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ME 570 Robot Motion Planning
9/26/2021

Homework #2 Report

Using 24 Hour Late Pass

Question 1.1:

- **$R_1(\theta)$** in this case would represent the counterclockwise rotation about the X-axis because the first column of the rotation matrix is the x unit vector.
- **$R_2(\theta)$** in this case would represent the counterclockwise rotation about the Y-axis because the center column of the rotation matrix is the y unit vector.
- **$R_3(\theta)$** in this case would represent the counterclockwise rotation about the Z-axis because the last column of the rotation matrix is the z unit vector.
- **$R_4(\theta)$** in this case would represent the clockwise rotation about the Z-axis because the last column of the rotation matrix is the unit vector and the signs of the values have been flipped (when compared to the counterclockwise rotation matrices).

Question 2.1:

1. The coordinates of the point given in B_1, denoted as ${}^{B1}p$ can be represented by the full equation below, which will give the coordinates of point p in reference to the world frame (${}^{W}p$).

$$ {}^{W}p = {}^{W}R_{B1}\,{}^{B1}p + {}^{W}T_{B1} $$

The given information is that ${}^{W}T_{B1} = 0$ and the rotation matrix ${}^{W}R_{B1}$ can be represented as:

$$ {}^{W}R_{B1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} $$

Therefore, the equation for ${}^{W}p$ can be written as:

$$ {}^{W}p = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} {}^{B1}p $$

2. The coordinates of the point given in B_2, denoted as ${}^{B2}p$ can be represented by the full equation below, which will give the coordinates of point p in reference to the world frame (${}^{W}p$).

$$ {}^{W}p = {}^{W}R_{B2}\,{}^{B2}p + {}^{W}T_{B2} $$

In this case we need to find ${}^{W}R_{B2}$ and ${}^{W}T_{B2}$ from the given information.

$$ {}^{W}R_{B2} = {}^{W}R_{B1}\,{}^{B1}R_{B2} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} $$

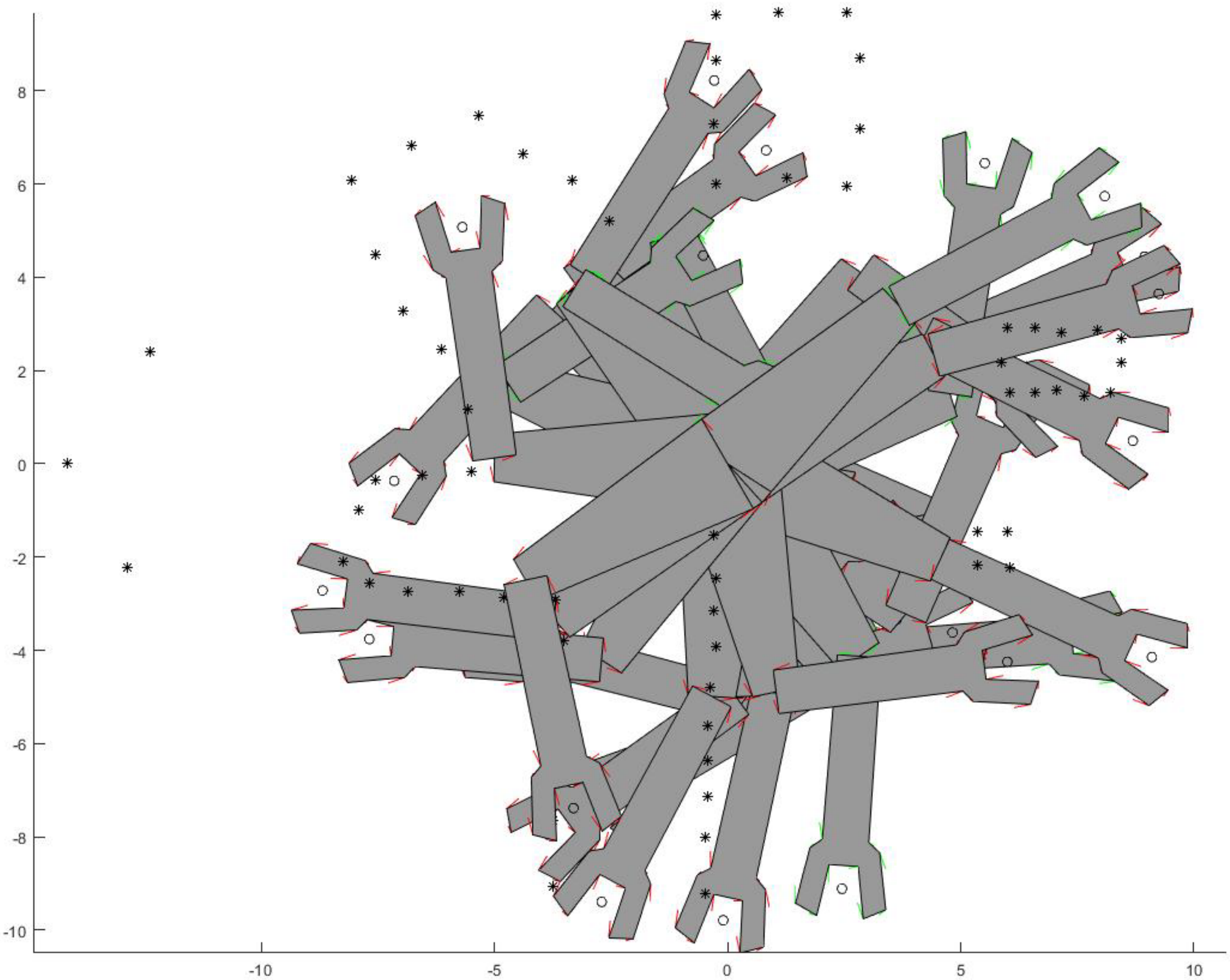
$$ {}^{W}T_{B2} = {}^{W}R_{B1}\,{}^{B1}T_{B2} + {}^{W}T_{B1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} + 0 $$

Combining all the terms we would get the expression:

$$^{w}p = \begin{bmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & -sin\theta_2 \\ sin\theta_2 & cos\theta_2 \end{bmatrix} + \begin{bmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Question 2.2:

Below is the figure generated from **twolink_plotCollision_test():**



Obstacles represented as points are shown as black asterisks. Each configuration of the two-link manipulator is represented by a green outline if the configuration is not in collision with an object and a red outline if it is.

