QFT Notes

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1 KG Equation

2 Dirac Field

2.1 Lorentz Invariance in Wave Eqns

- If $\mathcal{D}\phi = 0$ is **relativistically invariant**, then after performing a Lorentz transformation we get the same result. Similarly, for a Lorentz scalar, a transformation leaves extrema invariant.

- Lorentz transformation on coordinates:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

Lorentz transformation on scalar field (active transformation view):

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

Lorentz transformation on derivative of scalar field:

$$\partial_\mu \phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x)) = \frac{\partial \phi(\Lambda^{-1}x)}{\partial x^\mu} = (\Lambda^{-1})^\nu_\mu \partial_\nu(\phi(\Lambda^{-1}x))$$

Lorentz transformation for metric:

$$g^{\mu\nu} \to (\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$$

Lorentz transformation for Lagrangian density:

$$\mathcal{L} \to \mathcal{L}(\Lambda^{-1}x)$$

Lorentz transformation on vector field:

$$\phi^\mu(x) \to \Lambda^\mu_\nu \phi^\nu(\Lambda^{-1}x)$$

Rotation of vector field:

$$\phi^i(x) \to R^{ij}\phi^j(R^{-1}x)$$

Lorentz transformation of general tensor field:

$$T^{a_1\cdots a_n}_{b_1\cdots b_m}(x) \to \Lambda^{a_1}_{c_1}\cdots\Lambda^{a_n}_{c_n}\Lambda^{d_1}_{b_1}\cdots\Lambda^{d_m}_{b_m}T^{c_1\cdots c_n}_{d_1\cdots d_m}(\Lambda^{-1}x)$$

2.2 Lorentz Group and Representation Theory

For an $n-$component multiplet or quantity, Φ_a, the corresponding transformation law representation for a Lorentz transformation is

$$\Phi_a(x) = M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)$$

Specifically, **a Lorentz transformation must be given a representation as an element of the general linear group** $GL_n(\mathbb{R})$.

The **Lie Algebra** of a continuous group consists of all of the transformations that lie *infinitesimally* close to the identity of the group. This algebra is generated by a basis called the **generators** of the algebra.

For the Lie algebra of the 3 dimensional rotation group, $\mathfrak{so}(3)$, these generators are the infinitesimals rotations in the $x-$, $y-$, and $z-$directions. The set of generators follow a set of algebraic **relations** that define the **algebra** for all elements. These relations are given by the commutator or Lie bracket:

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

The rotation operator about some angle $\theta = \theta^i \hat{x}_i$ is given by

$$R = e^{i\theta^i J^i}$$

The matrix exponential determines the algebra of rotation operators via the commutator relations. A set of matrices satisfying the commutation relations produces a representation of the rotation group $SO(3)$.

2.2.1 Lorentz Group Representation

Lorentz Algebra

Rotations in 4-dimensions are given by generalizing the cross-product to 4-dimensions:

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

The LA is

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= J^{\mu\nu} J^{\rho\sigma} - J^{\rho\sigma} J^{\mu\nu} \\
&= i^2 \left[(x^\mu \partial^\nu - x^n u \partial^\mu)(x^\rho \partial^\sigma - x^\sigma \partial^\rho) - (x^\rho \partial^\sigma - x^\sigma \partial^\rho)(x^\mu \partial^\nu - x^n u \partial^\mu) \right] \\
&= i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})
\end{aligned}$$

A representation that satisfies this algebra is

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$

Conventional Representation of LT

The conventional representation for the LT basis is

$$\mathcal{J}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^{01} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \mathcal{J}^{02} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{J}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^{03} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

A general LT on an $n-$component multiplet is given by

$$V^\alpha = \left(\eta^\alpha_\beta - \frac{i}{2} \omega_{\mu\nu} (\mathcal{J}^{\mu\nu})^\alpha_\beta \right) V^\beta$$

so that ω_{ij} gives the rotations about the x_i coordinates and $\omega_{0\nu}$ and $\omega_{\mu 0}$ give the boosts in the x_i directions.

2.3 The Dirac Equation

Summary:

Any basis of the dirac algebra is a basis under conjugation. Similarly, Lorentz transformations of a basis of the dirac algebra is still a basis. Thus, a Lorentz transformation can be performed as conjugation. Using infinitesimal Lorentz transformations and the representation of Lorentz transformations on spinors, it can be shown that $[\gamma^a, S^{\mu\nu}] = i(\eta^{a\mu}\gamma^\nu - \eta^{a\nu}\gamma^\mu)$. This says that gamma matrices transform as vectors under the spinor representation.

Dirac algebra and Basis

The elements that form the basis of the Dirac algebra are the 16 elements

$$\begin{aligned}&\mathbb{1},\\&\gamma^\mu,\\&\gamma^\mu\gamma^\nu,\\&\gamma^\mu\gamma^\nu\gamma^\sigma,\\&\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\end{aligned}$$

They abide by the algebra given by the anti-commutator

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g_{\mu\nu} \otimes \mathbb{1}_{n\times n}$$

Spinor Representation of Lorentz algebra

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Using LA rep and DA, it is possible to show that

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

Weyl/Chiral Rep of DA

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Boost and Rotation Generators in Chiral Rep

$$S^{01} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Dirac Spinors

If ψ is a $4-$component field that transforms under the boosts and rotations according to the generators, then it is called a **Dirac spinor**.

LA Spinor Representation Action on Gamma Matrices

The Spinor representation of the Lorentz algebra acts on the gamma matrices like they were standard Lorentz 4-vectors

$$[\gamma^a, S^{\mu\nu}] = (\mathcal{J}^{\mu\nu})^a_b\gamma^b$$

giving

$$\begin{aligned}\Lambda_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}^{-1} &= \big(1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\big)\gamma^\mu\big(1 + \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\big)\\&= \big(1 - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\big)\gamma^\mu\end{aligned}$$

The Dirac Equation

The Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

The operator $(-i\gamma^\mu\partial_\mu - m)$ yields the KG equation

$$\begin{aligned}
0 &= (-i\gamma^\mu\partial_\mu - m) \\
&= (i\gamma^\nu\partial_\nu - m)\psi \\
&= (\gamma^\mu\gamma^\nu\partial_\mu\partial_n u + m^2)\psi \\
&= (\partial^2 + m^2)\psi
\end{aligned}$$

This is due to

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \otimes \mathbb{1}$$

so

$$\begin{aligned}
(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu &= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu \\
&= 2\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu \\
&= 2g^{\mu\nu}\partial_\mu\partial_\nu \\
&= 2\partial^2
\end{aligned}$$

Dirac Lagrangian and Dirac Spinor Products

Lorentz scalar is defined by

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$$

This is due to

$$\begin{aligned}
\bar{\psi} &= \psi^\dagger\Lambda^\dagger\gamma^0 \\
&= \psi^\dagger(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger)\gamma^0 \\
&= \psi^\dagger\gamma^0(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger)
\end{aligned}$$

The last equality comes from $S^{\mu\nu}$ commuting for non-zero μ, and ν. For 0 μ or ν, they anticommute. Since $S^{\mu\nu}$ is antisymmetric, these effects cancel out leaving the total product invariant under commutation.

