1. Find a symmetric matrix $\mathbf{H}$, vector $\vec{b}$, and constant c so that

$$q(x_1, x_2, x_3) = 4x_1^2+4x_1x_2+2x_1x_3+x_2^2+6x_2x_3+3x_3^2-2x_1+x_2-x_3+5 = \frac{1}{2}\vec{x}^T\mathbf{H}\vec{x}+\vec{x}^T\vec{b}+c.$$

Solution: Notice that $\mathbf{H}$ is the Hessian of q, so we can obtain the entries of $\mathbf{H}$ by taking second partial derivatives of q. The solution is

$$\mathbf{H} = \begin{bmatrix} 8 & 4 & 2 \\ 4 & 2 & 6 \\ 2 & 6 & 6 \end{bmatrix}. \tag{1}$$

The linear part of q is $-2x_1 + x_2 - x_3 = \vec{x}^T\vec{b}$ where $\vec{b} = (-2, 1, -1)^T$. The constant is $c = 5$.

2. Suppose that you have a real finite-dimensional vector space V whose elements are functions of x (e.g. polynomials or trig functions), and you have a basis for that space $\vec{v}_1, \dots, \vec{v}_m$. (These 'vectors' are functions.) Assume that you have an inner product on V $\langle\cdot,\cdot\rangle$. You have some function f such that $\langle\vec{v}_i, f\rangle$ exists for all i. You want to find $\vec{v} \in V$ that minimizes $\|f - \vec{v}\|^2 = \langle f - \vec{v}, f - \vec{v}\rangle$. Explain why a solution always exists and give a formula for the solution.

Solution: Let $\vec{v} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$. Consider the function

$$q(x_1, \dots, x_m) = \|f - \vec{v}\|^2 = \langle f - \vec{v}, f - \vec{v}\rangle = \|f\|^2 - 2\sum_{i=1}^{m} x_i\langle f, \vec{v}_i\rangle + \sum_{i=1}^{m}\sum_{j=1}^{m} x_ix_j\langle\vec{v}_i, \vec{v}_j\rangle.$$

Notice that this is a quadratic function

$$q(x_1, \dots, x_m) = \vec{x}^T\mathbf{M}\vec{x} - 2\vec{x}^T\vec{b} + c$$

where

$$m_{ij} = \langle\vec{v}_i, \vec{v}_j\rangle, \quad b_i = \langle f, \vec{v}_i\rangle, \quad c = \|f\|^2.$$

The matrix $\mathbf{M}$ is clearly a Gram matrix, and since the vectors used in its construction are linearly independent, $\mathbf{M}$ is symmetric and positive definite. The quadratic function q therefore has a unique minimizer that occurs at the critical point, which is the solution of

$$\nabla q = 2\mathbf{M}\vec{x} - 2\vec{b} = \vec{0}.$$

The solution of $\mathbf{M}\vec{x} = \vec{b}$ gives the coordinates of $\vec{v}$ with respect to the basis $\vec{v}_1, \dots, \vec{v}_m$. Explicitly,

$$\vec{v} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$$

where the x_i are the entries of

$$\vec{x} = \mathbf{M}^{-1}\vec{b}.$$

3. Consider the problem of optimizing a quadratic function

$$q(\vec{x}) = \frac{1}{2}\vec{x}^T\mathbf{K}\vec{x} + \vec{x}^T\vec{b} + c$$

subject to a linear equality constraint

$$\vec{d}^T\vec{x} = e \quad \text{i.e.} \quad g(\vec{x}) = \vec{d}^T\vec{x} - e = 0$$

where we can assume that $\mathbf{K}$ is symmetric without loss of generality, and also assume $\vec{d} \neq \vec{0}$.

(a) Apply the method of Lagrange multipliers and write the optimality condition as a linear system of equations for $\vec{x}$ and λ.

Solution: The two conditions are

$$\begin{aligned} \mathbf{K}\vec{x} + \vec{b} &= \lambda\vec{d} \\ \vec{d}^T\vec{x} &= e \end{aligned}$$

The unknowns are $\vec{x}$ and λ. Putting the above linear system of equations into the standard form yields

$$\left[\begin{array}{c|c} \mathbf{K} & -\vec{d} \\ \hline -\vec{d}^T & 0 \end{array}\right]\begin{pmatrix} \vec{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} -\vec{b} \\ -e \end{pmatrix}.$$

(b) The coefficient matrix in (a) should be symmetric. Explain why it cannot be positive definite even if $\mathbf{K}$ is positive definite.

Solution: There is a zero on the diagonal, which prevents the matrix from being positive definite. In this case $\vec{e}_n^T\mathbf{A}\vec{e}_n = 0$.

(c) Assuming that $\mathbf{K}$ is positive definite, use block Gaussian elimination to find an explicit expression for the solution $\vec{x}$.

Solution: Multiply the top row by $\vec{d}^T\mathbf{K}^{-1}$ and add to the bottom row to obtain

$$\left[\begin{array}{c|c} \mathbf{K} & -\vec{d} \\ \hline & -\vec{d}^T\mathbf{K}^{-1}\vec{d} \end{array}\right]\begin{pmatrix} \vec{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} -\vec{b} \\ -e - \vec{d}^T\mathbf{K}^{-1}\vec{b} \end{pmatrix}.$$

Since $\vec{d} \neq \vec{0}$ and $\mathbf{K}$ is positive definite, we know that $\vec{d}^T\mathbf{K}^{-1}\vec{d} \neq 0$. So divide the bottom row by $-\vec{d}^T\mathbf{K}^{-1}\vec{d}$ which gives us the solution for λ

$$\lambda = \frac{e + \vec{d}^T\mathbf{K}^{-1}\vec{b}}{\vec{d}^T\mathbf{K}^{-1}\vec{d}}.$$

Now substitute this value into the top row, and move it to the RHS, which yields

$$\mathbf{K}\vec{x} = -\vec{b} + \lambda\vec{d}.$$

Since $\mathbf{K}$ is positive definite, the solution is (formally)

$$\vec{x} = \mathbf{K}^{-1}\left(-\vec{b} + \lambda\vec{d}\right).$$