Question 4.1: We can show that $\boldsymbol{R(\theta)}$ is a rotation (i.e., $\boldsymbol{R(\theta) \in SO(2)}$) and that $\boldsymbol{\phi_{circle}(\theta) \in S^1}$ for all $\boldsymbol{\theta \in R}$ by the definition of a unit circle, which is defined in cartesian coordinates by $\boldsymbol{x = cos\theta}$ and $\boldsymbol{y = sin\theta}$. If we expand the equation for the circle given in the prompt we get:

$$\phi_{circle}(\theta) = R(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$$

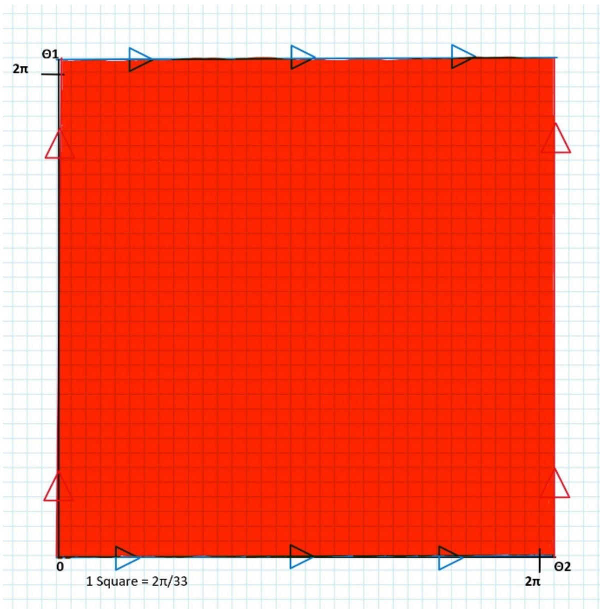
Since **cosθ** and **sinθ** are both continuous and therefore continuously differentiable, there exists some number in **R** for all possible values of theta.

Question 5.1: The atlas for the torus can be represented with 2 charts U_1 and U_2 where:

$$\mathbf{U_1 = [\ (-0.2\ ,\ 2\pi+2)\ x\ (-0.2,2(-0.2\ ,\ 2\pi+2)\ ]}$$

$$\mathbf{U_2 = [\ (0\ ,\ 2\pi+2)\ x\ (0\ ,\ 2\pi+2)\ ]}$$

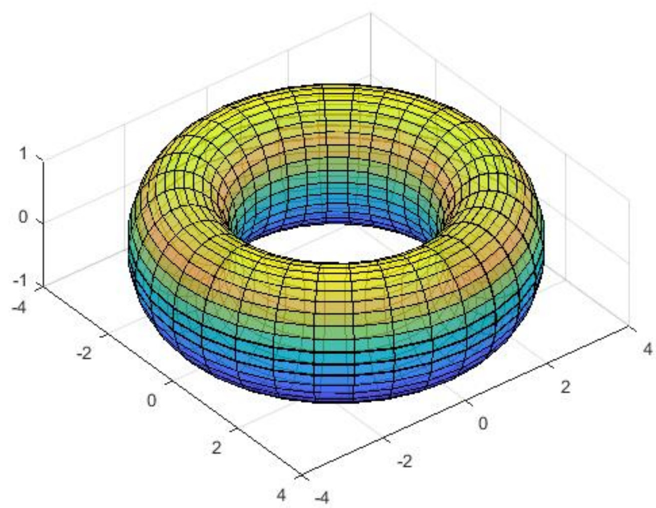
This would ensure that the charts overlap and that there wouldn't be a discontinuity when theta equals either **2π** or **0**, ensuring a smooth continuous mapping.

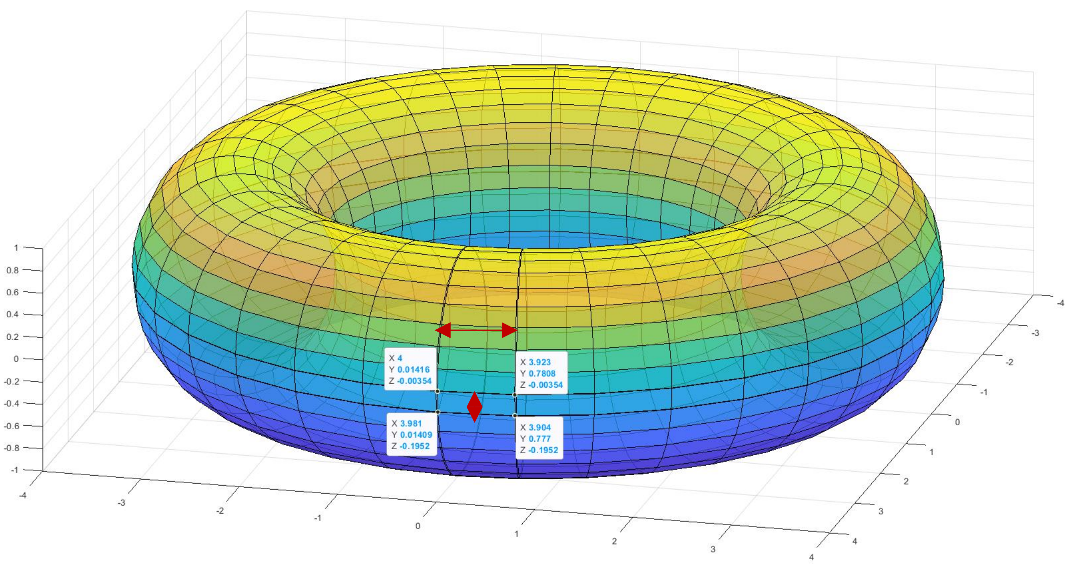


In this figure, the top two edges and the side two edges of the torus are "glued together" in that if you were to continue off the right edge you would "come back" on the left edge. Similarly, if you went "off" the top you would come back "on" the bottom.