The Lorentz invariant Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(-i\partial\!\!\!/ - m)\psi$$

Weyl Spinors

Break up LA rep into two 2−dimensional reps to get left-handed and right-handed Weyl spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Using Weyl/Chiral representation of the Dirac algebra, it is clear that

$$\begin{aligned}
\psi_L &\to (1 - i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2})\psi_L, \\
\psi_R &\to (1 - i\boldsymbol{\theta}\cdot\tfrac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\tfrac{\boldsymbol{\sigma}}{2})\psi_R
\end{aligned}$$

The Weyl equations are for massless Dirac particles

$$i(\partial_0\sigma^0 - \boldsymbol{\sigma}\cdot\nabla)\psi_L = 0,$$

and

$$i(\partial_0\sigma^0 + \boldsymbol{\sigma}\cdot\nabla)\psi_R = 0$$

Pauli Matrices Notation

Rewriting the Pauli matrices with indices

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

the gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

the Dirac equation is

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

and the Weyl equations are

$$i\sigma\cdot\partial\psi_R = 0, \qquad i\bar{\sigma}\cdot\partial\psi_L = 0$$

2.4 Free Particle Solutions to Dirac Equation

Solutions are plane waves (due to KG equation)

$$\psi(x) = u(p)e^{-ip\cdot x}, \qquad \text{where } p^2 = m^2$$

Plugging this into the D equation gives

$$(\gamma^\mu p_\mu - m)u(p) = 0$$

with rest frame equation

$$(m\gamma^0 - m)u(p_0) = \begin{pmatrix} -m & m \\ m & -m \end{pmatrix} u(p_0) = 0$$

which has solutions

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

where

$$\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad \xi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.4.1 Boosting Solutions to General Frames

For 4-vectors

$$\begin{aligned}
\begin{pmatrix} E \\ \mathbf{p} \end{pmatrix} &= \exp\big(\eta_i \mathcal{J}^{0i}\big) \begin{pmatrix} m \\ \mathbf{0} \end{pmatrix} \\
&= \exp\big(\eta_1 \mathcal{J}^{01}\big) \exp\big(\eta_2 \mathcal{J}^{02}\big) \exp\big(\eta_3 \mathcal{J}^{03}\big) \begin{pmatrix} m \\ \mathbf{0} \end{pmatrix} \\
&= \big(\mathbb{1} + \eta_1 \mathcal{J}^{01} + \frac{1}{2}\eta_1^2 (\mathcal{J}^{01})^2 + \frac{1}{3!}\eta_1^2 (\mathcal{J}^{01})^3 + \cdots\big)\times \\
&\quad \big(\mathbb{1} + \eta_2 \mathcal{J}^{02} + \frac{1}{2}\eta_2^2 (\mathcal{J}^{02})^2 + \frac{1}{3!}\eta_1^3 (\mathcal{J}^{02})^3 + \cdots\big)\times \\
&\quad \big(\mathbb{1} + \eta_3 \mathcal{J}^{03} + \frac{1}{2}\eta_3^2 (\mathcal{J}^{03})^2 + \frac{1}{3!}\eta_3^3 (\mathcal{J}^{03})^3 + \cdots\big) \begin{pmatrix} m \\ \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \cosh\eta_1 + \cosh\eta_2 + \cosh\eta_3 & \sinh\eta_1 & \sinh\eta_2 & \sinh\eta_3 \\ \sinh\eta_1 & \cosh\eta_1 & 0 & 0 \\ \sin\eta_2 & 0 & \cosh\eta_2 & 0 \\ \sinh\eta_3 & 0 & 0 & \cosh\eta_3 \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= m \begin{pmatrix} (\cosh\eta_1 + \cosh\eta_2 + \cosh\eta_3) \\ \sinh\eta_1 \\ \sinh\eta_2 \\ \sinh\eta_3 \end{pmatrix}
\end{aligned}$$

For the column vector $u(p)$,

$$\begin{aligned}
u(p) &= \exp\bigl(\boldsymbol{\eta}\cdot\mathbf{S}\bigr)u(p_0)\\
&= \exp\bigl(-\frac{\eta_1}{2}S^{01}\bigr)\exp\bigl(-\frac{\eta_2}{2}S^{02}\bigr)\exp\bigl(-\frac{\eta_3}{2}S^{03}\bigr)u(p_0)\\
&= \left(\sum_{i=1}^{3}\left[\cosh\bigl(\tfrac{1}{2}\eta_i\bigr)\begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & \mathbb{1}_2\end{pmatrix} - \sinh\bigl(\tfrac{1}{2}\eta_i\bigr)\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\right]\right)\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{pmatrix}\sum_{i=1}^{3} e^{\eta_i/2}\bigl(\frac{1-\sigma^i}{2}\bigr) + e^{-\eta_i/2}\bigl(\frac{1+\sigma^i}{2}\bigr) & 0\\ 0 & \sum_{i=1}^{3} e^{\eta_i/2}\bigl(\frac{1+\sigma^i}{2}\bigr) + e^{-\eta_i/2}\bigl(\frac{1-\sigma^i}{2}\bigr)\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&= \begin{pmatrix}\sum_{i=1}^{3}\left[\sqrt{E+p^i}\bigl(\frac{1-\sigma^i}{2}\bigr) + \sqrt{E-p^i}\bigl(\frac{1+\sigma^i}{2}\bigr)\right]\xi\\ \sum_{i=1}^{3}\left[\sqrt{E+p^i}\bigl(\frac{1+\sigma^i}{2}\bigr) + \sqrt{E-p^i}\bigl(\frac{1-\sigma^i}{2}\bigr)\right]\xi\end{pmatrix}\\
&= \begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix}
\end{aligned}$$

2.4.2 Helicity Eigenstates

For up eigenstates of $\boldsymbol{\sigma}$,

$$u(p) = \begin{pmatrix}\sqrt{E-|\mathbf{p}|}\binom{1}{0}\\ \sqrt{E+|\mathbf{p}|}\binom{1}{0}\end{pmatrix} \underset{\text{large boost}}{\longrightarrow} \sqrt{2E}\begin{pmatrix}0\\ \binom{1}{0}\end{pmatrix}$$

Similarly, for down eigenstates of $\boldsymbol{\sigma}$,

$$u(p) = \begin{pmatrix}\sqrt{E+|\mathbf{p}|}\binom{0}{1}\\ \sqrt{E-|\mathbf{p}|}\binom{0}{1}\end{pmatrix} \underset{\text{large boost}}{\longrightarrow} \sqrt{2E}\begin{pmatrix}\binom{0}{1}\\ 0\end{pmatrix}$$

