QFT Ch. 3 Exercises

Ian Haines

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1 Lorentz Group and Algebra

1. The Lorentz algebra has the following commutation relations

$$[J^{\alpha\beta}, J^{\gamma\delta}] = i\big(g^{\beta\gamma}J^{\alpha\delta} - g^{\alpha\gamma}J^{\beta\delta} - g^{\beta\delta}J^{\alpha\gamma} + g^{\alpha\delta}J^{\beta\gamma}\big) \tag{1}$$

where

$$J^{\mu\nu} = i\big(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\big) \tag{2}$$

The generators of rotations and boosts are respectively defined as

$$L^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i} \tag{3}$$

The commutation relations can be found as follows:

$$[L^i, L^l] = \tfrac{1}{2}\big(\epsilon^{ijk}J^{jk}\epsilon^{lmn}J^{mn} - \epsilon^{lmn}J^{mn}\epsilon^{ijk}J^{jk}\big) \tag{4}$$

For $i = l$, the commutator is clearly 0. If $i = 1$ and $l = 2$, we get

$$\begin{aligned}
[L^1, L^2] &= \tfrac{1}{4}\big((J^{23} - J^{32})(J^{31} - J^{13}) - (J^{31} - J^{13})(J^{23} - J^{32})\big) \\
&= \tfrac{1}{4}\big(J^{23}J^{31} - J^{31}J^{23} + J^{13}J^{23} - J^{23}J^{13} + J^{31}J^{32} - J^{32}J^{31} + J^{32}J^{13} - J^{13}J^{32}\big) \\
&= \tfrac{1}{4}\big([J^{23}, J^{31}] + [J^{13}, J^{23}] + [J^{31}, J^{32}] + [J^{32}, J^{13}]\big) \\
&= \tfrac{1}{4}\big(2[J^{23}, J^{31}] + 2[J^{13}, J^{23}]\big) \\
&= [J^{23}, J^{31}] \\
&= ig^{33}J^{21} \\
&= iJ^{12} \\
&= iL^3
\end{aligned}$$

We can generalize this:

$$[L^i, L^j] = i\epsilon^{ijk}J^{ij} = i\epsilon^{ijk}L^k \tag{5}$$

The boosts commutation relations are

$$\begin{aligned}
[K^i, K^j] &= [J^{0i}, J^{0j}] \\
&= -[J^{0i}, J^{j0}] \\
&= -ig^{00}J^{ij} \\
&= -iL^k \\
&= -i\epsilon^{ijk}L^k
\end{aligned}$$

So

$$[K^i, K^j] = -i\epsilon^{ijk}L^k \tag{6}$$

The commutation relations between boosts and rotations are

$$\begin{aligned}
[L^i, K^j] &= \tfrac{1}{2}\epsilon^{ijk}[J^{jk}, J^{0j}] \\
&= \tfrac{i}{2}\epsilon^{ijk}(-g^{kj}J^{j0} + g^{jj}J^{k0}) \\
&= \tfrac{i}{2}\epsilon^{ijk}(g^{jj}J^{k0}) \\
&= \tfrac{i}{2}\epsilon^{ijk}J^{0k} \\
&= i\epsilon^{ijk}K^k
\end{aligned}$$

$$[L^i, K^j] = i\epsilon^{ijk}K^k \tag{7}$$

Consider the following linear combinations:

$$\mathbf{J}_+ = \tfrac{1}{2}(\mathbf{L} + i\mathbf{K})$$

and

$$\mathbf{J}_- = \tfrac{1}{2}(\mathbf{L} - i\mathbf{K})$$

We can compute the commutators of these operators

$$\begin{aligned}
[\mathbf{J}_+, \mathbf{J}_-] &= \mathbf{J}_+\mathbf{J}_- - \mathbf{J}_-\mathbf{J}_+ \\
&= \tfrac{1}{4}\big[(\mathbf{L} + i\mathbf{K})(\mathbf{L} - i\mathbf{K}) - (\mathbf{L} - i\mathbf{K})(\mathbf{L} + i\mathbf{K})\big] \\
&= \tfrac{1}{4}\big[\mathbf{L}^2 - i\mathbf{L}\cdot\mathbf{K} + i\mathbf{K}\cdot\mathbf{L} + \mathbf{K}^2 - \mathbf{L}^2 - i\mathbf{L}\cdot\mathbf{K} + i\mathbf{K}\cdot\mathbf{L} - \mathbf{K}^2\big] \\
&= \tfrac{i}{2}\big[\mathbf{K}\cdot\mathbf{L} - \mathbf{L}\cdot\mathbf{K}\big] \\
&= 0
\end{aligned}$$

where the last equality comes from the fact that L^i and K^j commute with themselves and each other as long as $i = j$.