Question 5.2:

I had some difficulty with figuring out how to have MATLAB draw the charts and the torus on the same figure, but the second figure below illustrates the points where the two charts would overlap on the torus at the 4 intersections of the overlapping "edges" of the charts.

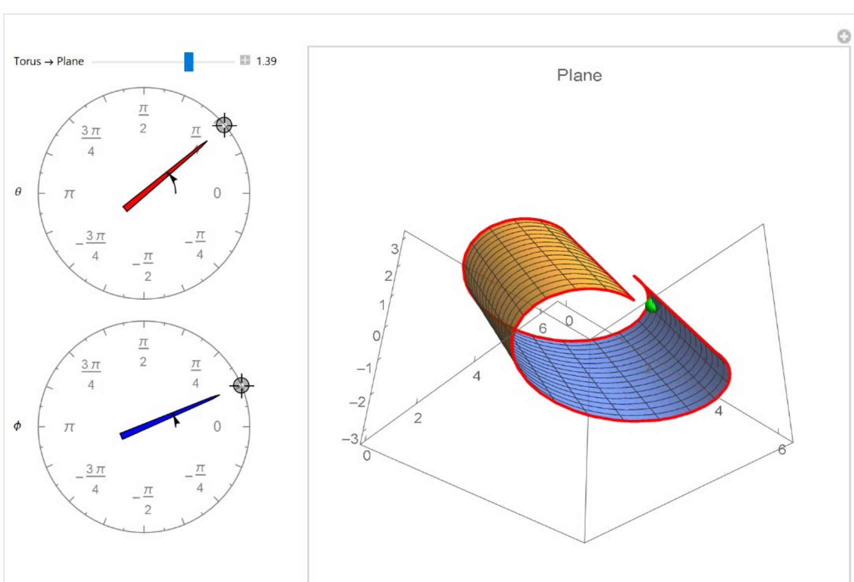




Each box represents the coordinates of a point where the "edges" of the chars overlap to ensure that the representation of the manifold is an atlas.

Another visualization I found was from Wolfram Alpha Demonstrations[1] which has an effective way to visualize the two charts in 3D space as it was transformed between its planar form and the torus form. The dials on the left allowed to change the position of the reference point, and the slider bar moved between torus and plane.

Chart for a Torus



[1] https://demonstrations.wolfram.com/ChartForATorus/

Question 5.3:

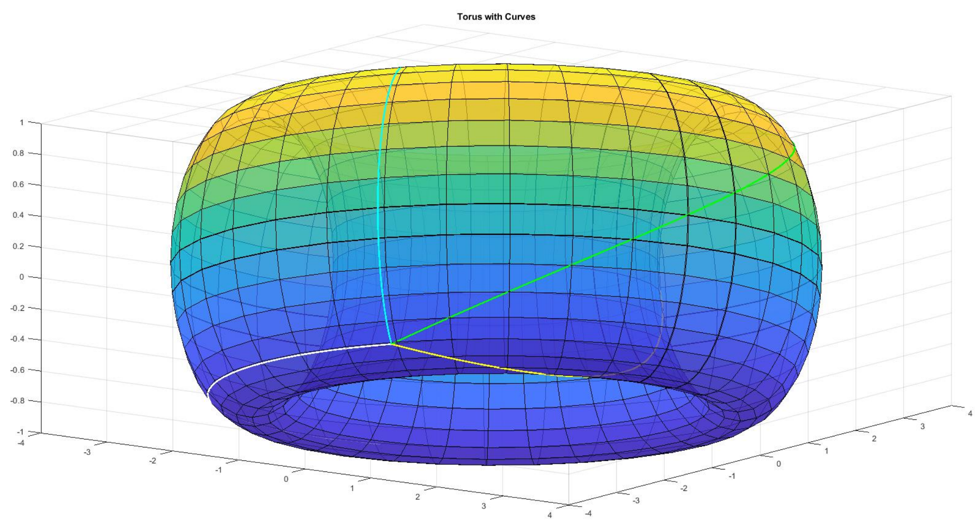
Each chart of the torus represents a one-to-one mapping, whose domains cover the entire configuration space and they are both C^{∞}-related. All of these factors mean that together they form an atlas that covers the entirety of the possible configurations.

Question 5.4:

The tangent line to any point on a straight line would be the line itself. So, the derivative which would give the slope of the tangent line should give the same slope.

$$\frac{d}{dt}\theta(t) = \dot{\theta} = \begin{bmatrix} \frac{d}{dt}(\mathbf{A1t + B1}) \\ \frac{d}{dt}(\mathbf{A2t + B2}) \end{bmatrix} = \begin{bmatrix} A1 \\ A2 \end{bmatrix}$$

Plot of Torus with four curves:



Question 6.1: Let the velocity of the end effector point be:

$$\dot{X} = \frac{d}{dt}({}^{W}\boldsymbol{p}_{eff})$$

And the angular velocities of the joints be:

$$\dot{\theta} = \begin{matrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{matrix}$$

Since the length of each link was given as **5,** we can write the equation for the Jacobian below:

$$J(q) = \begin{bmatrix} -5sin\theta_1 - 5sin(\theta_1 + \theta_2) & -5sin(\theta_1 + \theta_2) \\ 5cos\theta_1 - 5cos(\theta_1 + \theta_2) & 5cos(\theta_1 + \theta_2) \end{bmatrix}$$

And the velocity of the end effector can be calculated below by multiplying angular velocity by the Jacobian:

$$\dot{X} = J(q)\,\dot{\theta} = \begin{bmatrix} -5sin\theta_1 - 5sin(\theta_1 + \theta_2) & -5sin(\theta_1 + \theta_2) \\ 5cos\theta_1 + 5cos(\theta_1 + \theta_2) & 5cos(\theta_1 + \theta_2) \end{bmatrix} \bullet \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Question 6.2:

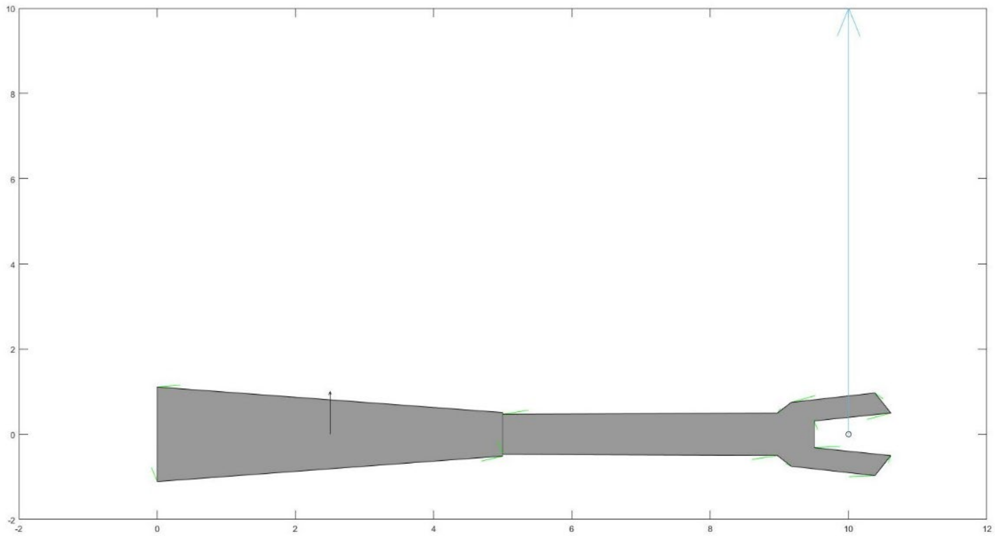


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{1}\\\mathbf{0}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\mathbf{0}\\\mathbf{0}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{0}\\\mathbf{10}\end{matrix}$

At first this didn't seem like it was correct, but then I realized that the angular velocity of the first joint was parameterized by 1, which means that although the resulting position was the same as the starting position, this indicated that it had completed a **2π** rotation with the arm fully extended (length 10) which would give the effector a lot of velocity (magnitude 10) in the upward direction.

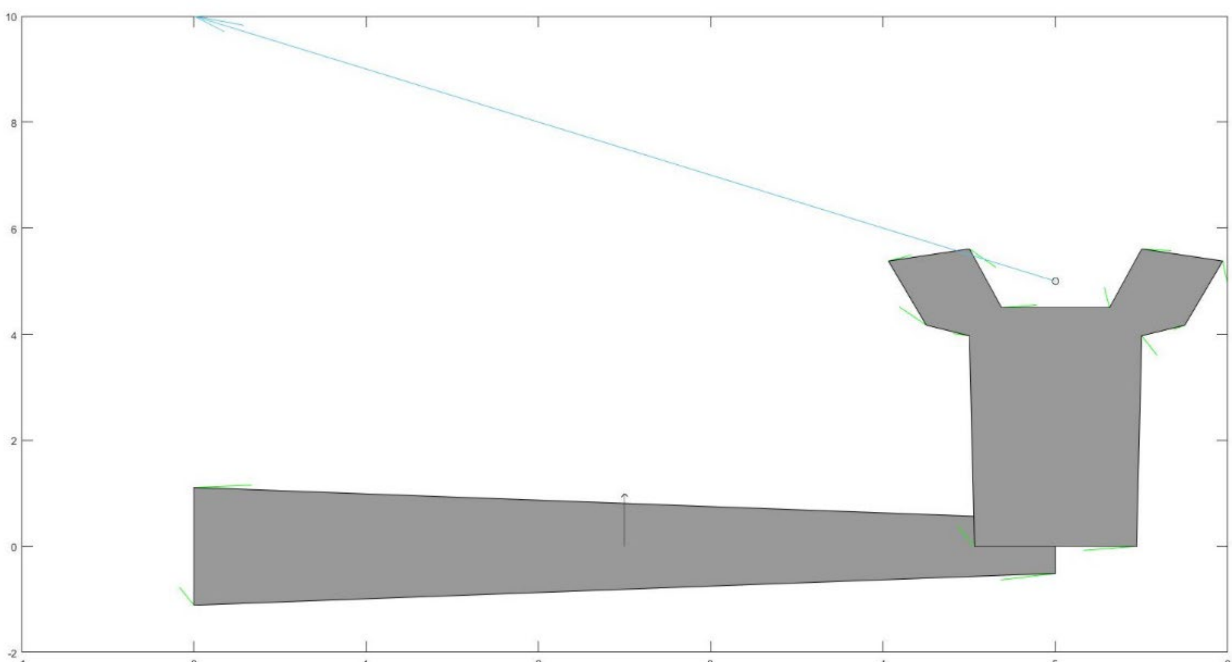


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{1}\\\mathbf{0}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\mathbf{0}\\\boldsymbol{\pi/2}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{-5}\\\mathbf{5}\end{matrix}$

Like the previous figure there has been a rotation of the first joint by **2π** the difference is that the second joint had already been rotated an additional **π/2** which caused the velocity of the end effector to have a horizontal component of -5 creating the vector upward to the left.

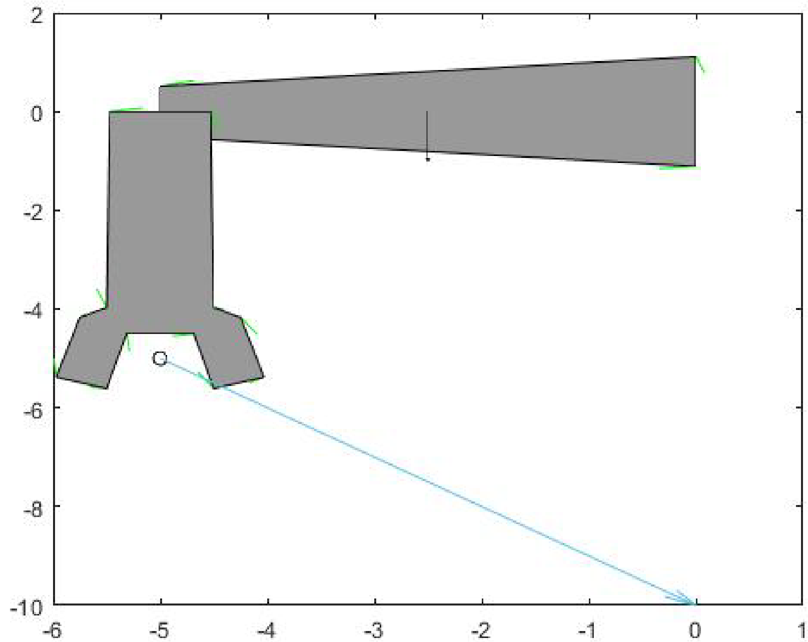


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{1}\\ \mathbf{0}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\boldsymbol{\pi}\\ \boldsymbol{\pi/2}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{5}\\ \mathbf{-5}\end{matrix}$

This configuration is similar to the previous one except the first joint rotated $\boldsymbol{\pi}$ instead.

