**Center Manifold Theory in the PDE Setting**

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## 1 Introduction: Motivation through ODEs

The continuous evolution of several interacting components (variables) is often described by a *dynamical system*, typically within the setting of $n$-dimensional Euclidean space $\mathbb{R}^n$. In particular, if we are able to completely specify the state of a system at time $t$ by a continuously differentiable $n$-dimensional function $x(t) \equiv (x_1(t), \cdots, x_n(t)) \in \mathbb{R} \to \mathbb{R}^n$, then we may describe the time-evolution via a **dynamical system** as follows:

$$\dot{x}(t) \quad = \quad f(x(t), t; \alpha), \tag{1}$$

where $\dot{}$ denotes differentiation with respect to time, $f : \mathbb{R}^{n+1+k} \to \mathbb{R}^n$ is a vector-valued function with some smoothness conditions and $k$ is the dimension of the real vector $\alpha \equiv (\alpha_1, \cdots, \alpha_k)$, whose components are taken to be constant (with respect to time). The vector $\alpha$ is called the *set of parameters* of the system, and it is typically the case that the topology of a trajectory $\{x(t)\}_{t\in\mathbb{R}}$ with initial condition $x(0) = x_0$ is determined by special values of and/or relationships among the parameter values $\alpha_j$.

If $f$ does not depend explicitly on $t$, the system is called *autonomous*– otherwise, the system is *non-autonomous*. Consider an autonomous system with fixed parameters $\alpha$, which we denote $f_\alpha$. Then $f_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defines a vector field on $\mathbb{R}^n$. If in addition $f_\alpha$ is smooth enough, we may assert the *existence and uniqueness* of solutions of (1); in brief, given an initial condition $x_0 \in \mathbb{R}^n$ at time $t = 0$, the set of points $\{x(t) = x_0 + \int_0^t f(x(t))dt\}$ defines a unique trajectory in $\mathbb{R}^n$ for all $t$ in a small enough neighborhood of 0.

Consider a system where most of the 'interesting' dynamics happens in a big enough neighborhood of the origin. Our system can be quite complicated even we restrict to small neighborhoods (picture). But we can sometimes find an invariant subspace *tangent to the center eigenspace* on which the local dynamical behavior is relatively simple.

## 2 Review on Semigroups

### 2.1 Basic Definitions

Let $X$ be a complex-valued Banach space and let $A : \text{Dom}(A) \to X$ be a linear operator. Recall the *resolvent set* $\rho(A)$ of $A$ is the set of all complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda I - A) : \text{Dom}(A) \to X$ is bijective and such that $(\lambda I - A)^{-1}$ is bounded in operator norm (i.e. in $B(X, X)$). Recall also that the *spectrum* $\sigma(A)$ of $A$ is the complement of $\rho(A)$ in $\mathbb{C}$.

We may also define the *point-spectrum* $P\sigma(A)$ of a linear operator $A$ as the set of all $\lambda \in \mathbb{C}$ such that $Ay = \lambda y$ has a nonzero solution $y \in \text{Dom}(A)$. The number $\lambda$ is called an eigenvalue of $A$ and the nonzero solution $y$ associated to $\lambda$ is called an (associated) eigenvector.

We immediately obtain the following theorem giving the spectrum some structure in the case of compact operators, and more interestingly, relating the operator spectrum to the point spectrum:

Let $X$ be a Banach space and $T \in B(X, X)$ compact. Then $\sigma(T)$ is a countable set with no accumuluation points apart from possibly zero. Furthermore, each nonzero $\sigma(T)^\dagger \equiv \sigma(T)\backslash\{0\} \subseteq P\sigma(T)$, and each member of $\sigma(T)^\dagger$ has a finite-dimensional space of associated eigenvectors.

Recall that a *strongly continuous semigroup* $\{T(t)\}_{t\geq 0}$ of continuous (i.e. bounded) operators on $X$ is a family of continuous mappings $T(t) : X \to X$ such that

1. $T(0)x = x$ for all $x \in X$,

2. $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$, and

3. $T(\cdot)x : [0, \infty) \to X$ is continuous (in $t$) for any fixed $x$.