2. We can rewrite the angular momentum operators by taking linear combinations:

$$\mathbf{L} = \mathbf{J}_+ + \mathbf{J}_- \tag{8}$$

and

$$\mathbf{K} = -i(\mathbf{J}_+ - \mathbf{J}_-) \tag{9}$$

The infinitesimal transformation law is

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K})\Phi \tag{10}$$

For a particle in the $\left(\frac{1}{2}, 0\right)$ representation, we know $\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$. Additionally, $\mathbf{J}_- = 0$. Hence, we have

$$\begin{aligned}
\Phi &\to \big(1 - i\boldsymbol{\theta}\cdot(\mathbf{J}_+ + \mathbf{J}_-) + (i)^2\boldsymbol{\beta}\cdot(\mathbf{J}_+ - \mathbf{J}_-)\big)\Phi \\
&= (1 - i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2})\Phi
\end{aligned}$$

which is how a left-handed Weyl spinor transforms. Similarly, in the $\left(0, \frac{1}{2}\right)$, we see that the transformation law is

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2})\Phi \tag{11}$$

which is the transformation law for a right-handed Weyl spinor.

3. For the $(\frac{1}{2}, \frac{1}{2})$ representation, we use the trick given by P&S: use the unitary transformation $\boldsymbol{\sigma}^* = -\sigma^2\boldsymbol{\sigma}\sigma^2$ to transform ψ_L into $\psi_L^T\sigma^2$. This turns the transformation law from $1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}$ to $1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}$. Hence, we can represent the object that transforms as the $(\frac{1}{2}, \frac{1}{2})$ representation as a 2×2 matrix

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} \tag{12}$$

The left-hand side of the matrix is the RH representation and the right-hand side of the matrix is the transposed LH representation. We can show that the Lorentz transformation of the matrix gives the Lorentz transformation of the 4-vector (V^0, V^1, V^2, V^3). To see this, consider the following expression:

$$
\begin{aligned}
&(1 - i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2})\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}(1 + i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2}) \\
&= \begin{pmatrix} 1 - i\theta_3 + \beta^3 & -i\theta_1 - \theta_2 + \beta_1 - i\beta_2 \\ -i\theta_1 + \theta_2 + \beta_1 + i\beta_2 & 1 + i\theta_3 - \beta_3 \end{pmatrix}\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} 1 + i\theta_3 + \beta^3 & i\theta_1 + \theta_2 + \beta_1 - i\beta_2 \\ i\theta_1 - \theta_2 + \beta_1 + i\beta_2 & 1 - i\theta_3 - \beta_3 \end{pmatrix} \\
&=
\end{aligned}
$$

2 Gordon Identity

The Gordon identity is

$$
\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p) \tag{13}
$$

Observe that we can use the Dirac algebra anticommutation relations to get the following:

$$
\begin{aligned}
\bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p) &= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} - (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\frac{(p'-p)_\nu}{4m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{g^{\mu\nu}p'_\nu + g^{\nu\mu}p_\nu}{2m} - (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\frac{(p'-p)_\nu}{4m}\right]u(p) \\
&= \bar{u}(p')\left[\gamma^\nu\gamma^\mu\frac{p'_\nu}{2m} + \gamma^\mu\gamma^\nu\frac{p_\nu}{2m}\right]u(p)
\end{aligned}
$$

This implies

$$
\begin{aligned}
\bar{\psi}(p')\left[\gamma^\nu\gamma^\mu\frac{p'_\nu}{2m} + \gamma^\mu\gamma^\nu\frac{p_\nu}{2m}\right]\psi(p) &= \frac{1}{2m}\left[\partial_\nu\bar{\psi}(p')\gamma^\nu\gamma^\mu\psi(p) + \bar{\psi}(p')\gamma^\mu\gamma^\nu\partial_\nu\psi(p)\right] \\
&= \frac{1}{2m}\left[\bar{\psi}(p')m\gamma^\mu\psi(p) + \bar{\psi}(p')\gamma^\mu m\psi(p)\right] \\
&= \bar{\psi}(p')\gamma^\mu\psi(p)
\end{aligned}
$$

which completes the proof.

