This document contains linear-algebra questions covering background material for the APPM PhD program at CU-Boulder. For each question simply indicate how confident you are of being able to answer it correctly. All vectors are lists of real numbers unless specified otherwise.

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1. Can you find, by hand, an expression that describes all solutions of a system of linear equations, whenever at least one solution exists?

2. Can you find the rank of a $m \times n$ matrix by hand?

3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix with rank n and $\mathbf{1}$ is an $n \times n$ matrix of ones. Prove that

$$\mathbf{A}\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\right)$$

   has rank exactly $n - 1$.

4. Suppose that you are given $\mathbf{A}$, $\mathbf{B}$, and $\vec{x}$. Do you know how to evaluate the expression $(\mathbf{A} + \mathbf{B}^{-1})^{-1}\vec{x}$ without computing the inverse of any matrices?

5. Can you find a basis for the range (aka image, aka column space), the co-range (aka co-image, aka row space), the kernel (aka null space), and the co-kernel (aka left hand null space) of an $n \times m$ matrix by hand?

6. Suppose that you are given P pairs of vectors $(\vec{x}_1, \vec{b}_1)$, ..., $(\vec{x}_P, \vec{b}_P)$ all of which satisfy

$$\mathbf{A}\vec{x}_p = \vec{b}_p, \quad p = 1, \dots, P$$

   where $\mathbf{A}$ is an $m \times n$ matrix that is not given to you. If you are given a vector $\vec{b}$ that is in the span of $\{\vec{b}_1, \dots, \vec{b}_P\}$, can you find a solution $\vec{x}$ to $\mathbf{A}\vec{x} = \vec{b}$? (The question is not asking whether a solution exists, which it does. It's asking if you know how to find it without knowing the matrix $\mathbf{A}$.)

7. Consider the linear operator $\mathcal{L} : f \mapsto xf'$ acting on the space of polynomials of degree ≤ 5. Can you find a matrix representation of this operator with respect to the monomial basis?

8. Consider the linear, non-autonomous system of ordinary differential equations

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \mathbf{L}(t)\vec{x}.$$

   Prove that there is a matrix $\mathbf{\Pi}(t)$ such that $\mathbf{\Pi}(t)\vec{x}_0$ is the solution to the differential equation at time t starting from initial condition $\vec{x}(t = 0) = \vec{x}_0$.

9. Do you know the definitions of the p norms of vectors and matrices for $1 \leq p \leq \infty$?

10. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Can you prove that $\mathbf{A}^T\mathbf{A}$ is non-negative definite? Could you state necessary and sufficient conditions on $\mathbf{A}$ so that $\mathbf{A}^T\mathbf{A}$ is positive definite?

11. Define $f : (\vec{x}, \vec{y}) \mapsto \vec{x}^T\mathbf{K}\vec{y}$ for a square matrix $\mathbf{K}$. Under what conditions on $\mathbf{K}$ does $f(\vec{x}, \vec{y})$ define an inner product on $\mathbb{R}^n$?

12. Give an expression for the adjoint of $\mathbf{A}$ with respect to the inner product defined by $\vec{x}^T\mathbf{K}\vec{y}$.

13. Given two vectors in $\mathbb{R}^n$, can you find the Euclidean angle between them?

14. Given an $m \times n$ matrix $\mathbf{A}$ and a vector $\vec{b}$ in $\mathbb{R}^m$, can you find the orthogonal projection of $\vec{b}$ onto the range of $\mathbf{A}$? (Assume orthogonality is defined with respect to the dot product.)

15. Given a nonzero $m \times n$ matrix $\mathbf{A}$, can you find an *orthonormal* basis for its range? (The question is not asking whether such a basis exists; it is asking whether you know how to find one.)

16. Can you prove the following statement: If $\mathbf{A}$ is skew-symmetric, then
$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$$
is well-defined and is an orthogonal matrix with determinant equal to 1?

17. Prove that the eigenvalues of $\mathbf{A}$ are also eigenvalues of $\mathbf{A}^T$ without making reference to the properties of the determinant.

18. Suppose that there is an invertible matrix $\mathbf{S}$ such that $\mathbf{SA} = \mathbf{BS}$. Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

19. Give an example of a 2×2 matrix that is not diagonalizable.

20. Suppose that $\mathbf{A}$ is symmetric and that $\mathbf{A} = \mathbf{LU}$ is an LU factorization of $\mathbf{A}$. If the diagonal values of $\mathbf{U}$ are all positive, what can you conclude, if anything, about the eigenvalues of $\mathbf{A}$?

21. Prove that $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$ for every $\mathbf{A} \in \mathbb{R}^{n \times n}$ and every operator norm $\|\cdot\|$. $\rho(\mathbf{A})$ is the spectral radius of $\mathbf{A}$.

22. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ have full column rank. Prove that $\|\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\|_2 = 1/\sigma_n$ where σ_n is the n^{th} singular value of $\mathbf{A}$. (Note that $\mathbf{A}^{-1}$ is not defined, so your proof cannot use $\mathbf{A}^{-1}$.)

23. Prove the following: the iteration defined by $\vec{x}_{k+1} = \mathbf{B}\vec{x}_k$, where $\mathbf{B} \in \mathbb{R}^{n \times n}$ and $\vec{x}_k \in \mathbb{R}^n$, converges to $\vec{0}$ for every initial condition $\vec{x}_0$ if and only if $\rho(\mathbf{B}) < 1$ where $\rho(\mathbf{B})$ is the spectral radius of $\mathbf{B}$.

24. Prove that if $\mathbf{A}$ is skew symmetric, then $e^{\mathbf{A}}$ is orthogonal.

25. Prove the following: If $\|\mathbf{F}\| < 1$ where $\|\cdot\|$ is an operator norm, then
$$(\mathbf{I} - \mathbf{F})^{-1} - \mathbf{I} = \mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}.$$

Computing questions:

1. Can you write up homework solutions in LaTeXand print them out?

2. Can you use `pip` or `conda` (or `mamba`) to set up and manage Python environments?

3. Can you use a Jupyter notebook?

4. Can you use `numpy` or `scipy` to perform numerical computations? (Or Matlab or Julia)

5. Can you plot a mathematical function (e.g. using `matplotlib.pyplot`), save it as a file, include it in a LaTeXdocument, and print it out as pdf?