Both eigenstates are eigenstates of the helicity operator

$$h \equiv \frac{1}{2}\hat{p}\cdot\mathbf{S} = \frac{1}{2}\hat{p}_i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

States with $h = +\frac{1}{2}$ are called *right-handed* and states with $h = -\frac{1}{2}$ are called *left-handed*. **HELICITY IS NOT LORENTZ INVARIANT QUANTITY!**

2.4.3 Normalization of Dirac Spinors

Just like with the Dirac spinor field, the column vector of 2-component spinors themselves are normalized similarly

$$\bar{u}(p) = u^\dagger(p)\gamma^0$$

so

$$\begin{aligned}
\bar{u}(p) &= u^\dagger(p)\gamma^0 u(p)\\
&= \begin{pmatrix}\xi^\dagger\sqrt{p\cdot\sigma} & \xi^\dagger\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\begin{pmatrix}\xi\sqrt{p\cdot\bar{\sigma}}\\ \xi\sqrt{p\cdot\sigma}\end{pmatrix}\\
&= 2m\xi^\dagger\xi
\end{aligned}$$

where the 2-component spinors are normalized so that $\xi^\dagger\xi = 1$

2.4.4 Summary

There are two lin-ind solutions

$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}, \quad \text{where } s = 1, 2$$

normalized according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^r s \quad \text{or} \quad u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs}$$

For negative frequency solutions

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \quad \text{where} \quad s = 1, 2$$

with the normalization condition

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs} \quad \text{or} \quad v^{r\dagger}(p)v^s(p) = +2E_{\mathbf{p}}\delta^{rs}$$

The u's and v's are orthogonal to each other because the two-component spinors are orthogonal to each other, so that

$$\bar{v}^r(p)u^s(p) = \bar{u}^r(p)v^s(p) = 0$$

But

$$v^{r\dagger}(p)u^s(p) \neq 0 \neq u^{r\dagger}(p)v^s(p)$$

However

$$v^{r\dagger}(\mathbf{p})u^s(-\mathbf{p}) = u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = 0$$

2.4.5 Spin Sums

Sum over polarizations is desirable. Thus

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\xi^\dagger & \sqrt{p\cdot\sigma}\xi^\dagger \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

In compact notation this is

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma\cdot p + m$$

and for the negative frequency solutions

$$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma\cdot p - m$$

2.5 Dirac Matrices and Dirac Bilinears

In general, we want to know how products of spinors transform or what they behave like under transformations. This requires looking at how elements of the Dirac algebra behave under transformations. There are 5 types of elements that form the basis for the Dirac algebra:

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\frac{i}{2}[\gamma^\mu, \gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1
		16

where

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

which has the following properties

$$(\gamma^5)^\dagger = \gamma^5,$$
$$(\gamma^5)^2 = \mathbb{1},$$
$$\{\gamma^5, \gamma^\mu\} = 0$$

with the last property implying $[\gamma^5, S^{\mu\nu}] = 0$. In the Dirac basis,

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

so left-handed (right-handed) spinors are eigenstates with eigenvalue -1 (1).

2.5.1 Dirac Field Bilinears

The two currents are **Dirac bilinears** formed from the product of two spinor fields

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

$$j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$

Using the Dirac equation, this current is always conserved

$$\begin{aligned} \partial_\mu j^\mu(x) &= ip_\mu\bar{\psi}(x)\gamma^\mu\psi(x) + \bar{\psi}(x)\gamma^\mu(-ip_\mu)\psi(x) \\ &= im\bar{\psi}(x)\psi(x) - im\bar{\psi}(x)\psi(x) \\ &= 0 \end{aligned}$$

As for the second current, it is not conserved due to the anti-commutation between γ^μ and γ^5. In the case that the field has massless excitations, it is conserved. In this case it is called an *axial vector current.* These two currents are Noether currents under the transformations

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x) \quad \text{and} \quad \psi(x) \longrightarrow e^{i\alpha\gamma^5}\psi(x)$$

where the second transformation is called the *chiral transformation* because it changes the chirality of the particle. The operators are typically formed

$$j^\mu_L = \bar{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi, \quad j^\mu_R = \bar{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi$$

These operators give the electric charge of left-handed and right-handed particles, respectively.

2.5.2 Fierz Identities

The Pauli matrices obey the following

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$$

where α, β, γ, and δ are spinor indices and ϵ is the antisymmetric matrix. Note: α, γ transform in the Lorentz representation of ψ_L and β, δ transform in the Lorentz representation of ψ_R.

Products of Dirac bilinears obey the following then

$$\begin{aligned} (\bar{u}_{1R}\sigma^\mu\bar{u}_{2R})(\bar{u}_{3R}\sigma_\mu\bar{u}_{4R}) &= \bar{u}_{1R\alpha}(\sigma^\mu)_{\alpha\beta}\bar{u}_{2R\beta}\bar{u}_{3R\gamma}(\sigma^\mu)_{\gamma\delta}\bar{u}_{4R\delta} \\ &= 2\epsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\epsilon_{\beta\delta}\bar{u}_{2R\beta}\bar{u}_{4R\delta} \\ &= 2\epsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\epsilon_{\beta\delta}\bar{u}_{4R\delta}\bar{u}_{2R\beta} \\ &= -(\bar{u}_{1R}\sigma^\mu\bar{u}_{4R})(\bar{u}_{3R}\sigma_\mu\bar{u}_{2R}) \end{aligned}$$

Similarly, for left-handed spinors and $\bar{\sigma}^\mu$ there is an identical relation. Additional identities

$$\epsilon_{\alpha\beta}(\sigma^\mu)_{\beta\gamma} = (\bar{\sigma}^{\mu T})_{\alpha\beta}\epsilon_{\beta\gamma}$$

and

$$\bar{\sigma}^\mu\sigma_\mu = 4$$

2.6 Quantizing the Dirac Field

Conjugate momenta to ψ is

$$\begin{aligned}\pi(x) &= \frac{\partial \mathcal{L}}{\partial \partial_0 \psi} \\ &= \frac{\partial}{\partial(\partial_0\psi)}\left[\psi^\dagger(i\gamma^0\gamma^\mu\partial_\mu + m\gamma^0)\psi\right] \\ &= \frac{\partial}{\partial(\partial_0\psi)}\left[\psi^\dagger(i\partial_0 - i\gamma^0\boldsymbol{\gamma}\cdot\nabla + m\gamma^0)\psi\right] \\ &= i\psi^\dagger\end{aligned}$$