3 Spinor Products

We define k_0^μ and k_1^μ to be fixed 4-vectors such that $k_0^2 = 0$, $k_1^2 = -1$, and $k_0^\mu k_{1\mu} = 0$. We let u_{L0} be the left-handed spinor for a fermion with momentum k_0. Additionally, we let $u_{R0} = \not{k}_1 u_{L0}$. So, for any lightlike 4-momentum p, we define

$$
u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0} \quad \text{and} \quad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0} \tag{14}
$$

1. First, we show that

$$
\not{k}_0 u_{R0} = 0 \tag{15}
$$

By definition, we have

$$\begin{aligned}
\not{k}_0 u_{R0} &= \not{k}_0 \not{k}_1 u_{L0} \\
&= \gamma^\mu (k_0)_\mu \gamma_\nu (k_1)^\nu u_{L0} \\
&= \tfrac{1}{2} g^{\mu\nu} (k_0)_\mu (k_1) \nu u_{L0} \\
&= \tfrac{1}{2} (k_0)^\mu (k_1)_\mu u_{L0} \\
&= 0
\end{aligned}$$

Similarly, for lightlight p, we have

$$\begin{aligned}
\not{p} u_L(p) &= \frac{1}{\sqrt{2p \cdot k_0}} (\not{p})^2 u_{R0} \\
&= \frac{1}{\sqrt{2p \cdot k_0}} \gamma^\mu p_\mu \gamma^\nu p_\nu u_{R0} \\
&= \frac{1}{2\sqrt{2p \cdot k_0}} g^{\mu\nu} p_\mu p_\nu u_{R0} \\
&= 0
\end{aligned}$$

This identity also holds for $\not{p} u_R(p)$

2. To construct the left-handed and right-handed spinors, we must use the definition of the Dirac spinors:

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} \tag{16}$$

We understand that the square root is the square root of a matrix. Since u_{L0} is the left-handed spinor with momentum $k_0 = (E, 0, 0, -E)$, we have

$$\begin{aligned}
p \cdot \sigma &= p^\mu \sigma_\mu \\
&= g^{\mu\nu} p_\nu \sigma_\mu \\
&= p_0 \sigma_0 - p_3 \sigma_3 \\
&= E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - (-E) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

The eigenvalue for this matrix is $2E$ and so the square root of the matrix is

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{2E} & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

and the eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, for $\sqrt{p \cdot \bar{\sigma}}$, we have

$$\sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2E} \end{pmatrix} \tag{18}$$

which has eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For the spinor to be left-handed, it needs to have an eigenvalue of $-\frac{1}{2}$ of the helicity operator, $h = \frac{1}{2} \hat{p} \cdot \mathbf{S}$. This operator is a vector operator and can be expanded to be

$$h = \tfrac{1}{2} \hat{p}^3 \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \tfrac{1}{2}(-E) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{19}$$

Hence, we need $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and so we have

$$u_{L0} = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

Since $k_1 = (0, 1, 0, 0)$, we have $\not{k}_1 = -\gamma^1 (k_1)_1 = -\gamma^1$. Hence,

$$u_{R0} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{21}$$

It is easy to see that this is a right-handed spinor using the helicity operator. Using the chiral basis for the gamma matrices, we have

$$u_L(p) = \frac{1}{\sqrt{p \cdot k_0}} \begin{pmatrix} 0 & 0 & p_0 - p_3 & -p_1 + ip_2 \\ 0 & 0 & -p_1 - ip_2 & p_0 + p_3 \\ p_0 + p_3 & p_1 + ip_2 & 0 & 0 \\ p_1 + ip_2 & p_0 - p_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

and

$$u_R(p) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} 0 \\ 0 \\ p_0 + p_3 \\ p_1 + ip_2 \end{pmatrix} \tag{23}$$

