Contact singularities in multiple-timescale dynamical systems

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1 Abstract

In this paper, we classify contact singularities of the critical manifold for singularly-perturbed vector fields of the form

$$z' = H(z, \varepsilon). \tag{1}$$

Our main result is the derivation of computable, coordinate-free defining equations for contact folds and contact cusps. The computability of the defining equations depends on an algebraic factorization of the leading-order part of (1) which can in turn be computed explicitly in a wide variety of applied problems. We also compute contact singularities in several nonstandard examples.

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2 Introduction

Classifying the loss of normal hyperbolicity of the critical manifold is a fundamental step in the analysis of multiple-timescale dynamical systems. For systems in the so-called *standard* form

$$\begin{aligned} x' &= f(x, y, \varepsilon) \\ y' &= \varepsilon g(x, y, \varepsilon), \end{aligned} \tag{2}$$

the k-dimensional critical manifold lies inside the zero set of a smooth mapping $f(x, y, 0) : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Classical singularity theory of smooth mappings has been used to provide normal forms for fold and cusp singularities of f, where the fast variables $x \in \mathbb{R}^{n-k}$ play the role of unfolding parameters.

The purpose of this work is to provide a more general classification of loss of normal hyperbolicity of the critical manifold for the larger class (1) of multiple-timescale systems. The relationship between Eqs. (2) and (1) is that coordinate transformations placing (1) in the form (2) are now typically defined only locally; in other words, there is no globally defined coordinate splitting into 'slow' versus 'fast' directions. The main complication is that the fast fiber bundle is no longer unidirectional, requiring a sufficiently general notion of contact between two smooth manifolds in $\mathbb{R}^n$ such that the unfolding directions still locally lie along the remaining slow directions.

3 Multiple-timescale dynamical systems

Consider a smooth n-dimensional family of vector fields (1), formally expanded as a series in ε:

$$z' = h_0(z) + \varepsilon G(z, \varepsilon), \tag{3}$$

where $\varepsilon \ll 1$.

Assumption 1. For $\varepsilon = 0$, the set of equilibria of (3) is a single k-dimensional manifold, with $1 \leq k < n$, equal to the zero level set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$. We call S the *critical manifold*.

Any system (3) having a manifold of singularities S is called a *singular perturbation problem*. Note that the Jacobian Dh_0 evaluated along points $z \in S$ has at least k zero eigenvalues corresponding to the tangent space $T_z S$ by construction. These eigenvalues are called *trivial*, and the remaining $n - k$ eigenvalues along S are called *nontrivial*.
Definition. Given a family of vector fields (3), the system

$$z' = H(z, 0) = h_0(z) \tag{4}$$

is called the *layer problem* of (3).

Definition. The set $S_n \subset S$ denotes the subset where all nontrivial eigenvalues of Dh evaluated along S_n are nonzero.

Assumption. We assume that there is a nonempty set $F \subset S - S_n$, denoted the *contact set*, consisting only of *simple* eigenvalue crossings. Equivalently, $z \in F$ if and only if rank $Dh_0(z) = n - k - 1$.

Remark. We may also consider subsets where crossings of complex-conjugate pairs of eigenvalues cross the imaginary axis. This set is related to the existence of delayed Hopf and singular Hopf bifurcations. We analyze this set in a later paper.

Returning to the set S_n, we may construct a pointwise-defined splitting of the tangent bundle along $z \in S_n$:

$$T_z\mathbb{R}^n \;\; = \;\; T_zS \oplus N_z, \tag{5}$$

where N_z is called the *linear fast fiber* at basepoint $z \in S_n$ identified with the quotient space $T_z\mathbb{R}^n/T_zS$. We may define the *tangent bundle of* S via the natural construction

$$TS \;\; = \;\; \cup_{z \in S_n} T_zS.$$

The corresponding bundle

$$N \;\; = \;\; \cup_{z \in S_n} N_z$$

is called the *(linear) fast fiber bundle.*

Remark. Along S, we may define a natural projection operator

$$\Pi^S : TS \oplus N \to TS.$$

Given a point $z \in S_n$, the map $\Pi^S|_{z \in S_n}$ can be characterized geometrically as an oblique projection onto T_zS along parallel translates of the fast fiber N_z.

Remark. The flow near S defines a locally invariant fast foliation of the layer problem in a tubular neighborhood B of S. We denote this foliation by $\mathcal{F}$. For $b \in B$, each fiber $\mathcal{F}_b = \mathcal{F}_z$ is tangent to the linear fast fiber N_z at the basepoint $z \in S$.
Recall from Assumption 1 that we have fixed a submersion f so that the critical manifold S is defined as its zero level set. We now assume a factorization of h_0 relative to this submersion:

Assumption. The function $h_0(z)$ can be factorized as follows:

$$h_0(z) \;\; = \;\; N(z)f(z),$$

where the ith column of the $n \times (n-k)$ matrix function $N(z)$, $N_i = (N_i^1 \;\cdots\; N_i^n)^T$ consists of smooth functions $N_i^k : \mathbb{R}^n \to \mathbb{R}$. Assume that $N(z)$ has full column rank $n - k$ for each $z \in S$, and furthermore that singularities of $N(z)f(z)$ for $z \notin S$ are isolated, if they exist.

