Statistical Data Analysis HW

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1.1 Moments of Uniform Distribution based on Definition

With the uniform distribution:

$$p_U(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & Otherwise \end{cases} \tag{1}$$

The 0^{th} moment:

$$M_0 = \int_{-\infty}^{\infty} x^0 p_U(x)dx = \int_a^b \frac{dx}{b-a} = 1. \tag{2}$$

The 1^{st} moment:

$$M_1 = \int_a^b \frac{xdx}{b-a} = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{1}{2}(a+b) = \mu, \tag{3}$$

where μ is the mean of the distribution. For the higher moments it makes more sense to compute the moments about the mean of the distribution. So for the 2^{nd} moment about μ:

$$\begin{aligned} M_2 = \int_a^b \frac{(x-\mu)^2 dx}{b-a} &= \frac{1}{3}\frac{1}{b-a}((b-\mu)^3 - (a-\mu)^3) \\ &= \frac{1}{3}\frac{1}{b-a}((\frac{b-a}{2})^3 - (\frac{a-b}{2})^3) \\ &= \frac{1}{3}\left(\frac{b-a}{2}\right). \end{aligned} \tag{4}$$

We can see that the 3rd moment will be zero as $(\frac{b-a}{2})^4 - (\frac{a-b}{2})^4 = 0$,

in general the odd moments will be zero. The kth moment will be:

$$\begin{aligned} M_k = \int_a^b \frac{(x^k-\mu)^2 dx}{b-a} &= \frac{1}{k+1}\frac{1}{b-a}\left((b-\mu)^{k+1} - (a-\mu)^{k+1})\right) \\ &= \frac{1}{k+1}\frac{1}{b-a}\left(\left(\frac{b-a}{2}\right)^{k+1}\left(1-(-1)^{k+1}\right)\right) \\ M_k &= \begin{cases} 0 & \text{k odd} \\ \frac{1}{k+1}\left(\frac{b-a}{2}\right)^k & \text{k even} \end{cases} \end{aligned} \tag{5}$$

1.2 Moments From the Characteristic function

$$\Phi = \int_{-\infty}^{\infty} e^{-ikx} p_U(x)dx = \int_a^b \frac{1}{b-a} e^{-ikx} dx$$

$$= \frac{i}{k(b-a)} e^{-ikx}\Big|_a^b \tag{6}$$

$$= \frac{i}{k(b-a)} \left[e^{-ikb} - e^{-ika}\right] \tag{7}$$

To make taking derivatives easier we re-write the exponentials as power series in k.

$$\Phi = \frac{i}{k(b-a)} \left[(1 - ikb + (-i)^2 \frac{k^2b^2}{2!} + \dots) - \left(1 - ika + (-i)^2 \frac{k^2a^2}{2!} + \dots\right)\right]$$

$$= \frac{1}{b-a} \left[(b-a) + (-i)k(b^2 - a^2)\frac{1}{2!} + \dots \frac{(-ik)^n}{(n+1)!}(b^{n+1} - a^{n+1})\right] \tag{8}$$

$$\tag{9}$$

Transforming to be around the mean: ($a \to a - \mu = \frac{a-b}{2}$, $b \to b - \mu = \frac{b-a}{2}$), and taking the nth derivative w.r.t k at $k = 0$ then gives us the nth moment:

$$(-i)^n M_n = \frac{d^n\Phi}{dk^n}\Big|_{k=0} = (-i)^n \frac{1}{(n+1)(b-a)} \left[\left(\frac{b-a}{2}\right)^{n+1} - (-1)^{n+1} \left(\frac{b-a}{2}\right)^{n+1}\right]$$

$$M_n = \begin{cases} 0 & \text{n odd} \\ \frac{1}{n+1}\left(\frac{b-a}{2}\right)^n & \text{n even} \end{cases} \tag{10}$$