1. Consider the space of polynomials of degree ≤ 3 with inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x$. Apply Gram-Schmidt to the monomial basis to obtain an orthogonal basis for the space. The resulting polynomials are called Legendre polynomials.

2. Consider the space V of functions f that are continuous on $[-\pi, \pi]$, with subspace W spanned by

$$\cos(nx), n = 0, \ldots, N; \sin(nx), n = 1, \ldots, N.$$

Find an expression for the orthogonal projection of a function $f \in V$ onto W.

3. Let $\mathbf{A}$ be a real skew-symmetric matrix. Without referring to eigenvalues, prove that

$$\mathbf{Q} = (\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$$

is

(a) Well-defined (i.e. the inverse exists), and

(b) An orthogonal matrix.