QFT Perturbation Theory Exercises

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1 Scalar Particle Decay

A Lagrangian for a theory of two scalar particles is given as follows:

$$\mathcal{L} = \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}M^2\Phi^2 + \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi \tag{1}$$

where the last term is an interaction term with coupling constant μ that allows for a Φ particle to decay into two ϕ particles, provided $M > 2m$. We are asked to find the lifetime of the Φ particle.

P&S tell us that

$$\mathrm{d}\Gamma = \frac{1}{2M}\frac{\mathrm{d}p_1^3\mathrm{d}p_2^3}{(2\pi)^6}\frac{1}{4E_{p_1}E_{p_2}}|\mathcal{M}(\Phi(0) \to p_1p_2)|^2(2\pi)^4\delta^{(4)}(p_\Phi - p_1 - p_2) \tag{2}$$

We can find the S-matrix element from first principles:

$$i\mathcal{M} = \langle \mathbf{p_1}\mathbf{p_2}|T\big\{\exp\big[-i\mu\int \mathrm{d}^4x\Phi(x)\phi(x)\phi(x)\big]\big\}|0\rangle \tag{3}$$

We contract ϕ with either $\langle\mathbf{p}_1|$ or $\langle\mathbf{p}_2|$ and Φ with $|0\rangle$.

Since

$$\langle\mathbf{p}_1| = \langle 0|a_{\mathbf{p}_1}\sqrt{2E_{\mathbf{p}_1}}, \tag{4}$$

$$|\mathbf{0}\rangle = \sqrt{2E_{\mathbf{0}}}b_{\mathbf{0}}^\dagger\langle 0|, \tag{5}$$

$$\phi(x) = \int\frac{\mathrm{d}p^3}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\big(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\big), \tag{6}$$

and

$$\Phi(x) = \int\frac{\mathrm{d}k^3}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\big(b_{\mathbf{k}}e^{-ik\cdot x} + b_{\mathbf{k}}^\dagger e^{ik\cdot x}\big) \tag{7}$$

we see

$$i\mathcal{M} = -2i\mu \tag{8}$$

where the factor of 2 comes from the fact that there are two ways to the contract the ϕ with the final momenta states. So

$$\mathrm{d}\Gamma = \frac{1}{2}\frac{1}{2M}\frac{\mathrm{d}p_1^3\mathrm{d}p_2^3}{(2\pi)^2}\frac{\mu^2}{E_{p_1}E_{p_2}}\delta(M - E_{p_1} - E_{p_2})\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2) \tag{9}$$

where the extra factor of $\frac{1}{2}$ comes from the fact that there are two identical bosons in the final state. Integrating over p_2 first yields

$$\Gamma = \int\frac{1}{4M}\frac{\mathrm{d}p_1^3}{(2\pi)^2}\frac{\mu^2}{E_{p_1}^2}\delta(M - 2E_{p_1}) \tag{10}$$

Converting to spherical coordinates, we have

$$\Gamma = \int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^\infty\mathrm{d}p_1\frac{\mu^2}{16\pi^2M}\frac{p_1^2}{E_{p_1}^2}\delta(M - 2E_{p_1}) = \int_0^\infty\mathrm{d}p_1\frac{\mu^2}{4\pi M}\frac{p_1^2}{E_{p_1}^2}\delta(M - 2E_{p_1}) \tag{11}$$

We can convert to energy coordinates via the transformation $p_1^2 = E_{p_1}^2 - m^2$ and $\mathrm{d}p_1 = E_{p_1}(E^2 - m^2)^{-\frac{1}{2}}\mathrm{d}E_{p_1}$ to get

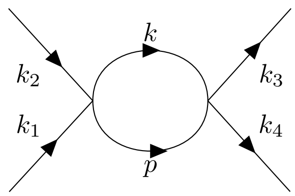
$$\Gamma = \int_m^\infty \mathrm{d}E_{p_1} \frac{\mu^2}{4\pi M} \frac{1}{E_{p_1}\sqrt{E_{p_1}^2 - m^2}} \delta(M - 2E_{p_1}) = \frac{\mu^2}{8\pi M} \frac{1}{\sqrt{1 - \frac{m^2}{M^2}}} \tag{12}$$

Thus, the lifetime is

$$\tau = \frac{8\pi M}{\mu^2} \sqrt{1 - \frac{m^2}{M^2}} \tag{13}$$

2 1-Loop Diagram

We consider the 1-loop diagram given below for the ϕ^4 theory. We are to (a) derive the amplitude of the diagram from first principles and (b) show that the amplitude diverges using a momentum cutoff Λ.