So the Dirac Hamiltonian density is

$$\begin{aligned}\mathcal{H}_{\text{Dirac}} &= \pi(x)\partial_0\psi - \mathcal{L} \\ &= i\psi^\dagger\gamma^0\gamma^0\partial_0\psi - \bar{\psi}(i\gamma^\mu\partial_\mu + m)\psi \\ &= i\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}(i\gamma^0\partial_0 - i\boldsymbol{\gamma}\cdot\nabla + m)\psi \\ &= \bar{\psi}(-i\boldsymbol{\gamma}\cdot\nabla + m)\psi\end{aligned}$$

Eigenstates of Hamiltonian are the Dirac spinors $u^s(p)e^{-ip\cdot x}$ and $v^s(p)e^{ip\cdot x}$ with eigenvalues $E_{\mathbf{p}}$. Creation and annihilation operators that diagonalize H are

$$a^s_{\mathbf{p}},\ a^{s\dagger}_{\mathbf{p}}$$

and

$$b^s_{\mathbf{p}},\ b^{s\dagger}_{\mathbf{p}}.$$

These operators are the Fourier coefficients of the plane waves. The Dirac field is

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}),$$

and the conjugate field is

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x} + b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x}).$$

The *anti*commutator for the Dirac field is

$$\begin{gathered}\{\psi_a(x), \bar{\psi}_b(y)\} = \delta^{(4)}(x-y)\delta_{ab}; \\ \{\psi_a(x), \psi_b(y)\} = \{\bar{\psi}_a(x), \bar{\psi}_{b(x)}\} = 0\end{gathered}$$

The *anti*commutators for the operator coefficients are

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$$

The Hamiltonian is

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s E_{\mathbf{p}}\left(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}\right)$$

One particle states are defined to be

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}}a^{s\dagger}_{\mathbf{p}}|0\rangle$$

with inner product

$$\langle\mathbf{p}, s|\mathbf{q}, r\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$$

Conjugation of $a^s_{\mathbf{p}}$ yields the transformed operator

$$U(\Lambda)a^s_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^s_{\Lambda\mathbf{p}}$$

where the same relation holds for the antifermion operator. Along with the above relation and the fact that

$$p \cdot x = \Lambda p \cdot \Lambda x = \tilde{p} \cdot \Lambda x,$$

we get

$$u^s(\Lambda^{-1}\tilde{p}) = \Lambda_{\frac{1}{2}}^{-1}u^s(\tilde{p}).$$

Thus,

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x).$$

2.6.1 Spin of Dirac Particle

The general Lorentz transform on a spinor is

$$\begin{aligned}
\Lambda_{\frac{1}{2}} &= \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \\
&= \exp\left[-\frac{1}{2}\left(\beta_j S^{0j} + i\theta_{4-i}\Sigma^i\right)\right] \\
&\approx 1 - \frac{1}{2}\beta_j S^{0j} - \frac{i}{2}\theta_{4-i}\Sigma^i
\end{aligned}$$

For a general Lorentz transformation acting on the coordinates, the coordinate transforms are

$$\begin{aligned}
t &\mapsto t - \beta_1 x - \beta_2 y - \beta_3 z \\
x &\mapsto x + \beta_1 t + y\theta_3 - z\theta_2 \\
y &\mapsto y + \beta_2 t - x\theta_3 + z\theta_1 \\
z &\mapsto z + \beta_3 t + x\theta_2 - y\theta_1
\end{aligned}$$

Thus, for rotations, if the field transforms as

$$\begin{aligned}
\delta\psi(x) &= \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) - \psi(x) \\
&= \left(1 - \frac{i}{2}\theta_{4-i}\Sigma^i\right)\psi(t, x + y\theta_3 - z\theta_2, y - x\theta_3 + z\theta_1, z + x\theta_2 - y\theta_1) - \psi(x)
\end{aligned}$$

This yields the following formula for how the field transforms

$$\left[\mathbf{x} \times (-i\nabla) + \frac{1}{2}\mathbf{\Sigma}\right]\psi$$

so that the angular momentum operator is

$$\mathbf{J} = \int \mathrm{d}^3x\psi^\dagger\left(\mathbf{x} \times (-i\nabla) + \tfrac{1}{2}\mathbf{\Sigma}\right)\psi$$

Angular momentum on fermions and antifermions is

$$J_z a_0^{s\dagger}\left|0\right\rangle = \pm\frac{1}{2}a_0^{s\dagger}\left|0\right\rangle \quad \text{and} \quad J_z b_0^{s\dagger}\left|0\right\rangle = \mp\frac{1}{2}b_0^{s\dagger}\left|0\right\rangle$$

where the upper sign is for $\xi^s = \begin{pmatrix}1\\0\end{pmatrix}$ and the lower sign is for $\xi^s = \begin{pmatrix}0\\1\end{pmatrix}$

2.6.2 Detour into Complex Analysis

A complex-valued function $f(z)$ can be rewritten as

$$f(z) = u(x, y) + iv(x, y), \quad \text{where } z = x + iy.$$

Additionally, $f(z)$ is complex differentiable at every point if and only if the **Cauchy-Riemann Equations** hold for all x and y:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Cauchy's Integral Theorem states that for an analytic function $f(z)$ and any closed loop C:

$$\oint_C f(z)\mathrm{d}z = 0$$

Proof sketch: Expand $f(z) = u(x, y) + iv(x, y)$ and $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$. Use Stokes' theorem to turn the line integral into a surface integral. Then, use the Cauchy-Riemann equations to see that the integral is 0.

Cauchy's Integral Formula states that for an analytic function (i.e., it can be represented as a power series) $f(z)$, a closed loop C, and an interior point z_0, the following formula holds:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = f(z_0)$$

Proof:
Let $z = z_0 + re^{i\theta}$. Then $\mathrm{d}z = ire^{i\theta}\mathrm{d}\theta$. Thus, the integral becomes

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\mathrm{d}\theta \\
&= \lim_{r\to 0} \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\mathrm{d}\theta \\
&= f(z_0)
\end{aligned}$$

Differentiating the integrand over and over again yields the following:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z$$

Taylor and Laurent Series
Using algebraic manipulations

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')}{z' - z}\mathrm{d}z' \\
&= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z' - z_0) - (z - z_0)}\mathrm{d}z' \\
&= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z' - z_0)[1 - (z - z_0)/(z' - z_0)]}\mathrm{d}z' \\
&= \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}} f(z')\mathrm{d}z' \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')}{(z' - z_0)^{n+1}}\mathrm{d}z' \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n
\end{aligned}$$

where in the 4th equality, the geometric series identity was used, and in the last equality, the definition of the n^{th} derivative of $f(z)$ was used.

