Homework 2

Thursday, February 25, 2021 4:07 PM

1. Asymptotic Comparison [50 pt]

In each of the following situations, indicate whether $f = O(g)$, or $f = \Omega(g)$, or both (i.e. $f = \Theta(g)$). Justify your choice. **[5 pt each]**

	$f(n)$	$g(n)$
a)	$n^{1/4}$	$n^{2/3}$
b)	$330n + \log n$	$n + (\log n)^2$
c)	$330 \log n$	$\log(n^2)$
d)	$n^{1.01}$	$n \log^2 n$
e)	$n^2 / \log n$	$n (\log n)^2$
f)	$(\log n)^{\log n}$	$n / \log n$
g)	$\sqrt{n}$	$(\log n)^3$
h)	$n^{1/2}$	$5^{\log_2 n}$
i)	$n2^n$	3^n
j)	$(\log n)^{\log n}$	$2^{(\log_2 n)^2}$

a) a) $n^{1/4}$ $n^{2/3}$

$f(n) = O(g_n)? \Rightarrow f(n) \leq c \cdot (g(n))$

$n^{1/4} \leq c \cdot n^{2/3}$ $c = 1$ $n_0 = 2$

$\log(n^{1/4}) \leq \log(n^{2/3})$

$\frac{1}{4}(\log(n)) \leq \frac{2}{3}(\log(n))$

$\frac{1}{4} \leq \frac{2}{3}$

$\boxed{f(n) = O(g(n))}$

$f(n) = \Omega(g(n))?$ $f(n) \geq c \cdot (g(n))$

$n^{1/4} \geq c \cdot n^{2/3}$ $c\ \frac{1}{2}$

$\log(n^{1/4}) \geq \log(\frac{1}{2} \cdot n^{2/3})$

$\frac{\frac{1}{4}\log n}{\log(n)} \geq \frac{\log \frac{1}{2} + \frac{2}{3}\log(n)}{\log(n)}$

$\frac{1}{4} \geq \frac{\log \frac{1}{2}}{\log n} + \frac{2}{3}$

$\uparrow$ not true

so $f(n) \neq \Omega(g(n))$

b) $f(n)$ $g(n)$

$330n + \log n$ $n + (\log n)^2$

$f(n) = O(g_n)?$ $f(n) \leq c \cdot g(n)$

$330n + \log n \leq c\left(n + (\log n)^2\right)$ $c = 331$

$\lim_{n\to\infty} \left(\frac{330n + \log n}{n} \leq \frac{331n + 331(\log n)^2}{n} \right)$

$\lim_{n\to\infty} \left(\frac{330\cancel{n}}{\cancel{n}} + \cancel{\frac{\log n}{n}}^{0} \leq \frac{331\cancel{n}}{\cancel{n}} + \cancel{\frac{331(\log n)^2}{n}}^{0} \right)$

$330 \leq 331$

$f(n) = O(g_n)$

$f(n) = \Omega(g(n))$

$330n + \log n \geq c\left(n + (\log n)^2\right)$ $c = 329$

$\lim_{n\to\infty} \left(\frac{330n + \log n}{n} \geq \frac{329n + (\log n)^2}{n} \right)$

$\lim_{n\to\infty} \left(\frac{330\cancel{n}}{\cancel{n}} + \cancel{\frac{\log n}{n}}^{0} \geq \frac{329\cancel{n}}{\cancel{n}} + \cancel{\frac{(\log n)^2}{n}}^{0} \right)$

$330 \geq 329$

$f(n) = \Omega(g_n)$

$\boxed{f(n) = \Theta(g(n))}$

c) c) $f(n)$ $g(n)$

$330 \log n$ $\log(n^2)$

$f(n) = O(g(n))?$ $f(n) \leq c \cdot g(n)$

$330 \log n \leq c \cdot \log(n^2)$ $c = 166$

$f(n) = \Omega(g(n))?$ $f(n) \geq c \cdot g(n)$

$330 \log n \geq c \cdot \log(n^2)$ $c = 164$

$f(n) = O(g(n))$: $f(n) \le c \cdot g(n)$

$$330 \log n \le c \cdot \log(n^2) \quad c = 166$$

$$\frac{330 \log(n)}{\log(n)} \le \frac{166 \cdot 2 \log(n)}{\log(n)}$$

$$330 \le 332$$

$$f(n) = O(g(n))$$

$f(n) = \Omega(g(n))$: $f(n) \ge c \cdot g(n)$

$$330 \log n \ge c \cdot \log(n^2) \quad c = 164$$

$$\frac{330 \log n}{\log n} \ge \frac{164 \cdot 2 \log(n)}{\log n}$$

$$330 \ge 328$$

$$f(n) = \Omega(g(n))$$

$$\boxed{f(n) = \Theta(g(n))}$$

d) d) $f(n)$ $n^{1.01}$ $g(n)$ $n \log^2 n$

$f(n) = O(g(n))$? $f(n) \le c \cdot g(n)$

$$n^{1.01} \le c \cdot n \log^2 n \quad c = 1$$

$$n^{1.01} \le 1 \cdot n \log(\log n)$$

$$\log(n^{1.01}) \le \log(n \log(\log n))$$

$$\frac{1.01 \cancel{\log n}}{\cancel{\log(n)}} \le \frac{\log(n(\log(\log n)))}{\log(n)}$$

