Wald Ch. 3 Solutions

Ian Haines

Jan. 18, 2022

1 Torsion

1.1 Torsion tensor

We want to show that given a non-torsion-free derivative operator, $\boldsymbol{\nabla}_a$, and a scalar field, f, there exists a tensor, T^c_{ab}, such that

$$\boldsymbol{\nabla}_a \boldsymbol{\nabla}_b f - \boldsymbol{\nabla}_b \boldsymbol{\nabla}_a f = -T^c_{ab} \boldsymbol{\nabla}_c f \tag{1}$$

Suppose $\boldsymbol{\nabla}_a$ is a derivative operator that is not torsion-free and $\widetilde{\boldsymbol{\nabla}}_a$ is a torsion-free derivative operator.

Let f be a scalar field and ω_b be a dual vector field. Consider the difference

$$\begin{aligned} \boldsymbol{\nabla}_a(f\omega_b) - \widetilde{\boldsymbol{\nabla}}_a(f\omega_b) &= \omega_b \boldsymbol{\nabla}_a f + f\boldsymbol{\nabla}_a \omega_b - \omega_b \widetilde{\boldsymbol{\nabla}}_a f - f\widetilde{\boldsymbol{\nabla}}_a \omega_b & (2) \\ &= f(\boldsymbol{\nabla}_a \omega_b - \widetilde{\boldsymbol{\nabla}}_a \omega_b) & (3) \end{aligned}$$

where the second equality comes from the condition that derivative operators act identically on scalar fields.

Suppose there exists another dual vector field, ω'_b, such that $\omega'_b(p) = \omega_b(p)$. Since ω is a dual vector field, $\omega : \mathbb{R}^n \to \mathbb{R}$. Hence, to first order we have

$$\omega_b(x) = \omega_b(p) + \sum_{\mu=1}^{n} (x^\mu - p^\mu) \frac{\partial \omega_b}{\partial x^\mu}. \tag{4}$$

So, for smooth functions $f_{(\mu)}$ which vanish at $x = p$ and smooth vector fields $\nu^{(\mu)}$, we have

$$\omega'_b(x) - \omega_b(x) = \sum_{\mu=1}^{n} f_{(\mu)}(x) \nu_b^{(\mu)}(x) \tag{5}$$

which vanishes at $x = p$ as we desire.

Consequently, the difference (where we dropped the explicit dependence)

$$\widetilde{\boldsymbol{\nabla}}_a(\omega'_b - \omega_b) - \boldsymbol{\nabla}_a(\omega'_b - \omega_b) = \sum_{\mu=1}^{n} f_{(\mu)}(\widetilde{\boldsymbol{\nabla}}_a \nu_b^{(\mu)} - \boldsymbol{\nabla}_a \nu_b^{(\mu)}) \tag{6}$$

vanishes at $x = p$.

Thus,

$$\widetilde{\boldsymbol{\nabla}}_a \omega'_b - \boldsymbol{\nabla}_a \omega'_b = \widetilde{\boldsymbol{\nabla}}_a \omega_b - \boldsymbol{\nabla}_a \omega_b \tag{7}$$

which shows that the difference between the derivative operators only depends upon the value of ω_b at p.

We see that $(\widetilde{\boldsymbol{\nabla}}_a - \boldsymbol{\nabla}_a)$ defines a tensor that takes dual vectors to $(0, 2)$ type tensors. Thus, it defines a tensor C^c_{ab} such that

$$\boldsymbol{\nabla}_a \omega_b = \widetilde{\boldsymbol{\nabla}}_a \omega_b - C^c_{ab} \omega_c \tag{8}$$

If we let $\omega_b = \boldsymbol{\nabla}_b f$, then we have

$$\boldsymbol{\nabla}_a \boldsymbol{\nabla}_b f = \widetilde{\boldsymbol{\nabla}}_a \widetilde{\boldsymbol{\nabla}}_b f - C^c_{ab} \boldsymbol{\nabla}_c f \tag{9}$$