Notation: A strongly continuous semigroup is also called a $C_0-$*semigroup*.

Recall that we can associate a $C_0$-semigroup with an *infinitesimal generator* $A : \text{Dom}(A) \to X$, defined by

$$\begin{aligned} \mathrm{Dom}(A) &= \left\{ x \in X : \lim_{t \to 0^+} \frac{1}{t}[T(t)x - x] \text{ exists} \right\} \\ Ax &= \lim_{t \to 0^+} \frac{1}{t}[T(t)x - x] \end{aligned}$$

### 2.2 Basic Theorems

We now state several useful theorems without proof, that form the underlying structure for our ultimate goal: a well-defined decomposition of the state space $C[((-\tau, 0), X)]$ into stable, unstable, and 'center' components.

Theorem:
Let $T \in B(Y, Y)$ be compact. If $\lambda_0 \in \sigma(T)$ with $\lambda_0 \neq 0$, and any $k \in \mathbb{Z}$ with $k \geq 0$, then we have that $\ker(\lambda_0 \text{ Id} - T)^k$ is always of positive finite dimension. Furthermore, there exists an $n_0 > 0$ such that

$$\ker(\lambda_0 \text{ Id} - T)^k = (\lambda_0 \text{ Id} - T)^{n_0}$$

for all $k \geq n_0$. For $k \leq n_0$, we have furthermore that

$$\ker(\lambda_0 \text{ Id} - T)^{k-1} \subsetneq \ker(\lambda_0 \text{ Id} - T)^k.$$

Theorem:
If $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup with infinitesimal generator $A$ on a Banach space $Y$, then

$$P\sigma(T(t)) = e^{tP\sigma(A)}(\cup\{0\})$$

(that is, the space might include $\{0\}$, but always includes at least $e^{tP\sigma(A)}$).

Furthermore, if $\{\lambda_n\}$ consists of all distinct points in $P\sigma(A)$ such that $e^{\lambda_n t} = \mu$, then $\ker(\mu \text{ Id} - T(t))^k$ is the closed linear extension of the (linearly independent) manifold $\ker(\lambda_n \text{ Id} - A)^k$.

For $T \in B(Y, Y)$, recall the result that the spectral radius $r(T)$ is given by

$$\begin{aligned} r(T) &= \sup_{\lambda_{\sigma(T)}} |\lambda| \\ &:= \lim_{n \to \infty} |T^n|^{1/n}. \end{aligned}$$

We have a nice estimate of the growth of a $C_0$-semigroup:

Theorem:

Let $\{T(t)\}$ be a $C_0$-semigroup on a Banach space $X$ such that for some $s > 0$, $r(T(s)) \neq 0$. Define $\beta = \ln \frac{\rho}{s}$. Then for all $\gamma > 0$, there exists a $K(\gamma) \geq 1$ such that

$$|T(t)x| \leq K(\gamma)e^{(\beta+\gamma)t}\,|x|$$

for all $t \geq 0$ and $x \in X$.

## 3 Abstract Solution Equation

The integral equation that we aim to study is the following:

$$u(t) = T(t)u(0) + \int_0^t T(t-s)F(u_s)ds,$$

where we take $t \geq 0$, $\{T(t)\}$ is $C_0$, and $F : C \to X$ is a bounded linear operator.

For this equation, we define

$$u(t)\phi = u_t\phi,$$

for the right hand side solving the solution with the initial condition

$$u_0\phi = \phi.$$

The infinitesimal generator $A_U :\ \mathrm{Dom}(A_U) \to C$ associated to this integral equation can be proven to be

$$A_U\phi = \dot{\phi}$$

with

$$\mathrm{Dom(A_U)} = \{\phi : \dot{\phi} \in C,\ \phi(0) \in\ \mathrm{Dom}(A_U),\ \dot{\phi}^{-}(0) = A_U\phi(0) + F(\phi)\}$$

## 4 Spectral Structure of the Solution Semigroup

Assume $T(t) : X \to X$ is compact for all $t \geq 0$ with infinitesimal generator $A_T$. Define

$$A(\lambda)x \quad = \quad A_T x - \lambda x + F(e^{\lambda} x)$$

where $x \in$ Dom$(A_T)$, where $(e^{\lambda} x) \in C$ is defined by

$$(e^{\lambda} x)(\theta) \quad = \quad e^{\lambda\theta} x$$

for $\theta \in [-\tau, 0]$.