Remark. The question of existence and uniqueness of such factorizations for an arbitrary system (3) satisfying the Assumptions, and over which regions of phase space, is a topic of active research. Local factorizations can be constructed explicitly in the case the $h_0(z)$ is a rational vector field in z. This includes a large variety of applied problems– notably, many chemical reaction networks can be modeled in this framework. In practice, these local factorizations can also be shown *a posteriori* to hold over large open sets of the phase space by ad hoc methods.

Two immediate results demonstrating the usefulness of this factorization is the following:

Lemma. For $z \in S_n$, the column vectors of $N(z)$ form a basis for the range of $Dh(z)$ and the row vectors of $Df(z)$ form a basis of the orthogonal complement of the kernel of $Dh(z)$ (i.e. a basis of the orthogonal complement of the tangent space $T_z S$).

Lemma. The nontrivial eigenvalues of the layer problem of (3) along S are encoded in the $(n-k)\times(n-k)$ matrix $DfN|_S$.

4 Structure of the contact set F.

From the definition we have the characterization

$$F \quad = \quad \{z \in S : \mathrm{rank}(DfN) = n-k-1\}.$$

Since DfN is a matrix of size $(n-k)\times(n-k)$, a necessary condition is (obviously)

$$\det(DfN)|_S \quad = \quad 0.$$

Geometrically, S is the set of points where a one-dimensional subspace of the fast fiber bundle locally aligns with the tangent space of S. In this setting it is straightforward to deduce the direction of tangency. Consider the following identity for the adjugate:

$$\mathrm{adj}(DfN)(DfN) \quad = \quad \det(DfN)I_{n-k}.$$

It turns out that $\mathrm{rank}(\mathrm{adj}(DfN)) = 1$ in the rank-one drop case. The adjugate therefore naturally encodes the one-dimensional left and right nullspaces of DfN: select a nonzero column vector r for the right nullspace and a nonzero row vector l for the left nullspace. The nonzero vector Nr therefore aligns with the direction of contact since $Df(Nr) = (DfN)r = 0$.

At points in $z \in F$, there exists a family of one-dimensional center manifolds C of the layer problem all tangent to the contact direction Nr at the basepoint z– indeed, by the center manifold theorem the k-jets of all curves in C passing through z are uniquely determined.

Our primary goal is to decompose the contact set F according to the singularity type. This requires a careful definition of contact between regular curves and smooth manifolds in $\mathbb{R}^n$.

5 Contact between submanifolds of $\mathbb{R}^n$

We begin by recalling the the basic objects of study in singularity theory, namely, smooth germs living in jet spaces.

Definition. The k-jet space $J^k(n,m)$ of smooth germs $f:\mathbb{R}^n\to\mathbb{R}^m$ is defined by

$$J^k(n,m) \;=\; \mathcal{M}_n\cdot\mathcal{E}(n,m)/\mathcal{M}_n^{k+1}\cdot\mathcal{E}(n,m),$$

where

$$\mathcal{E}(n,m) \;=\; (\mathcal{E}_n)^m$$

is the direct product of m copies of the set $\mathcal{E}_n$ of smooth germs from $\mathbb{R}^n$ to $\mathbb{R}$,

$$\mathcal{M}_n \;=\; \mathcal{E}_n\cdot\{x_1,\cdots,x_n\}$$

is the unique maximal ideal of germs vanishing at the origin, and

$$\mathcal{M}_n^k \;=\; \mathcal{E}_n\cdot\{x_1^{i_1},\cdots,x_n^{i_n}, i_1+\cdots+i_n=k\}$$

is the set of germs with vanishing partial derivates of order less than or equal to $k-1$ at the origin.

Remark: The set $J^k(n,m)$ may be identified with the set of polynomials of total degree less than or equal to k having no constant term.

We now define a notion of equivalence of map-germs due to Mather. This equivalence will be used to classify map classes according to their singularities.

Definition. The *contact group* $\mathcal{K}$ is the set of germs of diffeomorphisms of $(\mathbb{R}^n\times\mathbb{R}^m,(0,0))$ which can be written in the form

$$H(x,y) \;=\; (h(x),H_1(x,y)),$$

where h acts on the right (i.e. $h\cdot f=f\circ h^{-1}$) and $H_1(x,0)=0$ for x near 0. We say that f is $\mathcal{K}$-*equivalent* to g, denoted $f\sim_{\mathcal{K}} g$, if g lies in the group orbit of f.

Remark. Suppose $f,g\in\mathcal{M}_n\cdot\mathcal{E}(n,m)$ and $k=(h,H)\in\mathcal{K}$. Then $g=k\cdot f$ if and only if

$$(x,g(x)) \;=\; H(h^{-1}(x),f(h^{-1}(x))).$$

Observe that H sends the graph of f to the graph of g near 0 (i.e. the zero sets of $\mathcal{K}$-equivalent germs are diffeomorphic).

A definition of contact between two curves in $\mathbb{R}^2$ is a useful starting point to motivate contact in more general contexts.

