Math 207C Homework #1

Ian Gallagher

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Holmes Ex #1.1, 1.3, 1.5, 1.7, 1.12, 1.18 (a)(e)(h)(p), 1.19, 1.20, 1.21

Problem 1 (1.1)

(a) What values of α, if any, yield $f = O\big(\varepsilon^\alpha\big)$ as $\varepsilon \downarrow 0$ for each of the following functions?

(i) $f = \sqrt{1+\varepsilon^2}$

(ii) $f = \varepsilon \sin(\varepsilon)$

(iii) $f = \big(1 - e^\varepsilon\big)^{-1}$

(iv) $f = \ln(1+\varepsilon)$

(v) $f = \varepsilon \ln(\varepsilon)$

(vi) $f = \sin(1/\varepsilon)$

(vii) $f = \sqrt{x+\varepsilon}$, where $0 \geq x \geq 1$

(b) For the functions listed in (a), what values of α, if any, yield $f = o\big(\varepsilon^\alpha\big)$ as $\varepsilon \downarrow 0$?

Solution

(i) We can expand by plugging x^2 into the Taylor series for $\sqrt{1+x}$:

$$f = \sqrt{1+\varepsilon^2} = 1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \cdots .$$

Now, taking the ratio of limits we get,

$$\lim_{\varepsilon\downarrow 0} \frac{f}{\varepsilon^\alpha} = \lim_{\varepsilon\downarrow 0} \left(\varepsilon^{-\alpha} + \frac{1}{2}\varepsilon^{2-\alpha} - \frac{1}{8}\varepsilon^{4-\alpha} + \cdots\right) = \begin{cases} \infty, & \alpha > 0 \\ 1, & \alpha = 0 \\ 0, & \alpha < 0 \end{cases}$$

Therefore, $f = O(\varepsilon^\alpha)$ for $\alpha \leq 0$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 0$

(ii) Now, we can expand sine into its Taylor series

$$f = \varepsilon \sin(\varepsilon) = \varepsilon \left(\varepsilon - \frac{\varepsilon^3}{3!} + \cdots\right) = \varepsilon^2 - \frac{\varepsilon^4}{3!} + \cdots$$

For similar reasons as above, we have that $f = O(\varepsilon^\alpha)$ for $\alpha \leq 2$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 2$.

(iii) For this, we can use another Taylor series expansion $e^x = 1 + x + \frac{x^2}{2} + \cdots$ The following then holds

$$\begin{aligned} f &= \frac{1}{1-e^\varepsilon} \\ &= \frac{1}{-\varepsilon - \frac{\varepsilon^2}{2} - \cdots} \\ &= -\frac{1}{\varepsilon} \cdot \frac{1}{1 + \left(\frac{\varepsilon}{2} + \cdots\right)} \\ &= -\frac{1}{\varepsilon} - \frac{1}{2} + \cdots \end{aligned}$$

It follows that $f = O(\varepsilon^\alpha)$ for $\alpha \leq -1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < -1$.

(iv) Use the Taylor series expansion for $\ln(1+x)$:

$$f = \ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \cdots$$

Therefore, $f = O(\varepsilon^\alpha)$ for $\alpha \leq 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

(v) For $f = \varepsilon \ln(\varepsilon)$ Start by setting up the limit and then apply L'Hospital's rule:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \ln(\varepsilon)}{\varepsilon^\alpha} &= \lim_{\varepsilon \downarrow 0} \frac{\ln(\varepsilon)}{\varepsilon^{\alpha-1}} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\frac{1}{\varepsilon}}{(\alpha-1)\varepsilon^{\alpha-2}} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{(\alpha-1)\varepsilon^{\alpha-1}}. \end{aligned}$$

So we have that $f = O(\varepsilon^\alpha)$ for $\alpha < 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

(vi) Set up the limit

$$\lim_{\varepsilon \downarrow 0} \left| \frac{\sin(1/\varepsilon)}{\varepsilon^\alpha} \right| \leq \lim_{\varepsilon \downarrow 0} \varepsilon^{-\alpha}.$$

which is finite and thus $f = O(\varepsilon^\alpha)$ for $\alpha \leq 0$. It is zero when $\alpha < 0$, so $f = o(\varepsilon^\alpha)$ there.

(vii) If $x = 0$ then $f = \varepsilon^{1/2}$ and is $O(\varepsilon^\alpha)$ for $\alpha \leq 1/2$. Similarly, $f = o(\varepsilon^\alpha)$ for $\alpha < 1/2$.

For $x \neq 0$ then $\sqrt{x+\varepsilon} = \sqrt{x} + \frac{1}{2\sqrt{x}}\varepsilon + \cdots$ and we find that $f = O(\varepsilon^\alpha)$ for $\alpha \leq 1$ and $f = o(\varepsilon^\alpha)$ for $\alpha < 1$.

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Problem 2 (1.3)
This problem establishes some of the basic properties of the order symbols, some of which are used extensively in this book. The limit assumed here is $\varepsilon \downarrow 0$.

(a) If $f = o(g)$ and $g = O(h)$, or if $f = O(g)$ and $g = o(h)$, then show that $f = o(h)$. Note that this result can be written as $o(O(h)) = O(o(h)) = o(h)$.

(b) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f+g = O(|\phi_1|+|\phi_2|)$. Also, explain why the absolute signs are necessary. Note that this result can be written as $O(f) + O(g) = O(|f| + |g|)$.

(c) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $fg = O(\phi_1\phi_2)$. This result can be written as $O(f)O(g) = O(fg)$.

(d) Show that $O(O(f)) = O(f)$.

(e) Show that $O(f)o(g) = o(f)o(g) = o(fg)$.

Solution

(a) For the first case, the following holds for all δ_1 and some constants $C, \varepsilon_1, \varepsilon_2$

$$|f(\varepsilon)| \leq \delta_1 |g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_1$$

and

$$|g(\varepsilon)| \leq C|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_2$$

Therefore,

$$|f(\varepsilon)| \leq \delta|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$$

where $\delta = C\delta_1$. Since δ_1 is arbitrary, for any $\delta > 0$, we can set $\delta_1 = \delta/C$ and we may conclude that $f = o(h)$.

