Contact singularities in multiple-timescale dynamical systems

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1 Abstract

In this paper, we classify contact singularities of the critical manifold for singularly-perturbed vector fields of the form $z' = H(z, \varepsilon)$. Our main result is the derivation of computable, coordinate-free defining equations for contact singularities under an assumption that the leading-order term of the vector field admits a suitable factorization. This factorization can in turn be computed explicitly in a wide variety of applied problems. We demonstrate these computable criteria by locating contact folds and contact cusps in some nonstandard models of biochemical oscillators.

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2 Introduction

Classifying the loss of normal hyperbolicity of the critical manifold is a fundamental step in the analysis of multiple-timescale dynamical systems. For systems in the so-called *standard* form

$$\begin{aligned} x' &= \varepsilon g(x,y,\varepsilon) \\ y' &= f(x,y,\varepsilon), \end{aligned} \tag{1}$$

the k-dimensional critical manifold lies inside the zero set of a smooth mapping $f(x,y,0) : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Takens began the classification of singularities of low codimension from the point of view of constrained differential equations, corresponding to the $\varepsilon = 0$ limit [28, 29, 30]. Since then, much work has been done to extend this local analysis for $0 < \varepsilon \ll 1$, in the case where the slow variables $x \in \mathbb{R}^k$ play the role of unfolding parameters for folds [1, 14, 27] and cusps [2, 11].

The purpose of this work is to provide a more general classification of loss of normal hyperbolicity of the critical manifold for the larger class

$$z' = H(z,\varepsilon) \tag{2}$$

of multiple-timescale systems. The relationship between Eqs. (1) and (2) is that coordinate transformations placing (2) in the form (1) are now typically defined only locally; in other words, there is no globally defined coordinate splitting into 'slow' versus 'fast' directions. The main complication is that the fast fiber bundle is no longer unidirectional, requiring a sufficiently general notion of contact between two smooth manifolds in $\mathbb{R}^n$ such that the unfolding directions still locally lie along the remaining slow directions.

To illuminate the new complication, consider the following planar system, which admits a multiple-timescale structure for $\varepsilon \ll 1$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} (x-2)(y+x^2-1) \\ y+x^2-1 \end{pmatrix} + \varepsilon \begin{pmatrix} G_1(x,y,\varepsilon) \\ G_2(x,y,\varepsilon) \end{pmatrix} \tag{3}$$

where G_1, G_2 are smooth functions in their arguments. We may define the *layer problem* by taking $\varepsilon \to 0$. Let $f(x,y) = y + x^2 - 1$ denote the common factor in the leading-order part of the vector field. The set of equilibria of the *layer* problem is given by the curve $S = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\} = \{(x,y) \in \mathbb{R}^2 : y = 1 - x^2\}$. Solution curves of the layer flow define a local fast foliation of the critical manifold, and corresponding (linear) fast fibers may be defined along the critical manifold tangent to the layer flow. In stark contrast to the above case, however, the fast fibers are no longer unidirectional– both the slow and fast directions change as we traverse S (see Fig. 1). In particular, the apex of the parabola $(0,1)$ cannot reasonably be called a 'fold' of the layer flow, as the fast fibers are locally transverse to the parabola there!

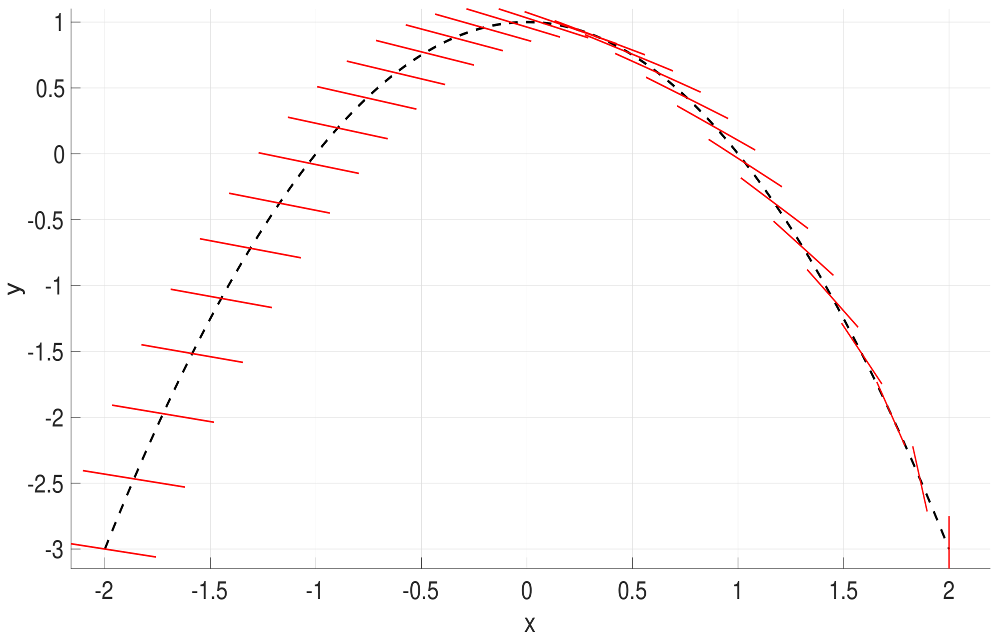


Figure 1: Critical manifold (dashed black curve) $\{(x, y) : y = 1 - x^2\}$ of the system (3) together with linear fast fibers (red line segments).