Definition. Let $\alpha,\beta:\mathbb{R}\to\mathbb{R}^2$ define two curves so that the curve β is equal to the zero set of a smooth function $F:\mathbb{R}^2\to\mathbb{R}$ and $\alpha(t_0)=\beta(t_0)$. Define $g(t)=(F\circ\alpha)(t)$. We say that α and β have contact of order c at $t=t_0$ if

$$\begin{aligned} g^{(i)}(t_0) &= 0 \text{ for } i = 0, \cdots, c \\ g^{(c+1)}(t_0) &\neq 0. \end{aligned}$$

The germ of the function g is called the *contact map* of α and β.

Remark. We may equivalently define contact between two curves by demanding that the first $c-1$ derivatives (but not the cth derivative) of $\alpha(t)$ and $\beta(t)$ coincide. The definition above in terms of a level set will be better suited to our purposes; we seek contact points on manifolds of equilibria defined via level sets.

In the case of contact between two submanifolds of $\mathbb{R}^n$, there is a natural generalization of the contact map. The following definition is due to Montaldi.

Definition. Suppose that a submanifold $M \subset \mathbb{R}^n$ is given locally as the image of some immersion-germ $g : (M, x) \to (\mathbb{R}^n, 0)$ and another submanifold $N \subset \mathbb{R}^n$ is given by the zero set of some submersion-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$. The *contact map* of M and N is the germ of the composite map $f \circ g$.

Two important details must be checked. First, 'the' contact map-germ of two submanifolds is well-defined (i.e. the choice of submersion/immersion-germs should not matter). We have

Lemma. For any pair of germs of the submanifold in $\mathbb{R}^n$, the $\mathcal{K}$-class of the contact map depends only on the submanifold-germs themselves and not on the choice of submersion and immersion germs (and therefore not on the contact map).

Having determined that the submanifold-germ pair determines the contact type completely, we also verify that the contact type of the submanifold-germ pair is determined by the equivalence class of the contact map between them.

Theorem. Suppose $g_1, g_2 : M_{1,2} \to \mathbb{R}^n$ are immersion-germs and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^k$ are submersion-germs (with $N_{1,2} = f_{1,2}^{-1}(0)$). Then $(M_1, N_1) \sim (M_2, N_2)$ iff $f_1 \circ g_1 \sim_{\mathcal{K}} f_2 \circ g_2$.

We can now consider the main setting of this paper: the contact of a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ with a submanifold given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ (with $1 \leq k < n$). The contact map is $g = f \circ \alpha : \mathbb{R} \to \mathbb{R}^{n-k}$. By Theorem, the contact type is well-defined by the $\mathcal{K}$-equivalence class of g.

Definition. A k-dimensional submanifold given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ and a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ have *contact of order c* if the corresponding contact map $g = f \circ \alpha$ admits an A_c-singularity.

Remark 1. In analogy to the previous definition of contact between two curves in $\mathbb{R}^2$, there is an equivalent formulation of this definition which is not as computable:

Alternate definition. A k-dimensional submanifold M given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ and a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ have *contact of order c at t_0* if there exists a curve $\beta : \mathbb{R} \to \mathbb{R}^n$ lying inside M such that

$$\begin{aligned} \alpha^{(i)}(t_0) &= \beta^{(i)}(t_0) \quad \text{for} \quad i = 0, \cdots, c \\ \alpha^{(c+1)}(t_0) &\neq \beta^{(c+1)}(t_0) \end{aligned}$$

The equivalence of these definitions follows from the characterization of the tangent space via equivalence classes of curves.

Remark 2. An analogous definition of contact for submanifolds of *equal* dimension can be used to define jet spaces and bundles. See Olver.

5.1 Contact folds and cusps

Let $\alpha(t) : \mathbb{R} \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ and let M be a k-dimensional submanifold in $\mathbb{R}^n$ given by the zero level set of a submersion: $M = f^{-1}(0)$. Let $\alpha(0) = x_0 \in M$ denote a contact point between α and M.

Definition. A *contact fold* is a contact of order-one between $\alpha(t)$ and M.

The contact map $f \circ \alpha : \mathbb{R} \to \mathbb{R}^{n-k}$ admits an A_1 singularity when

$$\begin{aligned} (f \circ \alpha)(0) &= 0 \\ (f \circ \alpha)'(0) &= 0 \\ (f \circ \alpha)''(0) &\neq 0. \end{aligned}$$

The first derivative condition may be simplified to

$$Df(x_0)\alpha'(0) = 0.$$

Note the geometric content of this condition: the tangent vector of α lies precisely inside the tangent space of M at the contact point.

The second-derivative condition may be simplified to

$$D^2 f(\alpha'(0), \alpha'(0)) + Df\alpha''(0) \neq 0$$

where the derivatives in f are all evaluated at x_0.

Definition. A *contact cusp* is a contact of order-two between $\alpha(t)$ and M.