$$\lim_{n \to \infty} \left(1.01 \le \frac{\log(n(\log(\cancel{\log n})^{0}))}{\cancel{\log(n)}} \right)$$

$$1.01 \le 0 \quad \leftarrow \text{false}$$

$$f(n) \ne O(g(n))$$

$f(n) = \Omega(g(n))$? $f(n) \ge c \cdot g(n)$

$$n^{1.01} \ge c \cdot n \log^2 n \quad c = 1$$

$$n^{1.01} \ge 1 \cdot n(\log(\log n))$$

$$\log(n^{1.01}) \ge \log(n(\log(\log n)))$$

$$\lim_{n \to \infty} \left(\frac{1.01 \cancel{\log n}}{\cancel{\log n}} \ge \frac{\log(n(\log(\cancel{\log n})^{0}))}{\cancel{\log n}} \right)$$

$$1.01 \ge 0$$

$$\boxed{f(n) = \Omega(g(n))}$$

e) e) $n^2/\log n$ $n(\log n)^2$

$f(n) = O(g(n))$?

$$f(n) \le c\, g(n)$$

$$\frac{n^2}{\log n} \le c\; n(\log n)^2 \quad c = 1$$

$$\frac{\frac{n^{\cancel{2}}}{\log n}}{\cancel{n}} \le \frac{\cancel{n}(\log n)^2}{\cancel{n}}$$

$$\cancel{\log n} \cdot \frac{n}{\cancel{\log n}} \le (\log n)^2 \cdot \log n$$

$$n \le (\log n)^3 \quad \leftarrow \text{false}$$

$$f(n) \ne O(g(n))$$

$f(n) = \Omega(g(n))$?

$$f(n) \ge c \cdot g(n)$$

$$\frac{n^2}{\log n} \ge c \cdot n(\log n)^2 \quad c = 1$$

$$\cancel{\log n} \cdot \frac{n^{\cancel{2}}/\cancel{\log n}}{\cancel{n}} \ge \frac{\cancel{n}(\log n)^2}{\cancel{n}} \cdot \log n$$

$$n \ge (\log n)^3 \leftarrow \text{true}$$

$$\boxed{f(n) = \Omega(g(n))}$$

f) f) $f(n)$ $(\log n)^{\log n}$ $g(n)$ $n/\log n$

$f(n) = O(g(n))$?

$$f(n) \le c \cdot g(n)$$

$$(\log n)^{\log n} \le c \cdot \frac{n}{\ldots} \quad c = 1$$

$f(n) = \Omega(g(n))$?

$$f(n) \ge c \cdot g(n)$$

$$(\log n)^{\log n} \ge c \cdot \frac{n}{\ldots} \quad c = 1$$

$f(n) \leq c \cdot g(n)$

$(\log n)^{\log n} \leq c \cdot \frac{n}{\log n} \qquad c=1$

$\log(\log n^{\log n}) \leq \log\left(\frac{n}{\log n}\right)$

$\frac{\cancel{\log n} \cdot \log(\log n)}{\cancel{\log n}} \leq \frac{\log n - \log(\log n)}{\log n}$

$\lim_{n\to\infty} \left(\log(\log n) \leq 1 - \frac{\log(\cancel{\log n})^{\to 0}}{\cancel{\log n}} \right)$

$\infty \leq 1 \leftarrow$ false

$f(n) \geq c \cdot g(n)$

$(\log n)^{\log n} \geq c \cdot \frac{n}{\log n} \qquad c=1$

$\log(\log n^{\log n}) \geq \log\left(\frac{n}{\log n}\right)$

$\frac{\cancel{\log n} \cdot \log(\log n)}{\cancel{\log n}} \geq \frac{\cancel{\log n}}{\cancel{\log n}} - \frac{\log(\log n)}{\log n}$

$\lim_{n\to\infty} \left(\log(\log n) \geq 1 - \frac{\log(\cancel{\log n})^{\to 0}}{\cancel{\log n}} \right)$

$\infty \geq 1 \leftarrow$ true

$\boxed{f(n) = \Omega(g(n))}$

g) g) $\sqrt{n}$ $(\log n)^3$

$f(n) = O(g(n))$?