Similarly, for all δ_1' and some constants $C', \varepsilon_1', \varepsilon_2'$

$$|f(\varepsilon)| \leq C'|g(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_1'$$

and

$$|g(\varepsilon)| \leq \delta_1'|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \varepsilon_2'$$

Therefore,

$$|f(\varepsilon)| \leq \delta'|h(\varepsilon)|, \text{ for } 0 < \varepsilon < \min\{\varepsilon_1', \varepsilon_2'\}$$

where $\delta' = C'\delta_1'$ and we may conclude that $f = o(h)$ in the same way.

(b) Assuming $f = O(\phi_1), g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| &\leq C_1|\phi_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq C_2|\phi_2(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(f+g)(\varepsilon)| &= |f(\varepsilon) + g(\varepsilon)| \\ &\leq |f(\varepsilon)| + |g(\varepsilon)| \\ &\leq C_1|\phi_1(\varepsilon)| + C_2|\phi_2(\varepsilon)| \\ &\leq C\left(|\phi_1(\varepsilon)| + |\phi_2(\varepsilon)|\right) \end{aligned}$$

holds for $C = \max\{C_1, C_2\}$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $f+g = O(|\phi_1|+|\phi_2|)$. The necessity of the absolute values are evident from the last inequality.

(c) Assuming $f = O(\phi_1), g = O(\phi_2)$, there exists constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned} |f(\varepsilon)| &\leq C_1|\phi_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq C_2|\phi_2(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(fg)(\varepsilon)| &= |f(\varepsilon)g(\varepsilon)| \\ &= |f(\varepsilon)||g(\varepsilon)| \\ &\leq C_1|\phi_1(\varepsilon)| \cdot C_2|\phi_2(\varepsilon)| \\ &= C|\phi_1(\varepsilon)\phi_2(\varepsilon)| \end{aligned}$$

holds for $C = C_1C_2$ and $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, $fg = O(\phi_1\phi_2)$.

(d) For this we need to show that

$$|O(f(\varepsilon)| \leq C|O(f(\varepsilon)|$$

for some constant C and ε in the same range as above. Equality holds for $C = 1$ and we are done.

(e) Assuming $f = O(\phi_1), g = o(\phi_2)$, there exists constants $C, \varepsilon_1, \varepsilon_2$ such that, for all $\delta' > 0$,

$$\begin{aligned} |f(\varepsilon)| &\leq C|\phi_1(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_1 \\ |g(\varepsilon)| &\leq \delta'|\phi_2(\varepsilon)|, \quad 0 < \varepsilon < \varepsilon_2 \end{aligned}$$

Therefore,

$$\begin{aligned} |(fg)(\varepsilon)| &= |f(\varepsilon)g(\varepsilon)| \\ &= |f(\varepsilon)||g(\varepsilon)| \\ &\leq C|\phi_1(\varepsilon)|\delta'|\phi_2(\varepsilon)| \\ &= \delta_1|\phi_1(\varepsilon)|\delta_2|\phi_2(\varepsilon)| \\ &= \delta|\phi_1(\varepsilon)\phi_2(\varepsilon)| \end{aligned}$$

holds for $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ and all possible choices δ_1, δ_2 where we left $\delta' = \frac{\delta_1\delta_2}{C}$ or $\delta' = \frac{\delta}{C}$ in the last line. This suffices to show that $O(f)o(g) = o(f)o(g) = o(fg)$.

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Problem 3

1.5 Suppose $f = o(\phi)$ for small ε, where f and ϕ are continuous.

(a) Give an example to show that it is not necessarily true that

$$\int_0^\varepsilon f \mathrm{d}\varepsilon = o\left(\int_0^\varepsilon \phi \mathrm{d}\varepsilon\right).$$

(b) Show that

$$\int_0^\varepsilon f \mathrm{d}\varepsilon = o\left(\int_0^\varepsilon |\phi| \mathrm{d}\varepsilon\right).$$

Solution

(a) This could be accomplished with functions that are oscillatory as ε approaches 0 and affect the decay rate after integrating. I thought about this for some time and think something with $\sin(1/\varepsilon)$ could work but couldn't figure it out.

(b) If $f = o(\phi)$ then $|f(\varepsilon)| \leq \delta|\phi(\varepsilon)|$ for all $\delta > 0$ and $0 < \varepsilon < \varepsilon_1$ for some ε_1. The following then holds

$$\begin{aligned}
\left|\int_0^\varepsilon f \mathrm{d}\varepsilon\right| &\leq \int_0^\varepsilon |f| \,\mathrm{d}\varepsilon \\
&\leq \int_0^\varepsilon |\delta\phi| \,\mathrm{d}\varepsilon \\
&\leq \delta \int_0^\varepsilon |\phi| \,\mathrm{d}\varepsilon
\end{aligned}$$

and we are done since this implies $f = o(\phi)$.

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Problem 4 (1.7)
Assuming $f \sim a_1\varepsilon^\alpha + a_2\varepsilon^\beta + \cdots$, find α, β (with $\alpha < \beta$) and nonzero a_1, a_2 for the following functions:

(a) $f = \dfrac{1}{1 - e^\varepsilon}$.

(b) $f = \left[1 + \dfrac{1}{\cos(\varepsilon)}\right]^{\frac{3}{2}}$.

(c) $f = 1 + \varepsilon - 2\ln(1+\varepsilon) - \dfrac{1}{1+\varepsilon}$.

(d) $f = \sinh\left(\sqrt{1+\varepsilon x}\right)$, for $0 < x < \infty$.

(e) $f = (1+\varepsilon x)^{\frac{1}{\varepsilon}}$, for $0 < x < \infty$.

(f) $f = \displaystyle\int_0^\varepsilon \sin\left(x + \varepsilon x^2\right) dx$.