Such a $\lambda$ is called a *characteristic value* if there is an $x \neq 0$ in the domain of the infinitesimal generator satisfying

$$A(\lambda)x \quad = \quad 0.$$

The *multiplicity* of the characteristic value is simply $\dim \ker A(\lambda)$.

The 'basic properties' section on semigroups immediately gives us the following very important spectral structure theorem of $A_U$:

Spectral Structure Theorem:

1. *Countability:* $\sigma(U(t))$ is a countable set, compact with 0 the only possible accumulation point, and $\sigma(U(t)) \backslash \{0\} \subset P\sigma(U(t))$.

2. *Point spectrum of the generator*: $P\sigma(U(t)) = e^{tP\sigma(A_U)}$ plus possibly $\{0\}$. The closed linear extension part also carried over exactly.

3. *Finite-dimensionality:* If $\lambda \in P\sigma(A_U)$, then $M_\lambda(A_U)$, where the latter set is the smallest subspace of $X$ containing $\ker(A_U - \lambda I)^k$ for all $k$.

4. *Invariance under the semigroup*: If $\mu \in P\sigma(U(t))$ and $\mu \neq 0$, then $\ker(U(t) - \mu I)^k$ is finite-dimensional for all $k$ and there exists a positive $n$ such that

$$M_\mu(U(t)) \quad = \quad \ker(U(t) - \mu I)^n.$$

Moreover,

$$U(t)M_\mu(U(t)) \subset M_\mu(U(t)).$$

5. *Growth Estimate*: Let $\beta \in \mathbb{R}$ with Re $\lambda \leq \beta$ for all characteristic values. Then for each $\gamma > 0$ there is a $K(\gamma) \geq 1$ such that

$$|U(t)\phi| \quad \leq \quad K(\gamma)e^{\beta+\gamma}\,|\phi|$$

for all $t \geq 0$.

Let's restrict ourselves to part 5 of the previous spectral structure theorem. It turns out that we can always find such a $\beta \in \mathbb{R}$ for $A_U$. This follows from the following two lemmas:

Lemma 1:

If $\omega \geq -\|F\|$ and Re $\lambda > \|F\| + \omega$ then $(A_U - \lambda I)^{-1}$ exists and has domain $C$.

Lemma 2:

If $\omega \geq -\|F\|$ and Re $\lambda > \|F\| + \omega$, then $\|(A_U - \lambda I)^{-1}\| \leq \frac{1}{\text{Re }\lambda - \|F\| - \omega}$.

This gives us the following theorem:

Theorem:

There exists a real $\beta$ such that Re $\lambda \leq \beta$ for all $\lambda \in \sigma(A_U)$. Moreover, if $\gamma$ is a real number then there exists only a finite number of $\lambda \in P\sigma(A_U)$ such that $\gamma \leq$ Re $\lambda$.

We now apply the richness of the structure we have developed to make a statement about the behavior of solutions $U(t)\phi$ satisfying the initial condition $U(0)\phi = \phi$.

Theorem:

Let $\beta$ be the smallest real number such that if $\lambda$ is a characteristic value of $A(\lambda)$ (i.e. satisfying $A(\lambda)x = 0$ for some $x \neq 0$ in $C$), then Re $\lambda \leq \beta$.