$f(n) \leq c \cdot g(n) \qquad c=1$

$\sqrt{n} \leq (\log n)^3$

$\log(n^{1/2}) \leq \log[(\log n)^3]$

$\frac{1/2 \log(n)}{\log(n)} \leq \frac{3 \log(\log n)}{\log(n)}$

$\lim_{n\to\infty} \left(1/2 \leq \frac{3 \log(\cancel{\log n})^{\to 0}}{\cancel{\log(n)}} \right)$

$1/2 \leq 0 \leftarrow$ false

$f(n) = \Omega(g(n))$?

$f(n) \geq c \cdot g(n) \qquad c=1$

$\sqrt{n} \geq (\log n)^3$

$\log(n^{1/2}) \geq \log[(\log n)^3]$

$\frac{1/2 \cancel{\log(n)}}{\cancel{\log(n)}} \geq \frac{3 \log(\log n)}{\log(n)}$

$\lim_{n\to\infty} \left(1/2 \geq \frac{3 \log(\cancel{\log n})^{\to 0}}{\cancel{\log(n)}} \right)$

$1/2 \geq 0 \leftarrow$ true

$\boxed{f(n) = \Omega(g(n))}$

h) h) $n^{1/2}$ $5^{\log_2 n}$

$f(n) = O(g(n))$?

$f(n) \leq c \cdot g(n)$

$n^{1/2} \leq 5^{\log_2 n} \qquad c=1$

$\log(n^{1/2}) \leq \log(5^{\log_2 n})$

$\frac{1/2 \cancel{\log(n)}}{\cancel{\log n}} \leq \frac{\cancel{\log(n)} \cdot \log(5)}{\cancel{\log(n)}}$

$1/2 \leq \log 5 \leftarrow$ true

$f(n) = O(g(n))$

$f(n) = \Omega(g(n))$?

$f(n) \geq c \cdot g(n) \qquad c = 1/5$

$n^{1/2} \geq 1/5 \cdot 5^{\log_2 n}$

$\log(n^{1/2}) \geq \log(1^{\log n})$

$\frac{1/2 \cancel{\log n}}{\cancel{\log n}} \geq \frac{\cancel{\log n} \cdot \log(1)}{\cancel{\log n}}$

$1/2 \geq 0$ true

$f(n) = \Omega(g(n))$

$\boxed{f(n) = \Theta(g(n))}$

i) i) $n2^n$ 3^n

$f(n) = O(g(n))$?

$f(n) \le c \cdot g(n)$ $c=1$

$n2^n \le 3^n$

$\log(n2^n) \le \log(3^n)$

$\log 2^n + \log(n) \le n\log(3)$

$\frac{n + \log n}{n} \le \frac{\cancel{n}\log 3}{\cancel{n}}$

$\lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}} + \cancel{\frac{\log n}{n}} \le \log(3)\right)$ $\to 0$

$1 \le \log(3)$ true

$f(n) = O(g(n))$

$f(n) = \Omega(g(n))$

$f(n) \ge c \cdot g(n)$ $c = 1/2$

$n2^n \ge 1/2\, 3^n$

$n2^n \ge 3/2^n$

$\log(n2^n) \ge \log(3/2^n)$

$\lim_{n\to\infty}\left(\cancel{\frac{\log n}{n}} + \frac{n}{\cancel{n}} \ge \frac{\cancel{n}\log(3/2)}{\cancel{n}}\right)$ $\to 0$

$1 \ge \log(3/2)$

$1 \ge \log 3 - 1$

$2 \ge \log 3$ ← true

$f_n = \Omega(g(n))$

$\boxed{f(n) = \Theta(g(n))}$

j) j) $(\log n)^{\log n}$ $2^{(\log_2 n)^2}$

$f(n) = O(g(n))$?

$f(n) \le c \cdot g(n)$ $c=1$

$(\log n)^{\log n} \le 2^{(\log_2 n)^2}$

$\log(\log n^{\log n}) \le \log(2^{(\log_2 n)^2})$

$\frac{\cancel{\log n}\,\log(\log n)}{\cancel{\log n}} \le \frac{(\log n)^{\cancel{2}} \cdot 1}{\cancel{\log n}}$

$\log(\log n) \le \log n$ ← true

$\boxed{f(n) = O(g(n))}$

$f(n) = \Omega(g(n))$?

$f(n) \ge c \cdot g(n)$ $c=1$

follow same pattern

$\log(\log n) \ge \log n$ ← false

$f(n) \ne \Omega(g(n))$