Math 258A Challenge #3

Ian Gallagher

April 28, 2025

(Worked jointly with Santiago Morales before writing up individual solutions).

Problem 1

Consider the nonlinear optimization problem

$$\max \quad \frac{x_1}{x_2 + 1}$$

subject to: $\quad x_1 - x_2 \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0.$

Use the KKT conditions to show that (4,2) is not optimal

The objective function and constraints for the given optimization problem are

$$f(x) = \frac{x_1}{x_2 + 1},$$
$$g_1(x) = x_1 - x_2 - 2 \leq 0,$$
$$g_2(x) = -x_1 \leq 0,$$
$$g_3(x) = -x_2 \leq 0.$$

The Lagrangian is therefore

$$L(x, \lambda) = f(x) - \sum_{i=1}^{3} \lambda_i g_i(x)$$

with $\lambda_i \geq 0$.

Checking for slack variables, we find that $g_1(4, 2) = 0$ while $g_2(4, 2) = -4$, $g_3(r, 2) = -2$. Complementarity conditions therefore require that $\lambda_2 = \lambda_3 = 0$. Checking the gradient condition, we have

$$\nabla f(4, 2) - \lambda_1 \nabla g_1(4, 2) = 0$$
$$(1/3, -4/9) - \lambda_1(1, -1) = 0.$$

Since $(1/3, -4/9)$ is not a scalar multiple of $(1, -1)$, there is no possible λ_1 that satisfies the equation. We may conclude that the point $(4, 2)$ is not optimal.

A solution that does satisfy the KKT conditions

To maximize the rational function f requires making the numerator as large as possible while simultaneously minimizing the denominator. This pushes any potential optimal solution onto the right boundary of the feasible region. The point $(2, 0)$ is a good guess since it will allow two degrees of freedom when evaluating the Lagrangian gradient condition.

That is, $(2,0)$ satisfies $g_1(x) = g_3(x) = 0$ and $g_2(x) = -2 < 0$. So $\lambda_2 = 0$ and we have

$$\nabla f(2,0) - \lambda_1 \nabla g_1(x) - \lambda_3 \nabla g_3(x) = (1, -2) - \lambda_1(1, -1) - \lambda_2(0, -1) = 0$$

Choosing $\lambda_1 = 1$ and $\lambda_3 = 1$ gives the desired equality, and this suffices to show that $(2,0)$ satisfies the KKT conditions.

Show this problem is not convex

Since we are maximizing our objective function, convexity will require that f be concave and that the feasible region is convex. The feasible region here is an intersection of linear inequalities, which is convex. The problem must be with the objective function. Take the points $A = (0, 2)$ and $B = (2, 0)$ inside the feasible region. Their midpoint is $C = (1, 1)$. Evaluating the objective function at these points gives

$$f(A) = \frac{0}{3} = 0,$$
$$f(B) = \frac{2}{1} = 2,$$
$$f(C) = \frac{1}{2} = 0.5.$$

Concavity would demand $f(C) \geq \frac{1}{2}f(A) + \frac{1}{2}f(B) = 1$, but we have shown this is not the case. Therefore, the given problem is not convex.

Optimality

As we have previously argued, any optimal solution must lie on the right boundary of the feasible region, where $x_1 - x_2 = 2$. Substituting this into the objective gives

$$f(x) = \frac{x_2 + 2}{x_2 + 1} = 1 + \frac{1}{x_2 + 1}.$$

This is a decreasing function of x_2 for $x_2 \geq 0$, so the maximum must occur at $x_2^* = 0$. The optimal solution is therefore $(x_1^*, x_2^*) = (2, 0)$ and we are done.