On the other hand, there appears to be 'fold-like' behavior with the parabola at two points. We can identify these points by formally rescaling time by $f(x, y) = y + x^2 - 1$ to obtain a desingularized layer problem:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x-2 \\ 1 \end{pmatrix} = N(x, y).$$

Evidently, the layer solutions approach the critical manifold in a tangent direction precisely when N lies in the kernel of Df:

$$\begin{aligned} DfN|_S &= 0 \\ 2x(x-2)+1 &= 0 \\ \Rightarrow x &= \frac{1}{2}(2 \pm \sqrt{2}). \end{aligned}$$

This calculation suggests that the classical defining equations for a fold singularity for slow-fast systems in the standard form may be generalized in a geometric, coordinate-independent manner, that is, without an explicit coordinate transformation into slow versus fast variables. Observe that the layer problem of this example can in fact be written in the factored form $z' = N(z)f(z)$. The geometric constraint

$$DfN|_S = 0$$

locates tangencies of the layer flow with the critical manifold, and immediately generalizes one of the classical fold conditions

$$D_y f(0,0,0) = 0$$

for standard slow-fast systems of the form (1). Indeed, note that $N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for planar systems in the standard form, and thus

$$DfN \;=\; \begin{pmatrix} D_x f & D_y f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = D_y f.$$

This computation suggests that the analogous second-order nondegeneracy criterion generalizing

$$D_{yy} f(0,0,0) \;\neq\; 0$$

will likely involve second derivatives of f, measuring curvature of the critical manifold, *as well as* derivatives of N, measuring the curvature of the fibers.

Our goal is to derive *computable* defining equations for contact singularities, and to clarify along the way that the unfolding scenarios are analogous to those of the classical case. Mather introduced the notion of contact singularities between equidimensional manifolds [18], but here we adopt the extended development of Montaldi's PhD thesis [19, 20, 21]. For our purposes we restrict to the particular case of contact between level sets of submersions and images of immersions, where the contact classes are easier to compute. We also briefly mention the complementary development of bifurcations without parameters [16], where classical bifurcation conditions have been phrased in terms of breakdown of normal hyperbolicity of manifolds of equilibria.

The present paper proceeds as follows: In Section 3, we give an account of multiple-timescale systems having a nonstandard slow-fast splitting. In Section 4, we give a rigorous definition of contact between a one-dimensional manifold and a k-dimensional manifold in $\mathbb{R}^n$, where $1 \leq k < n$, culminating in a full description of both the singularity classes and computable defining equations for contact folds and cusps. We conclude in Section 5 with examples. Technical details regarding jet spaces and contact equivalence are relegated to the Appendix (A).

Multilinear maps. We remind the reader of the standard notation and evaluation of multilinear maps $L : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^{n_d}$:

$$L(v_1, \cdots, v_k)_j \;=\; \sum_{l_k=1}^{n_k} \cdots \sum_{l_1=1}^{n_1} L_{l_1,\cdots,l_k,j} v_{1,l_1} \cdots v_{k,l_k}$$

for $j = 1, \cdots, n_d$.

3 Multiple-timescale dynamical systems

3.1 The nonstandard formulation

We begin by giving an abbreviated treatment of nonstandard multiple-timescale dynamical systems, following Fenichel's seminal work on geometric singular perturbation theory [4], and

Wechselberger's more recent treatment [32] which extends the framework to loss of normal hyperbolicity. Consider the family of vector fields (2), formally expanded in ε:

$$z' = h(z) + \varepsilon G(z, \varepsilon). \tag{4}$$

Definition 1. Given a family of vector fields (4), the system

$$z' = H(z, 0) = h(z) \tag{5}$$

is called the *layer problem* of (4).

Any system (4) such that the set of equilibria of the layer problem (5) contains a manifold of singularities S is called a *singular perturbation problem*. We focus on those families having a single manifold of equilibria:

Assumption 1. For $\varepsilon = 0$, the set of equilibria of (4) is a single k-dimensional manifold, with $1 \leq k < n$. We call S the *critical manifold*.

Note that the Jacobian Dh evaluated along points $z \in S$ has at least k zero eigenvalues corresponding to the tangent space T_zS by construction. These eigenvalues are called *trivial*, and the remaining $n - k$ eigenvalues along S are called *nontrivial*.

Definition 2. The set $S_n \subset S$ denotes the subset where all nontrivial eigenvalues of Dh evaluated along S_n are nonzero.

Along the set S_n, we may construct a pointwise-defined splitting of the tangent bundle along $z \in S_n$:

$$T_z\mathbb{R}^n = T_zS \oplus N_z, \tag{6}$$

where N_z is called the *linear fast fiber* at basepoint $z \in S_n$ identified with the quotient space $T_z\mathbb{R}^n/T_zS$. We may define the *tangent bundle of* S via the natural construction

$$TS = \cup_{z \in S_n} T_zS.$$

The corresponding bundle

$$N = \cup_{z \in S_n} N_z$$

is called the *(linear) fast fiber bundle*.

Along S, we may define a natural projection operator

$$\Pi^S : TS \oplus N \to TS.$$

Given a point $z \in S_n$, the map $\Pi^S|_{z \in S_n}$ can be characterized geometrically as an oblique projection onto T_zS along parallel translates of the fast fiber N_z.

The flow near S defines a locally invariant fast foliation of the layer problem in a tubular neighborhood B of S. We denote this foliation by $\mathcal{F}$. For $b \in B$, each fiber $\mathcal{F}_b = \mathcal{F}_z$ is tangent to the linear fast fiber N_z at the basepoint $z \in S$.

We now assume a factorization of h which captures the essential geometries of the slow and fast structures near to the critical manifold:

Assumption 2. The function $h(z)$ can be factorized as follows:

$$h(z) \;=\; N(z)f(z). \tag{7}$$

where the ith column of the $n \times (n-k)$ matrix function $N(z)$, $N_i = (N_i^1 \;\cdots\; N_i^n)^T$ consists of smooth functions $N_i^k : \mathbb{R}^n \to \mathbb{R}$. Assume that $N(z)$ has full column rank $n-k$ for each $z \in S$, and furthermore that singularities of $N(z)f(z)$ for $z \notin S$ are isolated, if they exist. We furthermore assume that the critical manifold S is equal to the zero level set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$:

$$S \;=\; f^{-1}(0).$$

Remark 1. The question of existence and uniqueness of factorizations of the form (7) for an arbitrary system (4) satisfying the Assumptions has only partial answers. Local factorizations can be constructed explicitly in the case the $h(z)$ is a rational vector field in z [6]. This includes a large variety of applied problems– notably, many chemical reaction networks can be modeled in this framework [25, 26]. In practice, these local factorizations can also be shown *a posteriori* to hold over large open sets of the phase space by ad hoc methods.