(g) $f = \displaystyle\sum_{n=1}^\infty \left(\frac{1}{2}\right)^n \sin\left(\frac{\varepsilon}{n}\right)$.

(h) $f = \displaystyle\prod_{k=0}^n (1+\varepsilon k)$ where n is a positive integer.

(i) $f = \displaystyle\int_0^\pi \frac{\sin(x)}{\sqrt{1+\varepsilon x}} \mathrm{d}x$.

Solution

(a) First we substitute the Taylor expansion for e^ε into the denominator:

$$\begin{aligned}
f &\sim \frac{1}{1 - \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right)} \\
&\sim \frac{1}{-\varepsilon - \frac{\varepsilon^2}{2} - \cdots} \\
&\sim -\frac{1}{\varepsilon} \cdot \frac{1}{1 + \left(\frac{\varepsilon}{2} + \cdots\right)} \\
&\sim -\frac{1}{\varepsilon} \cdot \left(1 - \frac{\varepsilon}{2} + \cdots\right), \text{ by Taylor expansion of } \frac{1}{1+x} \\
&\sim -\frac{1}{\varepsilon} + \frac{1}{2} + \cdots
\end{aligned}$$

Therefore, $\alpha = -1, a_1 = -1, \beta = 0, a_2 = 1/2$.

(b) Apply the Taylor expansion of $\cos(\varepsilon)$:

$$
\begin{aligned}
f &\sim \left(1 + \frac{1}{1 - \frac{\varepsilon^2}{2} + \cdots}\right)^{\frac{3}{2}} \\
&\sim \left(2 + \frac{\varepsilon^2}{2} + \cdots\right)^{\frac{3}{2}} \\
&\sim 2^{\frac{3}{2}} \left(1 + \frac{\varepsilon^2}{4} + \cdots\right)^{\frac{3}{2}} \\
&\sim 2^{\frac{3}{2}} \left(1 + \frac{3}{2}\frac{\varepsilon^2}{4} + \cdots\right) \\
&\sim 2^{\frac{3}{2}} \left(1 + \frac{3\varepsilon^2}{8} + \cdots\right)
\end{aligned}
$$

Therefore, $\alpha = 0, a_1 = 2^{\frac{3}{2}}, \beta = 2, a_2 = 2^{-\frac{3}{2}}3$.

(c) Apply the Taylor expansion of $\ln(1+x)$ and $\frac{1}{1+x}$:

$$
\begin{aligned}
f \sim 1 + \varepsilon - 2 &\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \cdots\right) \\
&- \left(1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \varepsilon^4 - \cdots\right) \\
\sim \frac{1}{3}\varepsilon^3 - \frac{1}{2}\varepsilon^4 + \cdots
\end{aligned}
$$

Therefore, $\alpha = 3, a_1 = \frac{1}{3}, \beta = 4, a_2 = -\frac{1}{2}$.

(d) Here it is easier to compute the Taylor coefficients directly:

$$
\begin{aligned}
f(x,0) &= \sinh(1) \\
\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(x,\varepsilon) &= \cosh\left(\sqrt{1+\varepsilon x}\right) \cdot \frac{x}{2\sqrt{1+\varepsilon x}} \\
\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(x,0) &= \frac{\cosh(1)x}{2}
\end{aligned}
$$

Therefore, $\alpha = 0, a_1 = \sinh(1), \beta = 1, a_2 = \frac{\cosh(1)x}{2}$.

(e) We can make the change of variables $y = \frac{1}{\varepsilon}$ to get

$$
\begin{aligned}
a_1 &= \lim_{\varepsilon \downarrow 0} (1 + \varepsilon x)^{\frac{1}{\varepsilon}} \\
&= \lim_{y \to \infty} \left(1 + \frac{x}{y}\right)^y \\
&= e^x
\end{aligned}
$$

Then we can find the next asymptotic term:

$$\begin{aligned}
a_2 &= \lim_{\varepsilon\downarrow 0} \frac{f(\varepsilon)-a_1}{\varepsilon} \\
&= \lim_{\varepsilon\downarrow 0} \frac{(1+\varepsilon x)^{\frac{1}{\varepsilon}} - e^x}{\varepsilon} \\
&= \lim_{y\to\infty} y\left[\left(1+\frac{x}{y}\right)^y - e^x\right] \\
&= \lim_{y\to\infty} y\left[\exp\left(y\ln(1+\tfrac{x}{y})\right) - e^x\right] \\
&= \lim_{y\to\infty} y\left[\exp\left(y\left(\frac{x}{y}-\frac{x^2}{2y^2}+\frac{x^3}{3y^3}-\cdots\right)\right) - e^x\right] \\
&= \lim_{y\to\infty} y\left[e^x\cdot\exp\left(-\frac{x^2}{2y}+\frac{x^3}{3y^2}-\cdots\right) - e^x\right] \\
&= \lim_{y\to\infty} y\left[e^x\left(1-\frac{x^2}{2y}+\frac{x^3}{3y^2}-\cdots\right) - e^x\right] \\
&= \lim_{y\to\infty} e^x\left(-\frac{x^2}{2}+\frac{x^3}{3y}-\cdots\right) \\
&= -\frac{1}{2}e^x x^2
\end{aligned}$$

Therefore, $\alpha = 0, a_1 = e^x, \beta = 1, a_2 = -\frac{1}{2}e^x x^2$.

