it out

$$\begin{aligned} f &= 1 + \varepsilon \sum_{k=0}^n k + \cdots \\ &= 1 + \varepsilon \frac{n(n+1)}{2} + \cdots \end{aligned}$$

Therefore, $\alpha = 0, a_1 = 1, \beta = 1, a_2 = \frac{n(n+1)}{2}$.

- (i) Again we can apply an expansion and then integrate term by term

$$\begin{aligned} f &\sim \int_0^\pi \frac{\sin(x)}{\sqrt{1+\varepsilon x}} dx \\ &\sim \int_0^\pi \sin(x) \left(1 - \frac{\varepsilon x}{2} + \cdots\right) dx \\ &\sim \int_0^\pi \sin(x) dx - \frac{\varepsilon}{2} \int_0^\pi x \sin(x) dx + \cdots \\ &\sim [-\cos(x)]_0^\pi - \frac{\varepsilon}{2} [-x \cos(x) + \sin(x)]_0^\pi + \cdots \\ &\sim 2 - \frac{\pi}{2} \varepsilon. \end{aligned}$$

Therefore, $\alpha = 0, a_1 = 2, \beta = 1, a_2 = -\frac{\pi}{2}$.

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Problem 5 (1.12)

Suppose $f(\varepsilon) \sim a_0\phi_0(\varepsilon) + a_1\phi_1(\varepsilon) + \cdots$ and $g(\varepsilon) \sim b_0\phi_0(\varepsilon) + b_1\phi_1(\varepsilon) + \cdots$ as $\varepsilon \downarrow \varepsilon_0$, where $\phi_0, \phi_1, \phi_2, \dots$ is an asymptotic sequence as $\varepsilon \downarrow \varepsilon_0$.

- (a) Show that $f + g \sim (a_0 + b_0)\phi_0 + (a_1 + b_1)\phi_1 + \cdots$ as $\varepsilon \downarrow \varepsilon_0$.
- (b) Assuming $a_0b_0 \neq 0$, show that $fg \sim a_0b_0\phi_0^2$ as $\varepsilon \downarrow \varepsilon_0$. Also, discuss the possibilities for the next term in the expansion.
- (c) Suppose that $\phi_i\phi_j = \phi_{i+j}$ for all i, j . In this case show that

$$fg \sim a_0b_0\phi_0 + (a_0b_1 + a_1b_0)\phi_1 + (a_0b_2 + a_1b_1 + a_2b_0)\phi_2 + \cdots \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

- (d) Under what conditions on the exponents will the following be an asymptotic sequence satisfying the condition in part (c): $\phi_0 = (\varepsilon - \varepsilon_0)^\alpha$, $\phi_1 = (\varepsilon - \varepsilon_0)^\beta$, $\phi_2 = (\varepsilon - \varepsilon_0)^\gamma$, ...?

Solution

- (a) The above form asymptotic expansions for f and g up to the n^{th} term if and only if

$$f = \sum_{k=0}^m a_k \phi_k + o(\phi_m) \quad \text{for } m = 0, 1, 2, \dots, n \quad \text{as } \varepsilon \downarrow \varepsilon_0,$$

and

$$g = \sum_{k=0}^m b_k \phi_k + o(\phi_m) \quad \text{for } m = 0, 1, 2, \dots, n \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

It follows that

$$\begin{aligned} f + g &= \sum_{k=0}^m (a_k + b_k) \phi_k + 2o(\phi_m) \\ &= \sum_{k=0}^m (a_k + b_k) \phi_k + o(\phi_m) \end{aligned}$$

for $m = 0, 1, 2, \dots, n$ since the constant factor of 2 does not affect the little-oh asymptotic order term.