The following two results immediately demonstrate the usefulness of this factorization:

Lemma 1. *For $z \in S_n$, the column vectors of $N(z)$ form a basis for the range of $Dh(z)$ and the transposes of the row vectors of $Df(z)$ form a basis of the orthogonal complement of the kernel of $Dh(z)$ (i.e. a basis of the orthogonal complement of the tangent space T_zS).*

Lemma 2. *The nontrivial eigenvalues of the layer problem of (4) along S are encoded in the $(n-k) \times (n-k)$ matrix $DfN|_S$.*

Proofs. See [32, 17].

3.2 The contact set

We are concerned with studying the subset $S - S_n$, where the local tangent splittings (6) break down due to the alignment of the fast fiber bundle with the critical manifold along a one-dimensional subspace.

Definition 3. The *contact set* $F \subset S - S_n$ is the set of *simple* zero eigenvalue crossings (i.e., $z_0 \in F$ if and only if rank $Dh(z_0) = n - k - 1$.

Assumption 3. The contact set F is nonempty.

From Lemmas 1–2 we have the characterization

$$F \;=\; \{z \in S : \operatorname{rank}(DfN) = n - k - 1\}. \tag{8}$$

Since DfN is a matrix of size $(n-k) \times (n-k)$, a necessary condition is

$$\det(DfN)|_S \;=\; 0. \tag{9}$$

Geometrically, S is the set of points where a one-dimensional subspace of the fast fiber bundle locally aligns with the tangent space of S. In this setting it is straightforward to deduce the direction of tangency.

Lemma 3. *For $z_0 \in F$, let $r \in \mathbb{R}^{n-k}$ be any nontrivial column of adj$(Df(z_0)N(z_0))$. Then the contact direction at z_0 is $N(z_0)r$.*

Proof. Recall the following identity for the adjugate:

$$(DfN)\text{adj}(DfN) = \det(DfN)I_{n-k}.$$

If $\text{rank}(DfN) = n-k-1$, then $\text{rank}(\text{adj}(DfN)) = 1$. Select a nonzero column vector r of adj(DfN). Then $Df(Nr) = (DfN)r = 0$. Note that $N(z_0)r$ is a nonzero vector since N is assumed to have maximal rank along points in S. □

Remark 2. We may also construct local projections onto the contact direction by selecting a nonzero row l of adj(DfN). Note that if $\text{rank}(DfN) < n-k-1$, then $\text{adj}(DfN) = 0$, so these results are not generalizable to the case of higher-dimensional tangencies of the fast fibers with the critical manifold.

4 Contact between submanifolds of $\mathbb{R}^n$

Our primary goal is to classify points in the contact set F according to their singularity type. To do this, we must rigorously define a notion of contact between two submanifolds of $\mathbb{R}^n$.

We begin with the most elementary setting: contact between two smooth regular curves $\alpha(t)$ and $\beta(t)$ in $\mathbb{R}^2$ sharing a common point $\alpha(0) = \beta(0) = z_0$. One candidate definition of 'contact of order c at z_0' between α and β is fairly straightforward: the first c derivatives of α and β coincide at $t = 0$, but the $(c+1)$st derivatives do not. We have

$$\begin{aligned} \alpha^{(i)}(0) &= \alpha^{(i)}(0) \text{ for } i = 0, \cdots, c \\ \alpha^{(c+1)}(0) &\neq \alpha^{(c+1)}(0). \end{aligned}$$

We will instead use a slightly different notion of contact for two curves in the plane which turns out to be more natural for our setting. We are ultimately concerned with contact between the fast fibers and the critical manifold of system (4), where the critical manifold is defined as the level set of a submersion. This motivates the following definition, where we assume that one of the curves lies inside the zero level set of a smooth function.

Definition 4. Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}^2$ define two regular curves so that the image of β is equal to the zero set of a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha(0) = \beta(0)$. Note that $(F \circ \alpha) : \mathbb{R} \to \mathbb{R}$. We say that α and β *have contact of order c at $t = 0$* if

$$\begin{aligned} (F \circ \alpha)^{(i)}(0) &= 0 \text{ for } i = 0, \cdots, c \\ (F \circ \alpha)^{(c+1)}(0) &\neq 0. \end{aligned}$$

Example: contact of order two between curves.
Suppose $\alpha(t)$ and $\beta(t)$ have contact of order 1, with F as in Def. 4. Then

$$\begin{aligned} F(\alpha(0)) &= 0 \\ Df(\alpha(0))\alpha'(0) &= 0 \\ D^2f(\alpha'(0), \alpha'(0)) + Df(\alpha(0))\alpha''(0) &\neq 0. \end{aligned}$$

The first condition specifies that the contact point $\alpha(0)$ lies in the zero set of F (i.e. on the curve β). The second condition specifies that the tangent vector $\alpha'(0)$ of α at the contact point lies inside the tangent space of the curve β, given by ker Df, at the contact point.

To compare the third condition, first observe that $\beta(t)$ lies inside the zero level set of F by construction, and thus $F(\beta(t)) = 0$ over an interval of t. Differentiating both sides twice and evaluating the result at $t = 0$, we obtain the identity

$$D^2 f(\beta'(0), \beta'(0)) \;\; = \;\; -Df(\beta(0))\beta''(0).$$

Using this identity, $\alpha(0) = \beta(0)$, and $\alpha'(0) = \beta'(0)$ in the third condition, we have

$$Df(\beta(0))(\alpha''(0) - \beta''(0)) \;\; \neq \;\; 0,$$

and thus

$$\alpha''(0) \;\; \neq \;\; \beta''(0).$$

This demonstrates the connection between the two definitions of contact above.