Additionally, we can see that

$$\bar{u}_L(p) = \frac{1}{\sqrt{p_0 + p_3}} (0, 0, -p_1 - ip_2, p_0 + p_3) \tag{24}$$

and

$$\bar{u}_R(p) = \frac{1}{\sqrt{p_0 + p_3}} (p_0 + p_3, p_1 - ip_2, 0, 0) \tag{25}$$

3. For lightlike p_1, p_2, we define the spinor products

$$s(p, q) = \bar{u}_R(p) u_L(q), \qquad t(p, q) = \bar{u}_L(p) u_R(q) \tag{26}$$

Using our definitions from the previous subproblem, we have

$$s(p, q) = \frac{1}{\sqrt{(p_0 + p_3)(q_0 + q_3)}} \big[(p_0 + p_3)(-q_1 + iq_2) + (p_1 - ip_2)(q_0 + q_3)\big] \tag{27}$$

and

$$t(p, q) = \frac{1}{\sqrt{(p_0 + p_3)(q_0 + q_3)}} \big[(-p_1 - ip_2)(q_0 + q_3) + (p_0 + p_3)(q_1 + iq_2)\big] \tag{28}$$

We see that

$$\big(s(q, p)\big)^* = \frac{1}{\sqrt{(p_0 + p_3)(q_0 + q_3)}} \big[(q_0 + q_3)(-p_1 - ip_2) + (q_1 + iq_2)(p_0 + p_3)\big] = t(p, q) \tag{29}$$

Similarly, we see that

$$s(q, p) = \frac{1}{\sqrt{(p_0 + p_3)(q_0 + q_3)}} \big[(q_0 + q_3)(-p_1 + ip_2) + (q_1 - iq_2)(p_0 + p_3)\big] = -s(p, q) \tag{30}$$

Putting these two results together, we see

$$(s(p,q))^* = -t(p,q) \tag{31}$$

Hence

$$|s(p,q)|^2 = (s(p,q))(s(p,q))^* = -s(p,q)t(p,q) \tag{32}$$

Expanding this out, we get

$$\begin{aligned}
|s(p,q)|^2 &= -\frac{1}{(p_0+p_3)(q_0+q_3)}\big[(p_0+p_3)(-q_1+iq_2)+(p_1-ip_2)(q_0+q_3)\big] \\
&\qquad\qquad\qquad \times \big[(q_0+q_3)(-p_1-ip_2)+(q_1+iq_2)(p_0+p_3)\big] \\
&= -\frac{1}{(p_0+p_3)(q_0+q_3)}\big[-(p_0+p_3)^2(q_1^2+q_2^2)-(q_0+q_3)^2(p_1^2+p_2^2) \\
&\qquad - (p_0+p_3)(q_0+q_3)(-p_1q_1-iq_1p_2+iq_2p_1-q_2p_2) \\
&\qquad\qquad + (p_0+p_3)(q_0+q_3)(p_1q_1+ip_1q_2-ip_2q_1+p_2q_2)\big] \\
&= -\big[-(p_0+p_3)(q_0-q_3)-(q_0+q_3)(p_0-p_3)+2p_1q_1+2p_2q_2\big] \\
&= 2p\cdot q
\end{aligned}$$

4 Majorana Fermions

1. P&S tell us that a relativistic equation for a massive 2-component fermion field, $\chi_a(x)$ with $a=1,2$, that transforms as a left-handed Weyl spinor can be written as follows:

$$i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0 \tag{33}$$

We must first verify that this equation is Lorentz invariant. Additionally, P&S tell us that this term transforms like a RH Weyl spinor:

$$\sigma^2\chi_L^*(x) \to \Lambda_R\sigma^2\chi_L^*(\Lambda^{-1}x) \tag{34}$$

The left-handed Weyl spinor transforms as

$$\chi(x) \to \chi'(x) = \Lambda_L\chi(\Lambda^{-1}x) \tag{35}$$

So,

$$\begin{aligned}
i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) &\to i\bar{\sigma}^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu\Lambda_L\chi(\Lambda^{-1}x) - \Lambda_R im\sigma^2\chi^*(\Lambda^{-1}x) && (36)\\
&= \Lambda_L\Lambda_L^{-1}i\bar{\sigma}^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu\Lambda_L\chi(\Lambda^{-1}x) - \Lambda_R im\sigma^2\chi^*(\Lambda^{-1}x) && (37)\\
&= \Lambda_L i\bar{\sigma}^\rho\Lambda^\mu_\rho(\Lambda^{-1})^\nu_\mu\partial_\nu\chi(\Lambda^{-1}x) - \Lambda_R im\sigma^2\chi^*(\Lambda^{-1}x) && (38)\\
&= \Lambda_L i\bar{\sigma}\cdot\partial\chi(\Lambda^{-1}x) - \Lambda_R im\sigma^2\chi^*(\Lambda^{-1}x) && (39)
\end{aligned}$$

This is just how the last component of the Dirac equation in 2-component form transforms under a Lorentz transformation:

$$\begin{aligned}
(i\partial\!\!\!/ - m)\psi &= \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} && (40)\\
&= \begin{pmatrix} -m\psi_L + i\sigma\cdot\partial\psi_R \\ i\bar{\sigma}\cdot\partial\psi_L - m\psi_R \end{pmatrix} && (41)\\
&= \begin{pmatrix} -m\psi_L + i\sigma\cdot\partial\psi_R \\ i\bar{\sigma}\cdot\partial\psi_L - m\psi_R \end{pmatrix} && (42)\\
&= \begin{pmatrix} -m\psi_L + i\sigma\cdot\partial\psi_R \\ i\bar{\sigma}\cdot\partial\psi_L - m\psi_R \end{pmatrix} && (43)\\
&\to \begin{pmatrix} \Lambda_L m\psi_L + \Lambda_R i\sigma\cdot\partial\psi_R \\ \Lambda_L i\bar{\sigma}\cdot\partial\psi_L - \Lambda_R m\psi_R \end{pmatrix} && (44)
\end{aligned}$$

Since the Dirac equation is Lorentz invariant, the Majorana equation must be as well. Additionally, this allows us to rewrite the Dirac spinor as a Majorana spinor

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} \tag{45}$$

Using this spinor, the Dirac equation in 2-component form is

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} = \begin{pmatrix} -m\psi_L - m\sigma\cdot\partial\sigma^2\psi_L^* \\ i\bar{\sigma}\cdot\partial\psi_L - im\sigma^2\psi_L^* \end{pmatrix} \tag{46}$$

Hence, by multiplying the right-hand side by the complex conjugate, we get

$$\begin{pmatrix} -m & -i\sigma\cdot\partial \\ -i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} -m\psi_L - m\sigma\cdot\partial\sigma^2\psi_L^* \\ i\bar{\sigma}\cdot\partial\psi_L - im\sigma^2\psi_L^* \end{pmatrix} = \begin{pmatrix} \partial^2+m^2 & 0 \\ 0 & \partial^2+m^2 \end{pmatrix}\begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} \tag{47}$$

which shows that the Majorana equation implies the Klein-Gordon equation.

2. P&S give us the classical action

$$S = \int \mathrm{d}^4x \Big[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\Big] \tag{48}$$

Taking the complex conjugate, we have

$$S^* = \int \mathrm{d}^4x \Big[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\Big]^* \tag{49}$$

$$= \int \mathrm{d}^4x \Big[(\chi^\dagger i\bar{\sigma}\cdot\partial\chi)^* - \frac{im}{2}\big((\chi^T\sigma^2\chi)^* - (\chi^\dagger\sigma^2\chi^*)^*\big)\Big] \tag{50}$$

Observe that each term in the action is a product of two components of χ or χ^*. Hence, we can use the definition of the complex conjugate of Grassmann numbers to see that:

$$\left(\chi^T\sigma\chi\right)^* = \left(\begin{pmatrix}\chi_1 & \chi_2\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix}\right)^* \tag{51}$$

$$= \left(-i\chi_1\chi_2 + i\chi_2\chi_1\right)^* \tag{52}$$

$$= -i\chi_2^*\chi_1^* + i\chi_1^*\chi_2^* \tag{53}$$

$$= \begin{pmatrix}\chi_1^* & \chi_2^*\end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix}\chi_1^* \\ \chi_2^*\end{pmatrix} \tag{54}$$