Since there are two vertices in the diagram, the amplitude must be of order λ^2. Thus, we have

$$iT = i\Big(-\frac{i\lambda}{4!}\Big)^2 \int \frac{\mathrm{d}^4x}{(2\pi)^4} \phi(x)\phi(x)\phi(x)\phi(x) \int \frac{\mathrm{d}^4y}{(2\pi)^4} \phi(y)\phi(y)\phi(y)\phi(y) \tag{14}$$

The initial state is $|\mathbf{k}_1\mathbf{k}_2\rangle_{\mathrm{in}}$ and the final state is ${}_{\mathrm{out}}\langle\mathbf{k}_3\mathbf{k}_4|$. So, the scattering amplitude with wick contractions is

$${}_{\mathrm{out}}\langle\mathbf{k}_3\mathbf{k}_4| i\Big(-\frac{i\lambda}{4!}\Big)^2 \int \frac{\mathrm{d}^4x}{(2\pi)^4} \phi(x)\phi(x)\phi(x)\phi(x) \int \frac{\mathrm{d}^4y}{(2\pi)^4} \phi(y)\phi(y)\phi(y)\phi(y) |\mathbf{k}_1\mathbf{k}_2\rangle_{\mathrm{in}} \tag{15}$$

There are $4 \cdot 3$ possible ways to contract the outgoing states with the $\phi(x)$ fields. There are $4 \cdot 3$ possible ways to contract the two remaining $\phi(x)$ with the $\phi(y)$. Finally, there are 2 possible ways to contract the incoming states with the $\phi(y)$. This yields a symmetry factor of $\frac{1}{2}(4!)^2$.

We will use position-space representations to arrive at the scattering amplitude. The contraction of states with fields produces

$$\langle\mathbf{k}|\phi(x) = \langle 0|e^{ik\cdot x} \tag{16}$$

and

$$\phi(y)|\mathbf{k}\rangle = e^{-ik\cdot y}|0\rangle. \tag{17}$$

The field-field contractions produce propagators as usual

$$\phi(x)\phi(y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{18}$$

So, the amplitude is

$$\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4x}{(2\pi)^4} \int \frac{\mathrm{d}^4y}{(2\pi)^4} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} e^{-i(k_1+k_2)\cdot y} e^{i(k_3+k_4)\cdot x} \tag{19}$$

Rearranging the exponentials yields

$$\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4x}{(2\pi)^4} \int \frac{\mathrm{d}^4y}{(2\pi)^4} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} e^{-i(p+k-k_3-k_4)\cdot x} e^{-i(k_1+k_2-p-k)\cdot y} \tag{20}$$

Integrating over x and y yields the delta functions which enforce momentum conservationd to give

$$\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} \delta^{(4)}(p + k - k_3 - k_4) \delta^{(4)}(k_1 + k_2 - p - k) \tag{21}$$

Integrating over p (and ignoring the overall delta function factor which just enforces conservation of momentum between incoming and outgoing states), we get

$$\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k_1 + k_2 - k)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} \tag{22}$$

We can't integrate this directly because it is not simply a function of k^2. First, we must use Feynman parameterization to combine the denominators. Let $K = k_1 + k_2$ and $M = m^2 - i\epsilon$. Then, the integral becomes

$$\begin{aligned}
\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(K-k)^2 - M} \frac{1}{k^2 - M} &= \frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{\big(k^2 - M - x[(K-k)^2 - k^2]\big)^2} &(23)\\
&= \frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{\big(k^2 - 2xkK + xK^2 - M\big)^2} &(24)\\
&= \frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{\big((k - xK)^2 - M + K^2 x(1-x)\big)^2} &(25)
\end{aligned}$$