(f) Since $\sin(x+\varepsilon x^2)$ is integrable on $[0,\varepsilon]$, we can integrate its asymptotic terms to get the desired equation. Computing the first two terms from the Taylor formulas shows $b_1 = f(x,0) = \sin(x)$ and the second term is computed as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(x,\varepsilon) &= x^2\cos(x+\varepsilon x^2) \\
b_2 &= x^2\cos(x).
\end{aligned}$$

So we have $\sin(x+\varepsilon x^2) \sim \sin(x) + \varepsilon x^2\cos(x)$. Integrating term by term yields

$$\begin{aligned}
f &\sim -\cos(x)|_0^\varepsilon + \varepsilon\left[x^2\sin(x)+2x\cos(x)-2\sin(x)\right]|_0^\varepsilon + \cdots \\
&\sim 1-\cos(\varepsilon)+\varepsilon^3\sin(\varepsilon)+2\varepsilon^2\cos(\varepsilon)-2\varepsilon\sin(\varepsilon)+\cdots \\
&\sim 1-(1-\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{24}\cdots)+\varepsilon^3\left(\varepsilon-\frac{\varepsilon^3}{6}+\cdots\right)+2\varepsilon^2\left(1-\frac{\varepsilon^2}{2}+\cdots\right)-2\varepsilon\left(\varepsilon-\frac{\varepsilon^3}{6}+\cdots\right)+\cdots \\
&\sim \frac{1}{2}\varepsilon^2+\frac{7}{24}\varepsilon^4+\cdots
\end{aligned}$$

Therefore, $\alpha = 2, a_1 = \frac{1}{2}, \beta = 4, a_2 = \frac{7}{24}$.

(g) Starting by plugging in the Taylor series for $\sin\left(\frac{\varepsilon}{n}\right)$ to get

$$\begin{aligned}
f &\sim \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{\varepsilon}{n}-\frac{\varepsilon^3}{6n^3}+\cdots\right) \\
&\sim \ln(2)\varepsilon - \frac{\varepsilon^3}{6}\sum_{n=1}^{\infty}\frac{1}{2^n n^3}+\cdots
\end{aligned}$$

Therefore, $\alpha = 1, a_1 = \ln(2), \beta = 3, a_2 = -\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{2^n n^3}$.

(h) The expansion will be finite and exact in this case and we can just multiply it out

$$\begin{aligned} f &= 1 + \varepsilon \sum_{k=0}^{n} k + \cdots \\ &= 1 + \varepsilon \frac{n(n+1)}{2} + \cdots \end{aligned}$$

Therefore, $\alpha = 0, a_1 = 1, \beta = 1, a_2 = \frac{n(n+1)}{2}$.

(i) Again we can apply an expansion and then integrate term by term

$$\begin{aligned} f &\sim \int_0^{\pi} \frac{\sin(x)}{\sqrt{1+\varepsilon x}} \mathrm{d}x \\ &\sim \int_0^{\pi} \sin(x) \left(1 - \frac{\varepsilon x}{2} + \cdots\right) \mathrm{d}x \\ &\sim \int_0^{\pi} \sin(x) \mathrm{d}x - \frac{\varepsilon}{2} \int_0^{\pi} x \sin(x) \mathrm{d}x + \cdots \\ &\sim [-\cos(x)]_0^{\pi} - \frac{\varepsilon}{2} [-x\cos(x) + \sin(x)]_0^{\pi} + \cdots \\ &\sim 2 - \frac{\pi}{2}\varepsilon. \end{aligned}$$

Therefore, $\alpha = 0, a_1 = 2, \beta = 1, a_2 = -\frac{\pi}{2}$.

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Problem 5 (1.12)
Suppose $f(\varepsilon) \sim a_0\phi_0(\varepsilon) + a_1\phi_1(\varepsilon) + \cdots$ and $g(\varepsilon) \sim b_0\phi_0(\varepsilon) + b_1\phi_1(\varepsilon) + \cdots$ as $\varepsilon \downarrow \varepsilon_0$, where $\phi_0, \phi_1, \phi_2, \ldots$ is an asymptotic sequence as $\varepsilon \downarrow \varepsilon_0$.

(a) Show that $f + g \sim (a_0 + b_0)\phi_0 + (a_1 + b_1)\phi_1 + \cdots$ as $\varepsilon \downarrow \varepsilon_0$.

(b) Assuming $a_0 b_0 \neq 0$, show that $fg \sim a_0 b_0 \phi_0^2$ as $\varepsilon \downarrow \varepsilon_0$. Also, discuss the possibilities for the next term in the expansion.

(c) Suppose that $\phi_i\phi_j = \phi_{i+j}$ for all i, j. In this case show that

$$fg \sim a_0b_0\phi_0 + (a_0b_1 + a_1b_0)\phi_1 + (a_0b_2 + a_1b_1 + a_2b_0)\phi_2 + \cdots \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

(d) Under what conditions on the exponents will the following be an asymptotic sequence satisfying the condition in part (c): $\phi_0 = (\varepsilon - \varepsilon_0)^\alpha,\ \phi_1 = (\varepsilon - \varepsilon_0)^\beta,\ \phi_2 = (\varepsilon - \varepsilon_0)^\gamma,\ \ldots$?

Solution

(a) The above form asymptotic expansions for f and g up to the n^{th} term if and only if

$$f = \sum_{k=0}^{m} a_k\phi_k + o(\phi_m) \quad \text{for } m = 0, 1, 2, \ldots, n \quad \text{as } \varepsilon \downarrow \varepsilon_0,$$

and

$$g = \sum_{k=0}^{m} b_k\phi_k + o(\phi_m) \quad \text{for } m = 0, 1, 2, \ldots, n \quad \text{as } \varepsilon \downarrow \varepsilon_0.$$

It follows that

$$\begin{aligned} f + g &= \sum_{k=0}^{m} (a_k + b_k)\phi_k + 2o(\phi_m) \\ &= \sum_{k=0}^{m} (a_k + b_k)\phi_k + o(\phi_m) \end{aligned}$$

for $m = 0, 1, 2, \ldots, n$ since the constant factor of 2 does not affect the little-oh asymptotic order term.

(b) Similarly, we have

$$\begin{aligned} fg &= \left(\sum_{k=1}^{m} a_k\phi_k + o(\phi_m)\right)\left(\sum_{k=1}^{m} b_k\phi_k + o(\phi_m)\right) \\ &= a_0b_0\phi_0^2 + (a_0b_1 + a_1b_0)\phi_0\phi_1 + a_1b_1\phi_1^2 + (a_0b_2 + a_2b_0)\phi_0\phi_2 + \cdots + o(\phi_m) \end{aligned}$$

again for $m = 0, 1, \ldots, n$. From this, we can confirm that $fg \sim a_0b_0\phi_0^2$. The next term in the expansion would then depend on the relative ordering of higher terms like $\phi_0\phi_2$ versus ϕ_1^2.