This gives us

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = \widetilde{\nabla}_a \widetilde{\nabla}_b f - \widetilde{\nabla}_b \widetilde{\nabla}_a f - C^c_{ab} \nabla_c f + C^c_{ba} \nabla_c f \tag{10}$$

Because $\widetilde{\nabla}_a$ is torsion-free, we see that

$$\boxed{\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T^c_{ab} \nabla_c f} \tag{11}$$

where T^c_{ab} is non-zero because ∇_a is not torsion-free and so $C^c_{ab} \neq C^c_{ba}$.

1.2 Commutator of vector fields for a general connection

Let X^a and Y^a be two smooth vector fields. Let ∇_a be a covariant derivative associated with a general connection. Let $[\cdot, \cdot]$ denote the commutator of two vector fields. Then we have

$$\begin{aligned}
[X, Y](f) &= X^a \nabla_a (Y^b \nabla_c f) - Y^a \nabla_a (X^b \nabla_c f) && (12)\\
&= (X^a \nabla_a Y^b) \nabla_c f + Y^b X^a \nabla_a - (Y^a \nabla_a X^b) \nabla_b f - X^a Y^b \nabla_b \nabla_a f && (13)\\
&= (X^a \nabla_a Y^b - Y^a \nabla_a X^b) \nabla_b f - T^c{}_{ab} \nabla_c f X^a Y^b && (14)
\end{aligned}$$

Hence,

$$\boxed{T^c{}_{ab} X^a Y^b = X^a \nabla_a Y^c - Y^a \nabla_a X^c - [X, Y]^c} \tag{15}$$

1.3 Uniqueness of compatible connection with torsion

We are given a general metric. Associated with this metric is a connection and therefore a covariant derivative. We enforce the criterion that the metric is compatible. In other words,

$$\nabla_a g_{bc} = 0 \tag{16}$$

The covariant derivative operator is defined by the action of the partial derivative operator ∂_a, the torsion tensor $T^a{}_{bc}$, and the connection coefficients $C^a{}_{bc}$. We wish to determine the connection coefficients in terms of the partial derivative operator and the torsion tensor.

We know from the action of the covariant derivative on a general tensor that

$$\nabla_a g_{bc} = \partial_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} \tag{17}$$

$$\nabla_b g_{ac} = \partial_b g_{ac} - C^d{}_{ba} g_{dc} - C^d{}_{bc} g_{ad} \tag{18}$$

and

$$\nabla_c g_{ab} = \partial_c g_{ab} - C^d{}_{ca} g_{db} - C^d{}_{cb} g_{ad} \tag{19}$$

Because of the compatibility criterion, we have

$$0 = \partial_a g_{bc} - \partial_b g_{ac} - \partial_c g_{ab} - T^c{}_{ab} - T^b{}_{ac} + C^a{}_{bc} + C^a{}_{cb} \tag{20}$$

where $T^c{}_{ab} = C^c{}_{ab} - C^c{}_{ba}$.

Using the antisymmetry of the torsion tensor, we have

$$2C_{abc} = \partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc} + T_{cab} + T_{bac} + T_{abc} \tag{21}$$

Thus, we have

$$\boxed{C^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} + T_{cdb} + T_{bdc} + T_{dbc})} \tag{22}$$

2 Symmetries of the Riemann Curvature Tensor

2.1 A useful identity

We wish to show that

$$R_{abcd} = R_{cdab} \tag{23}$$

Consider the equations

$$\begin{aligned} R_{[abc]d} &= R_{abcd} - R_{bacd} + R_{bcad} - R_{cbad} + R_{cabd} - R_{acbd} \quad &(24)\\ &= 0 &(25)\end{aligned}$$

and

$$\begin{aligned} R_{[cad]b} &= R_{cadb} - R_{acdb} + R_{adcb} - R_{dacb} + R_{dcab} - R_{cdab} \quad &(26)\\ &= 0 &(27)\end{aligned}$$