Now, performing the substitution $k \mapsto k + xK$ and letting $\Delta = K^2 x(1-x)$, gives

$$\frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{\big(k^2 - M + \Delta\big)^2} = \frac{i}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{\big((k_0)^2 - \mathbf{k}^2 - m^2 + i\epsilon + \Delta\big)^2} \tag{26}$$

which has poles at $\pm\sqrt{\mathbf{k}^2 - m^2 \mp i\epsilon + \Delta}$. For negative (positive) frequency solutions, the pole is in the upper (lower) half of the complex plane. So, we can perform a wick rotation $k_0 \mapsto ik_4$. This yields the Euclidean integral with $k_E = -k_4^2 - \mathbf{k}^2$

$$\begin{aligned}
\frac{i}{2}(-i\lambda)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \frac{1}{\big(k_E^2 + m^2 - \Delta\big)^2} &= \frac{i}{16\pi^2}(-i\lambda)^2 \int_0^1 \mathrm{d}x \int_0^\Lambda \mathrm{d}k_E^2 \frac{k_E^2}{\big(k_E^2 + m^2 - \Delta\big)^2} &(27)\\
&= \frac{i}{16\pi^2}(-i\lambda)^2 \int_0^1 \left[\ln\left(\frac{\Lambda^2}{m^2 - \Delta}\right) - 1\right] &(28)\\
&= \frac{i}{16\pi^2}(-i\lambda)^2 \left(\ln\big(\Lambda^2\big) - 1 - \int_0^1 \ln\big(m^2 - \Delta\big)\right) &(29)\\
&= \frac{i}{16\pi^2}(-i\lambda)^2 \left(\ln\big(\Lambda^2\big) - 1 - \int_0^1 \ln\big(m^2 - K^2 x(1-x)\big)\right) &(30)\\
&\approx \frac{-i}{16\pi^2}\lambda^2 \ln\big(\Lambda^2\big) &(31)
\end{aligned}$$

where we've assumed that $\Lambda^2 \gg m^2 - \Delta$. We can see that as $\Lambda \to \infty$ the value of the integral diverges.

3 Linear Sigma Model

The interaction of pions at low energy is described by a phenomenological model called the linear sigma model. The model consists of N Klein-Gordon fields coupled via ϕ^4 interaction. Specifically, $\Phi^i(x)$, $i = 1, \cdots, N$, are N scalar fields governed by the Hamiltonian

$$H = \int \mathrm{d}^3 x \Big(\tfrac{1}{2}\big(\Pi^i\big)^2 - \tfrac{1}{2}\big(\nabla\Phi^i\big)^2 + V(\Phi^2)\Big), \tag{32}$$

where $\Phi^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V(\Phi^2) = \tfrac{1}{2}m^2(\Phi^i)^2 + \tfrac{\lambda}{4}\big((\Phi^i)^2\big)^2 \tag{33}$$

is invariant under rotations of $\mathbf{\Phi}$.

a) Suppose $m^2 > 0$. We can see that the non-interacting part is just the Klein-Gordon Hamiltonian:

$$H = H_{\text{Klein-Gordon}} + \int \mathrm{d}^3x \tfrac{\lambda}{4}\big((\Phi^i)^2\big)^2 \tag{34}$$

so that in the limit that λ goes to zero, we recover N free Klein-Gordon fields. For each field, we can expand it in a sum over Fourier modes:

$$\Phi^i(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big(a_{i_{\mathbf{p}}}e^{-ip\cdot x} + a^\dagger_{i_{\mathbf{p}}}e^{ip\cdot x}\Big) \tag{35}$$

where $a^\dagger_{i_{\mathbf{p}}}$ and $a_{i_{\mathbf{p}}}$ are the creation and annihilation operators for Φ^i. By definition

$$\contraction{}{\Phi^i(x)}{}{\Phi^j(y)}\Phi^i(x)\Phi^j(y) = \begin{cases}[\Phi^{i+}(x), \Phi^{j-}(y)] & \text{for } x^0 > y^0 \\ [\Phi^{j+}(y), \Phi^{i-}(x)] & \text{for } y^0 > x^0\end{cases} \tag{36}$$

