**Math 258A (De Loera)** **Homework Problems**

INSTRUCTIONS: Here I register the homework problems each week. Try to do as many as you can each week, do not try to do all in one shot the night before the midterm. Be organized and methodic. You do not need to hand written solutions but they will appear in the take-home midterms.

- **Weeks 1,2: (Motivation, Applications, Linear Algebra and Multivariate Calculus)**

1. (Some Linear Algebra) $A$ is a matrix in $\mathbb{R}^{m\times n}$ with $m \leq n$.
   a. Give the definition of the rank of $A$. What is the largest possible rank of $A$?
   b. Let us denote by $\mathbf{a}_1, \dots, \mathbf{a}_m$ the rows of $A$, *i.e* $\mathbf{a}_i \in \mathbb{R}^n$ and $A = [\mathbf{a}_1 \dots \mathbf{a}_m]^\intercal$. Show that:

$$\text{rank}(A) < m \Leftrightarrow \exists \mathbf{x} \in \mathbb{R}^m \setminus \{0\},\ \sum_{i=1}^{m} \mathbf{a}_i x_i = 0$$

2. Consider the matrix $A = uv^T$ , with $u \in R^n, v \in R^m$. Find the nullspace and range of A. Explain how to compute an singular value decomposition of A.
3. (Inner Products/Norms)
   For $\mathbf{x}$ and $\mathbf{y}$ two vectors of $\mathbb{R}^d$, we write $\mathbf{x}^\intercal\mathbf{y} = \sum_{i=1}^d x_i y_i$ their inner product. $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\intercal\mathbf{x}}$ denotes the Euclidean norm of $\mathbf{x}$.
   a. Let $\mathbf{a} \in \mathbb{R}^d$. Show that:

$$\mathbf{a} = 0 \Longleftrightarrow \forall \mathbf{x} \in \mathbb{R}^d,\ \mathbf{a}^\intercal\mathbf{x} = 0$$
$$\mathbf{a} \geq 0 \Longleftrightarrow \forall \mathbf{x} \geq 0,\ \mathbf{a}^\intercal\mathbf{x} \geq 0$$

   b. Let $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d$. Show that:

$$\mathbf{x}^\intercal\mathbf{y} = \frac{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2}{2} = \frac{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2}{2}$$

   c. Deduce the parallelogram law:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

   d. Let us denote by $B_2(0,1)$ the unit ball of $\mathbb{R}^d$, $B_2(0,1) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\| \leq 1\}$ and let us consider $\mathbf{v} \in \mathbb{R}^d$. Show that:

$$\max_{\mathbf{x}\in B_2(0,1)} \mathbf{v}^\intercal\mathbf{x} = \|\mathbf{v}\|$$

4. (eigenvectors and singular values)
   (a) Show that the Frobenius norm of a matrix $A$ depends only on its singular values. More precisely, show that

$$||A||_F = ||\sigma||_2$$

   , where $\sigma := (\sigma_1, \dots, \sigma_r)$ is the vector formed with the singular values of $A$, and $r$ is its rank.

(b) Consider a vector $u \in R^n$, which is normalized ($||u||_2 = 1$), and the associated symmetric matrix $P = I - uu^T$ . (a) Show that P is positive semi-definite. (b) Show that the eigenvalues of P are 1 and 0. (c) Using the fact that, starting from a vector $u$, you can find $n-1$ vectors $u_2, u_3, \dots, u_n$ such that $(u, u_2, \dots, u_n)$ forms an orthonormal basis of $R^n$, find an eigenvalue decomposition of $P$.