The contact map admits an A_2 singularity when

$$\begin{aligned}(f\circ\alpha)(0) &= 0\\ (f\circ\alpha)'(0) &= 0\\ (f\circ\alpha)''(0) &= 0\\ (f\circ\alpha)'''(0) &\neq 0.\end{aligned}$$

The third-derivative condition may be simplified to

$$D^3f(\alpha'(0),\alpha'(0),\alpha'(0)) + 3D^2f(\alpha'(0),\alpha''(0)) + Df\alpha'''(0) \neq 0.$$

In the setting of multiple-timescale dynamical systems, the derivatives of $\alpha(t)$ at arbitrary order can be evaluated in terms of derivatives of $N(z)$ by observing that the k-jets (i.e. Taylor series expansions) are defined by the flow. For example, we have

$$\begin{aligned}\lambda(0) &= x_0\\ \lambda'(0) &= Nr\\ \lambda''(0) &= DN(Nr,r)\\ \lambda'''(0) &= D^2N(Nr,Nr,r)+DN(DN(Nr,r),r),\end{aligned}$$

etc. To ensure that the unfolding directions are restricted to those locally tangent to the critical manifold simply requires projecting by the left nullvector l.

5.2 Contact points of order one; nonstandard folds

The defining equation and genericity condition for a fold at $z\in F$ are:

$$\begin{aligned}\mathrm{rank}(Df|_z) &= n-k\\ (l(D^2f(Nr,Nr)+DfDN(Nr,r)))_z &\neq 0,\end{aligned}$$

where l, r are nonzero left and right nullvectors of $\mathrm{adj}(DfN)$ (note: l is a row and r is a column vector).

Remark. Compare these defining equations to the computation of the A_1 singularity of the contact map in the previous section.

The first equation says that the critical manifold is still locally regular at the point of tangency. This is satisfied immediately when f is a submersion. We will see that this condition places us in the appropriate setting to read off the normal form of a folded saddle-node immediately. We will also show that the second condition is sufficient to provide a locally parabolic contact between the fibres and the critical manifold at $z\in F$.

We first give a proof using coordinate changes. The first step is to write down the coordinate transformation $x = M(u, y)$ with inverse $u = L(x, y)$ (fixing y) transforming the nonstandard form of the layer problem locally into standard form (i.e. straightens the fibers nearby). We have

$$\begin{bmatrix} u' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ N^y(M(u,y),y) \end{bmatrix} f(M(u,y),y) = \begin{bmatrix} 0 \\ \tilde{f}(M(u,y),y) \end{bmatrix}.$$

Then we apply another coordinate transformation so that we can identify the dynamics along the tangency. Specifically, the local fast variable $y \in \mathbb{R}^{n-k}$ is further decomposed into the parts

$$y = \tilde{r}v + \tilde{P}w = \begin{bmatrix} \tilde{r} & \tilde{P} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix},$$

where $(v, w) \in \mathbb{R} \times \mathbb{R}^{n-k-1}$ is a new set of coordinates so that the 'horizontal' coordinate v measures the component of y along the tangency defined by the right nullvector $\tilde{r}$. The inverse transformation is given by

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \tilde{l} \\ \tilde{Q} \end{bmatrix} y.$$

The result of these two coordinate transformations gets us to

$$\begin{aligned} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \tilde{h}_0(u,v,w) &= \begin{bmatrix} 0 \\ \tilde{l} \\ \tilde{Q} \end{bmatrix} \tilde{f}(u, \tilde{r}v + \tilde{P}w) \\ &= \begin{bmatrix} 0 \\ l \\ Q \end{bmatrix} f(M(u, \tilde{r}v + \tilde{P}w), \tilde{r}v + \tilde{P}w), \end{aligned}$$

where $(u, v, w) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$.

Remarks:
- The vectors l and r denote the left (resp. right) nullvectors of $\mathrm{adj}(DfN)$, which have the geometric meaning of defining the one-dimensional subspace along which the fast fiber bundle aligns with the critical manifold.

- The vectors $\tilde{l} = l(N^y)^{-1}$ and $\tilde{r} = N^y r$ are reoriented versions of l and r after the coordinate transformation (i.e. these are the corresponding nullvectors of $D_y\tilde{f}|_S$). Similarly, we have $\tilde{P} = N^y P$ and $\tilde{Q} = Q(N^y)^{-1}$.

We first obtain the defining equation and transversality condition for generic contact of order one. Let us first write down the Jacobian $\tilde{J} = D\tilde{h}_0$ along S. We have

$$\tilde{J}|_S = \begin{bmatrix} O_{k,k} & O_{k,1} & O_{k,n-k-1} \\ lD_xf(D_xL)^{-1} & lDfNr & lDfNP \\ QD_xf(D_xL)^{-1} & QDfNr & QDfNP \end{bmatrix}.$$

On F, the (2,2), (3,2), and $(2,3)$ (block) entries are further annihilated because l and r are precisely the nullvectors of DfN on the set F of contact points of S:

$$\tilde{J}|_F = \begin{bmatrix} O_{k,k} & O_{k,1} & O_{k,n-k-1} \\ lD_xf(D_xL)^{-1} & O_{1,1} & O_{1,n-k-1} \\ QD_xf(D_xL)^{-1} & O_{n-k-1,1} & QDfNP \end{bmatrix}.$$