By the commutativity of the annihilation and creation operators for different species of fields, we see that

$$\contraction{}{\Phi^i(x)}{}{\Phi^j(y)}\Phi^i(x)\Phi^j(y) = \delta^{ij}D_F(x-y) \tag{37}$$

To get the right vertex factor, we use the rules for the ϕ^4 theory and apply the Wick contraction rule we just deduced. A generic interaction has the form

$$-\frac{i\lambda}{c}\int \mathrm{d}^4w\,\Phi^i(w)\Phi^j(w)\Phi^k(w)\Phi^l(w) \tag{38}$$

where $c = 4$ if all field species are identical and $c = 2$ if there are two species of fields. We can perform the following contractions ij and kl, ik and jl, and il and jk. In momentum-space, this is

$$-\frac{i\lambda}{c}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{39}$$

If there is only one species of field, then there are 4! possible ways to place the fields on the lines. Since $c = 4$ in this case, we see the vertex equals $-2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) = -6i\lambda$

Now, if two of the fields are of species i and the other two are of species j, then we either have $\Phi^i\Phi^i \to \Phi^j\Phi^j$, $\Phi^i\Phi^j \to \Phi^i\Phi^j$, or $\Phi^j\Phi^j \to \Phi^i\Phi^i$. In each case, there are (2!)(2!) combinations since we can interchange each identical field with the other one. Thus, the vertex equals $-2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) = -2i\lambda$.

Using these facts along with the fact that the masses of each field species are identical, we can easily compute the differential scattering cross-section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2E_{\text{cm}}} \tag{40}$$

For $\pi^1\pi^1 \to \pi^1\pi^1$,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}(\pi^1\pi^1 \to \pi^1\pi^1)|^2}{64\pi^2E_{\text{cm}}} = \frac{9\lambda^2}{16\pi^2E_{\text{cm}}}. \tag{41}$$

For $\pi^1\pi^2 \to \pi^1\pi^2$ and $\pi^1\pi^1 \to \pi^2\pi^2$,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}(\pi^1\pi^2 \to \pi^1\pi^2)|^2}{64\pi^2E_{\text{cm}}} = \frac{|\mathcal{M}(\pi^1\pi^1 \to \pi^2\pi^2)|^2}{64\pi^2E_{\text{cm}}} = \frac{\lambda^2}{16\pi^2E_{\text{cm}}}. \tag{42}$$

b) Supposing $m^2 = -\mu^2 < 0$ and $\phi^i(x) = \pi^i(x)$ and $\phi^N(x) = v + \sigma(x)$, we see that

$$V(\Phi(x)^2) = \frac{1}{2}m^2(\Phi^i(x))^2 + \frac{\lambda}{4}\big((\Phi^i(x))^2\big)^2 \tag{43}$$

$$= -\frac{1}{2}\mu^2\Big[(\pi^i(x))^2 + \big(v^2 + 2v\sigma(x) + \sigma^2(x)\big)\Big] + \frac{\lambda}{4}\Big[(\pi^i(x))^2 + \big(v^2 + 2v\sigma(x) + \sigma^2(x)\big)\Big]^2 \tag{44}$$

We can minimize $V(\Phi^2)$ with respect to Φ^N and then solve for v:

$$0 = \frac{\mathrm{d}V}{\mathrm{d}\Phi^N} = -\mu^2\Phi^N(x) + \lambda\big((\pi^i(x))^2 + (\Phi^N(x))^2\big)\Phi^N(x) \tag{45}$$

$$= \Phi^N(x)\Big(-\mu^2 + \lambda(\pi^i(x)) + \lambda(\Phi^N(x))^2\Big) \tag{46}$$