Now, consider a point z enclosed by a circular contour C_1 which is connected to another circular contour C_2, thus forming an annulus. The **Laurent Series** can be obtained by using the same argument as above to get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z)^{n+1}} \mathrm{d}z'$$

Singularities

An **isolated point** z_0 of a function $f(z)$ is a point such that $f(z)$ is analytic at all points neighboring z_0 but not z_0 itself. Since $f(z)$ is analytic at all points neighboring z_0, $f(z)$ can be rewritten as a Laurent series centered at z_0. This leads to two possibilities:

1. if the most negative power of $z - z_0$ is finite $-n$, then z_0 is said to be a **pole** of **order** n;
2. if the most negative power of $z - z_0$ is infinite, then z_0 is said to be an **essential singularity**.

To find poles, simply test values of n such that the limit

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

exists. If $f(z)$ is analytic throughout $\mathbb{C}$ except for isolated poles, it is called **meromorphic**.

For a function $f(z)$, any point z_0 such that $f'(z_0)$ is not well-defined is a **branch point**. The **order** of the branch point is the number of times that a circular path must go around the point before the function returns to its original value. To prevent this multi-valuedness, there is a line called the **branch line** or **branch cut** which extends from one branch point to another or a branch point to infinity. This branch cut prevents a closed path from crossing it. If the order of a branch point is n, then there are n **branches** of that function corresponding to the n possible values. The **principal branch** is the commonly agreed upon branch that is used and the value of that branch is called the **principal value**.

Analytic Continuation

Let D_1 and D_2 be two subsets of the complex plane $\mathbb{C}$ such that $D_1 \cap D_2$ is non-empty. If two functions $f_1 : D_1 \to \mathbb{C}$ and $f_2 : D_2 \to \mathbb{C}$ agree on some subset of the intersection $I \subset D_1 \cap D_2$, that is $f_1(z) = f_2(z)$ for all $z \in I$, then both functions are equivalent to each other and $f_1(z) = f_2(z) = f(z)$ on $D_1 \cup D_2$. This is known as **analytic continuation.**

The Calculus of Residues

Let $f(z)$ be a function with a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Then for a circular contour C around an isolated singular point, the following holds

$$a_n \oint_C (z - z_0)^n \mathrm{d}z = \begin{cases} 0, & n \neq -1 \\ 2\pi i a_{-1}, & n = -1 \end{cases}$$

The coefficient a_{-1} is called the **residue** of $f(z)$ at $z = z_0$. In general

$$\oint_C f(z) \mathrm{d}z = 2\pi i \sum_n a_{-1,n},$$

where n is an index for the total number of residues of $f(z)$ enclosed by the path C. For a pole z_0 of order n, there is a formula for the residue of $f(z)$:

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \Big((z - z_0)^n f(z) \Big) \right]$$

Cauchy Principal Value For integrals where the contour runs over a singularity, the **P.V.** or **Cauchy Principal Value** is

$$P\int f(x)\mathrm{d}x = \lim_{\delta\to 0^+}\left(\int^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0+\delta} f(x)\mathrm{d}x\right)$$

2.6.3 Another Brief Detour into Green's Functions

The **kernel** of the transformation between electric charge density, $\rho(\mathbf{r}_2)$, and the electric potential, $\psi(\mathbf{r}_1)$, is the **Green's function**

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}$$

so that

$$\psi(\mathbf{r}_1) = \int \mathrm{d}^3 r_2 G(\mathbf{r}_1,\mathbf{r}_2)\rho(\mathbf{r}_2)$$

What a Green's function does is tell us the response of the solution to a given DE to a source term. If $\mathcal{L}$ is some Hermitian (self-adjoint) differential operator and $y = g(x)$ is a function such that

$$\mathcal{L}y = f(x),$$

then we define the Green's function to be such that

$$\mathcal{L}G(x,t) = \delta(x-t).$$

Hence,

$$\begin{aligned}
\mathcal{L}_x y &= \mathcal{L}_x \int G(x,t)f(t)\mathrm{d}t \\
&= \int \mathcal{L}_x\big(G(x,t)f(t)\big)\mathrm{d}t \\
&= \int \big(\mathcal{L}_x G(x,t)\big)f(t)\mathrm{d}t \\
&= \int \delta(x-t)f(t)\mathrm{d}t \\
&= f(x)
\end{aligned}$$

Properties of Green's Functions

The $x-$derivative of $G(x,t)$ is generally discontinuous at the point $x = t$ which corresponds to the Green's function being an impulse. Using the fact that $\mathcal{L}G(x,t) = \delta(x-t)$ and the eigenfunction expansion of $G(x,t)$, we get the symmetric expression

$$G(x,t) = G^*(t,x).$$

Form of Green's Functions

The Green's function has the form

$$G(x,t) = \begin{cases} Ay_1(x)y_2(t), & x < t \\ Ay_2(x)y_1(t), & x > t \end{cases}$$

where $A = \Big[p(t)\big(y_1'(t)y_2(t) - y_2'(t)y_1(t)\big)\Big]^{-1}$

Initial Value Problems

For initial value problems, it may be the case that the only linear combination of the homogeneous solutions is the trivial solution. The effect of this is that the symmetry of the Green's function is lost. This lack of symmetry is due to the lack of the problem being a Sturm-Liouville problem.

Change of Variables

For systems with inhomogeneous boundary conditions, $y(a) = c_1$ and $y(b) = c_2$, a simple change of variables is all that is needed to get a homogeneous system:

$$u = y - \frac{c_1(b-x) + c_1(x-a)}{b-a}$$

Integral Equations

Consider the eigenvalue problem

$$\mathcal{L}y(x) = \lambda y(x).$$

Using the Green's function, we can rewrite this as

$$y(x) = \lambda \int_a^b G(x,t)y(t)\mathrm{d}t$$

Operating on both sides with $\mathcal{L}$ yields

$$\begin{aligned}
\mathcal{L}_x y(x) &= \lambda \mathcal{L}_x \int_a^b G(x,t)y(t)\mathrm{d}t \\
&= \lambda \int_a^b \mathcal{L}_x G(x,t)y(t)\mathrm{d}t \\
&= \lambda \int_a^b \delta(x-t)y(t)\mathrm{d}t \\
&= \lambda y(x)
\end{aligned}$$

This allows us to convert an **unbounded** differential equation into a **bounded** integral equation, where the Green's function is the **kernel** of the integral equation and incorporates the boundary conditions.