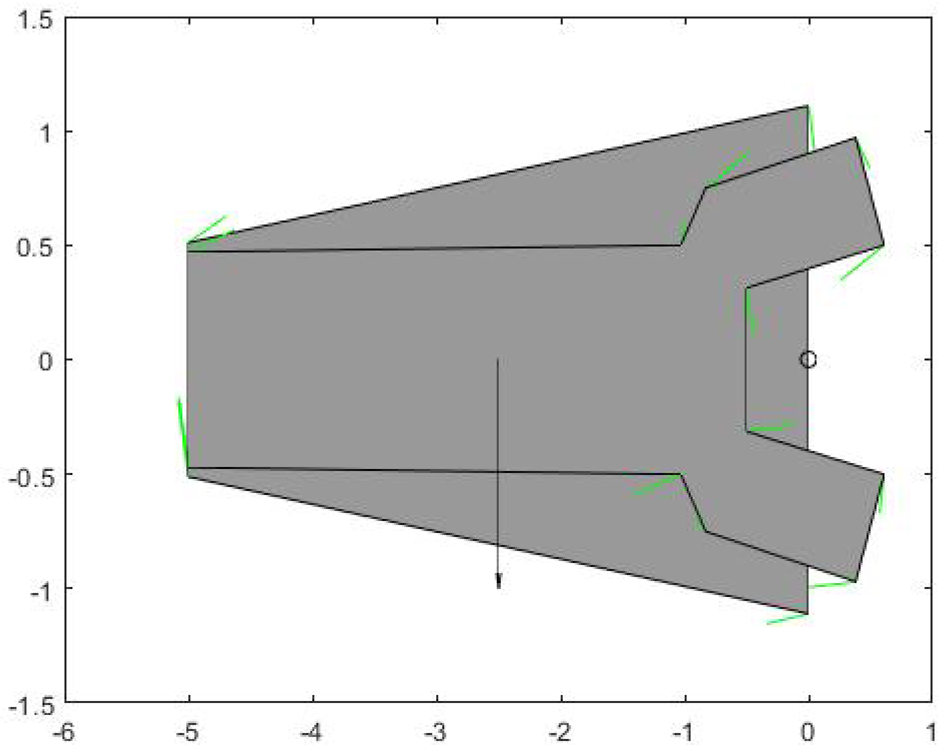


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{1}\\ \mathbf{0}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\boldsymbol{\pi}\\ \boldsymbol{\pi}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix}$

This configuration doesn't have a net velocity for the end effector because the end effector has not moved. This is because while the first joint has rotated, the configuration of the second joint already placed the end effector in it's final position by being folded in on itself already.

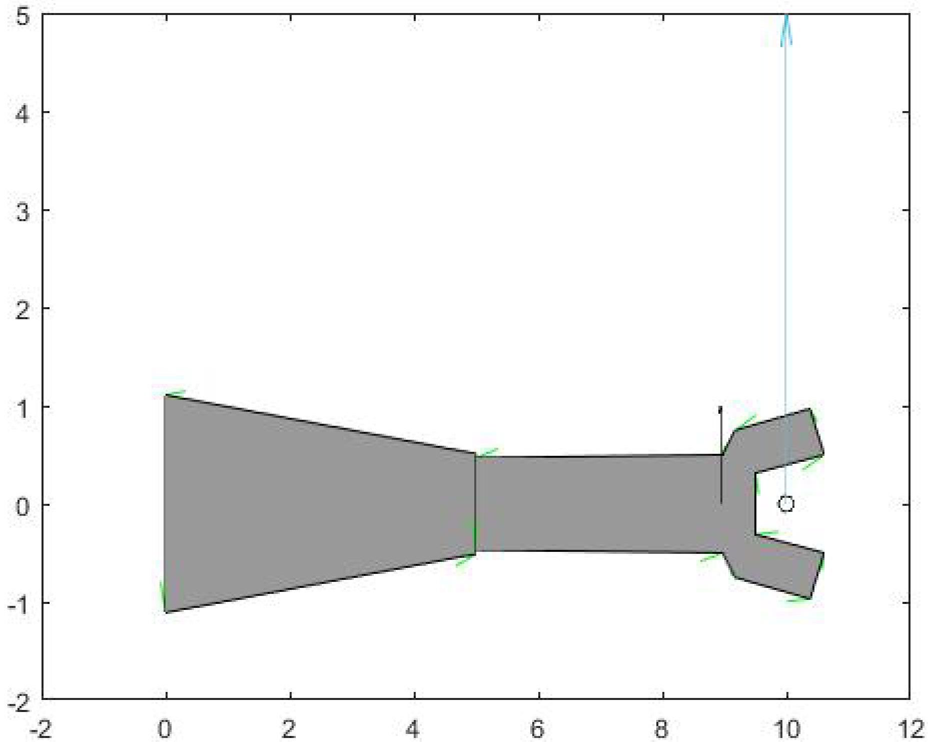


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{0}\\ \mathbf{1}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{0}\\ \mathbf{5}\end{matrix}$

In this configuration, the first joint remained stationary but the second joint completed a complete revolution. The difference in magnitude from the first figure's configuration is because the length being rotated was only one link long, instead of both. Thus, the magnitude is 5 instead of 10.