This has solutions at $\Phi^N(x) = 0$ and $\Phi^N(x) = \pm\sqrt{\frac{\mu^2 - \lambda(\pi^i(x))^2}{\lambda}}$. Thus,

$$v = -\sigma \pm \sqrt{\frac{\mu^2 - \lambda(\pi^i(x))^2}{\lambda}} \tag{47}$$

We see that when we take the fields $\pi^i(x) = \sigma(x) = 0$, we get

$$v = \pm\frac{\mu}{\sqrt{\lambda}} \tag{48}$$

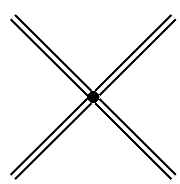
Plugging this back in yields

$$V(\Phi^i(x)) = \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\lambda}\mu\sigma^3 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{4}(\pi^i)^4 + \sqrt{\lambda}\mu\sigma(\pi^i)^2 + \frac{\lambda}{2}\sigma^2(\pi^i)^2 \tag{49}$$

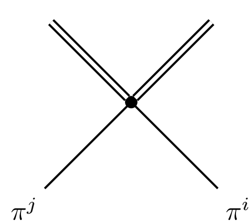
which we can see has a massive σ field and $N-1$ massless pion fields. Additionally, it has multiple couplings: σ^3, σ^4, π^4, $\sigma\pi^2$, and $\sigma^2\pi^2$ which all vanish as $\lambda \to 0$. The Feynman rules are:

$\overbracket{\sigma\sigma}$ =  $= \frac{i}{p^2 - 2\mu^2}$

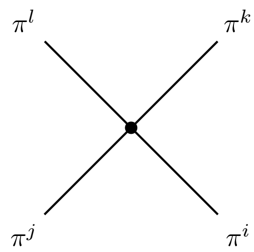
$\overbracket{\pi^i\pi^j}$ = i j $= \frac{i\delta^{ij}}{p^2}$



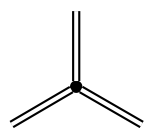
$= -6i\lambda$



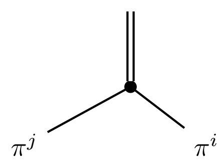
$= -2i\lambda\delta^{ij}$



$$= -6i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$$



$$= -6i\lambda v$$

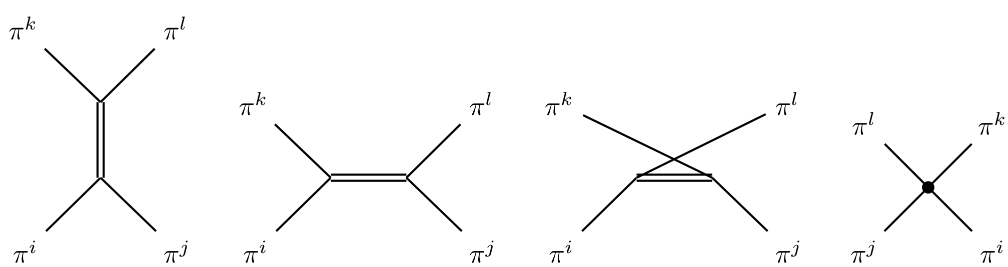


$$= -2i\lambda v\delta^{ij}$$

c) Now, we are to calculate the scattering amplitude for

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

There are four diagrams that contribute to this process: s, t, and u channels, and a quartic interaction vertex.



The amplitude for the s-channel diagram is

$$i\mathcal{M} = \left(-2i\lambda v\delta^{ij}\frac{i}{p^2 - 2\mu^2}(-2i\lambda v\delta^{kl})\right)\delta^{(4)}(p - p_1 - p_2)\delta^{(4)}(p_3 + p_4 - p) \tag{50}$$

The amplitude for the t-channel diagram is

$$i\mathcal{M} = \left(-2i\lambda v\delta^{ik}\frac{i}{p^2 - 2\mu^2}(-2i\lambda v\delta^{jl})\right)\delta^{(4)}(p - p_1 - p_3)\delta^{(4)}(p_2 + p_4 - p) \tag{51}$$

The amplitude for the u-channel diagram is

$$i\mathcal{M} = \left(-2i\lambda v\delta^{il}\frac{i}{p^2 - 2\mu^2}(-2i\lambda v\delta^{jk})\right)\delta^{(4)}(p - p_1 - p_4)\delta^{(4)}(p_2 + p_3 - p) \tag{52}$$