(c) This follows directly from the previous expansion:

$$\begin{aligned} fg &\sim a_0b_0\phi_0^2 + (a_0b_1 + a_1b_0)\phi_0\phi_1 + a_1b_1\phi_1^2 + (a_0b_2 + a_2b_0)\phi_0\phi_2 + \cdots \\ &= a_0b_0\phi_{0+0} + (a_0b_1 + a_1b_0)\phi_{0+1} + a_1b_1\phi_{1+1} + (a_0b_2 + a_2b_0)\phi_{0+2} + \cdots \\ &= a_0b_0\phi_0 + (a_0b_1 + a_1b_0)\phi_1 + (a_1b_1 + a_0b_2 + a_2b_0)\phi_2 + \cdots. \end{aligned}$$

(d) For this condition to hold, the powers would have to follow an arithmetic sequence starting at 0. So $\alpha = 0, \beta, \gamma = 2\beta, 3\beta \ldots$

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Problem 6 (1.18)
Find a two-term asymptotic expansion, for small ε, of each solution x of the following equations:

(a) $x^2 + x - \varepsilon = 0$

(e) $\varepsilon x^3 - 3x + 1 = 0$

(h) $x^2 + \varepsilon\sqrt{2+x} = \cos(\varepsilon)$

(p) $xe^{-x} = \varepsilon$

Solution Find a two-term asymptotic expansion, for small ε, of each solution x of the following equations:

(a) $x^2 + x - \varepsilon = 0$
First, look at the $O(1)$ terms $x_0^2 + x_0 = 0$. It follows that $x_0 = 0, -1$. Let's proceed first with $x_0 = 0$. Then we can examine the $O(\varepsilon)$ equation:

$$\varepsilon^2 x_1^2 + \varepsilon x_1 - \varepsilon \sim \varepsilon x_1 - \varepsilon = 0$$

and it follows that $x_1 = 1$. This first solution's asymptotic expansion is then $x_r \sim \varepsilon$.

For the other root, we get

$$(-1 + \varepsilon x_1)^2 + (-1 + \varepsilon x_1) - \varepsilon \sim 1 - 2\varepsilon x_1 + \varepsilon^2 x_1^2 - 1 + \varepsilon x_1 - \varepsilon \sim -\varepsilon x_1 - \varepsilon = 0$$

which implies $x_1 = -1$ and $x_l \sim -1 - \varepsilon$

(e) $\varepsilon x^3 - 3x + 1 = 0$
Looking at the naive $O(1)$ terms yields a single solution $a_0 = 1/3$. Assuming $x \sim 1/3 + a_1\varepsilon^\alpha$, this yields,

$$\begin{aligned}\varepsilon(1/3 + a_1\varepsilon^\alpha)^3 - 3(1/3 + a_1\varepsilon^\alpha) + 1 &\sim 0 \\ \frac{1}{27}\varepsilon - 3a_1\varepsilon^\alpha &\sim 0 \\ \Rightarrow \alpha = 1, a_1 &= \frac{1}{81}\end{aligned}$$

So we have $x \sim \frac{1}{3} + \frac{1}{81}\varepsilon$ for this root.

For the other roots, assume $x \sim \lambda + r, \lambda \gg 1, r \ll \lambda$. Plug this into the equation and we get

$$\begin{aligned}\varepsilon(\lambda + r)^3 - 3(\lambda + r) + 1 &\sim 0 && \text{plugging in} \\ \varepsilon\lambda^3 - 3\lambda &\sim 0 && \text{taking leading order terms} \\ \Rightarrow \lambda &= \pm\sqrt{3}\varepsilon^{-1/2} && \text{solve for first term} \\ 3\varepsilon\lambda^2 r - 3r + 1 &\sim 0 && \text{take next order terms} \\ 9r - 3r + 1 &= 0 && \text{plug in } \lambda \\ \Rightarrow r &= -\frac{1}{6} && \text{solve for second term}\end{aligned}$$

These two roots are then $x \sim \pm\sqrt{3}\varepsilon^{-1/2} - \frac{1}{6}$.

(h) $x^2 + \varepsilon\sqrt{2+x} = \cos(\varepsilon)$
Consider the leading order terms and get $a_0 = \pm 1$. Then we can assume $x \sim \pm_1 + a_1\varepsilon^\alpha$ and plug this into the equation and drop higher order terms to get

$$\begin{aligned}\pm 2a_1\varepsilon^\alpha + \varepsilon\sqrt{2 \pm 1 + a_1\varepsilon^\alpha} &\sim 0 \\ \pm 2a_1\varepsilon^\alpha + \varepsilon\sqrt{2 \pm 1} &\sim 0 \\ \Rightarrow \alpha = 1, a_1 = -\frac{\sqrt{2 \pm 1}}{\pm 2} &= -\frac{\sqrt{3}}{2}, \frac{1}{2}\end{aligned}$$

Therefore, we get the two roots $x_l \sim 1 - \frac{\sqrt{3}}{2}\varepsilon$ and $x_r \sim -1 + \frac{1}{2}\varepsilon$.

