1. Let $\mathbf{A}$ and $\mathbf{B}$ both be $n \times n$ and assume that there is an invertible matrix $\mathbf{S}$ such that $\mathbf{SAS}^{-1}$ and $\mathbf{SBS}^{-1}$ are both diagonal (but not necessarily equal). Prove that $\mathbf{AB} = \mathbf{BA}$.

2. Suppose that $\mathbf{A}$ is symmetric and positive definite, while $\mathbf{B}$ is symmetric. Prove that there is a real invertible matrix $\mathbf{S}$ such that

   $$\mathbf{A} + \mathbf{B} = \mathbf{S}\,(\mathbf{I} + \mathbf{D})\,\mathbf{S}^T$$

   where $\mathbf{D}$ is diagonal.

3. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$. Prove that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

4. Suppose that $\mathbf{L} \in \mathbb{R}^{n\times n}$ is a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. (Note that m can be $\leq n$.) Prove that

   $$\mathbf{L} = \lambda_1 \mathbf{P}_1 + \cdots + \lambda_m \mathbf{P}_m$$

5. Prove that if $\mathbf{A}$ is invertible (and, for simplicity, real), then there is an orthogonal matrix $\mathbf{U}$ and a symmetric and positive definite matrix $\mathbf{E}$ such that $\mathbf{A} = \mathbf{UE}$. This is analogous to writing complex numbers in polar form $x + \mathrm{i}y = \rho e^{\mathrm{i}\theta}$. Show that if $\mathbf{A}$ is square but not invertible then there is a similar factorization but with $\mathbf{E}$ being symmetric and non-negative definite.

6. Let $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$ be diagonalizable. Find a symmetric positive definite matrix $\mathbf{K}$ that defines an inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \mathbf{K} \vec{y}$ such that $\mathbf{A}$ commutes with its $\mathbf{K}$-adjoint.

7. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ have full column rank. Prove that $\|\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\|_2 = 1/\sigma_n$ where σ_n is the n^{th} singular value of $\mathbf{A}$. (Note that $\mathbf{A}^{-1}$ is not defined, so your proof cannot use $\mathbf{A}^{-1}$.)

   where $\mathbf{P}_i$ is an orthogonal projection matrix that projects orthogonally onto the nullspace of $\mathbf{L} - \lambda_i \mathbf{I}$.