or $PGL_2(\mathbb{F}_q)$

Consider conjugation action of $GL_2(\mathbb{F}_q)$ on $M_2(\mathbb{F}_q)$

For matrices, $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ← Rational Canonical Forms for matrices

$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ← with min. poly $X^2 - 1$ or X^2

want to compute size of their orbits using orbit coset correspondence theorem

$|G_x| = (q-1)^2 = q^2 - 2q + 1$

$G_x = \{ A \in GL_2(\mathbb{F}_q) : A \cdot x = x \}$

$= \{ A \in GL_2(\mathbb{F}_q) : A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$

Clearly, $sx + tI$ commutes with x for all $s, t \in \mathbb{F}_q$, so if $sx + tI \in GL_2(\mathbb{F}_q)$, then $sx + tI \in G_x$

So, when is $\det(sx + tI) = 0$

$\det(sx + tI) = \begin{pmatrix} t & s \\ s & t \end{pmatrix} = t^2 - s^2$

0 when $t^2 = s^2$, so when $t = \pm s$.

So we have found $q^2 - 2q + 1$ elements of G_x. If these are all of them, then

$[G : G_x] = \frac{(q^2-1)(q^2-q)}{(q-1)^2} = q(q+1)$

$\Rightarrow \#\,\text{orbit}(x) = q(q+1)$

Similarly, we can compute

$$G_y = \{ B \in GL_2(\mathbb{F}_q) : ByB^{-1} = y \}$$

Now, $sy + tI$ commutes with y $\forall s,t \in \mathbb{F}_q$, so if $sy + tI \in GL_2(\mathbb{F}_q)$, then $sy + tI \in G_x$

$\det(sy + tI) = \det\begin{pmatrix} t & 0 \\ s & t \end{pmatrix} = t^2$, which is 0 iff $t = 0$.

We have therefore found $q^2 - q = q(q-1)$ elements of G_y. If these are all of them, then $[G : G_y] = \frac{(q^2-1)(q^2-q)}{q^2-q} = q^2 - 1$

$\Rightarrow \#\text{orbit}(x) = q^2 - 1$