If $\beta < 0$, then for all $\phi \in C$, we have

$$\|U(t)\phi\| \to 0 \qquad \text{as} \qquad t \to \infty.$$

If $\beta = 0$ then *there exists* a $\phi \in C\backslash\{0\}$ such that

$$\|U(t)\phi\| = \|\phi\| \qquad \text{for all} \qquad t \geq 0.$$

If $\beta > 0$ then *there exists* a $\phi \in C\backslash\{0\}$ such that

$$\|U(t)\phi\| \to \infty \qquad \text{as} \qquad t \to \infty.$$

Let's now introduce some dynamical systems language. A solution of $A(\lambda)x = 0$ is called *stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\phi\| < \delta$ then $\|U(t)\phi\| < \varepsilon$ for all $t \geq 0$. It is called *asymptotically stable* if it is stable and also $\lim_{t\to\infty}\|U(t)\phi\| = 0$ for all $\phi \in C$. If the above two possibilities don't hold, we say the zero solution is *unstable*.

Our last theorem then tells us that the zero solution is asymptotically stable *iff* $\beta < 0$! Otherwise it is unstable if $\beta > 0$.

## 5 Decomposition of the State Space into Invariant Subspaces

### 5.1 The Resolvent Operator

Let $A$ be a closed linear operator with a nonempty resolvent set. Then the *resolvent operator*

$$R(\lambda; A) \quad = \quad (A - \lambda I)^{-1}$$

is analytic as a function of $\lambda$ on the resolvent of $A$. So if $\lambda_0$ is an isolated singular point, we may express $R$ has a Laurent expansion:

$$R(\lambda; A) \quad = \quad \sum_{n=0}^{\infty}(\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty}(\lambda - \lambda_0)^{-n} B_n,$$

where

$$\begin{aligned} A_n &= \frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda_0)^{-(n+1)} R(\lambda; A) d\lambda, \\ B_n &= \frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda_0)^{n-1} R(\lambda; A) d\lambda \end{aligned}$$

(this is just complex analysis), with $\Gamma$ a counterclockwise circle centered at $\lambda_0$ in the complex plane with radius smaller than $\delta$, the maximum radius in which the representation is valid.

## 5.2 Computation of Center, Stable, and Unstable Subspaces

Let $B : \mathrm{Dom}\,(B) \subset X \to X$ be a closed operator on a Banach space $X$.

Assume that $\sigma(B)$ can be separated into several parts $\Sigma_0, \Sigma_1, \cdots, \Sigma_s$, where for all $1 \leq j \leq s$, each $\Sigma_j$ is bounded and enclosed by a closed curve $\Gamma_j$ running in $\rho(B)$ and lying outside one another, whereas $\Sigma_0$ may be unbounded. Then recall from functional analysis that

$$P_j \equiv \frac{1}{2\pi i} \int_{\Gamma_i} (\lambda I - B)^{-1} d\lambda$$

is a projection. We also have the following basic results:

1. $P_i P_j = \delta_{ij} P_j$
2. $P_i(\,\mathrm{Dom}\,(B)) \subset \mathrm{Dom}\,(B)$
3. $X = \oplus_{0 \leq k \leq s} M_k$, where $M_i = P_i(Y)$ and $P_0 = I - (P_1 + P_2 + \cdots + P_s)$
4. $B(\,\mathrm{Dom}\,(B) \cap M_i) \subset M_i$
5. $B : M_i \to M_i$ is bounded for $1 \leq i \leq s$
6. $\sigma(B|_{M_i}) = \Sigma_i$, $0 \leq i \leq s$.

Returning as usual to our generator $A_U$ for the solution semigroup $\{U(t)\}_{t \geq 0}$, our prior results tell us that $\sigma(A_U)$ can be decomposed as follows:

$$\begin{aligned} \Sigma_1 &= \{\lambda \in P\sigma(A_U);\ \mathrm{Re}\ \lambda > 0\} \\ \Sigma_2 &= \{\lambda \in P\sigma(A_U);\ \mathrm{Re}\ \lambda < 0\} \\ \Sigma_3 &= \sigma(A_U) \backslash (\Sigma_1 \cup \Sigma_2) \end{aligned}$$