Observe that $R_{cadb} - R_{acdb} = -R_{cabd} + R_{acbd}$. Thus, we have

$$R_{abcd} - R_{bacd} + R_{bcad} - R_{cbad} = -(R_{adcb} - R_{dacb} + R_{dcab} - R_{cdab}) \tag{28}$$

Notice that the left-hand side is just the right-hand side with the indices a and c swapped and b and d swapped. Hence,

$$\boxed{R_{abcd} = R_{cdab}} \tag{29}$$

2.2 Independent components from symmetries

We wish to show that the Riemann curvature tensor has $n^2(n^2-1)/12$ independent components using the known symmetries of it.

The first symmetry says

$$R_{abcd} = -R_{bacd} \tag{30}$$

and the second symmetry says

$$R_{abcd} = -R_{abdc} \tag{31}$$

These tell us that R must be antisymmetric in the first and second and third and fourth indices. Hence, the first and second indices can only have

$$n^2 - \frac{n(n-1)}{2} - n = \frac{n(n-1)}{2} \tag{32}$$

possible unique values. The second term comes from the fact that the swapped indices for off-diagonal terms are related by a negative sign. The third term comes from the fact that the diagonal must be filled with 0's due to the antisymmetry.

Thus, we see that we have

$$\left[\frac{n(n-1)}{2}\right]^2 = \frac{n^2(n-1)^2}{4} \tag{33}$$

components coming from the first two symmetries. Now, the totally antisymmetric property tells us

$$R_{[abc]d} = 0 \tag{34}$$

Hence, our number of independent components decreases by an

$$\frac{n \times (n-1) \times (n-2) \times n}{3!} = \frac{n^2(n-1)(n-2)}{6} \tag{35}$$

where the numerator comes from the possible combinations and the 3! comes from the number of permutations.

Thus, we have

$$\boxed{\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n^2-1)}{12}} \tag{36}$$

3 Low dimensional curvature

3.1 Two-dimensional Riemann tensor

We wish to show that in 2-dimensions that the Riemann curvature tensor reduces to

$$R_{abcd} = R g_{a[c} g_{d]b} \tag{37}$$

We do this by showing that the right-hand side expression generates the vector space of tensors having the symmetries of the RCT.

Observe that

$$g_{a[c} g_{d]b} = \frac{1}{2}(g_{ac} g_{db} - g_{ad} g_{cb}) \tag{38}$$

Due to the symmetry of the metric tensor, this expression is antisymmetric with respect to interchanging a, b and c, d. So, we only need to show that the antisymmetrization of a, b, c forces the tensor to vanish.

After performing some algebra, we see this is just

$$g_{[a[c} g_{|d]|b]} = (g_{ac} g_{db} - g_{ad} g_{cb}) + (g_{ba} g_{dc} - g_{bd} g_{ac}) + (g_{cb} g_{da} - g_{cd} g_{ba}) \tag{39}$$

Using the symmetry of the metric tensor, we see that each term on the right-hand side cancels leaving us with the desired symmetry property.

Now, in $n = 2$ dimensions, we have 1 independent component: R_{1212}. However, $C_{1212} = C^1{}_{212} g_{11} = 0 g_{11} = 0$. So, the Weyl tensor must vanish for $n = 2$ dimensions. The middle term in equation (3.2.28) in Wald also vanishes when plugging in $1, 2, 1$ and 2 for a, b, c and d. Thus, the right-hand side of (37) generates the vector space of tensors with the symmetries of the RCT.