The amplitude for the quartic diagram is

$$i\mathcal{M} = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\delta^{(4)}(p_3 + p_4 - p_1 - p_2) \tag{53}$$

When the undetermined propagator momentum is integrated over in the first three diagrams at threshold ($\mathbf{p}_i = 0$), we get

$$\text{sum of s, t, and u diagrams} = 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\delta^{(4)}(p_3 + p_4 - p_1 - p_2) \tag{54}$$

which when added to the quartic interaction diagram yields a vanishing amplitude

4 Rutherford Scattering

The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int \mathrm{d}^3 x e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

a) We wish to show that the lowest order T-matrix element is

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\cdot\tilde{A}_\mu(p'-p),$$

where $\tilde{A}_\mu(p'-p)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

The matrix element is

$$\langle p'|iT|p\rangle = \langle p'| - ie\int \mathrm{d}t \int \mathrm{d}^3 x\bar{\psi}\gamma^\mu\psi A_\mu|p\rangle = \langle p'| - ie\int \mathrm{d}^4 x\bar{\psi}\gamma^\mu\psi A_\mu|p\rangle$$

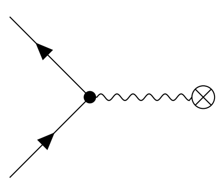
Performing our wick contractions of the Dirac fields with the incoming and outgoing momentum states yields

$$\langle p'| - ie\int \mathrm{d}^4 x\bar{\psi}\gamma^\mu\psi A_\mu|p\rangle = -ie\int \mathrm{d}^4 x\bar{u}(p')\gamma^\mu u(p)e^{i(p'-p)\cdot x}A_\mu(x)$$

Moving the terms that aren't dependent upon x out of the integral and recognizing the integral is a four-dimensional Fourier transform gives

$$\boxed{\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p)}$$

b) Now, we have a new Feynman rule for computing $\mathcal{M}$:

 $= -ie\gamma^\mu\tilde{A}_\mu(\mathbf{q})$

The derivation for the differential scattering cross-section is identical to P&S's derivation but with one species of particle incoming and outgoing. This yields

$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2(2\pi)\delta(E_i - E_f), \tag{55}$$

where v_i is the particle's initial velocity. In this expression, there is no dependence upon the vector aspect of p_f, only a dependence upon magnitude. Thus,

$$\mathrm{d}^3 p_f = \mathrm{d}\Omega\mathrm{d}p_f p_f^2$$

So,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{v_i}\frac{1}{2E_i}\int_0^\infty \frac{\mathrm{d}p_f}{(2\pi)^3}\frac{p_f^2}{2\sqrt{p_f^2+m^2}}|\mathcal{M}(p_i\to p_f)|^2(2\pi)\delta(E_i-E_f)$$

Rewriting the energy delta function in terms of momentum, we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{v_i}\frac{1}{2E_i}\int_0^\infty \frac{\mathrm{d}p_f}{(2\pi)^2}\frac{p_f^2}{2\sqrt{p_f^2+m^2}}|\mathcal{M}|^2\delta\left(\sqrt{p_i^2+m^2}-\sqrt{p_f^2+m^2}\right)$$

The derivative of the argument of the delta function is

$$\frac{\mathrm{d}f(p_f)}{\mathrm{d}p_f} = \frac{p_f}{\sqrt{p_f^2+m^2}}$$

Thus,

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{v_i}\frac{1}{(2\pi)^2}\frac{p_i}{4\sqrt{p_i^2+m^2}}|\mathcal{M}|^2}$$

c) Specializing to the non-relativistic case, we have $p_i \ll m$ and $p_i = mv_i$. Hence, we get the expression

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4(2\pi)^2}|\mathcal{M}|^2$$

Now, the square modulus of the invariant matrix element has the form

$$|\mathcal{M}|^2 = \frac{Z^2e^2m^2}{16p^4(1-\cos(\theta))^2}$$

where we used the Coulomb potential for the vector potential and the non-relativistic approximation of the spinor products. Using a half-angle identity, we get

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2Z^2}{4m^2v^4\sin^4(\theta/2)}}$$

where $\alpha = e/2\pi$ in natural units.