(p) $xe^{-x} = \varepsilon$
For the O(1) term, $xe^{-x} = 0$ has a single solution $x_0 = 0$. Plugging in $x \sim a_1\varepsilon^\alpha$ gives

$$\begin{aligned} a_1\varepsilon^\alpha e^{-a_1\varepsilon^\alpha} &= \varepsilon \\ a_1\varepsilon^\alpha\left(1 - a_1\varepsilon^\alpha + \cdots\right) &= \varepsilon \\ a_1\varepsilon^\alpha &\sim \varepsilon \\ \Rightarrow \alpha &= 1, a_1 = 1 \end{aligned}$$

Now, plug in $x \sim \varepsilon + a_2\varepsilon^\beta$ and get

$$\begin{aligned} (\varepsilon + a_2\varepsilon^\beta)(1 - \varepsilon - a_2\varepsilon^\beta) &\sim \varepsilon \\ a_2\varepsilon^\beta - \varepsilon^2 &= 0 \\ \Rightarrow \beta &= 2, a_2 = 1 \end{aligned}$$

So we get the asymptotic expansion at 0 $x_l \sim \varepsilon + \varepsilon^2$. There is another solution away from 0 that we can consider in this problem. The trick involves carefully taking the log of both sides of the equation to get $\ln(x) - x = \ln(\varepsilon)$. For leading order terms, $\ln(x) \ll x$ and we get a first term in the expansion $b_0 = -\ln(\varepsilon)$. Assuming now that $x \sim -\ln(\varepsilon) + \mu$, we get

$$\ln\left(-\ln(\varepsilon) + \mu\right) - \left(-\ln(\varepsilon) + \mu\right) = \ln(\varepsilon) \Rightarrow \ln\left(-\ln(\varepsilon) + \mu\right) = \mu$$

Now, we can do some reworking of the logarithm terms

$$\mu = \ln(-\ln(\varepsilon) + \mu) = \ln\left(-\ln(\varepsilon)(1 - \frac{\mu}{\ln(\varepsilon)})\right) = \ln(-\ln\varepsilon) + \ln(1 - \frac{\mu}{\ln\varepsilon} \sim \ln(-\ln\varepsilon)$$

Where the last term is valid due to the blow up of $\ln\varepsilon$. So altogether, we get the asymptotic expansion $x_r \sim -\ln(\varepsilon) + \ln(-\ln\varepsilon)$.

■

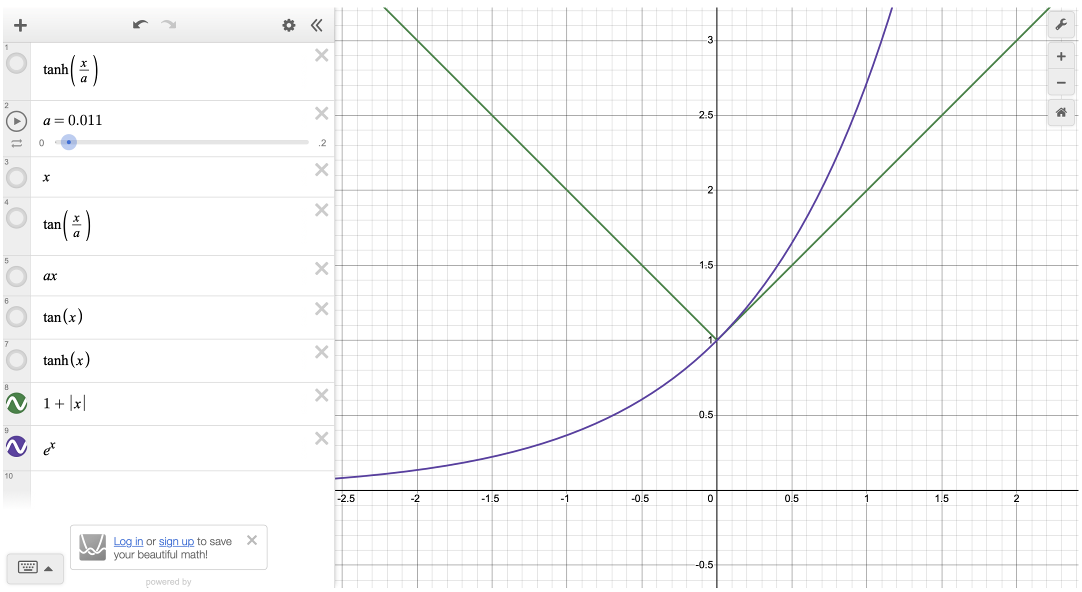
Problem 7 (1.19)
This problem considers the equation $1 + \sqrt{x^2 + \varepsilon} = e^x$.

(a) Explain why there is one real root for small ε.

(b) Find a two-term expansion of the root.

Solution

(a) There is one root for small ε because $1 + |x|$ and e^x cross at $x = 0$. The following image illustrates this:



(b) It is helpful to rework the equation somewhat before attempting to find the asymptotic expansion. We can make the following simplification:

$$\varepsilon = (e^x - 1)^2 - x^2 = (e^x - 1 - x)(e^x - 1 + x)$$

Plugging in $x \sim a_1 \varepsilon^\alpha$ gives

$$\begin{aligned} \varepsilon &\sim \left(\frac{1}{2} a_1^2 \varepsilon^{2\alpha}\right)(2a_1\varepsilon^\alpha) \\ \varepsilon &= a_1^3 \varepsilon^{3\alpha} \\ &\Rightarrow \alpha = 1/3, a_1 = 1 \end{aligned}$$

We can proceed with plugging in $x \sim \varepsilon^{1/3} + a_2 \varepsilon^\beta$, but first not that the left hand side can be expanded as follows:

$$(e^x - 1 - x)(e^x - 1 + x) = \left(\frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(2x \frac{x^2}{2} + \cdots\right) = x^3 + x^4/3 + x^4/4 + \cdots = x^3 + \frac{7}{12}x^4 + \cdots$$

This helps identify the next terms in the expansion:

$$\begin{aligned} 0 &= 3a_2 \varepsilon^{2/3+\beta} + \frac{7}{12}\varepsilon^{4/3} \\ \Rightarrow \beta &= 2/3, a_2 = -\frac{7}{36} \end{aligned}$$

So the asymptotic expansion is $x \sim \varepsilon^{1/3} - \frac{7}{36}\varepsilon^{2/3}$.

■

Problem 8 (1.20)
In this problem you should sketch the functions in each equation and then use this to determine the number and approximate location of the real-valued solutions. With this, find a three-term asymptotic expansion, for small ε, of the nonzero solutions.