4 Affine parameters and affine transformations

1. We wish to show that any curve satisfying

   $$T^a \boldsymbol{\nabla}_a T^b = \alpha T^b$$

   can be reparameterized to satisfy

   $$T^a \boldsymbol{\nabla}_a T^a = 0.$$

   Using the definition of the covariant derivative, we have

   $$T^a \boldsymbol{\nabla}_a T^b = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} \frac{\mathrm{d}x^\sigma}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = \alpha \frac{\mathrm{d}x^\mu}{\mathrm{d}t}.$$

   Now, suppose that $t = f(t^*)$. Using the chain rule, we have

   $$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t^*} \frac{\mathrm{d}t^*}{\mathrm{d}t},$$

   and

   $$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^{*2}} \left[\frac{\mathrm{d}t^*}{\mathrm{d}t}\right]^2 + \frac{\mathrm{d}x^\mu}{\mathrm{d}t^*} \frac{\mathrm{d}^2 t^*}{\mathrm{d}t^2}.$$

   Plugging these in to the geodesic equation yields

   $$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^{*2}} \left[\frac{\mathrm{d}t^*}{\mathrm{d}t}\right]^2 + \frac{\mathrm{d}x^\mu}{\mathrm{d}t^*} \frac{\mathrm{d}^2 t^*}{\mathrm{d}t^2} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t^*} \frac{\mathrm{d}x^\nu}{\mathrm{d}t^*} \left[\frac{\mathrm{d}t^*}{\mathrm{d}t}\right]^2 = \alpha \frac{\mathrm{d}x^\mu}{\mathrm{d}t^*} \frac{\mathrm{d}t^*}{\mathrm{d}t}.$$

   Hence, for any α, we can reparameterize the curve with a function $t = f(t^*)$ such that

   $$\frac{\mathrm{d}^2 t^*}{\mathrm{d}t^2} = \alpha \frac{\mathrm{d}t^*}{\mathrm{d}t}.$$

   This yields the affine condition.

2. Now because of the second derivative in the LHS of the equation above, we see that t^* will be an affine parameter, (i.e., $\alpha = 0$), if and only if $t^* = at + b$.

5 Christoffel Symbols and Geodesics in Spherical Coordinates

1. The Christoffel symbols are given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right).$$

In this case, the metric is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta. \end{pmatrix}$$

So, we have

$$\Gamma^{1}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix},$$

$$\Gamma^{2}{}_{\mu\nu} = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix},$$

and lastly

$$\Gamma^{3}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1/r \\ 0 & 0 & \cot\theta \\ 1/r & \cot\theta & 0 \end{pmatrix}.$$

2. The geodesic equation says

$$\frac{\mathrm{d}^2 x^{\sigma}}{\mathrm{d}s^2} + \sum_{\mu,\nu}\Gamma^{\sigma}{}_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} = 0.$$

So, the r component equation says

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0,$$

the θ component equation says

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0,$$

and the ϕ component equation says

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + 2\cot\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0.$$

A line from a point $P = (x_1, y_1, z_1)$ to a point $Q = (x_2, y_2, z_2)$ is parameterized as follows:

$$\gamma(s) = \big((x_2 - x_1)s + x_1, (y_2 - y_1)s + y_1, (z_2 - z_1)s + z_1\big).$$

We show that this curve is a solution to the geodesic equations above. From the definition of the spherical coordinates transformation,

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\big[(x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1)\big]s + \big(x_1 + y_1 + z_1\big)}{r},$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \frac{\Big[(x' + y' + z')(x^2 + y^2 + z^2) - (x + y + z)^2\Big](x^2 + y^2)}{r^3(x^2 + y^2)},$$

$$r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = \frac{\left[z'(x^2+y^2+z^2) - z(x+y+z)\right]^2}{r^3(x^2+y^2)},$$

and

$$r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = \frac{(y'x - x'y)^2(x^2+y^2+z^2)}{r^3(x^2+y^2)}.$$

Plugging this into the r component equation, we find it to be true. Thus, the solutions to the geodesic equation are straight lines.

6 Curvature of a Lorentz Metric

We have a manifold with a Lorentz Metric

$$\mathrm{d}s^2 = \Omega(x,t)(-\mathrm{d}t^2 + \mathrm{d}x^2),$$

or

$$g_{\mu\nu} = \begin{pmatrix} -\Omega & 0 \\ 0 & \Omega \end{pmatrix}.$$

We want to compute the curvature using both the component method and the tetrad method.