5. a. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^n$, we define the function $f_{\mathbf{x},\mathbf{d}} : \mathbb{R} \to \mathbb{R}$ by:

$$f_{\mathbf{x},\mathbf{d}}(\lambda) = f(\mathbf{x} + \lambda\mathbf{d})$$

Express the first and second derivative of $f_{\mathbf{x},\mathbf{d}}$ in terms of the gradient and Hessian of $f$.

b. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and let $\mathbf{x}$ be a local minimum of $f$, *i.e* there exists $\epsilon > 0$ such that:

$$\|\mathbf{y} - \mathbf{x}\| \leq \epsilon \Rightarrow f(\mathbf{y}) \geq f(\mathbf{x})$$

show that $\nabla f(\mathbf{x}) = 0$. **Hint:** remember the Taylor expansion of $f$ at $\mathbf{x}$:

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{h}^\intercal \nabla f(\mathbf{bx}) + o(\|\mathbf{h}\|)$$

c. Let $\mathbf{a} \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d \times d}$. What are the gradients of $f(\mathbf{x}) = \mathbf{a}^\intercal\mathbf{x}$, $g(\mathbf{x}) = \|\mathbf{x}\|^2$ and $h(\mathbf{x}) = \mathbf{x}^\intercal M\mathbf{x}$?

6. Let $f : R^n - R$ continuous differentiable function

(a) Let $x_0$ be a point where $\nabla f(x_0) \neq 0$. What is the meaning for $\nabla f(x_0)$ for $S = \{x \in R^n : f(x) = f(x_0)\}$?

(b) Suppose that there exist $L > 0$ such that $||\nabla f(x) - \nabla f(x')|| \leq L||x - x'||$ for all pairs $(x, x')$ Show that $|f(x + a) - f(x) - \langle \nabla f(x), a \rangle| \leq \frac{L||a||^2}{2}$ holds for all pairs (x,a).

7. Let $M_n(R)$ the space of $n \times n$ real matrices. It has a scalar product $< M, N >= tr(M^T N)$.

(a) Show that the set $\Omega$ of invertible matrices is an open set.

(b) For the following maps prove or disprove the maps are differentiable (It may not be advisable to do too many calculations):

i. trace

ii. determinant

iii. $g : \Omega \to R$ with $g(A) := \ln(|\det(A)|)$.

iv. $f_{-1} : \Omega \to \Omega$ with $f_{-1}(A) := A^{-1}$.

v. For an integer $p$, set $f_p(A) := A^p$.

8. **Modeling optimal facility location of a warehouse:**

A number of communities want to build a central warehouse. The location of each of the communities is described by city A $(3, 9)$, city B $(10, 6)$, city C $(0, 12)$ and city D $(4, 9)$. Goods will be delivered to cities by plane, the price of the delivery is proportional to the distance between the warehouse and the city. City A will need 20 deliveries, City B 14, City C 8, and City D 24. Write an optimization model that when solved would find the location of the warehouse (i.e., explicit coordinates), so that the total cost of the deliveries is minimized.

9. **Geometry Challenge via non-linear optimization models:** Your challenge is packing $m$ of spheres in a box of minimal area. The spheres have a given radius $r_i$, and the problem is to determine the precise location of the centers $x_i$. The constraints in this problem are that the spheres should not overlap, and should be contained in a square of center 0 and half-size R. The objective is to minimize the area of the containing box.

- Show that two spheres of radius $r_1$, $r_2$ and centers $x_1$, $x_2$ respectively do not intersect if and only if $||x_1 - x_2||_2$ exceeds a certain number, which you will determine.
- Formulate the sphere packing problem as an optimization model. Is the formulation you have found convex optimization?
- Write (on anything SCIP, Python, MATLAB, or etc) code to solve the packing problem of five and six circular disks of the same radius inside a square of half-size R. What is the optimal size if the disks have radius 1?
- Do some drawings using MATLAB of the packings you have discovered. Is the solution unique?

10. **Select a final project!** As we saw in class, optimization problems are everywhere! You are supposed to chose a concrete optimization problem you will present for a final. You need to (a) write a (max) 2 page proposal (with references), a problem statement, description, motivation and background. You will submit this to Prof. De Loera (no later than April 28). If approved, you will (b) Design a model and find the optimal solutions (depending on your problem you may need to write computer code to do so (c) Write latex slides for an oral presentation explaining your solution in max 30 mins (about 15 slides). You can do this in teams of 3-4 students.