(a) $x = \tanh\left(\frac{x}{\varepsilon}\right)$,

(b) $x = \tan\left(\frac{x}{\varepsilon}\right)$.

Solution

(a) The function $\tanh(x/\varepsilon)$ has two roots away from 0 as shown in the included Figure 1. To find an expansion for this, we make the change of variables $y = x/\varepsilon$ and get $\varepsilon y = \tanh y$. Then we can assume that $y = \lambda + r$ with $\lambda \gg 1$ and $r \ll \lambda$ (Solutions are large after making the transformation). Plugging this in gives

$$\begin{aligned}\varepsilon\lambda + \varepsilon r &= \tanh(\lambda + r)\\ &= \frac{e^{2(\lambda+r)} - 1}{e^{2(\lambda+r)} + 1}\\ &= 1 - \frac{2}{e^{2(\lambda+r)} + 1}\end{aligned}$$

For $\lambda \gg 1$, we get $\varepsilon\lambda = 1 \Rightarrow \lambda = \varepsilon^{-1}$. Now we check the next order:

$$\begin{aligned}\varepsilon r &\sim -\frac{2}{e^{2(\lambda+r)} + 1}\\ &\sim -2e^{-2r}e^{-2/\varepsilon}\\ &\sim -2e^{-2/\varepsilon}\\ \Rightarrow y &\sim \varepsilon^{-1} - 2\varepsilon^{-1}e^{-2/\varepsilon}\\ \Rightarrow x_r &\sim 1 - 2e^{-2/\varepsilon}\end{aligned}$$

By symmetry, we can also find the other root $x_l \sim -1 + 2e^{-2/\varepsilon}$.

(b) The function $\tan(x/\varepsilon)$ has an infinite number of roots away from 0 as shown in the included Figure 2. To find an expansion for this, we make the same change of variables to get $\varepsilon y = \tan(y)$. Then we guess that $y = a_0 + a_1\varepsilon^\alpha$, but we can observe that $a_0 = n\pi$ for $n \in \mathbb{Z}$. Plugging this in gives

$$\varepsilon n\pi = \tan(n\pi + a_1\varepsilon^\alpha) = \tan(a_1\varepsilon^\alpha) \sim a_1\varepsilon^\alpha.$$

So $\alpha = 1, a_1 = n\pi$. Now we can check the next order:

$$\varepsilon n\pi = \tan(n\pi + \varepsilon n\pi + a_2\varepsilon^\beta) \sim a_2\varepsilon^\beta$$

$\Rightarrow \beta = 2, a_2 = n\pi^2/3$ and we get $y \sim n\pi(1 + \varepsilon + \varepsilon^2)$, $x \sim n\pi(\varepsilon + \varepsilon^2 + \varepsilon^3)$.