$$= -\chi^\dagger\sigma^2\chi^* \tag{55}$$

It is clear that $\chi^T\sigma\chi = \left(\left(\chi^T\sigma\chi\right)^*\right)^* = -\left(\chi^\dagger\sigma^2\chi^*\right)^*$. A similar result holds for the term containing the derivatives since the derivative acts on the right for transposed fields. Hence

$$\int \mathrm{d}^4x \Big[(\chi^\dagger i\bar{\sigma}\cdot\partial\chi)^* - \frac{im}{2}\big((\chi^T\sigma^2\chi)^* - (\chi^\dagger\sigma^2\chi^*)^*\big)\Big] \tag{56}$$

$$= \int \mathrm{d}^4x \Big[i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\Big] \tag{57}$$

$$= S \tag{58}$$

By varying this action with respect to both χ and χ^*, the Majorana equation can be determined. As usual, we vary the action so that the first order variation vanishes:

$$0 = \delta S \tag{59}$$

$$= \int \mathrm{d}^4x \Big[(\delta(\partial_\mu\chi))i\chi^\dagger\bar{\sigma} + (\delta\chi)\frac{im}{2}\chi^T\sigma^2 - \frac{im}{2}(\delta\chi^*)\chi^\dagger\sigma^2\Big] \tag{60}$$

$$= \int \mathrm{d}^4x \left[(\delta\chi)\Big(-i\partial_\mu\chi^\dagger\bar{\sigma}^\mu + \frac{im}{2}\chi^T\sigma^2\Big) - \frac{im}{2}(\delta\chi^*)\chi^\dagger\sigma^2\right] \tag{61}$$

This tells us that

$$-i\partial_\mu \chi^\dagger \bar{\sigma}^\mu + \tfrac{im}{2}\chi^T\sigma^2 = 0, \quad -\tfrac{im}{2}\chi^\dagger\sigma^2 = 0 \tag{62}$$

Taking the hermitian conjugate of the left equation and the transpose of the right equation yields

$$i\bar{\sigma}\cdot\partial\chi - \tfrac{im}{2}\sigma^2\chi^* = 0, \quad -\tfrac{im}{2}\sigma^2\chi^* = 0 \tag{63}$$

Adding these two equations together yields the Majorana equation as desired.

3. Recall that a Dirac spinor can be written in a chiral 2-component basis

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{64}$$

We let $\psi_L(x) = \chi_1(x)$ and $\psi_R(x) = i\sigma^2\chi_2^*(x)$, so

$$\psi(x) = \begin{pmatrix} \chi_1(x) \\ i\sigma^2\chi_2^*(x) \end{pmatrix} \tag{65}$$

and

$$\bar{\psi}(x) = \begin{pmatrix} -i\chi_2^T(x)\sigma^2 \\ \chi_1^\dagger(x) \end{pmatrix}^T \tag{66}$$

The Dirac lagrangian is

$$\mathscr{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{67}$$

So, plugging in the Majorana spinor, we have

$$\begin{aligned}
\mathscr{L}_{\text{Majorana}} &= \begin{pmatrix} -i\chi_2^T(x)\sigma^2 \\ \chi_1^\dagger(x) \end{pmatrix}^T \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \chi_1(x) \\ i\sigma^2\chi_2^*(x) \end{pmatrix} &(68)\\
&= \begin{pmatrix} -i\chi_2^T(x)\sigma^2 & \chi_1^\dagger(x) \end{pmatrix} \begin{pmatrix} -m\chi_1(x) - (\sigma\cdot\partial)\sigma^2\chi_2^*(x) \\ i\bar{\sigma}\cdot\partial\chi_1(x) - im\sigma^2\chi_2^*(x) \end{pmatrix} &(69)\\
&= im\chi_2^T(x)\sigma^2\chi_1(x) + i\chi_2^T(x)\bar{\sigma}\cdot\partial\chi_2^*(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x) - im\chi_1^\dagger(x)\sigma^2\chi_2^*(x) &(70)\\
&= im\big(\chi_2^T(x)\sigma^2\chi_1(x) - \chi_1^\dagger(x)\sigma^2\chi_2^*(x)\big) + i\chi_2^T(x)\bar{\sigma}\cdot\partial\chi_2^*(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x) &(71)\\
&= im\big(\chi_2^T(x)\sigma^2\chi_1(x) - \chi_1^\dagger(x)\sigma^2\chi_2^*(x)\big) + i\chi_2^\dagger(x)\bar{\sigma}\cdot\partial\chi_2(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x) &(72)\\
&= im\big(\chi_2^T(x)\sigma^2\chi_2^*(x) - \chi_1^\dagger(x)\sigma^2\chi_1(x)\big) + i\chi_2^\dagger(x)\bar{\sigma}\cdot\partial\chi_2(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x) &(73)\\
&= im\big(\chi_2^\dagger(x)\sigma^2\chi_2(x) - \chi_1^\dagger(x)\sigma^2\chi_1(x)\big) + i\chi_2^\dagger(x)\bar{\sigma}\cdot\partial\chi_2(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x) &(74)
\end{aligned}$$

