

INVESTIGATING THE STRUCTURE OF $M_2(\mathbb{F}_q)$

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ABSTRACT. We present and prove a count of the maximal commutative subalgebras of $M_2(\mathbb{F}_q)$, as well as counts for the individual isomorphism classes.

1. INTRODUCTION

For a finite field of $q = p^m$ elements, it is possible to count the number of vector subspaces of \mathbb{F}_q^n of a given dimension. These counts arise in problems involving the number of points of \mathbb{P}_q^n , the Grassmannian, and further generalizations.

This paper is meant to address a similar problem. Namely, the structure and count of the maximal commutative subalgebras of $M_2(\mathbb{F}_q)$. Such sub-algebra's...

2. IDENTIFYING MAXIMAL COMMUTATIVE SUB-ALGEBRAS

3. PLANES IN $M_2(\mathbb{F}_q)$

4. PLANE COUNTS

Lemma 1. *Let $A \in M_2(k)$. Let $S = \det(kA)$. Then*

$$S = \begin{cases} \{0\} & \text{if } \det(A) = 0 \\ k^{\times^2} & \text{if } \det(A) \in k^{\times^2} \\ k^{\times} - k^{\times^2} & \text{if } \det(A) \in k^{\times} - k^{\times^2} \end{cases}$$

Proof. Let $A \in M_2(k)$. Then $\det(\lambda A) = \det(\lambda I) \det(A) = \lambda^2 \det(A)$ where $\lambda \in k$.

Case (1) Suppose $\det(A) = 0$. Then $\lambda^2 \det(A) = 0$ for all $\lambda \in k$ and we have $S = 0$.

Case (2) Suppose $\det(A) = \alpha^2$ where $\alpha \in k^{\times}$. Then $\lambda^2 \det(A) = \lambda^2 \alpha^2 = (\lambda \alpha)^2 \in k^{\times^2}$. It follows that $S \subseteq k^{\times^2}$. Now, let $\sigma \in k^{\times}$. Then $\det(\frac{\sigma}{\alpha} A) = \frac{\sigma^2}{\alpha^2} \alpha^2 = \sigma^2$. So $\sigma^2 \in S$ and we have shown $k^{\times^2} \subseteq S$. Therefore, $S = k^{\times^2}$.

Case (3) Suppose $\det(A) = \beta$ where $\beta \in k^{\times} - k^{\times^2}$. Then $\lambda^2 \det(A) = \lambda^2 \beta \in k^{\times} - k^{\times^2}$. It follows that $S \subseteq k^{\times} - k^{\times^2}$. Since $|k^{\times}/k^{\times^2}| = 2$, if $\sigma, \gamma \notin k^{\times^2}$, we know there exists $\lambda \in k^{\times}$ such that $\sigma \lambda^2 = \gamma$. So $k^{\times} - k^{\times^2} \subseteq S$ is clear and we have shown $S = k^{\times} - k^{\times^2}$.

□

Theorem 2. *This is a theorem.*

$$\begin{aligned} (1) \quad & |[\mathbb{F}_{q^2}]| = \binom{q}{2} \\ (2) \quad & |[\mathbb{F}_q^2]| = \binom{q+1}{2} \\ (3) \quad & |[\mathbb{F}_{q\text{nil}}^2]| = q+1 \end{aligned}$$

Theorem 3. *We shall consider the conjugation action of $GL_2(\mathbb{F}_q)$ on $M_2(\mathbb{F}_q)$ for the matrices,*

$$\begin{aligned} x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

These are the rational canonical forms for matrices with minimum polynomials $X^2 - 1$ and X^2 respectively. We want to compute the size of their orbits, G_x, G_y , using the orbit coset correspondence theorem. Now,

$$\begin{aligned} G_x &= \{A \in GL_2(\mathbb{F}_q) \mid A \cdot x = x\} \\ &= \{A \in GL_2(\mathbb{F}_q) \mid Ax A^{-1} = x\} \\ &= \{A \in GL_2(\mathbb{F}_q) \mid Ax = xA\} \end{aligned}$$

Clearly, matrices of the form $sx + tI$ for $s, t \in \mathbb{F}_q$, so if $sx + tI \in GL_2(\mathbb{F}_q)$, then $sx + tI \in G_x$. These are also the only possible matrices in G_x since these are the elements of the maximal commutative subalgebra containing x [requires result citation/prior inclusion](#). It now suffices to determine when $\det(sx + tI) = 0$.

$$\det(sx + tI) = \begin{vmatrix} t & s \\ s & t \end{vmatrix} = t^2 - s^2$$

Therefore, $\det(sx + tI) = 0$ when $t^2 = s^2$, so when $t = \pm s$.