Math 258A Challenge #5

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Problem 5

Prove that given a finite collection of vertical segments in the plane, if for every three segments there is a line intersecting them, there exists a line intersecting them all.

Proof. Let $S_1, S_2, \ldots, S_n$ be the finite collection of vertical segments in the the plane. For each S_i, consider the space of lines passing through that segment, parameterized by slope and y-intercept. In particular, define $L_i = \left\{(m, b) \in \mathbb{R}^2 : y = mx + b \text{ s.t } (x_i, mx_i + b) \in S_i\right\}$. The set L_i is a closed, bounded, convex set. For closure, it must contain its limit points since the original line segment is closed. The line segment being finite length also guarantees that the set of legal slopes and y-intercepts must be bounded.

To prove convexity, take two lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ in L_i. The line $y = (m_1 x + b_1) + \theta\,(m_2 x + b_2)$,$\theta \in [0, 1]$, is the line corresponding to the convex combination of the two points in the parameter space. We have $y(x_i) = y_1 + \theta(y_2 - y_1)$, where y_1 and y_2 are the y-coordinates of the two lines at x_i. Since S_i is a convex line segment, this y-coordinate is in the segment.

By the assumption that every three line segments, S_i, S_j, S_k have a line intersecting them, we have that the intersection $L_{ijk} = L_i \cap L_j \cap L_k$ is non-empty. That is, the intersection of any three of the closed bounded convex sets is non-empty. By Helly's theorem, with $d = 2$, we have the intersection $L = \bigcap_{i=1}^n L_i$ is non-empty. Picking a point $(m, b) \in L$ gives the desired line $y = mx + b$ that intersects all the segments at once. $\square$

Problem 13 (estimation of probability distribution)

A random variable ξ has possible values $\xi_1, \ldots, \xi_n$, but the corresponding probabilities $p_1, \ldots, p_n$ are unknown. Formulate the problem of finding $p_1, \ldots, p_n$ such that the variance of ξ is maximized, the expected value of ξ is between α and β, the probabilities sum to one and no probability is less than $0.01/n$. Reformulate the resulting model as a minimization problem and check convexity.

Solution

The maximization problem can be formulated as follows:

$$
\begin{aligned}
\underset{p_1, \cdots p_n}{\text{maximize}} \quad & \mathrm{Var}(\xi) = \mathrm{E}[\xi^2] - \mathrm{E}[\xi]^2 \\
\text{subject to} \quad & p_1 + \cdots + p_n = 1, \\
& p_i \geq .01/n, \quad i = 1, \ldots, n, \\
& \mathrm{E}[\xi] \geq \alpha, \\
& \mathrm{E}[\xi] \leq \beta
\end{aligned}
\tag{1}
$$

The minimization problem is the same but with the objective function negated:

$$
\begin{aligned}
\underset{p_1,\dots p_n}{\text{minimize}} \quad & \mathrm{E}[\xi]^2 - \mathrm{E}[\xi^2] \\
\text{subject to} \quad & p_1 + \cdots + p_n = 1, \\
& \mathrm{E}[\xi] \geq \alpha, \\
& \mathrm{E}[\xi] \leq \beta, \\
& p_i \geq .01/n, \quad i = 1, \dots, n
\end{aligned}
\tag{2}
$$

The feasible region is the intersection of a set of affine equalities and inequalities and is therefore convex. The objective function is a positive quadratic function in the variables p_i and is therefore convex. Therefore, this minization problem is convex.