A trivial observation is that our definition of contact is well-defined with respect to smooth reparametrizations of $\alpha(t)$ and $\beta(t)$. Less obvious technical issues that must be resolved are that (1) contact of a particular order should not depend on a particular choice of function F, and (2) the notion of contact is inherently local, so the domains of α and F should not matter outside of small neighborhoods of the contact point z_0.

Remark 3. These issues motivate the use of *germs of smooth functions* and *jet spaces*, which are the natural objects that we will use to define contact in the general case. We give a broad description of these terms, and relegate proper definitions to the Appendix (A). Fix a basepoint $z_0 \in \mathbb{R}^n$. The germ of a function f at z_0 is defined from the equivalence class of all smooth functions $g : \mathbb{R}^n \to \mathbb{R}^m$ that are equal to f on a common neighborhood of z_0. The collection of germs has the structure of a ring in the space of smooth functions. We may define the *k-jet space at z_0*, denoted $J^k(n, m)$, by taking a quotient of this ring by the ideal of all germs that vanish to order k. The k-jet space may be identified with the set of polynomials of total degree less than or equal to k. The k-jet of a germ f, denoted $J^k f$, is, roughly speaking, the element of $J^k(n, m)$ that may be identified with the truncated Taylor polynomial of order k under some suitable local coordinate transformation. The natural advantage of using k-jets is that contact can be defined in a coordinate-independent manner.

We now rigorously define contact between two submanifolds of $\mathbb{R}^n$, following the treatment of Montaldi [19, 20, 21] and the presentation of Izumiya et. al. [10]. The first step is to define a suitable generalization of contact order.

Definition 5. Suppose (M_1, N_1) and (M_2, N_2) are two pairs of submanifolds of $\mathbb{R}^n$ with $\dim(M_i) = m$ and $\dim(N_i) = d$. We say that *the contact of M_1 and N_1 at y_1 is of the same type as the contact of M_2 and N_2 at y_2* if there is a germ of a diffeomorphism $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ so that $\Phi(M_1) = M_2$ and $\Phi(N_1) = N_2$.

Our objective is to relate this 'generalized contact order' to a generalized version of the map $F \circ \alpha$ in Def. 4 in a suitable setting. We have

Definition 6. Suppose that a submanifold $M \subset \mathbb{R}^n$ is given locally as the image of some immersion-germ $g : (M, x) \to (\mathbb{R}^n, 0)$ and another submanifold $N \subset \mathbb{R}^n$ is given by the zero set of some submersion-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$. The *contact map* of M and N near x is the germ of the composite map $f \circ g$ near x.

This definition should be compared with Def. 4. The regular curve α defines an immersion in small neighborhoods of the contact point.

Remark 4. In analogy to the previous definition of contact between two curves in $\mathbb{R}^2$, there is an equivalent formulation of this definition which is not as computable:

Alternate definition. A k-dimensional submanifold M given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ and a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ have *contact of order c at t_0* if there exists a curve $\beta : \mathbb{R} \to \mathbb{R}^n$ lying inside M such that

$$\begin{aligned} \alpha^{(i)}(t_0) &= \beta^{(i)}(t_0) \quad \text{for} \quad i = 0, \cdots, c \\ \alpha^{(c+1)}(t_0) &\neq \beta^{(c+1)}(t_0) \end{aligned}$$

The equivalence of these definitions follows from the characterization of the tangent space via equivalence classes of curves.

Remark 5. For submanifolds of *equal* dimension, analogous definitions of contact have been used to study contact between locally defined center manifolds [31], between center manifolds and center subspaces [22], and to define jet bundles [24].

We now present two important results.

Lemma 4. *For any pair of submanifolds in $\mathbb{R}^n$, the contact class of the contact map depends only on the submanifold-germs themselves and not on the choice of submersion and immersion germs (and therefore not on the contact map).*

Proof. See [20] or Lemmas 4.1 and 4.2 in [10].

This lemma ensures that 'the' contact map-germ of two submanifolds is well-defined. The contact class of a smooth germ is the equivalence class of all germs whose zero sets are diffeomorphic (the complete definition is given in the appendix).

Theorem 1. *Suppose $g_1, g_2 : M_{1,2} \to \mathbb{R}^n$ are immersion-germs and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^k$ are submersion-germs (with $N_{1,2} = f_{1,2}^{-1}(0)$). Then the pairs (M_1, N_1) and (M_2, N_2) have the same contact type iff $f_1 \circ g_1$ and $f_2 \circ g_2$ lie in the same contact class.*

Proof. See [20] or Theorem 4.1 in [10].
This result provides the required connection to classical singularity theory: the contact class of a pair of submanifolds is completely determined by the singularities of the contact map between them.

4.1 A_c contact singularities and their unfoldings

We are finally in a position to consider the main setting of this paper: the contact of a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ with a submanifold given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ (with $1 \le k < n$). The contact map is $g = f \circ \alpha : \mathbb{R} \to \mathbb{R}^{n-k}$.

By Theorem 1, the contact type is well-defined by the contact class of g. The maps $h : \mathbb{R} \to \mathbb{R}^{d \ge 1}$ which are stable with respect to the contact class are the well-understood and classified A_k singularities (see for eg. [9, 10]). For completeness we write the definition

Definition 7. A critical point p of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is *of type* A_c if it is locally equivalent to $x_1^{c+1} + x_2^2 + \cdots + x_n^2$.

Stable maps are finitely-determined [18], so are contact-equivalent (see Def. 12 in the Appendix) to the germ of a map $\tilde{h}(t) = (t^{c+1}, 0, \cdots, 0)$ for some constant c. In this way we readily obtain a suitable analogue of the A_c singularity classes for contact equivalence:

Definition 8. We say that a smooth map $h : \mathbb{R} \to \mathbb{R}^{n-k}$ *has an* A_c *singularity* if it is contact-equivalent to $\tilde{h}(t) = (t^{c+1}, 0, \cdots, 0)$. The derivative conditions for an A_c singularity are

$$\begin{aligned} h^{(i)}(0) &= 0 \text{ for } i = 0, \cdots, c \\ h^{(c+1)} &\neq 0. \end{aligned}$$

If $h = f \circ \alpha$ is a contact map between a k-dimensional submanifold given by the zero set of a submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ and a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$, we say that the submanifold and the curve *make contact of order* c *at* z_0 if h admits an A_c-singularity at z_0.