Near F we expand the right-hand side of the (one-dimensional) v' equation. We have

$$\begin{aligned} v' &= \tilde{l}\tilde{f}(u, \tilde{r}v + \tilde{P}w) \\ &= lf(M(u, \tilde{r}v + \tilde{P}w), \tilde{r}v + \tilde{P}w) \\ &= lD_xf(D_xL)^{-1}u + l(D^2f(Nr, Nr) + DfDN(Nr, r))v^2 + \cdots \end{aligned}$$

(ignoring the remaining cross-terms of order two and the higher-order terms).
Ok, so here we need to remark on a few facts again:

- The coefficient of the vector-valued component u is $lD_xf(D_xL)^{-1}$, which *must be non-trivial* by the assumption that the critical manifold is still regular near contact points (i.e. $\mathrm{rank}(Df|_F) = n-k$). To convince oneself of this, stare at $\tilde{J}|_F$ and observe that $QDfNP$ has full rank $n-k-1$ since DfN encodes the nontrivial eigenvalues along S and Q, P have maximal rank.

Thus $lD_xf(D_xL)^{-1}u$ plays the role of a unfolding parameter (say α), but with the parameter axis lying along the nullvector of $lD_xf(D_xL)^{-1}$.

- The second-order derivatives must be pointwise-defined and they satisfy the basic tensorial multilinearity properties. For $i = 1, \cdots, n-k$,

$$\begin{aligned} [D^2f]_i(Nr, Nr) &= \sum_{l,m=1}^{n} \sum_{j,s=1}^{n-k} \frac{\partial^2 f_i}{\partial z_l \partial z_m}(N_{mj}r_j)(N_{ls}r_s) \\ [DfDN]_i(Nr, r) &= \sum_{l,m=1}^{n} \sum_{j,s=1}^{n-k} \frac{\partial f_i}{\partial z_l}\frac{\partial N_{lj}}{\partial z_m}(N_{ms}r_s)r_j \end{aligned}$$

5.3 Contact points of order two; nonstandard cusps

We review the singularity theory of the generic cusp. We have the two-parameter unfolding

$$\dot{x} \;=\; f(x,\alpha), \tag{6}$$

$x \in \mathbb{R}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with an equilibrium point at $p = (x,\alpha) = (0,0)$. Assume that

$$\begin{array}{lr} \text{(i)} & f_x(p) = f_{xx}(p) = 0 \\ \text{(ii)} & f_{xxx}(p) \neq 0 \\ \text{(iii)} & (f_{\alpha_1} f_{x\alpha_2} - f_{\alpha_2} f_{x\alpha_1})(p) \neq 0. \end{array} \tag{7}$$

Then we can find smooth invertible coordinate changes in the extended phase space so that the system 6 is transformed into

$$\dot{\eta} \;=\; \beta_1 + \beta_2\eta + \eta^3 + O(\eta^4).$$

Some remarks:

- from a generic point of view the high-order error is irrelevant topologically (i.e. the stable germ is the union of $V_{\pm}(\beta_1, \beta_2, \eta) = \beta_1 + \beta_2\beta \pm \eta^3$).
- In n-dimensions just repeat this on the center manifold. Then you pick up some hyperbolic flow transverse to the center manifold precisely corresponding to the existence of eigenvalues with positive/negative real part off the center manifold.
- The first inequality is called the 'nondegeneracy' condition and the second inequality is called the 'transversality condition'. These two can be expressed more compactly by specifying instead that the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F : (x,\alpha) \mapsto (f, f_x, f_{xx})(x,\alpha)$$

be regular at the cusp point.

Defining equations for a nonstandard cusp.

In our context, the unfolding parameter β_1 is defined by specifying that the matrix coefficient of the leading-order term $lD_xf(D_xL)^{-1}u$ is nonzero. However, the coefficient of v is still zero on the contact set, so we need to go to the quadratic cross-terms to recover a suitable unfolding parameter. Let us clarify the issue by introducing an arbitrary smooth scalar-valued function $k : \mathbb{R}^n \to \mathbb{R}$ with $z = (u, v, w)^T$ and $k(0,0,0) = 0$. The expansion becomes

$$\begin{aligned} k(u,v,w) &= k(0,0,0) + Dk \cdot z + Hk(z,z) + \cdots \\ &= 0 + D_u k u + D_v k v + D_w k w + 2(D_{uv} k u + D_{wv} k w) v + D_{vv} k v^2 + D_{vvv} k v^3 + \cdots . \end{aligned}$$

In our setting, we have $D_v k = 0$ and $D_u k \neq 0$ on the set F. Using this expansion, we can read off the defining equations for a cusp at a point $p \in F$:

(i) $D_{vv} k(p) = 0$.

(ii) $D_{vvv} k(p) \neq 0$.

(iii) The $2 \times (n-1)$ matrix

$$C(z) = \begin{bmatrix} D_u k & D_w k \\ D_{uv} k & D_{wv} k \end{bmatrix}$$

has full rank at the cusp point: rank $C(p) = 2$.

The third condition is a sufficient condition for a generic two-parameter unfolding of the cusp.