Example: Bessel's Equation

We can construct the Green's function for the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (k^2x^2 - 1)y = 0$$

The first step is to solve the differential equation and then use the general solution to construct the Green's function. We divide both sides by x. Although *a priori* we cannot rule out $x = 0$, we will see that it does not matter. This yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} + (k^2 - x^{-2})y = 0.$$

Next, we see that because the lowest power on x is -2 that $x = 0$ is a regular singular point. Hence, we can perform a series expansion of the solution about the point $x = 0$:

$$\begin{aligned}
y &= x^s \sum_{j=0}^{\infty} a_j x^j \\
y' &= x^s \sum_{j=0}^{\infty} a_j (s+j) x^{j-1} \\
y'' &= x^s \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{j-2}
\end{aligned}$$

Plugging these into the differential equation and looking at the coefficient for x^{s-2} yields the following indicial equation for a_0:

$$s^2 - 1 = 0$$

which has the solutions $s = \pm 1$. We also get an indicial equation for a_1 by looking at the coefficient for x^{s-1}:

$$s(s+2) = 0$$

which does not have a solution for $s = \pm 1$. Hence, $a_1 = 0$. For $s = 1$, we get the following equation

$$\sum_{j=0}^{\infty} a_j j(j+1)x^{j-1} + a_j(j+1)x^{j-1} - a_j x^{j-1} + k^2 a_j x^{j+1} = 0$$

From this equation we can get the following recurrence relation:

$$a_j = -\frac{k^2}{j(j+2)} a_{j-2}, \quad \text{where } j \geq 2$$

By plugging in various even values of j, we get the following

$$a_{2\ell} = \frac{(-1)^\ell k^{2\ell}}{[(2\ell)!!]^2(\ell+1)} a_0$$

Using the identity $(2l)!! = 2^\ell \ell!$, we get the following relation

$$a_{2\ell} = \frac{(-1)^\ell k^{2\ell}}{2^{2\ell}\ell!(\ell+1)!} a_0$$

Thus, the first solution is

$$y(x) = x\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2^{2\ell}\ell!(\ell+1)!} a_0 (kx)^{2\ell} = k^{-1} J_1(kx),$$

which is also known as the **first order Bessel function of the first kind**. The subscript on J is the order number. More generally for order ν

$$J_\nu(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu}$$

Differentiating with respect to ν yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\nu} J_\nu(x) &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell![\Gamma(\ell+\nu+1)]^2} \left(\frac{x}{2}\right)^{2\ell+\nu} (\ell+\nu)!\big(\gamma - H(\ell+\nu)\big) + \frac{(-1)^\ell}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu} \ln\left(\frac{x}{2}\right) \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu} \big(\gamma - H(\ell+\nu)\big) + \frac{(-1)^\ell}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu} \ln\left(\frac{x}{2}\right) \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+\nu+1)} \left(\frac{x}{2}\right)^{2\ell+\nu} \big(\gamma - H(\ell+\nu)\big) + \ln\left(\frac{x}{2}\right) J_\nu(x)
\end{aligned}$$

where $H(n) = \sum_{i=1}^{n} \frac{1}{k}$ is the n-th **Harmonic number**. Similarly,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\nu} J_{-\nu}(x) &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell-1}}{\ell![\Gamma(\ell-\nu+1)]^2} \left(\frac{x}{2}\right)^{2\ell-\nu} (\ell-\nu)!\big(\gamma - H(\ell-\nu)\big) - \frac{(-1)^\ell}{\ell!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell-\nu} \ln\left(\frac{x}{2}\right) \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell-1}}{\ell!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell-\nu} \big(\gamma - H(\ell-\nu)\big) - \frac{(-1)^\ell}{\ell!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell-\nu} \ln\left(\frac{x}{2}\right) \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell-1}}{\ell!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell-\nu} \big(\gamma - H(\ell-\nu)\big) + \ln\left(\frac{x}{2}\right) J_\nu(x) \\
&= \frac{2}{x} \frac{\gamma}{\Gamma(1-\nu)} + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell-\nu} \big(\gamma - H(\ell-\nu)\big) + \ln\left(\frac{x}{2}\right) J_\nu(x) \\
&= \frac{2}{x} \frac{\gamma}{\Gamma(1-\nu)} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell+1)!\Gamma(\ell-\nu+1)} \left(\frac{x}{2}\right)^{2\ell+2-\nu} \big(\gamma - H(\ell+1-\nu)\big) + \ln\left(\frac{x}{2}\right) J_\nu(x)
\end{aligned}$$

The **Bessel function of the second kind of order** ν is defined as follows:

$$Y_\nu(x) = \lim_{\alpha\to\nu} \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

For $\nu = 1$, we must use l'Hospital's rule:

$$\begin{aligned}
Y_\nu(x) &= \lim_{\alpha\to\nu} \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)} \\
&= \lim_{\alpha\to\nu} \frac{J'_\alpha\cos(\alpha\pi) - \pi J_\alpha(x)\sin(\alpha\pi) - J'_{-\alpha}(x)}{\pi\cos(\alpha\pi)} \\
&= \frac{-J'_1(x) - J'_{-1}(x)}{-\pi} \\
&= \frac{J'_1(x) + J'_{-1}(x)}{\pi}
\end{aligned}$$

Plugging in our definitions for the derivatives of the Bessel functions of the first kind, we get

$$\begin{aligned}
Y_1(x) &= \frac{1}{\pi}\left[\frac{2}{x} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+2)}\left(\frac{x}{2}\right)^{2\ell+1}\big(2\gamma - H(\ell+1) - H(\ell)\big) + 2\ln\left(\frac{x}{2}\right)J_\nu(x)\right] \\
&= \frac{2}{\pi}J_1(x)\left(\gamma + \ln\tfrac{x}{2}\right) + \frac{2}{\pi}\left[\frac{1}{x} - \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!\Gamma(\ell+2)}\left(\frac{x}{2}\right)^{2\ell+1}\big(H(\ell+1) + H(\ell)\big)\right]
\end{aligned}$$

It can be shown that $Y_n(x)$ is linearly independent of $J_n(x)$ and also satisfies the Bessel equation. Hence, the homogeneous Bessel equation solution is:

$$y(x) = c_1 J_1(kx) + c_2 Y_1(kx)$$

where we set $n = 1$ since that was the order of the equation given. The boundary conditions allow us to determine what the $c_i's$ must be to give two linearly independent solutions, $y_1(x)$ and $y_2(x)$. The Green's function is defined as

$$G(x,t) = \begin{cases} \frac{1}{t^2W(t)}y_1(x)y_2(t), & 0 \le x < t \\ \frac{1}{t^2W(t)}y_2(x)y_1(t), & t < x \le 1 \end{cases}$$

where $W(t) = y_1(t)y'_2(t) - y'_1(t)y_2(t)$ is the Wronskian. Since $Y_1(0)$ is undefined, the first Dirichlet boundary condition tells us $c_2 = 0$ for $x < t$. The second condition tells us $y(1) = 0$, and so we get

$$y(1) = c_1 J_1(1) + c_2 Y_1(1) = 0$$

Hence, for $x > t$, $c_1 = Y_1(1)$ and $c_2 = -J_1(1)$. This yields

$$G(x,t) = \begin{cases} \frac{1}{t^2W(t)}J_1(kx)\big(Y_1(1)J_1(t) - J_1(1)Y_1(t)\big), & 0 \le x < t \\ \frac{1}{t^2W(t)}J_1(t)\big(Y_1(1)J_1(kx) - J_1(1)Y_1(kx)\big), & t < x \le 1 \end{cases}$$

Green's Functions in Higher Dimensions

For differential operators in more than 1 dimension, the following properties still hold true:

1. A homogeneous PDE $\mathcal{L}\psi(\mathbf{r}) = 0$ and its boundary conditions define a Green's function which is the solution to the equation

   $$\mathcal{L}G(\mathbf{r},\mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_1)$$

   subject to relevant boundary conditions.