In this Lagrangian, the mass term is $im\big(\chi_2^\dagger(x)\sigma^2\chi_2(x) - \chi_1^\dagger(x)\sigma^2\chi_1(x)\big)$ which couples the LH and RH 2-component Grassmann fields to each other.

4. The action of the previous theory is

$$S = \int \mathrm{d}^4x \Big[im\big(\chi_2^T(x)\sigma^2\chi_1(x) - \chi_1^\dagger(x)\sigma^2\chi_2^*(x)\big) + i\chi_2^T(x)\bar{\sigma}\cdot\partial\chi_2^*(x) + i\chi_1^\dagger(x)\bar{\sigma}\cdot\partial\chi_1(x)\Big] \tag{75}$$

We observe that this action is invariant under the global transformation

$$\chi_1(x) \to e^{i\alpha}\chi_1(x), \qquad \chi_2(x) \to e^{-i\alpha}\chi_2(x) \tag{76}$$

The divergence of the Noether current is

$$\begin{aligned}
\partial_\mu(\chi^\dagger\bar{\sigma}^\mu\chi) &= (\partial_\mu\chi^\dagger)\bar{\sigma}^\mu\chi + \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi &(77)\\
&= \chi^T\bar{\sigma}\cdot\partial\chi^\dagger + \chi^\dagger\bar{\sigma}\cdot\partial\chi &(78)\\
&= m(\chi^T\sigma^2\chi + \chi^\dagger\sigma^2\chi^*) &(79)
\end{aligned}$$

We can see that the Noether current is proportional to the mass term. Hence, it must be the case that $m = 0$ for this current to be conserved. Similarly, we can see that the divergence of the Noether current for the latter theory is

$$\partial_\mu(\chi_1^\dagger \bar{\sigma}^\mu \chi_1 - \chi_2^\dagger \bar{\sigma}^\mu \chi_2) = m\big[(\chi_1^T \sigma^2 \chi_1 + \chi_1^\dagger \sigma^2 \chi_1^*) - (\chi_2^T \sigma^2 \chi_2 + \chi_2^\dagger \sigma^2 \chi_2^*)\big] \tag{80}$$

which again is 0 if and only if $m = 0$. To construct a O(N) invariant theory, we can generalize the theory in part 2:

$$S_{\text{Majorana}} = \int \mathrm{d}^4x \sum_{i=1}^{N} \bigg(\chi_i^\dagger i\bar{\sigma} \cdot \partial \chi + \frac{im}{2}(\chi_i^T \sigma^2 \chi_i - \chi_i^\dagger \sigma^2 \chi_i^*)\bigg) \tag{81}$$

where the rotation transformation is

$$\chi_i \to \chi_i' = J_{ij}\chi_j \tag{82}$$

5. Now, we are told to quantize the classical Majorana theory using canonical quantization by (i) promoting the field $\chi(x)$ to a quantum field (or operator-valued distribution) which satisfies the canonical commutation relation

$$\{\chi_a(x), \chi_b^\dagger(y)\} = \delta_{ab}\delta(x-y) \tag{83}$$

and (ii) constructing a Hermitian Hamiltonian that is diagonalizable in terms of creation and annihilation operators. The CAR are

$$\{\chi_a(x), \chi_b^\dagger(y)\} = \delta_{ab}\delta(x-y) \tag{84}$$

and

$$\{\chi_a(x), \chi_b(y)\} = \{\chi_a^\dagger(x), \chi_b^\dagger(y)\} = 0 \tag{85}$$

