

Math 207C Homework #1

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Holmes Ex #1.1, 1.3, 1.5, 1.7, 1.12, 1.18 (a)(e)(h)(p), 1.19, 1.20, 1.21

Problem 1 (1.1)

(a) What values of α , if any, yield $f = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$ for each of the following functions?

(i) $f = \sqrt{1 + \varepsilon^2}$

(v) $f = \varepsilon \ln(\varepsilon)$

(ii) $f = \varepsilon \sin(\varepsilon)$

(vi) $f = \sin(1/\varepsilon)$

(iii) $f = (1 - e^\varepsilon)^{-1}$

(vii) $f = \sqrt{x + \varepsilon}$, where $0 \leq x \leq 1$

(iv) $f = \ln(1 + \varepsilon)$

(b) For the functions listed in (a), what values of α , if any, yield $f = o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$?

Solution

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Problem 2 (1.3)

This problem establishes some of the basic properties of the order symbols, some of which are used extensively in this book. The limit assumed here is $\varepsilon \downarrow 0$.

- (a) If $f = o(g)$ and $g = O(h)$, or if $f = O(g)$ and $g = o(h)$, then show that $f = o(h)$. Note that this result can be written as $o(O(h)) = O(o(h)) = o(h)$.
- (b) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f + g = O(|\phi_1| + |\phi_2|)$. Also, explain why the absolute signs are necessary. Note that this result can be written as $O(f) + O(g) = O(|f| + |g|)$.
- (c) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $fg = O(\phi_1\phi_2)$. This result can be written as $O(f)O(g) = O(fg)$.
- (d) Show that $O(O(f)) = O(f)$.
- (e) Show that $O(f)o(g) = o(f)o(g) = o(fg)$.

Problem 3

1.5 Suppose $f = o(\phi)$ for small ε , where f and ϕ are continuous.

(a) Give an example to show that it is not necessarily true that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon \phi d\varepsilon\right).$$

(b) Show that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon |\phi| d\varepsilon\right).$$

Problem 4 (1.7)

Assuming $f \sim a_1\varepsilon^\alpha + a_2\varepsilon^\beta + \dots$, find α, β (with $\alpha < \beta$) and nonzero a_1, a_2 for the following functions:

(a) $f = \frac{1}{1 - e^\varepsilon}$.

(b) $f = \left[1 + \frac{1}{\cos(\varepsilon)}\right]^{\frac{3}{2}}$.

(c) $f = 1 + \varepsilon - 2 \ln(1 + \varepsilon) - \frac{1}{1 + \varepsilon}$.

(d) $f = \sinh(\sqrt{1 + \varepsilon}x)$, for $0 < x < \infty$.

(e) $f = (1 + \varepsilon x)^{\frac{1}{\varepsilon}}$, for $0 < x < \infty$.

(f) $f = \int_0^\varepsilon \sin(x + \varepsilon x^2) dx$.

(g) $f = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{\varepsilon}{n}\right)$.

(h) $\int_0^\varepsilon \sin(x + \varepsilon x^2) dx$.

(i) $f = \int_0^1 \frac{dx}{\varepsilon + x(x - 1)}$.

Problem 5 (1.18)

Find a two-term asymptotic expansion, for small ε , of each solution x of the following equations:

(a) $x^2 + x - \varepsilon = 0$

(e) $\varepsilon x^3 - x + \varepsilon = 0$

(h) $x^2 + \varepsilon\sqrt{2 + x} = \cos(\varepsilon)$

(p) $xe^{-x} = \varepsilon$

Problem 6 (1.19)

This problem considers the equation $1 + \sqrt{x^2 + \varepsilon} = e^x$.

(a) Explain why there is one real root for small ε .

(b) Find a two-term expansion of the root.

Problem 7 (1.20)

In this problem you should sketch the functions in each equation and then use this to determine the number and approximate location of the real-valued solutions. With this, find a three-term asymptotic expansion, for small ε , of the nonzero solutions.

(a) $x = \tanh\left(\frac{x}{\varepsilon}\right)$,

(b) $x = \tan\left(\frac{x}{\varepsilon}\right)$.

Problem 8 (1.21)

To determine the natural frequencies of an elastic string, one is faced with solving the equation $\tan(\lambda) = \lambda$.

(a) After sketching the two functions in this equation on the same graph explain why there is an infinite number of solutions.

- (b) To find an asymptotic expansion of the large solutions of the equation, assume that $\lambda \sim \varepsilon^{-\alpha}(\lambda_0 + \varepsilon^\beta \lambda_1$. Find $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$ (note that λ_0 and λ_1 are nonzero and $\beta > 0$).