Homework 1

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EC 330

1. Sums [10 pt]

Provide a closed-form solution to the following problems. Make sure you show the steps.

a) $\sum_{i=0}^{\infty}(\frac{2}{11})^i$

$$\sum_{i=0}^{\infty}\left(\frac{2}{11}\right)^i = \left(\frac{2}{11}\right)^0 + \left(\frac{2}{11}\right)^1 + \left(\frac{2}{11}\right)^2 + \ldots + \left(\frac{2}{11}\right)^i$$

this is a geometric series with the form $\frac{1}{1-x}$

$$\sum_{i=0}^{\infty}\left(\frac{2}{11}\right)^i = \frac{1}{1-2/11} = \frac{1}{9/11} = \boxed{11/9}$$

b) $\sum_{i=1}^{N}(i^3 + 2i^2 - 4i + 6)$

using linearity principle:

$$\sum_{i=1}^{n}(i^3+2i^2-4i+6) = \sum_{i=1}^{n} i^3 + 2\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + 6$$

$$= \sum_{i=0}^{n} i^3 - 1 + 2\sum_{i=0}^{n} i^2 - 1 - 4\sum_{i=1}^{n} i + 6$$

$$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4$$

$$= \frac{n^2(n^2+2n+1)}{4} + \frac{n(2n^2+3n+1)}{6} - \frac{n^2+n}{2} + 4$$

$$= 6n^4 + 12n^3 + 6n^2 + 8n^3 + 12n^2 + 4n - 12n^2 - 12n + 96$$

$$= 6n^4 + 20n^3 + 6n^2 - 12n + 96$$

$$\boxed{\sum_{i=1}^{n}(i^3+2i^2-4i+6) = 3n^4 + 10n^3 + 3n^2 - 6n + 48}$$

however at large values of n, n^4 dominates the rest so $O(n^4)$

using formulas for arithmetic series and sums of squares and cubes:

$$\sum_{K=1}^{n} K = \tfrac{1}{2}(n(n+1)) = \Theta(n^2)$$

$$\sum_{K=0}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{K=0}^{n} K^3 = \frac{n^2(n+1)^2}{4}$$

2. Exponents and Logs [10 pt]

Simplify the following expressions. Make sure you show the steps.

a) $log_x x^{55x}$

b) $log_{55}(55^{55} \cdot 55)$

a) $log_x x^{55x}$

$\log_x x^{55x} \rightarrow$ using the log rule: $\log_b(b^k) = k$

$$\boxed{\log_x x^{55x} = 55x}$$

b) $log_{55}(55^{55} \cdot 55)$

$\rightarrow$ using log rules: $\log_b(M \cdot N) = \log_b M + \log_b N$

$\log_b(b^x) = x$ $\quad \log_b(b) = 1$

b) $\log_{55}(55^{55} \cdot 55)$

$\longrightarrow$ using log rules: $\log_b(M \cdot N) = \log_b M + \log_b N$

$\log_b(b^x) = x \qquad \log_b(b) = 1$

$$\log_{55}(55^{55} \cdot 55) = \log_{55} 55^{55} + \log_{55} 55$$
$$= 55 + 1$$

$$\boxed{\log_{55}(55^{55} \cdot 55) = 56}$$

3. Combinatorics [5 pt]

How many integral solutions of $x_1 + x_2 + x_3 = 13$ satisfy $x_1 \geq 5$, $x_2 \geq 0$ and $x_3 \geq -3$?

given the values of the other variables: x_1 must be $5 \leq x_1 \leq 16$

x_2 must be $0 \leq x_2 \leq 11$

x_2 must be $-3 \leq x_3 \leq 8$

$$(x-5) + x_2 + (x_3+3) = 13$$
$$x_1 + x_2 + x_3 = 15 \qquad k=3$$
$$n = 15$$

$$\frac{(n+k-1)!}{k!\,(n-1)!} \qquad \frac{(15+3-1)!}{3!\,(15-1)!} \qquad \frac{17!}{3!\,14!} \qquad \frac{17 \cdot \cancel{16}^{8} \cdot \cancel{15}^{5} \cdot \cancel{14!}}{\cancel{3 \cdot 2} \cdot 1 \cdot \cancel{14!}} = \boxed{680}$$

4. Induction [10 pt]

Consider Fibonacci numbers $F_0, F_1, F_2, \ldots$ defined by the following rule

$$F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}.$$

In this problem, we will confirm that this sequence grows exponentially fast and obtain bounds on this growth.

a) Use induction to prove that $F_n \geq 2^{0.5n}$ for $n \geq 2$.

b) Find a constant $c < 1$ such that $F_n \leq 2^{cn}$ for all $n \geq 0$.

Proof by Induction

- Essential ingredients:
  - Inductive hypothesis H(n)
  - Show the base case H(1) is true.
  - Show that H(k) true implies H(k+1) is true
  - By induction, H(n) is true.

a) $F(n) = F_{n-1} + F_{n-2}$

Initial base case hypothesis: $\underline{H(2)}$: $F(1) + F(0) \geq 2^{\frac{1}{2} \cdot 2}$

$$1 + 1 \geq 2$$
$$2 \geq 2 \quad \text{true}$$

assuming $H(k)$ is true

$n = k \qquad F(k) \geq 2^{k/2}$

$n = k+1 \qquad F(k+1) \geq 2^{\frac{k+1}{2}}$

$$F(k) + F(k-1) \geq 2^{k+\frac{1}{2}}$$

$$2^{k/2} + 2^{k-\frac{1}{2}} \geq 2^{k+\frac{1}{2}} \quad \text{true}$$

So by induction $F(n) \geq 2^{n/2}$

b) Find a constant $c < 1$ such that $F_n \leq 2^{cn}$ for all $n \geq 0$.

if $F(n) = 2^{\frac{1}{2}n}$

$$F_n \leq 2^{cn}$$
$$2^{\frac{1}{2}n} \leq 2^{cn}$$
$$\log_2\left(2^{\frac{1}{2}n}\right) \leq \log_2\left(2^{cn}\right)$$
$$\frac{\frac{1}{2}n}{n} \leq \frac{cn}{n}$$
$$\frac{1}{2} \leq c$$

$$\boxed{\text{so } F(n) \leq 2^{cn} \text{ for } c \geq 0.5}$$

5. Program Understanding [5 pt]

What is the value of sum after the double-loop exits in the following program? Express your answer as a function of n. Show your steps.

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```
int sum = 0;
for (int i = n; i > 0; i--) {
    for (int j = n - i; j < n; j++) {
        sum = sum + 2;
    }
}
```

this function adds 2 $n(n-1)$ times because for each value of i from $0 \to n$ it adds 2 j times where $j = n-1$

$$\boxed{\text{so the value of sum would be } 2 \cdot \left(n \cdot \left(\tfrac{n+1}{2}\right)\right)}$$