We can find the Hamiltonian by finding the Hamiltonian density

$$\begin{aligned}
\mathscr{H} &= \pi\dot{\chi} - \mathscr{L} &(86)\\
&= \frac{\partial \mathscr{L}}{\partial \partial_0 \chi}\partial_0\chi - \mathscr{L} &(87)\\
&= \chi^\dagger i\partial_0\chi - \Big[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\Big] &(88)\\
&= \frac{1}{2}\Big[\chi^T\big(i\boldsymbol{\sigma}\cdot\nabla\chi^* - im\sigma^2\chi\big) - \chi^\dagger\big(i\boldsymbol{\sigma}\cdot\nabla\chi + im\sigma^2\chi^*\big)\Big] &(89)
\end{aligned}$$

Thus,

$$H = \int \mathrm{d}^3x \frac{1}{2}\Big[\chi^T\big(i\boldsymbol{\sigma}\cdot\nabla\chi^* - im\sigma^2\chi\big) - \chi^\dagger\big(i\boldsymbol{\sigma}\cdot\nabla\chi + im\sigma^2\chi^*\big)\Big] \tag{90}$$

The Majorana spinor can be found from the Dirac spinor by imposing the condition that the RH Weyl spinor is the charge conjugated LH Weyl spinor

$$\psi_D = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \psi_M = \begin{pmatrix}\chi\\ i\sigma^2\chi^*\end{pmatrix} \tag{91}$$

The Dirac spinor is

$$\psi_D = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\Big(a_{\mathbf{p}}^s u^s(p)e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger}v^s(p)e^{ip\cdot x}\Big) \tag{92}$$

Putting this in components, we have

$$\psi_D = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a^1_{\mathbf{p}} e^{-ip\cdot x} \begin{pmatrix} \sqrt{\sigma \cdot p}\xi^1 \\ \sqrt{\bar{\sigma} \cdot p}\xi^1 \end{pmatrix} + a^2_{\mathbf{p}} e^{-ip\cdot x} \begin{pmatrix} \sqrt{\sigma \cdot p}\xi^2 \\ \sqrt{\bar{\sigma} \cdot p}\xi^2 \end{pmatrix} \right. \tag{93}$$

$$\left. + b^{1\dagger}_{\mathbf{p}} e^{ip\cdot x} \begin{pmatrix} \sqrt{\sigma \cdot p}\eta^1 \\ -\sqrt{\bar{\sigma} \cdot p}\eta^1 \end{pmatrix} + b^{2\dagger}_{\mathbf{p}} e^{ip\cdot x} \begin{pmatrix} \sqrt{\sigma \cdot p}\eta^2 \\ -\sqrt{\bar{\sigma} \cdot p}\eta^2 \end{pmatrix} \right] \tag{94}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \begin{pmatrix} (a^1_{\mathbf{p}} e^{-ip\cdot x} + b^{2\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\sigma \cdot p} \\ (a^2_{\mathbf{p}} e^{-ip\cdot x} + b^{1\dagger} e^{ip\cdot x})\sqrt{\sigma \cdot p} \\ (a^1_{\mathbf{p}} e^{-ip\cdot x} - b^{2\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\bar{\sigma} \cdot p} \\ (a^2_{\mathbf{p}} e^{-ip\cdot x} - b^{1\dagger} e^{ip\cdot x})\sqrt{\bar{\sigma} \cdot p} \end{pmatrix} \tag{95}$$

where $\eta^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\eta^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The constraint tells us

$$(a^1_{\mathbf{p}} e^{-ip\cdot x} + b^{2\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\sigma \cdot p} = (-b^1_{\mathbf{p}} e^{-ip\cdot x} + a^{2\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\bar{\sigma} \cdot p}, \tag{96}$$

and

$$(a^2_{\mathbf{p}} e^{-ip\cdot x} + b^{1\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\sigma \cdot p} = (-b^2_{\mathbf{p}} e^{-ip\cdot x} + a^{1\dagger}_{\mathbf{p}} e^{ip\cdot x})\sqrt{\bar{\sigma} \cdot p} \tag{97}$$