2. For inhomogeneous PDEs, $\mathcal{L}\psi(\mathbf{r}) = f(\mathbf{r})$, the solution is given by

   $$\psi(\mathbf{r} = \int G(\mathbf{r},\mathbf{r}_1)f(\mathbf{r}_1)\mathrm{d}^3r_1$$

   where the region of integration is over the space relevant to the problem.

3. When the differential operator is Hermitian and defines an eigenvalue problem, $G(\mathbf{r}, \mathbf{r}_1)$ will be symmetric and can be represented as a sum of products of eigenfunctions.

4. G will be continuous at all $\mathbf{r}$ except at $\mathbf{r} = \mathbf{r}_1$. At $\mathbf{r} = \mathbf{r}_1$, G has the characteristic delta function singularity behavior.

Green's Function Form

Like with the 1 dimensional case, the Green's function will be defined by the equation

$$\mathcal{L}_1 G(\mathbf{r}, \mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_1)$$

In the case of higher dimensions, $\mathcal{L}$ is a self-adjoint operator if it has the form

$$\mathcal{L} = \nabla \cdot \left[p(\mathbf{r_1})\nabla\right] + q(\mathbf{r_1})$$

That the differential equation is self-adjoint is obvious

$$\begin{aligned} \mathcal{L}\psi(\mathbf{r}_1) &= \Big(\nabla \cdot \left[p(\mathbf{r_1})\nabla\right] + q(\mathbf{r_1})\Big)\psi(\mathbf{r}_1) \\ &= \Big[p(\mathbf{r}_1)\nabla^2 + \left[\nabla p(\mathbf{r}_1)\right]\nabla + q(\mathbf{r}_1)\Big]\psi(\mathbf{r}_1) \\ &= \sum_{i=1}^{3}\left[p(\mathbf{r}_1)\frac{\partial^2\psi}{\partial x_i^2} + \frac{\partial p}{\partial x_i}\frac{\partial \psi}{\partial x_i}\right] + q(\mathbf{r}_1)\psi(\mathbf{r}_1) \end{aligned}$$

Additionally, if at the boundaries the inner-products are equal:

$$\langle \phi | \mathcal{L}\psi \rangle = \langle \phi\mathcal{L} | \psi \rangle \,,$$

then the differential equation is not only self-adjoint but also Hermitian. The Green's function is defined so that

$$\nabla_1^2 G(\mathbf{r}, \mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_1)$$

with $\lim_{r_1 \to \infty} G(\mathbf{r}, \mathbf{r}_1) = 0$. Integrating both sides and using Stokes' theorem yields

$$\begin{aligned} \int_\Omega \mathrm{d}^3 r_1 \nabla \cdot \nabla G(\mathbf{r}, \mathbf{r}_1) &= \int_{\partial\Omega} \nabla G(\mathbf{r}, \mathbf{r}_1) \cdot \mathrm{d}\mathbf{\Sigma} \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta a^2 \sin(\theta) \frac{\mathrm{d}G}{\mathrm{d}r_1}\bigg|_{r_1 = a} \\ &= 1 \end{aligned}$$

where a is some arbitrary constant that can be varied. So,

$$\frac{\mathrm{d}G}{\mathrm{d}r_1} = \frac{1}{4\pi a^2}$$

In this case, we assumed that the coordinate $\mathbf{r}$ in $G(\mathbf{r}, \mathbf{r}_1)$ was centered at the origin. In the more general case, we get

$$\frac{\mathrm{d}G}{\mathrm{d}r_1} = \frac{1}{4\pi r_1^2}$$

or

$$G(\mathbf{r}, \mathbf{r}_1) = -\frac{1}{4\pi}\frac{1}{|\mathbf{r} - \mathbf{r}_1|}$$

The previous case was 3 dimensional, as in, the source was a point source and so there were 3 dimensions to move relative to the source. In the next case, we consider a 2 dimensional scenario. This corresponds not to a infinitesimally small point, but an infinitesimally thin line source. Again, the first integral is the same:

$$\begin{aligned}\int_\Omega \mathrm{d}^3 \nabla \cdot \nabla G(\boldsymbol{\rho}, \boldsymbol{\rho}_1) &= \int_{\partial\Omega} \nabla G(\boldsymbol{\rho}, \boldsymbol{\rho}_1) \cdot \mathrm{d}\boldsymbol{\Sigma} \\ &= \int_0^{2\pi} \mathrm{d}\phi \rho_1 \frac{\mathrm{d}G}{\mathrm{d}\rho_1} \\ &= 1\end{aligned}$$

Hence

$$\frac{\mathrm{d}G}{\mathrm{d}\rho_1} = \frac{1}{2\pi\rho_1}$$

and

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}_1) = -\frac{1}{2\pi} \ln |\boldsymbol{\rho} - \boldsymbol{\rho}_1|$$

We can now discuss the Klein-Gordon and Dirac propagator.

2.7 Propagators

The Klein-Gordon equation is

$$(\partial^2 + m^2)\psi(x) = 0$$

Clearly, the operator is self-adjoint. Because the operator is self-adjoint, we can formulate this as an eigenvalue problem

$$\partial^2 \psi(x) = -m^2 \psi(x)$$

We expect the solutions to be plane waves:

$$\psi_p(x) = e^{ip\cdot x}$$

where we use the 4-momentum p because

$$p^2 = \eta^{\mu\nu} p_\nu p_\mu = p^\mu p_\mu = E_{\mathbf{p}}^2 - |\mathbf{p}|^2 = m^2$$

Hence, the eigenfunctions for our original operator will have the same functional form. To formulate the Green's functions for this differential equation, we need to use the Completeness relation

$$\delta(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x} e^{ip\cdot y} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)}$$

Since the integrand of the delta function is an exponential function, we expect that the Green's function will also have an exponential function in it:

$$\partial^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (-p^2) e^{-ip\cdot(x-y)}$$

Hence, using the Klein-Gordon equation, we get that

$$G(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

If we integrate over the time coordinate, we can relate this back to the commutator for the KG field:

$$\begin{aligned}G(x,y) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{(2\pi)} \frac{-1}{p^0 p_0 - |\mathbf{p}|^2 - m^2} e^{-ip^0(x_0 - y_0)} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{(2\pi)} \frac{-1}{p^0 p_0 - E_{\mathbf{p}^2}} e^{-ip^0(x_0 - y_0)} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{(2\pi)} \frac{-1}{(p^0 + E_{\mathbf{p}})(p_0 - E_{\mathbf{p}})} e^{-ip^0(x_0 - y_0)}\end{aligned}$$

