1. Suppose that $\mathbf{A}$ is not diagonalizable. Construct a matrix $\mathbf{E}$ such that for sufficiently small $\epsilon > 0$ the matrix $\mathbf{A} + \epsilon\mathbf{E}$ is diagonalizable.

2. Prove that if $\mathbf{A}$ is skew symmetric, then $e^{\mathbf{A}}$ is orthogonal.

3. (a) Consider the linear, autonomous system of ODEs

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \mathbf{A}\vec{x}, \ \vec{x}(t=0) = \vec{b}.$$

   Write an explicit expression for a matrix $\mathbf{L}(t)$ such that

$$\vec{x}(t) = \mathbf{L}(t)\vec{b}.$$

   (b) Consider the problem of finding a direction for the vector $\vec{b}$ that will lead to the largest $\vec{x}$ (amplitude measured using the 2-norm) at a fixed time T. Explain how to find both the forcing vector $\vec{b}$ and the solution vector $\vec{x}$ using the SVD of $\mathbf{L}$.

   (c) Suppose that $\mathbf{A}$ is symmetric. How is the optimal vector $\vec{b}$ from (b) related to the eigenvectors of $\mathbf{A}$?

4. Prove the following: If $\|\mathbf{F}\| < 1$ where $\|\cdot\|$ is an operator norm, then

$$(\mathbf{I} - \mathbf{F})^{-1} - \mathbf{I} = \mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}.$$

5. Consider the matrix $\vec{u}\vec{v}^T$ where $\vec{u}$ and $\vec{v}$ are unit vectors and $\vec{u} \cdot \vec{v} \neq 0$.

   (a) What are the eigenvalues and eigenvectors of this matrix?

   (b) Is the matrix diagonalizable?

   (c) Find an explicit expression for a matrix $\mathbf{X}$ such that $\mathbf{X}^2 = \mathbf{I} + \vec{u}\vec{v}^T$.

   (d) Find an explicit expression for $(\mathbf{I} + \vec{u}\vec{v}^T)^{-1}$ assuming that $\vec{u} \cdot \vec{v} \neq -1$.

6. Prove the following: the iteration defined by $\vec{x}_{k+1} = \mathbf{B}\vec{x}_k$, where $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\vec{x}_k \in \mathbb{R}^n$, converges to $\vec{0}$ for every initial condition $\vec{x}_0$ if and only if $\rho(\mathbf{B}) < 1$ where $\rho(\mathbf{B})$ is the spectral radius of $\mathbf{B}$.