Let $\alpha(t) : \mathbb{R} \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ and let M be a k-dimensional submanifold in $\mathbb{R}^n$ given by the zero level set of a submersion: $M = f^{-1}(0)$. Let $\alpha(0) = z_0 \in M$ denote a contact point between α and M.

The contact map $f \circ \alpha : \mathbb{R} \to \mathbb{R}^{n-k}$ admits an A_1 singularity when

$$\begin{aligned} (f \circ \alpha)(0) &= 0 \\ (f \circ \alpha)'(0) &= 0 \\ (f \circ \alpha)''(0) &\neq 0. \end{aligned}$$

The first derivative condition may be simplified to

$$Df(z_0)\alpha'(0) = 0.$$

Note the geometric content of this condition: the tangent vector of α lies precisely inside the tangent space of M at the contact point.

The second-derivative condition may be simplified to

$$D^2 f(\alpha'(0), \alpha'(0)) + Df\alpha''(0) \neq 0$$

where the derivatives in f are all evaluated at z_0. We remind the reader here of the evaluation of the multilinear forms in these expressions; for $i = 1, \cdots, n-k$,

$$(D^2 f(\alpha'(0), \alpha'(0)))_i \quad = \quad \sum_{j,k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k} \alpha'_j(0)\alpha'_k(0).$$

The contact map admits an A_2 singularity when

$$\begin{aligned}(f \circ \alpha)(0) &= 0\\ (f \circ \alpha)'(0) &= 0\\ (f \circ \alpha)''(0) &= 0\\ (f \circ \alpha)'''(0) &\neq 0.\end{aligned}$$

The third-derivative condition may be simplified to

$$D^3 f(\alpha'(0), \alpha'(0), \alpha'(0)) + 3D^2 f(\alpha'(0), \alpha''(0)) + Df\alpha'''(0) \quad \neq \quad 0. \tag{10}$$

4.2 Computing the contact order

At points in $z_0 \in F$, the center manifold theorem provides local families of one-dimensional center manifolds $W^c(z_0)$ of the layer problem all tangent to the contact direction Nr at the basepoint z_0. We may thus define regular curves passing pass through z_0 with nonzero speed. Our goal is to evaluate high-order derivatives of a given curve segment $\alpha(t)$ at $\alpha(0) = z_0$, in terms of derivatives of N and f.

There turn out to be technical complications when $1 \leq \dim\ S < n-1$. It is instructive to first consider the case $\dim S = n-1$. Here, the factorization $h_0(z) = N(z)f(z)$ consists of the term $N(z)$ of size $n \times 1$ and the scalar function $f(z)$. Let $z_0 \in F$ and let B_{z_0} denote an open ball centered at z_0 such that $N(z)$ has full rank for all $z \in B_{z_0}$ and $f(z) \neq 0$ for all $z \in B_{z_0} - S$. For points $z \in B_{z_0} - S$, the vector field is a nonzero multiple of the vector field $N(z)$. Solutions of the desingularized layer problem

$$z' \quad = \quad N(z)$$

defined in B_c consist of regular curves. In particular, each such curve crosses S with nonzero speed. Away from S, the tangent vectors of the solution curves are aligned with the original vector field h_0 everywhere (possibly with a change in orientation).

Crucially, in the codimension-1 case it is straightforward to compute high-order derivatives of solution curves intersecting S nontrivially. Let $\alpha(t)$ denote a solution curve of the desingularized layer problem with the property that $\alpha(0) = z_0 \in S$. Then

$$\begin{aligned}\alpha'(0) &= N(z_0)\\ \alpha''(0) &= DN(z_0)N(z_0)\\ \alpha'''(0) &= D^2N(z_0)(N(z_0), N(z_0)) + DN(z_0)DN(z_0)N(z_0),\end{aligned}$$

etc. For example, if $z_0 \in F$, then $\alpha'(0) = N(z_0) \neq 0$, whereas $(f \circ \alpha)'(0) = DfN(z_0) = 0$, giving contact order of at least one—as expected.

For critical manifolds of dimension $1 < k < n$, it is less obvious how to desingularize the layer flow since f is no longer a scalar function— the components $f_i(\alpha(t))$ grow and shrink independently along a curve $\alpha(t)$ lying in the center manifold. We sidestep this technical issue by first making a local coordinate transformation which straightens the fibers locally.

Lemma 5. *In the following, we evaluate all quantities at $z_0 \in F$. Recall that r is a right nullvector of $Df(z_0)N(z_0)$.The defining equations for contact of order-one between the fast fiber bundle and the critical manifold are*

$$\begin{aligned} DfNr &= 0 \\ D^2f(Nr, Nr) + DfDN(Nr, r) &\neq 0. \end{aligned}$$

The defining equations for contact of order-two between the fast fiber bundle and the critical manifold are

$$\begin{aligned} DfNr &= 0 \\ D^2f(Nr, Nr) + DfDN(Nr, r) &= 0 \\ D^3f(Nr, Nr, Nr) + 3D^2f(Nr, DN(Nr, r)) + & \\ Df(D^2N(Nr, Nr, r) + DN(DN(Nr, r), r)) &\neq 0. \end{aligned}$$

Generally, the defining equations for contact of order-k at a point $z_0 \in F$ may be computed by repeatedly differentiating the map $(f \circ \delta)(t)$, where $\delta(t) = z_0 + Nrt$, and evaluating the derivatives at $t = 0$.