To evaluate the p^0 integral, we can extend this analytically into the complex plane. The integrand has two poles at $\pm E_{\mathbf{p}}$ along the real axis. Additionally, the exponential vanishes for complex p^0 depending on whether or not $x_0 < y_0$ or $x_0 > y_0$. If $x_0 > y_0$, then we need to integrate along a contour in the lower half of the complex plane. Otherwise, we must integrate along a contour in the upper half of the complex plane. Assuming $x_0 > y_0$, we have four possible choices for contours then: two CW circles including both poles in the contour, two CCW circles excluding both poles, a CCW circle excluding $-E_{\mathbf{p}}$ and a CW circle including $E_{\mathbf{p}}$, and a CW circle including $-E_{\mathbf{p}}$ and a CCW circle excluding $E_{\mathbf{p}}$. In the case where both poles are excluded, we get 0 for the integral. In the case where both poles are included, we get

$$\oint \frac{\mathrm{d}p^0}{(2\pi)} \frac{-1}{(p^0 + E_{\mathbf{p}})(p_0 - E_{\mathbf{p}})} e^{-ip^0(x_0 - y_0)} = -i\left(\frac{1}{2E_{\mathbf{p}}} e^{iE_{\mathbf{p}}(x_0 - y_0)} - \frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(x_0 - y_0)} \right)$$

where the minus sign comes from the CW direction of the contours. So, the total expression for $G(x, y)$ is

$$\begin{aligned} G(x,y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} i \left(\frac{-1}{2E_{\mathbf{p}}} e^{iE_{\mathbf{p}}(x_0 - y_0)} + \frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(x_0 - y_0)} \right) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{i}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)}\Big|_{p_0 = E_{\mathbf{p}}} - e^{-ip\cdot(x-y)}\Big|_{p_0 = -E_{\mathbf{p}}} \right) \end{aligned}$$

This can be related back to the commutator for the KG field using the definition for the field

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \langle 0 | \Big[\big(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}\big)\big(a_{\mathbf{p}} e^{-ipy} + a_{\mathbf{p}}^\dagger e^{ipy}\big) - \big(a_{\mathbf{p}} e^{-ipy} + a_{\mathbf{p}}^\dagger e^{ipy}\big)\big(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}\big) \Big] | 0 \rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \langle 0 | \big([a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] e^{-ip\cdot(x-y)} - [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] e^{ip\cdot(x-y)} \big) | 0 \rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\ &= -iG(x,y) \end{aligned}$$

where we've assumed that $x_0 > y_0$. Due to the $-i$ factor, we redefine the Green's function to be the one that satisfies the differential equation

$$\partial^2 G(x,y) + m^2 G(x,y) = -i\delta(x-y)$$

Hence,

$$G(x,y) = \Theta(x_0 - y_0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

where the factor $\Theta(x_0 - y_0)$ is the Heaviside step function. Notice that this agrees with our notion that if $x_0 < y_0$, then the contour picks up no poles and thus vanishes. This Green's function, then, is known as the **retarded propagator** or **retarded Green's function**.

2.7.1 Feynman Prescription

As opposed to including both poles in the contour, we want something that more resembles our intuition from 1 dimensional Green's functions. This implies that we want non-vanishing propagators for both $x_0 < y_0$ and $x_0 > y_0$. To do so, we perform a trick by Feynman and redefine $p_0 = \pm(E_{\mathbf{p}} - i\epsilon)$ where $\epsilon \ll 1$ is some small parameter which we will take the vanishing limit of later on. This further implies that we want positive energy when $x_0 > y_0$ and negative energy when $x_0 < y_0$. So, we will choose a CW contour for $-E_{\mathbf{p}}$ and a CCW contour for $E_{\mathbf{p}}$. This yields

$$G_F(x,y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 - i\epsilon} e^{-ip\cdot(x-y)}$$

This is the correct form because it yields

$$G_F(x,y) = \begin{cases} G(x,y), & x_0 > y_0 \\ G(y,x), & x_0 < y_0 \end{cases}$$

which is the correct form for a Green's function.

2.7.2 Dirac Propagator

The Dirac propagator is easier to find after knowing the KG propagator. Since the Dirac equation is

$$(i\not{\partial} - m)\psi(x) = 0,$$

we are trying to find the Green's function such that

$$(i\not{\partial} - m)G_D(x, y) = -i\delta(x - y)$$

We know, however, that

$$(i\not{\partial} - m)(i\not{\partial} + m) = -(\partial^2 - m^2)$$

and so the propagator is

$$G_D(x, y) = (i\not{\partial} + m)G(x, y) = \int \frac{\mathrm{d}^4p}{(2\pi)} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

Just like in the KG equation case, we can relate the Dirac propagator to the anticommutator of spinor fields:

$$\begin{aligned}
\langle 0| \{\psi_a(x), \bar{\psi}_b(y)\} |0\rangle = \langle 0| \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} \sum_{s,r} \Bigg[& \left(a^s_\mathbf{p} u^s(p) e^{-ipx} + b^{s\dagger}_\mathbf{p} v^s(p) e^{ipx}\right)\left(a^{r\dagger}_\mathbf{q} \bar{u}^r(q) e^{iqy} + b^r_\mathbf{q} \bar{v}^r(q) e^{-iqy}\right) + \\
& \left(a^{r\dagger}_\mathbf{q} \bar{u}^r(q) e^{iqy} + b^r_\mathbf{q} \bar{v}^r(q) e^{-iqy}\right)\left(a^s_\mathbf{p} u^s(p) e^{-ipx} + b^{s\dagger}_\mathbf{p} v^s(p) e^{ipx}\right)\Bigg] |0\rangle \\
&= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} \sum_{s,r} \left(\{a^s_\mathbf{p}, a^{r\dagger}_\mathbf{q}\} u^s(p)\bar{u}^r(q) e^{-i(px-qy)} + \{b^r_\mathbf{q}, b^{s\dagger}_\mathbf{p}\} \bar{v}^r(q) v^s(p) e^{-i(qy-px)}\right) \\
&= \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6} \frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{q})\delta^{rs}}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} \sum_{s,r} \left(u^s(p)\bar{u}^r(q) e^{-i(px-qy)} + v^s(p)\bar{v}^r(q) e^{-i(qy-px)}\right) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left((\not{p} + m) e^{-ip\cdot(x-y)} + (\not{p} - m) e^{-ip\cdot(y-x)}\right) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left((i\not{\partial} + m) e^{-ip\cdot(x-y)} - (i\not{\partial} + m) e^{-ip\cdot(y-x)}\right) \\
&= (i\not{\partial} + m) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)}\right) \\
&= (i\not{\partial} + m) G(x, y)
\end{aligned}$$