∎

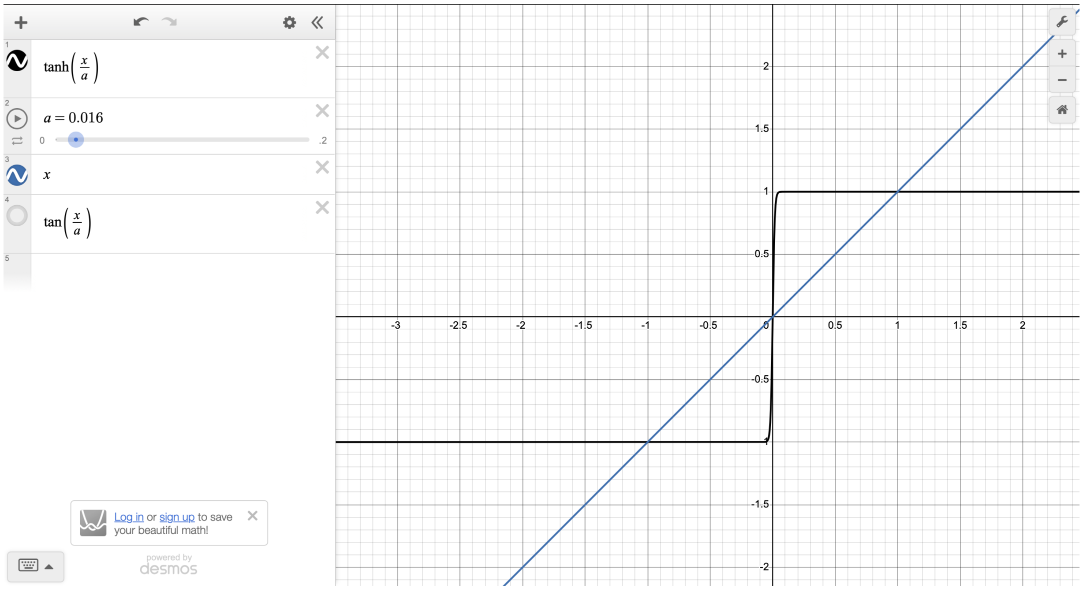


Figure 1: Graph of $\tanh(x/\varepsilon)$ and x

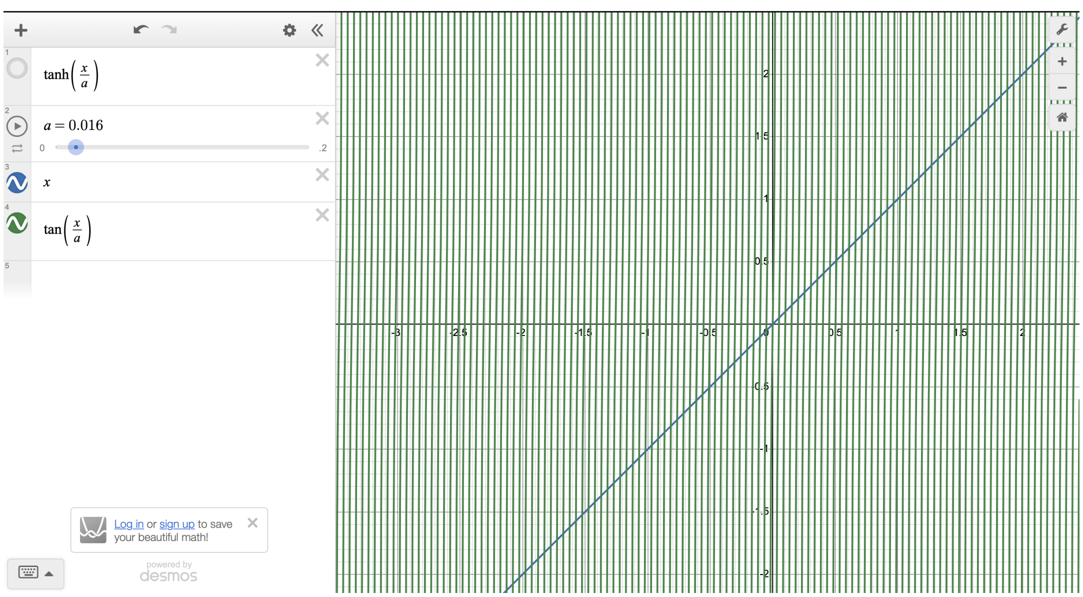


Figure 2: Graph of $\tan(x/\varepsilon)$ and x

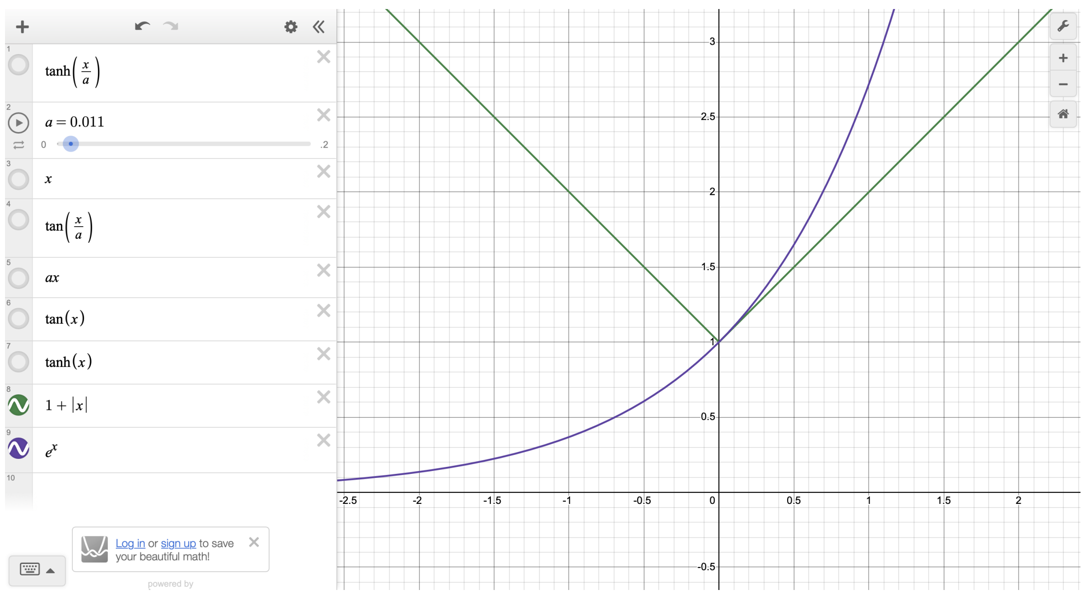
Problem 9 (1.21)
To determine the natural frequencies of an elastic string, one is faced with solving the equation $\tan(\lambda) = \lambda$.

(a) After sketching the two functions in this equation on the same graph explain why there is an infinite number of solutions.

(b) To find an asymptotic expansion of the large solutions of the equation, assume that $\lambda \sim \varepsilon^{-\alpha}(\lambda_0 + \varepsilon^{\beta}\lambda_1$. Find $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$ (note that λ_0 and λ_1 are nonzero and $\beta > 0$).

Solution

(a) As can be seen in Figure a, since $\tan(x)$ is periodic and has full range, there are infinitely many solutions to the equation since the line $y = \lambda$ will cross each branch of the $\tan(x)$ function. The crossing points for large λ will be close to the vertical asymptotes of the tangent function at $\lambda_k = \frac{1}{2}(2k-1)\pi$ for $k \in \mathbb{Z}$.



(b) For values of λ near λ_k, we can expand sine and cosine to get

$$\begin{aligned}
\sin(\lambda) &\sim 1 - \frac{1}{2}(\lambda - \lambda_k)^2 \\
\cos(\lambda) &\sim 0 - (\lambda - \lambda_k) + \frac{1}{6}(\lambda - \lambda_k)^3 \\
\Rightarrow \tan(\lambda) &\sim \frac{1 - \frac{1}{2}(\lambda - \lambda_k)^2}{-(\lambda - \lambda_k) + \frac{1}{6}(\lambda - \lambda_k)^3} \\
&\sim -\frac{1}{\lambda - \lambda_k} \cdot \frac{1 - \frac{1}{2}(\lambda - \lambda_k)^2}{1 - \frac{1}{6}(\lambda - \lambda_k)^2} \\
&\sim -\frac{1}{\lambda - \lambda_k}\left[1 - \frac{1}{3}(\lambda - \lambda_k)^2\right]
\end{aligned}$$

For large solutions, we know from the picture that $\lambda \sim \lambda_k$, so we can assume $\lambda \sim \lambda_k + \mu$ where

$\mu \ll \lambda_k$. Then the previous calculation gives

$$\lambda_k + \mu \sim \frac{-1}{\mu} \sim (1 - \frac{1}{3}\mu^2)$$
$$\lambda_k\mu + \mu^2 \sim -1 + \frac{1}{3}\mu^2$$

which, after dropping higher order terms, implies that $\mu = -\frac{1}{\lambda_k}$. Therefore, we have the asymptotic expansion up to two terms $\lambda \sim \lambda_k - \frac{1}{\lambda_k}$.

■