Proof. There exists a local coordinate transformation $u = L(z)$ which places the system (4) in standard form by straightening the fibers, with local coordinates (u, y) (see [4], [32] for the full treatment, or [12] for the straightening step beginning with the system in standard form). If we let $x = M(u, y)$ denote the inverse of L, we have

$$\begin{aligned} \begin{pmatrix} u' \\ y' \end{pmatrix} &= \begin{pmatrix} \mathbb{O}_{k,n-k} \\ N^y(M(u, y), y) \end{pmatrix} f(M(u, y), y) \\ &= \begin{pmatrix} \mathbb{O}_{k,n-k} \\ I_{n-k,n-k} \end{pmatrix} \tilde{f}(u, y). \end{aligned}$$

Here, $u \in \mathbb{R}^k$ is the local slow variable and $y \in \mathbb{R}^{n-k}$ is the local fast variable, so that z_0 is placed at $(u, y) = (0, 0)$. The direction of contact can be made explicit through another change of variable

$$y = N^y rv + N^y Pw,$$

where P is an $(n-k) \times (n-k-1)$ matrix chosen so that $\mathrm{Col}\begin{pmatrix} r & P \end{pmatrix}$ provides a basis of $\mathbb{R}^k$. Let l and Q satisfy the identity

$$\begin{pmatrix} l \\ Q \end{pmatrix} \begin{pmatrix} r & P \end{pmatrix} = I_{n-k,n-k}.$$

Let $\tilde{r} = N^y r$ and $\tilde{P} = N^y P$. After these two coordinate transformations, we have

$$\begin{aligned}
\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} &= \begin{pmatrix} \mathbb{O}_{k,n-k} \\ lf(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) \\ Qf(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{O}_{k,n-k} \\ I_{n-k,n-k} \end{pmatrix} \begin{pmatrix} lf(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) \\ Qf(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{O}_{k,n-k} \\ I_{n-k,n-k} \end{pmatrix} \tilde{f}(u,v,w),
\end{aligned}$$

where by slight abuse of notation we use $\tilde{f}$ again to denote the second factor. We observe that in particular the one-dimensional center manifold has been locally straightened, containing the regular curve

$$\tilde{\alpha}(t) = \begin{pmatrix} \mathbb{O}_{k,1} \\ 1 \\ \mathbb{O}_{n-k-1,1} \end{pmatrix} t.$$

The straightening transformations simultaneously deform the critical manifold locally, reflected in the transformation from f to $\tilde{f}$. The key point is that this sequence of coordinate transformations *preserves the contact order* (by Theorem 1).

It remains to compute the defining equations for the A_k singularity classes of the deformed contact map $\tilde{f} \circ \tilde{\alpha}$. Observe that

$$\begin{aligned}
(\tilde{f} \circ \tilde{\alpha})(0) &= 0 \\
(\tilde{f} \circ \tilde{\alpha})^{(k)}(0) &= D^{(k)}\tilde{f}(0)(\tilde{\alpha}'(0), \cdots, \tilde{\alpha}'(0)) \\
&= D_{\underbrace{v \cdots v}_{k \text{ times}}}\tilde{f}(0)
\end{aligned}$$

since $\tilde{\alpha}^{(k)}(0) = 0$ for $k \geq 2$, and the subscript in the last line denotes that the partial derivative is taken k times with respect to the center variable v. We compute the first two derivatives of the contact map. We have

$$\begin{aligned}
D_v\tilde{f}(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) &= \begin{pmatrix} l \\ Q \end{pmatrix} D_xfD_yM\tilde{r} + D_yf\tilde{r} \\
&= \begin{pmatrix} l \\ Q \end{pmatrix} (D_xfN_x(N^y)^{-1} + D_yf)\tilde{r} \\
&= \begin{pmatrix} l \\ Q \end{pmatrix} (D_xfN_x + D_yfN^y)(N^y)^{-1}N^yr \\
&= \begin{pmatrix} l \\ Q \end{pmatrix} DfNr.
\end{aligned}$$

Generally, for any differentiable function

$$\zeta = \zeta(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w),$$

we have

$$D_v\zeta = D\zeta Nr.$$

This rule can be used to generate derivatives of arbitrary order. We have for instance the second-order derivative

$$D_{vv}\tilde{f}(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) = \begin{pmatrix} l \\ Q \end{pmatrix} (D^2f(Nr,Nr) + DfDN(Nr,r)),$$

the third-order derivative

$$\begin{aligned}
D_{vvv}\tilde{f}(M(u,\tilde{r}v+\tilde{P}w),\tilde{r}v+\tilde{P}w) &= \begin{pmatrix} l \\ Q \end{pmatrix} (D^3f(Nr,Nr,Nr) + 3D^2f(Nr,DN(Nr,r)) + \\
&\quad Df(D^2N(Nr,Nr,r) + DN(DN(Nr,r),r))),
\end{aligned}$$

and so on. The matrix $\begin{pmatrix} l \\ Q \end{pmatrix}$ stores the dual basis, and therefore does not affect the corresponding (non)zero conditions in the associated defining equations.

To prove the final statement of the lemma, use the chain rule to compute the kth derivative of $(f \circ \delta)(t)$, where $\delta(t) = z_0 + Nrt$, and compare the result to the computation of the kth derivative of the contact map as above. □

Remark 6. The 'recipe' for computing derivatives in the Lemma 5 above may be derived directly by making one more transformation so that the center manifold lies along the contact direction $N(z_0)r$ instead of along the unit vector e_k. The straightening transformation in the lemma above has been used to compute an unfolding of contact points of order-one, with the slow variables unfolding the contact point [32]. Lemma 5 establishes a rigorous relationship between that computation and the singularity classes of the contact map between the fast fibers and the slow manifold: the nondegeneracy condition for the A_k singularity of the contact map is (up to a trivial projection along the center flow) identical to the corresponding codimension-k folded singularity nondegeneracy condition.