These three conditions should be compared to the defining equations of the standard generic cusp in (7). In particular, consider the comparison to (7)(iii) in the minimal case, which is easily identified as a regularity condition of a Jacobian minor:

$$\det \begin{bmatrix} f_{\alpha_1} & f_{\alpha_2} \\ f_{\alpha_1,x} & f_{\alpha_2,x} \end{bmatrix} (p) \neq 0.$$

The analogy is that the tangency direction v plays the role of the unfolding variable x, and two linearly independent combinations of the remaining slow variables u_1, u_2 play the role of the unfolding parameters α_1, α_2.

We will discuss the computation of this condition in more detail later. First, we focus on the nonvanishing of the third-order derivative.

The third-order contribution (requirement (ii)).

We now suppose that $k = lf|_F$. The requirement (ii) can be written out:

$$\begin{aligned} l D_{vv}(DfNr)|_F &\neq 0 \\ l \cdot (D^3 f(Nr,Nr,Nr) + 3D^2 f(DN(Nr,r),Nr) + \\ + Df D^2 N(Nr,Nr,r) + Df DN(DN(Nr,r),r)) &\neq 0. \end{aligned}$$

Of interest here is the observation that the critical manifold can be quite flat (second- and higher-order derivatives of f vanish) and the system can still admit a cusp due to bending of the fibers. This already appears in the condition for the fold. Of more interest is that the fibers needn't bend that much either (i.e. the penultimate term above can vanish while the final term is nonzero).

Transversality: evaluating condition (iii).

Exploiting the fact that $D_v k = D_{vv} k = 0$ at the cusp (by definition), we have

$$\begin{aligned} \operatorname{rank}(C_0(p)) &= \operatorname{rank}\begin{pmatrix} Dk \\ D_v Dk \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} l \cdot Df \\ l \cdot (D^2 f(Nr, I) + Df DN(r, I)) \end{pmatrix}. \end{aligned}$$

This condition is completely coordinate-free.

6 Examples

6.0.1 Cusp normal form in a standard slow-fast system.

Here the straightening transformations are trivial. Consider the normal form of the singularly perturbed cusp in $\mathbb{R}^3$ in the standard case (ref: Broer, Krupa, Kaper, J. Diff. Eq. 2013):

$$\begin{aligned} x' &= \varepsilon(1 + \mathcal{O}(x, y, z, \varepsilon) \\ y' &= \varepsilon\mathcal{O}(x, y, z, \varepsilon) \\ z' &= (z^3 + yz + x) + \mathcal{O}(\varepsilon, xz, z^4). \end{aligned}$$

Here the slow variables are x, y and the fast variable is z. In terms of the Nf-splitting we have the right-hand side of the layer problem given by

$$h_0 = Nf = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (x + yz + z^3).$$

We can check the conditions for the cusp. Note first that

$$DfN = y + 3z^2.$$

It is not hard to show that the parabola $F = \{(x, y, z) : y + 3z^2 = 0\} \cap S$ on the cusp surface $S = \{(x, y, z) : x + yz + z^3 = 0\}$ consists of contact points of at least order one. In particular we have that $l = r = 1$ for this problem and thus

$$
\begin{aligned}
l(D^2f(Nr,Nr)+DfDN(Nr,r) &= Hf(N,N)+DfDN(N,N)\\
&= \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 6z\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix}+0\\
&= 6z,
\end{aligned}
$$

whence every point on F except the point $p=(0,0,0)$ is a fold point. At the point p, cusp condition (i) is satisfied.

The third-derivative condition (ii) reduces to checking the trivial computation

$$
\begin{aligned}
lD^3f(N,N,N) &= \sum_{j,k,l=1}^{3}\frac{\partial^3 f}{\partial z_j\partial z_k\partial z_m}N_jN_kN_m\\
&= 6\neq 0.
\end{aligned}
$$

The transversality condition (iii) bears further discussion. We are in the scenario where the fast fibers are locally one-dimensional so the matrix C in condition (iii) has the minimal size 2×2; there are no contributions D_wk, $D_{wv}k$ as there are no remaining unfolding directions. We must therefore study $D_{uv}k$. Luckily, the coordination transformation L is trivial, and the direction v of contact points along the z-axis. To clarify, we have $u=(x,y)$ and $v=z$. Therefore the matrix C becomes

$$
\begin{aligned}
C &= \begin{bmatrix}D_{x,y}f\\ D_z(D_{x,y}f)\end{bmatrix}\\
&= \begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.
\end{aligned}
$$

The rank of C is therefore equal to 2 at the cusp point $(0,0,0)$.

6.0.2 Three-component feedback oscillator.

Let us demonstrate the computability of the third derivative condition using the following system. The layer problem is given by

$$
\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix} = \begin{bmatrix}\alpha_1\left(\frac{1}{1+z^2}-x\right)\\ \alpha_2 x-1\\ \alpha_3(y-z)\end{bmatrix} y.
$$

Here we have $N(x, y, z)$ the vector function on the RHS and $f(x, y, z) = y$. The (regular part of the) critical manifold is given by

$$S = \{y = 0\}.$$

The nontrivial eigenvalues are encoded in the (in this case *scalar*) function $DfN|_S = \alpha_2 x - 1$. The fold curve

$$\tilde{F} = \{(x, y, z) \in S : x = 1/\alpha_2\}$$

therefore divides S into an attracting and repelling branch.