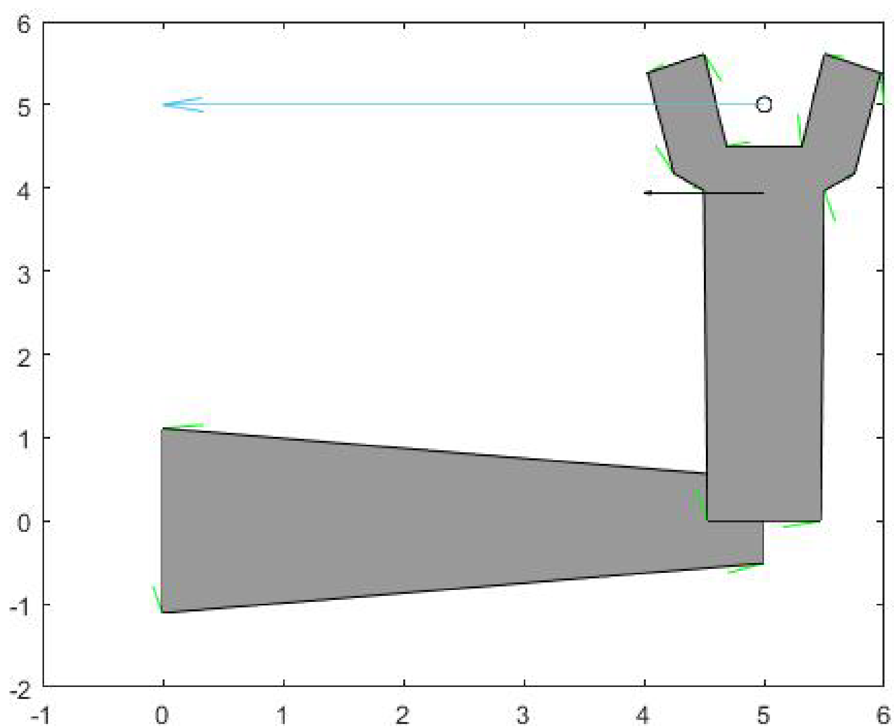


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix}\mathbf{0}\\ \mathbf{1}\end{matrix}$, $\boldsymbol{\theta} = \begin{matrix}\mathbf{0}\\ \boldsymbol{\pi/2}\end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix}\mathbf{-5}\\ \mathbf{0}\end{matrix}$

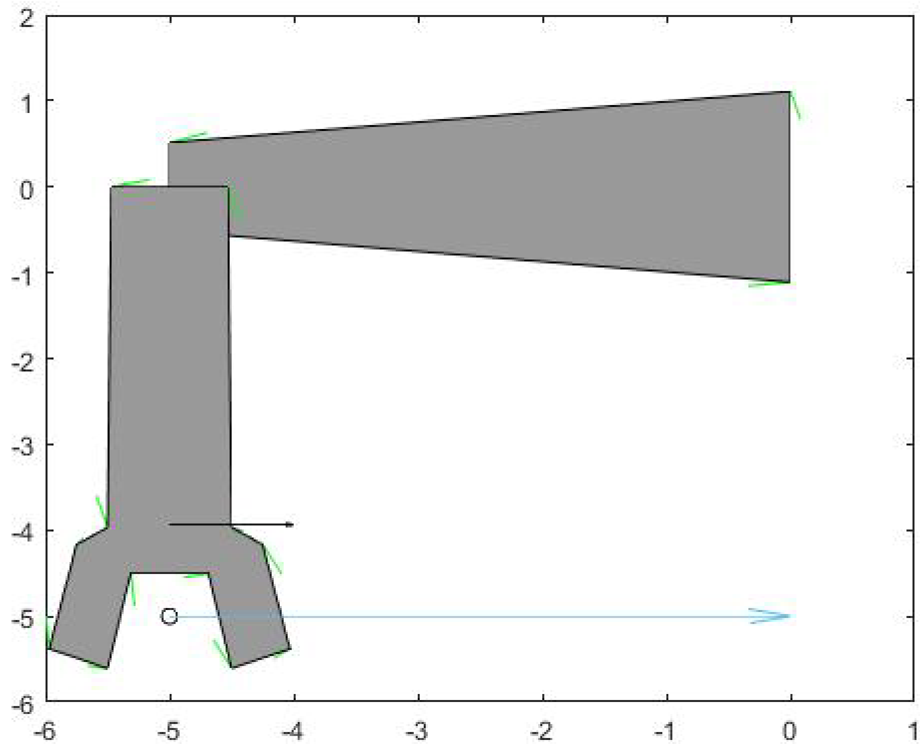


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix} \mathbf{0} \\ \mathbf{1} \end{matrix}$, $\boldsymbol{\theta} = \begin{matrix} \boldsymbol{\pi} \\ \boldsymbol{\pi/2} \end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix} \mathbf{5} \\ \mathbf{0} \end{matrix}$