4.3 Slow unfoldings of contact singularities; contact folds

From the point of view of geometric singular perturbation theory, the dynamical relevance of loss of normal hyperbolicity of the critical manifold is only manifested when the unfoldings occur *under local variation of the slow variables*. Unfoldings in these directions admit scenarios where the slow flow may cross fold points transversely. This is the basic ingredient in constructing persistent nontrivial connections between attracting and repelling slow manifolds.
Consider the case of an isolated fold point in the planar, standard slow-fast system

$$\begin{aligned} x' &= \varepsilon g(x,y,\varepsilon) \\ y' &= f(x,y,\varepsilon). \end{aligned} \tag{11}$$

Suppose the following conditions hold:

$$\begin{aligned} f(0,0,0) = g(0,0,0) &= 0 \\ f_y(0,0,0) &= 0 \\ f_{yy}(0,0,0) &\neq 0 \\ f_x(0,0,0) &\neq 0. \end{aligned}$$

If we have furthermore a 'slow dynamics transversality' condition $g(0,0,0) \neq 0$, then there exists [14] a smooth, local coordinate change $\phi(x,y) = (\xi,\eta)$ in which the system (11) is given by

$$\begin{aligned} \frac{d\xi}{dt} &= \eta + \xi^2 + \mathcal{O}(\xi^2, \xi\eta, \eta^2, \varepsilon) \\ \frac{d\eta}{dt} &= \varepsilon(\pm 1 + \mathcal{O}(\xi,\eta,\varepsilon)). \end{aligned}$$

The classical singularity theory of the generic fold map $f(\alpha,x) = \alpha + x^2$ is an essential ingredient in these results; in particular, the slow variable y plays the role of the parameter in the corresponding transversality condtion $f_x(0,0,0) \neq 0$. The question is how to write down coordinate-free defining equaitons of these equations *without* having to compute explicit local coordinate changes everywhere along the critical manifold.

We now consider computable criteria for such restricted unfoldings in the more general case of contact singularities.

Definition 9. Assume (4) exhibits a contact point $z_0 \in F$. We say that z_0 is a *contact fold* if it is a contact of order one that admits a versal (codimension-one) unfolding under local variation of the slow variables.

We now state the first main lemma of the paper: there are computable defining equations for a contact fold.

Lemma 6. *Defining equations for a contact fold at $z_0 \in F$ are:*

- *rank*$(Df(z_0)) = n - k$

- $l(D^2 f(Nr, Nr) + Df DN(Nr, r))|_{z_0} \neq 0,$

where l, r are nonzero left and right nullvectors of $Df(z_0)N(z_0)$.

Proof. The first local regularity condition is satisfied immediately when f is a submersion. We follow the proof given in [32] to compute the defining equation. Recall the transformed vector field with locally straightened fibers

$$\begin{aligned} \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} &= \begin{pmatrix} \mathbb{O}_{k,1} \\ \tilde{l} \\ \tilde{Q} \end{pmatrix} \tilde{f}(u, \tilde{r}v + \tilde{P}w) \\ &= \begin{pmatrix} \mathbb{O}_{k,1} \\ l \\ Q \end{pmatrix} f(M(u, \tilde{r}v + \tilde{P}w), \tilde{r}v + \tilde{P}w), \end{aligned} \tag{12}$$

where $(u, v, w) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$.

The Jacobian $\tilde{J} = D\tilde{h}_0$ along S is given by

$$\tilde{J}|_S = \begin{pmatrix} \mathbb{O}_{k,k} & \mathbb{O}_{k,1} & \mathbb{O}_{k,n-k-1} \\ lD_x f(D_x L)^{-1} & lDfNr & lDfNP \\ QD_x f(D_x L)^{-1} & QDfNr & QDfNP \end{pmatrix}.$$

On F, the (2,2), (3,2), and $(2,3)$ (block) entries are further annihilated because l and r are precisely the nullvectors of DfN on the set F of contact points of S:

$$\tilde{J}|_F = \begin{pmatrix} \mathbb{O}_{k,k} & \mathbb{O}_{k,1} & \mathbb{O}_{k,n-k-1} \\ lD_x f(D_x L)^{-1} & \mathbb{O}_{1,1} & \mathbb{O}_{1,n-k-1} \\ QD_x f(D_x L)^{-1} & \mathbb{O}_{n-k-1,1} & QDfNP \end{pmatrix}.$$

Near F we expand the right-hand side of the (one-dimensional) v' equation. We have

$$\begin{aligned} v' &= \tilde{l}\tilde{f}(u, \tilde{r}v + \tilde{P}w) \\ &= lf(M(u, \tilde{r}v + \tilde{P}w), \tilde{r}v + \tilde{P}w) \\ &= lD_x f(D_x L)^{-1} u + l(D^2 f(Nr, Nr) + DfDN(Nr, r))v^2 + \cdots, \end{aligned}$$

where we ignore the remaining cross-terms of order two and the higher-order terms.

The coefficient of the vector-valued component u is $lD_x f(D_x L)^{-1}$ which is nontrivial since rank $Df(z_0) = n - k$. Thus $lD_x f(D_x L)^{-1} u$ plays the role of an unfolding parameter, but with the parameter axis lying along a nullvector of $lD_x f(D_x L)^{-1}$. $\square$

4.4 Contact cusps

Definition 10. Assume (4) exhibits a contact point $z_0 \in F$. We say that z_0 is a *contact cusp* if it is a contact of order cusp that admits a versal (codimension-two) unfolding under local variation of the slow variables.

We recall the generic criteria for the unfolding of a cusp point [15]. Consider the smooth map

$$\dot{x} \;=\; f(x, \alpha_1, \alpha_2), \tag{13}$$

$x, \alpha_1, \alpha_2 \in \mathbb{R}$, with an isolated equilibrium point at $p = (x, \alpha_1, \alpha_2) = (0, 0, 0)$. Assume that

- $f_x(p) = f_{xx}(p) = 0$
- Nondegeneracy condition: $f_{xxx}(p) \neq 0$
- Parameter transversality condition: $(f_{\alpha_1} f_{x\alpha_2} - f_{\alpha_2} f_{x\alpha_1})(p) \neq 0$

Then we can find smooth invertible coordinate changes in the extended phase space so that the system 13 is transformed into

$$\dot{\eta} \;=\; \beta_1 + \beta_2 \eta + \eta^3 + O(\eta^4).$$

These nondegeneracy and parameter transversality conditions can be expressed more compactly by specifying instead that the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F : (x, \alpha_1, \alpha_2) \mapsto (f, f_x, f_{xx})(x, \alpha_1, \alpha_2)$$

be regular at the cusp point.