Most of the points on $\tilde{F}$ are fold points. We observe this by checking the defining and nondegeneracy conditions:

$$\begin{aligned}\operatorname{rank}(Df)|_{\tilde{F}} &= n - k = 1 \\ DfDNN|_{\tilde{F}} &= \alpha_1 \frac{\alpha_2 - (1 + z^2)}{1 + z^2}.\end{aligned}$$

(observe how simple the nondegeneracy condition becomes in the codimension-one case: we have $\operatorname{adj}(DfN) = 1$).
Thus, the parabolic coefficient is nontrivial everywhere on the contact set except where $z = \sqrt{\alpha_2 - 1}$. We call this distinguished point $C = \{(x, y, z) \in \tilde{F} : z = \sqrt{\alpha_2 - 1}\} = \{1/\alpha_2, 0, \sqrt{\alpha_2 - 1}\}$.

At this stage we are now able to classify the contact order of the point C. The flatness of the manifold implies that $D^2 f = 0$ and $D^3 f = 0$. We need only worry about the evaluations $DfD^2N(Nr, Nr, r)$ and $DfDNDNN$. The second derivative in the first term admits the following simple formula in the codimension-one case:

$$DfD^2(N, N, 1) = Df \begin{bmatrix} N^T HN_1 N \\ N^T HN_2 N \\ \vdots \\ N^T HN_n N \end{bmatrix},$$

where HN_i denotes the Hessian of the function $N_i(z)$. This piece evaluates to 0 (this can be read off from the fact that $N_2(z)$ is only linear, whereas $D_x f$ and $D_z f$ are 0). On the other hand, the last term is nontrivial:

$$DfDNDNN = \frac{2\alpha_1\alpha_3(\alpha_2 - 1)}{\alpha_2}.$$

As long as $\alpha_2 > 1$, C has contact-order of 2 (note that the point C exists for $\alpha_2 \geq 1$). To show that the unfolding is versal (and thus that we are in the scenario of a generic cusp), we need a final transversality condition.

Now we compute $C_0(p)$ 'by hand' in the three-component negative feedback problem, thereby completing our task of determining the existence of a contact cusp in that model.

The slow- and fast- dimensions are locally given by $u \in \mathbb{R}^2$ and $v \in \mathbb{R}$, and thus C_0 is 2×2 with the form

$$C_0 \quad = \quad \begin{bmatrix} lD_x f \\ l[D(D_x f)(Nr, I) + DfD_xN(r, I)] \end{bmatrix}.$$

We recall that $l = r = 1$. Furthermore, $D(D_x f) = 0$ since the critical manifold S is flat: $S = \{y = 0\}$. We should clarify that x in the above formula refers to any local choice of coordinate $z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ so that $N^y(z)$ is regular. At the cusp point $p =$, it is sufficient to choose "x"$= (x, y)$, where by abuse of notation, the right-hand side now refers to the variables in the original formulation of the oscillator problem. Altogether, C_0 has the form

$$C_0 \quad = \quad \begin{bmatrix} D_{x,y} f \\ DfD_{x,y}N \end{bmatrix}.$$

We have

$$\begin{aligned} D_{x,y} f \quad &= \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \\ D_{x,y} N \quad &= \quad \begin{bmatrix} -\alpha_1 & 0 \\ \alpha_2 & 0 \\ 0 & \alpha_3 \end{bmatrix}. \end{aligned}$$

So

$$C_0(p) \quad = \quad \begin{bmatrix} 0 & 1 \\ \alpha_2 & 0 \end{bmatrix}.$$

The rank of the cross-term is therefore equal to 2.

6.0.3 Nonstandard cusps in a mitotic oscillator model.

We demonstrate the existence of a cusp in Goldbeter's minimal model for the embryonic cell cycle. A formulation as a GSPT problem is given by (Kosiuk 2012, K & Szmolyan 2016)

$$\begin{aligned}
\frac{dX}{dt} &= \left(M(1-X)(\varepsilon+X) - \frac{7}{10}(\varepsilon+1-X)\right)F_\varepsilon(M)\\
\frac{dM}{dt} &= \left(\frac{6C}{1+2C}(1-M)(\varepsilon+M) - \frac{3}{2}M(\varepsilon+1-M)\right)F_\varepsilon(X)\\
\frac{dC}{dt} &= \frac{1}{4}(1-X-C)F_\varepsilon(X,M),
\end{aligned}$$

where

$$\begin{aligned}
F_\varepsilon(X,M) &= F_\varepsilon(X)F_\varepsilon(M)\\
F_\varepsilon(X) &= (\varepsilon+1-X)(\varepsilon+X)\\
F_\varepsilon(M) &= (\varepsilon+1-M)(\varepsilon+M).
\end{aligned}$$