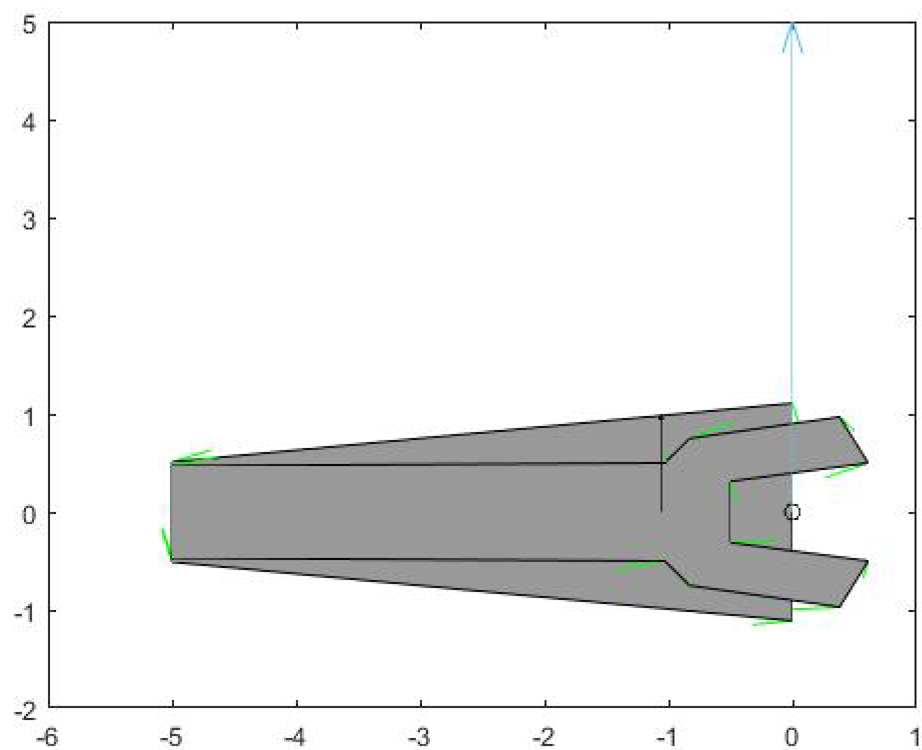


Figure for $\dot{\boldsymbol{\theta}} = \begin{matrix} \mathbf{0} \\ \mathbf{1} \end{matrix}$, $\boldsymbol{\theta} = \begin{matrix} \boldsymbol{\pi} \\ \boldsymbol{\pi} \end{matrix}$ and $\dot{\boldsymbol{X}} = \begin{matrix} \mathbf{0} \\ \mathbf{5} \end{matrix}$

This configuration's effector velocity is different from the other configuration that looks superficially similar because in this case, the second link was the link that completed the motion, meaning that the effector had actually moved (unlike in the first scenario). The first link was already in the $\boldsymbol{\pi}$ position, and the second link rotated $\boldsymbol{\pi}$ which would give the effector a velocity vector pointing upwards.

Question 6.3:

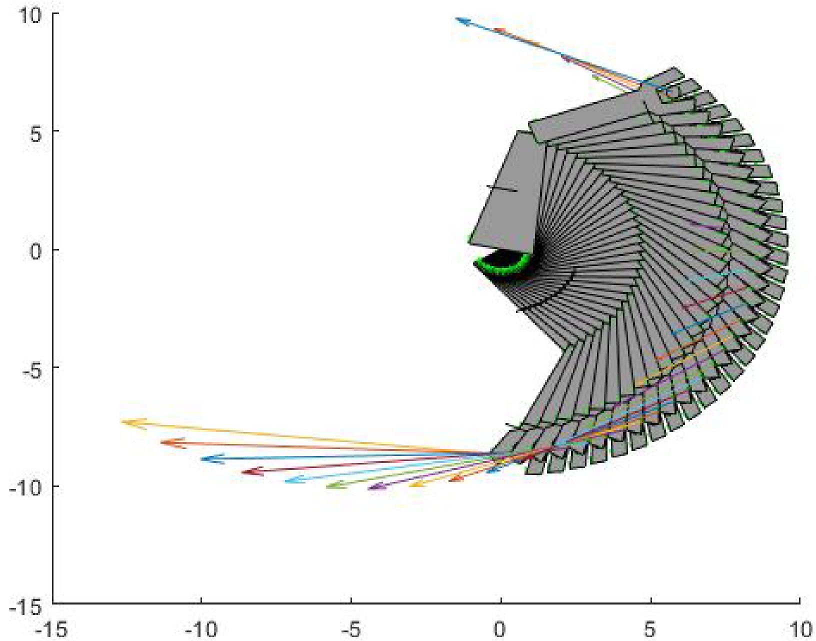


Figure for configuration a1

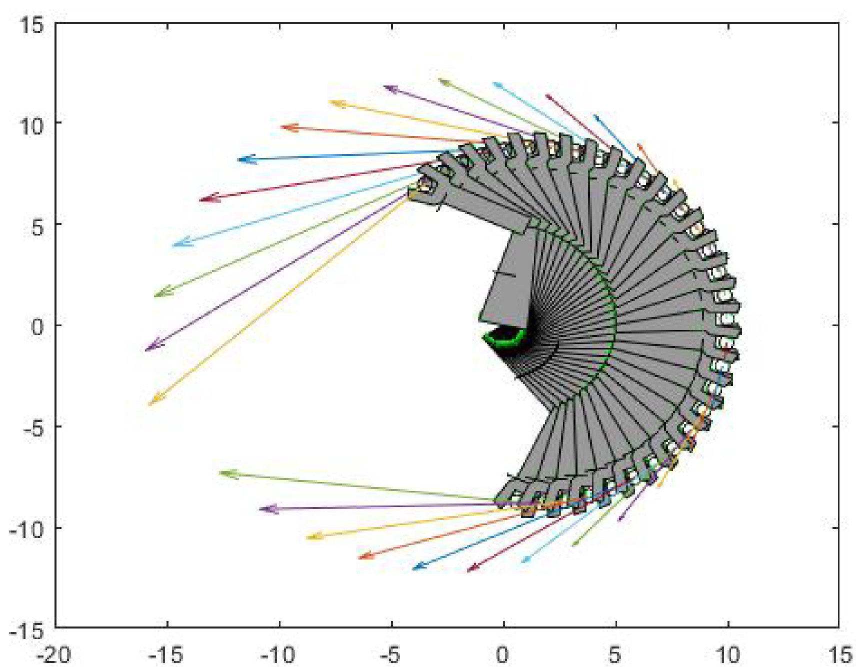


Figure for configuration a2

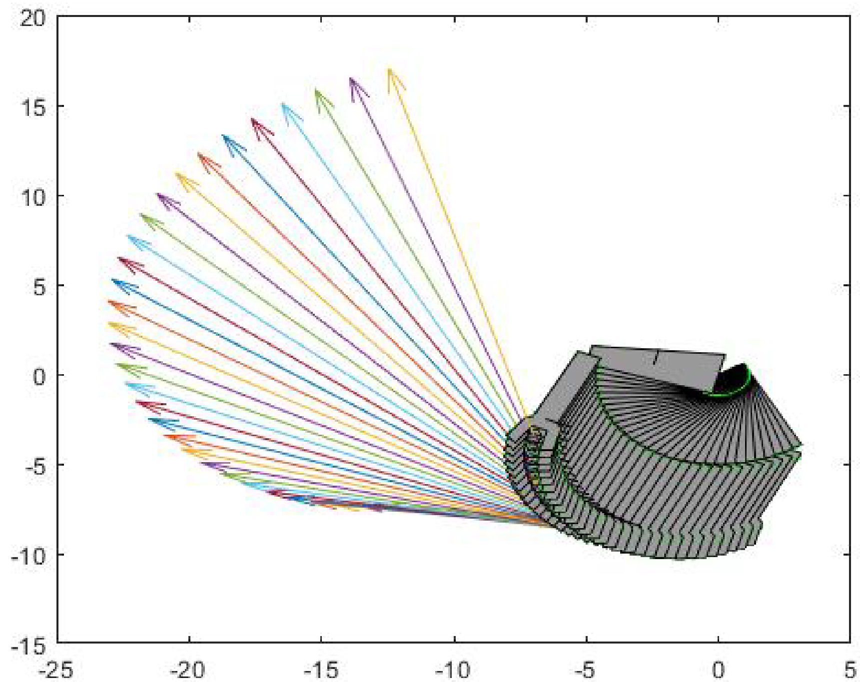


Figure for configuration a3

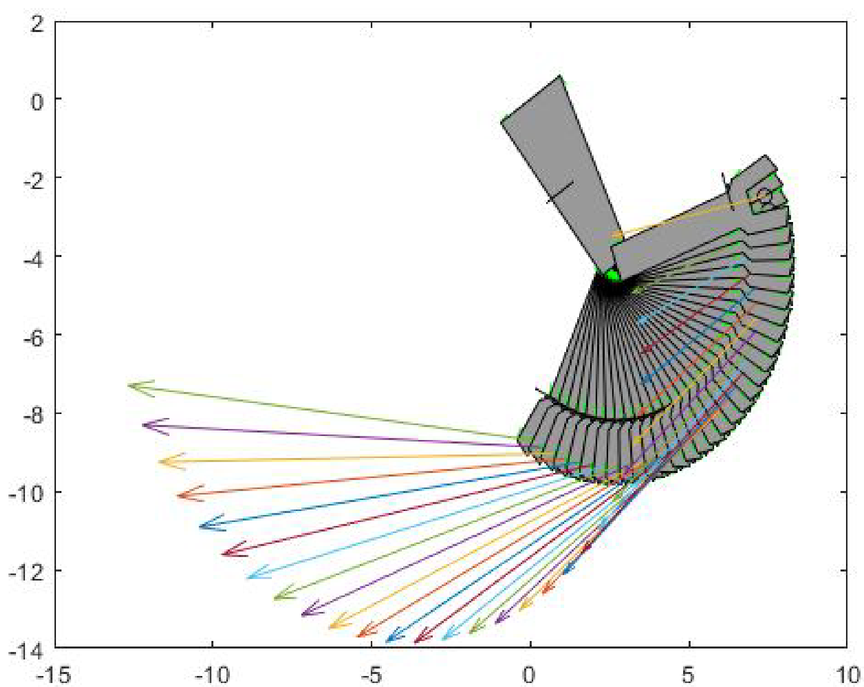
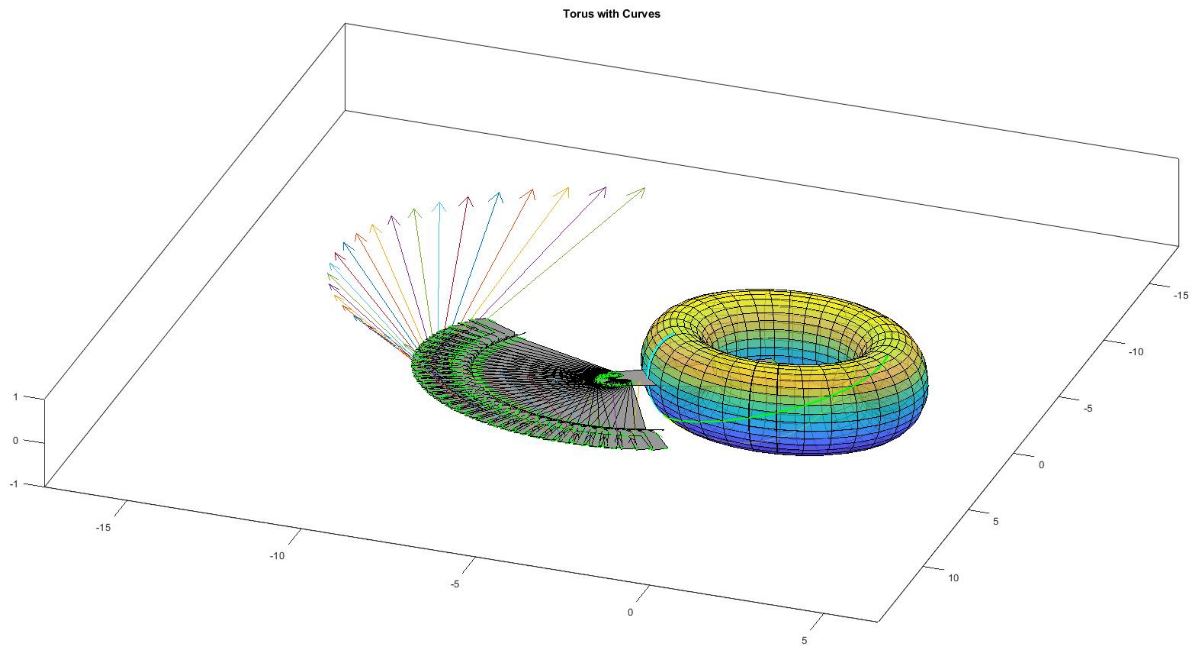


Figure for configuration a4

Question 6.4:

The tangents pushed through the torus represent the different angles of each of the joints of the configuration space that are represented by the above figures. The above figures expand on that by providing a representation of the velocity of the end effector as the robot moves through space.

Question 6.5:



This relates to the material that we covered in class regarding the tangent space. These tangents represent the tangent space to the torus manifold at any arbitrary point on the surface of the manifold. The tangents in the tangent space in this case would represent the velocity of the end effector caused by the different configurations and the angular rotations of the two joints of the two link manipulator.