1. Suppose that you are given $\mathbf{A}$, $\mathbf{B}$, and $\vec{x}$. Explain how to evaluate the expression $(\mathbf{A}+\mathbf{B}^{-1})^{-1}\vec{x}$ without computing the inverse of any matrices.

2. Without using the Wronskian, prove that the following three functions are linearly independent

$$p_1(x) = 1,\ p_2(x) = 1 - x,\ p_3(x) = 1 - x + x^2.$$

3. Consider the vector spaces of polynomials V of degree at most 2 and W of degree at most 3.

   (a) Find the matrix representation of the following linear function from V to W

   $$L[p] = 2x\frac{\mathrm{d}p}{\mathrm{d}x}$$

   with the monomial basis for both spaces.

   (b) Find the matrix representation of the same linear function with respect to the Chebyshev basis for both spaces: $1, x, 2x^2 - 1$, and for W also $4x^3 - 3x$.

4. Consider the function $L : \mathbb{R}^n \to \mathbb{R}^n$ that maps the initial condition to the final condition at time $t = \tau$ for a linear, non-autonomous, homogeneous system of differential equations:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \mathbf{A}(t)\vec{x},\quad \vec{x}(0) = \vec{x}_0.$$

   (a) Prove that this function is linear.

   (b) Infer from (a) that $\vec{x}(\tau) = \Pi(\tau)\vec{x}_0$ (no work required). Write down a system of ODEs that $\Pi(\tau)$ must solve (explain why).

5. Let $\vec{0} \neq \vec{v} \in \mathbb{R}^m$ and $\vec{0} \neq \vec{w} \in \mathbb{R}^n$. Prove that the matrix $\vec{v}\vec{w}^T$ has rank one.

6. Find the rank of this $n \times n$ matrix

$$\mathbf{I} - \frac{1}{n}\mathbf{1}$$

   where $\mathbf{1}$ is a matrix whose every entry is 1.

7. (Sherman-Morrison) Let $\mathbf{A}$ be an $n \times n$ invertible matrix and let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Without even *thinking* about using a determinant, prove

   (a) that $\mathbf{A} + \vec{u}\vec{v}^T$ is invertible iff $1 + \vec{v}^T\mathbf{A}^{-1}\vec{u} \neq 0$, and

   (b) in that case

   $$\left(\mathbf{A} + \vec{u}\vec{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{1}{1 + \vec{v}^T\mathbf{A}^{-1}\vec{u}}\mathbf{A}^{-1}\vec{u}\vec{v}^T\mathbf{A}^{-1}.$$

   (Hint: Use the expression given in (b) to prove one direction in (a). To prove the other direction, consider the vector $\mathbf{A}^{-1}\vec{u}$.)

8. Prove that if $\vec{v}_1, \ldots, \vec{v}_r$ are a basis for the corange of $\mathbf{A}$, then $\mathbf{A}\vec{v}_1, \ldots, \mathbf{A}\vec{v}_r$ are a basis for the range of $\mathbf{A}$.