The layer problem is given by

$$\begin{aligned}
\frac{dX}{dt} &= \left(M - \frac{7}{10}\right)F_0(X,M,C)\\
\frac{dM}{dt} &= \left(\frac{6C}{1+2C} - \frac{3}{2}\right)F_0(X,M,C)\\
\frac{dC}{dt} &= \frac{1}{4}(1-X-C)F_0(X,M,C),
\end{aligned}$$

for

$$F_0(X,M,C) = XM(1-X)(1-M).$$

The critical manifold S is given by regular two-dimensional subsets of the zero set $\{F_0(X,M,C) = 0\}$; in particular $S = \{X=0\} \cup \{X=1\} \cup \{M=0\} \cup \{M=1\}$. The one-dimensional linear fast fibers are spanned by the vector

$$N(X,M,C) = \begin{bmatrix} M - \frac{7}{10} \\ \frac{6C}{1+2C} - \frac{3}{2} \\ \frac{1}{4}(1-X-C) \end{bmatrix}$$

at points $(X,M,C) \in S$. In the sequel we denote $f = F_0$ so that we can read off the defining equations classifying the singularities along the contact set.

Let us record the following derivatives:

$$\begin{aligned} Df &= \begin{bmatrix} M(1-M)(1-2X) & X(1-X)(1-2M) & 0 \end{bmatrix} \\ D^2f &= \begin{bmatrix} 2M(M-1) & (1-2X)(1-2M) & 0 \\ (1-2X)(1-2M) & 2X(X-1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ DN &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{6}{(2C+1)^2} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix} \end{aligned}$$

Let us for the time being restrict to the subset $S' = \{X = 0\}$ of the critical manifold. We have

$$DfN|_{S'} = M(1-M)(M-7/10).$$

We read off that S' loses normal hyperbolicity along the lines $M = 0$, $M = 7/10$, and $M = 1$, and that S' is attracting on the subset $S'_a = S \cap \{0 < M < 7/10\}$ and repelling on the subset $S'_r = S \cap \{7/10 < M < 1\}$. We will not consider the degenerate lines $M = 0$ and $M = 1$. These define the 'corners' of S and require a more careful blow-up analysis (see Ilona's paper). We focus on the fold line $F = S' \cap \{M = 7/10\}$. Note that the critical manifold remains locally two-dimensional along this line.

The matrix DfN drops rank along F, giving us the trivial left- and right- nullvectors $l = r = 1$ of DfN.

Points of contact order one. We already have the pieces to check the second-derivative condition:

$$\begin{aligned} l(D^2f(Nr,Nr) + DfDN(Nr,r)) &= N^TD^2fN + DfDNN \\ &= 0 + \frac{63}{200}\frac{2C-1}{2C+1} \end{aligned}$$

along F. Thus, a line of fold points separates S'_a from S'_r, but there is a distinguished point $K = F \cap \{C = 1/2\} = \{(X, M, C) = (0, 7/10, 1/2)\}$ which has higher contact order.
Points of contact order two. We test the third-derivative condition:

$$\begin{aligned} & l \cdot (D^3f(Nr,Nr,Nr) + 2D^2f(DN(Nr,r),Nr) + DfD^2N(Nr,Nr,r) + DfDN(DN(Nr,r),r)) \\ = \;& D^3f(N,N,N) + 2D^2f(DNN,N) + DfD^2N(N,N,1) + DfDNDNN \\ = \;& 0 + 0 + 0 + \frac{63}{1000} \end{aligned}$$

when evaluated at K. It remains to test the rank of the following 2×2 matrix at K:

$$C_0 \quad = \quad \begin{bmatrix} D_x f \\ [D(D_x f)(N,I) + Df D_x N(1,I)] \end{bmatrix}.$$

Here "x" $= (x, y)$ is an adequate choice for the partial Jacobian evaluation. We have

$$\begin{aligned} D_{(x,y)}f &= \begin{bmatrix} M(1-M)(1-2X) & X(1-X)(1-2M) \end{bmatrix} \\ D(D_{(x,y)}f) &= \begin{bmatrix} 2M(M-1) & (1-2X)(1-2M) & 0 \\ (1-2M)(1-2X) & 2X(X-1) & 0 \end{bmatrix} \\ D_{(x,y)}N &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1/4 & 0 \end{bmatrix}. \end{aligned}$$

The first row is

$$D_{(x,y)}f \quad = \quad \begin{bmatrix} (7/10)(3/10) & 0 \end{bmatrix}$$

at K. The two terms in the second row are evaluated in turn. First we have

$$\begin{aligned} D(D_{(x,y)}f)(N,I) &= \begin{bmatrix} \alpha(2C-1) + \beta(10M-7) & \delta X(2C-1) & \gamma(10M-7) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned}$$

at K, where $\alpha, \beta, \delta, \gamma$ are some unspecified functions of X, M, C. The second term is

$$\begin{aligned} DfD_{(x,y)}N(1,I) &= \begin{bmatrix} 0 & M(1-M)(1-2X) \end{bmatrix} \\ &= \begin{bmatrix} 0 & (7/10)(3/10)(1) \end{bmatrix} \end{aligned}$$

at K.
Altogether, we have

$$C_0(K) \quad = \quad \begin{bmatrix} 21/100 & 0 \\ 0 & 21/100 \end{bmatrix},$$

and thus the cross-term matrix has maximal rank of 2, completing the identification of the cusp.