- (b) Similarly, we have

$$\begin{aligned} fg &= \left(\sum_{k=0}^m a_k \phi_k + o(\phi_m) \right) \left(\sum_{k=0}^m b_k \phi_k + o(\phi_m) \right) \\ &= a_0b_0\phi_0^2 + (a_0b_1 + a_1b_0)\phi_0\phi_1 + a_1b_1\phi_1^2 + (a_0b_2 + a_2b_0)\phi_0\phi_2 + \cdots + o(\phi_m) \end{aligned}$$

again for $m = 0, 1, \dots, n$. From this, we can confirm that $fg \sim a_0b_0\phi_0^2$. The next term in the expansion would then depend on the relative ordering of higher terms like $\phi_0\phi_2$ versus ϕ_1^2 .

- (c) This follows directly from the previous expansion:

$$\begin{aligned} fg &\sim a_0b_0\phi_0^2 + (a_0b_1 + a_1b_0)\phi_0\phi_1 + a_1b_1\phi_1^2 + (a_0b_2 + a_2b_0)\phi_0\phi_2 + \cdots \\ &= a_0b_0\phi_{0+0} + (a_0b_1 + a_1b_0)\phi_{0+1} + a_1b_1\phi_{1+1} + (a_0b_2 + a_2b_0)\phi_{0+2} + \cdots \\ &= a_0b_0\phi_0 + (a_0b_1 + a_1b_0)\phi_1 + (a_1b_1 + a_0b_2 + a_2b_0)\phi_2 + \cdots \end{aligned}$$

- (d) For this condition to hold, the powers would have to follow an arithmetic sequence starting at 0. So $\alpha = 0, \beta, \gamma = 2\beta, 3\beta, \dots$

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Problem 6 (1.18)

Find a two-term asymptotic expansion, for small ε , of each solution x of the following equations:

- (a) $x^2 + x - \varepsilon = 0$ First, look at the 0th order terms $x_0^2 + x_0 = 0$. It follows that $x_0 = 0, -1$. Let's proceed first with $x_0 = 0$. Then we can examine the first order equation

$$\varepsilon^2 x_1^2 + \varepsilon x_1 - \varepsilon \sim \varepsilon x_1 - \varepsilon = 0$$

and it follows that $x_1 = 1$. This first solution is then $x' \sim \varepsilon$.

For the other root, we get

$$(-1 + \varepsilon x_1)^2 + (-1 + \varepsilon x_1) - \varepsilon \sim 1 - 2\varepsilon x_1 + \varepsilon^2 x_1^2 - 1 + \varepsilon x_1 - \varepsilon \sim -\varepsilon x_1 - \varepsilon = 0$$

which implies $x_1 = -1$ and $x'' \sim -1 - \varepsilon$

- (e) $\varepsilon x^3 - x + \varepsilon = 0$ This becomes trickier since the standard zero order equation has a solution for the single root around $x_0 = 0$. Looking at the first
- (h) $x^2 + \varepsilon\sqrt{2+x} = \cos(\varepsilon)$
- (p) $xe^{-x} = \varepsilon$

Problem 7 (1.19)

This problem considers the equation $1 + \sqrt{x^2 + \varepsilon} = e^x$.

- (a) Explain why there is one real root for small ε .
- (b) Find a two-term expansion of the root.

Problem 8 (1.20)

In this problem you should sketch the functions in each equation and then use this to determine the number and approximate location of the real-valued solutions. With this, find a three-term asymptotic expansion, for small ε , of the nonzero solutions.

(a) $x = \tanh\left(\frac{x}{\varepsilon}\right),$

(b) $x = \tan\left(\frac{x}{\varepsilon}\right).$

Problem 9 (1.21)

To determine the natural frequencies of an elastic string, one is faced with solving the equation $\tan(\lambda) = \lambda$.

- (a) After sketching the two functions in this equation on the same graph explain why there is an infinite number of solutions.
- (b) To find an asymptotic expansion of the large solutions of the equation, assume that $\lambda \sim \varepsilon^{-\alpha}(\lambda_0 + \varepsilon^\beta \lambda_1)$. Find $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$ (note that λ_0 and λ_1 are nonzero and $\beta > 0$).