Math 258A Challenge #1

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Problem 9: Geometry Challenge via Non-linear Optimization Models

Your challenge is packing m spheres in a box of minimal area. The spheres have a given radius r_i, and the problem is to determine the precise location of the centers x_i. The constraints in this problem are that the spheres should not overlap, and should be contained in a square of center 0 and half-size R. The objective is to minimize the area of the containing box.

(a) Show that two spheres of radius r_1, r_2 and centers x_1, x_2 respectively do not intersect if and only if $\|x_1 - x_2\|_2$ exceeds a certain number, which you will determine.

(b) Formulate the sphere packing problem as an optimization model. Is the formulation you have found convex optimization?

(c) Write (in SCIP, Python, MATLAB, or any other environment) code to solve the packing problem of five and six circular disks of the same radius inside a square of half-size R. What is the optimal size if the disks have radius 1?

(d) Do some drawings using MATLAB of the packings you have discovered. Is the solution unique?

Solution

Initial setup of the optimization problem was discussed with both Santiago and Jared. Santiago also assisted with debugging a variable bound issue that was forcing all circles to have centers in the first quadrant.

(a) The distance between two spheres is $\|x_1 - x_2\|_2$. If we consider the line segment connecting the centers of the two spheres, we can see that the spheres do not intersect if and only if the distance between their centers is greater than the sum of their radii. Thus, we have:

$$\|x_1 - x_2\|_2 > r_1 + r_2$$

as the constraint to prevent intersections. If we allow spheres to be tangent at a point then we may relax this strict inequality to get

$$\|x_1 - x_2\|_2 \geq r_1 + r_2$$

(b) Our objective is to minimize R subject to the constraints that the spheres don't intersect (discussed in part a) and the constraint that the spheres are entirely contained within the box. The latter condition can be restated as $|x_{i,j} \pm r_i| \leq R$ for all $i \in \{1, \dots, m\}, j \in \{1, \dots n\}$ where n is the dimension of the space. Putting these together, we have the following minimization problem

$$\begin{aligned} \min \quad & R \\ \text{s.t.} \quad & \|x_i - x_j\|_2 \geq r_i + r_j \quad && \text{for } i, j \in \{1, \dots, m\}, i \neq r_j \\ & |x_{i,j} \pm r_i| \leq R && \text{for } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}. \end{aligned} \tag{1}$$

This is a non-convex optimization problem. One way to see this is in the case of two circles. If we have two placements x_1, x_2 of the circles, then linear interpolation of x_1 to x_2 and vice versa can lead to the intersection of the two circles.

(c) I implemented a solver for the given optimization problem using Gurobi. The code can be viewed at the following link and is also uploaded along with the writeup: https://gist.github.com/iangallagherm/9377faa6099e3c640b34df9183293e41

Based on the solutions I found, the minimum half-size R of the box containing the disks is:

- 5 disks: $R \approx 2.4142$
- 6 disks: $R \approx 2.6641$

(d) Visualizations of the solution for 5 disks and 6 disks are shown in Figures 1 and 2 respectively. The solution for 6 disks is not unique as it does not have a 90 degree rotation symmetry. The solution for 5 disks appears to be unique but I have not verified this.

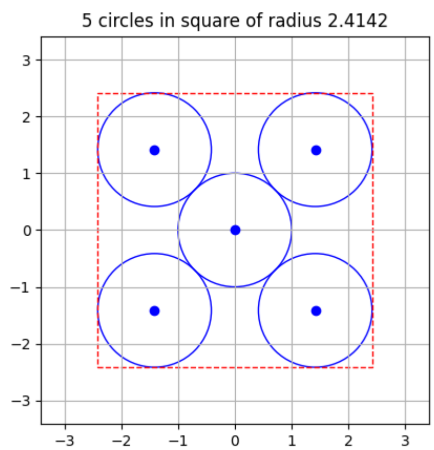

Figure 1: Packing of 5 disks

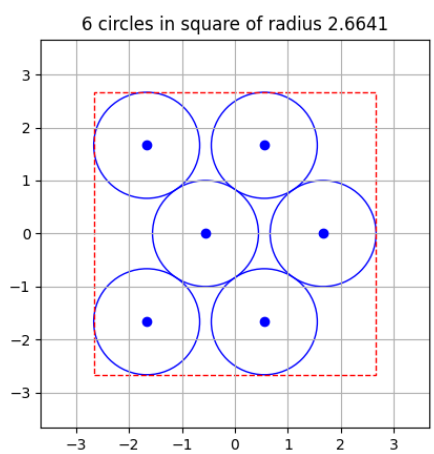

Figure 2: Packing of 6 disks

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