1. Using the coordinate method, we have

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^\nu}(\Gamma^\sigma{}_{\mu\rho}) - \frac{\partial}{\partial x^\mu}(\Gamma^\sigma{}_{\nu\rho}) + \sum_\alpha (\Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\alpha\mu}).$$

We need to calculate the Christoffel symbols from the metric. They are given by

$$\Gamma^1{}_{\mu\nu} = \begin{pmatrix} -\Omega_t/2\Omega & -\Omega_x/2\Omega \\ -\Omega_x/2\Omega & \Omega_t/2\Omega \end{pmatrix},$$

and

$$\Gamma^2{}_{\mu\nu} = \begin{pmatrix} \Omega_x/2\Omega & \Omega_t/2\Omega \\ \Omega_t/2\Omega & -\Omega_x/2\Omega \end{pmatrix},$$

where the subscripts correspond to partial derivatives with respect to the subscripted variable. Since $n = 2$, there is only 1 independent component. The independent component is given by

$$R_{121}{}^2 = \frac{\Omega_x^2 + \Omega_t^2}{2\Omega^2} - \frac{\Omega_{xx} + \Omega_{tt}}{2\Omega}.$$

Using the symmetries of the Riemann curvature tensor, we have for the other components:

$$\begin{aligned}
R_{111}{}^1 &= 0 \\
R_{111}{}^2 &= 0 \\
R_{112}{}^1 &= 0 \\
R_{121}{}^1 &= 0 \\
R_{211}{}^1 &= 0 \\
R_{112}{}^2 &= 0 \\
R_{121}{}^2 &= R_{121}{}^2 \\
R_{211}{}^2 &= -R_{121}{}^2 \\
R_{122}{}^1 &= -R_{121}{}^2 \\
R_{212}{}^1 &= R_{121}{}^2 \\
R_{221}{}^1 &= 0 \\
R_{222}{}^1 &= 0
\end{aligned}$$

$$R_{221}{}^{2} = 0$$
$$R_{212}{}^{2} = 0$$
$$R_{122}{}^{2} = 0$$
$$R_{222}{}^{2} = 0.$$

2. Now, we use the tetrad method to calculate the Riemann curvature tensor. To do this, we need to calculate the spin connection 1-forms $\omega_\mu{}^{ab}$. The formula is given by

$$\omega_\mu{}^{ab} = e_\nu{}^a \partial_\mu e^{\nu b} + e_\nu{}^a \Gamma^\nu{}_{\sigma\mu} e^{\sigma b}.$$

The vierbein is given by the formula

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab},$$

or

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab} e_\mu{}^a e_\nu{}^b \mathrm{d}x^\mu \mathrm{d}x^\nu.$$

Since $\mathrm{d}s^2 = \Omega(x,t)(-\mathrm{d}t^2 + \mathrm{d}x^2)$, we see that

$$e_\mu{}^a = \begin{pmatrix} \sqrt{\Omega} & 0 \\ 0 & \sqrt{\Omega} \end{pmatrix},$$

and

$$e^{\mu a} = \begin{pmatrix} \frac{-\sqrt{\Omega}}{\Omega} & 0 \\ 0 & \frac{\sqrt{\Omega}}{\Omega} \end{pmatrix}.$$

Plugging in the appropriate values, we find that

$$\omega_1{}^{ab} = \begin{pmatrix} 0 & -\,{}^{\Omega_x}\!/_{2\Omega} \\ {}^{\Omega_x}\!/_{2\Omega} & 0 \end{pmatrix},$$

$$\omega_2{}^{ab} = \begin{pmatrix} 0 & {}^{\Omega_t}\!/_{2\Omega} \\ -\,{}^{\Omega_t}\!/_{2\Omega} & 0 \end{pmatrix}.$$