Remark 7. In two-parameter families of n-dimensional flows $\dot{x} = f_{\alpha_1,\alpha_2}(x)$ with $x \in \mathbb{R}^n$, the corresponding conditions for a cusp bifurcation are that Df_{α_1,α_2} has one simple zero eigenvalue and $n-1$ eigenvalues with nonzero real part. Then there exist coordinate transformations locally placing the vector field in the normal form

$$\begin{aligned} \dot{u} &= \beta_1 + \beta_2 u \pm u^3 \\ \dot{y} &= Ay, \end{aligned}$$

where β_1, β_2 are parameters, $(u, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and A is an $(n-1) \times (n-1)$ hyperbolic matrix [15].

We now state and prove the analogous result to Lemma 6 for contact cusps of nonstandard slow-fast systems.

Lemma 7. *The defining equation and genericity condition for a contact cusp at $z_0 \in F$ are:*

- $\mathit{rank}(Df(z_0)) = n - k$
- $l(D^2 f(Nr, Nr) + Df\, DN(Nr, r))|_{z_0} = 0$
- $l \cdot (D^3 f(Nr, Nr, Nr) + 3D^2 f(DN(Nr, r), Nr) +$
  $Df\, D^2 N(Nr, Nr, r) + Df\, DN(DN(Nr, r), r)) \neq 0$

- *The* $2 \times n$ *matrix*

$$C_0(z_0) \quad = \quad \begin{pmatrix} lDf \\ l(D^2 f(Nr, I) + Df DN(r, I)) \end{pmatrix}$$

(where all terms on the right are evaluated at z_0*) has full rank of* 2*.*

Here, l *and* r *denote nonzero left and right nullvectors of* $Df(z_0)N(z_0)$*.*

Proof: Taylor-expanding the right-hand side of the v' equation (see Eq. (12)) near $(u, v, w) = (0, 0, 0)$, we have

$$\begin{aligned} (lf)(u, v, w) &= (lf)(0, 0, 0) + D(lf) \cdot z + H(lf)(z, z) + \cdots \\ &= 0 + D_u(lf)u + D_v(lf)v + D_w(lf)w + (D_{uv}(lf)u + D_{wv}(lf)w)v + \\ &\quad \frac{1}{2} D_{vv}(lf)v^2 + \frac{1}{6} D_{vvv}(lf)v^3 + \cdots . \end{aligned}$$

In our setting, we have $D_v(lf) = 0$ and $D_u(lf) \neq 0$ on the set F. Using this expansion, we can read off the defining equations for a cusp at a point $z_0 \in F$:

(i) $D_{vv}(lf)(z_0) = 0$.

(ii) $D_{vvv}(lf)(z_0) \neq 0$.

(iii) (Unfolding in the non-contact slow directions) The $2 \times k$ matrix

$$C(z) \quad = \quad \begin{pmatrix} D_u(lf) \\ D_{uv}(lf) \end{pmatrix}$$

has full rank at the contact point: rank $C(z_0) = 2$.

The third condition is a sufficient condition for a generic two-parameter unfolding of the cusp with the unfolding directions lying along the critical manifold. Note that we require $k \geq 2$. For contact between the fast fiber bundle and the critical manifold to be defined, we therefore require the system to be at least three-dimensional with a two-dimensional critical manifold.

These three conditions should be compared to the standard defining equations of the standard generic cusp. In particular, the transversality condition in the classical cusp unfolding is usually written as a regularity condition on a minor of the Jacobian of the map $(x, \alpha_1, \alpha_2) \mapsto (f(x, \alpha_1, \alpha_2), f_x(x, \alpha_1, \alpha_2), f_{xx}(x, \alpha_1, \alpha_2))$:

$$\det \begin{pmatrix} f_{\alpha_1} & f_{\alpha_2} \\ f_{\alpha_1, x} & f_{\alpha_2, x} \end{pmatrix} (z_0) \quad \neq \quad 0.$$

The analogy is that the tangency direction v plays the role of the unfolding variable x, and two linearly independent combinations of the remaining slow variables u_1, u_2 play the role of the unfolding parameters α_1, α_2.

Evaluating the nondegeneracy condition.

Differentiate three times and use the chain rule:

$$\begin{aligned}
lD_{vvv}(f)|_{z_0} &= lD_{vv}(DfNr)|_{z_0} \\
&= lD_v(D^2f(Nr, Nr) + DfDN(Nr, r))|_{z_0} \\
&= l \cdot (D^3f(Nr, Nr, Nr) + 3D^2f(DN(Nr, r), Nr) + \\
&\quad +Df(D^2N(Nr, Nr, r) + DN(DN(Nr, r), r)))|_{z_0}.
\end{aligned}$$

Evaluating the transversality condition.

We have $lD_uf = lD_xf(D_xL)^{-1} = lDf\hat{I}(D_xL)^{-1}$, where $\hat{I} = \begin{pmatrix} I_{k,k} \\ O_{n-k,k} \end{pmatrix}$. Then
We have

$$\begin{aligned}
\mathrm{rank}(C(z_0)) &= \mathrm{rank} \begin{pmatrix} lD_uf \\ lD_{uv}f \end{pmatrix} \\
&= \mathrm{rank} \begin{pmatrix} lDf\hat{I}(D_xL)^{-1} \\ lD_u(DfNr) \end{pmatrix} \\
&= \mathrm{rank} \begin{pmatrix} lDf\hat{I}(D_xL)^{-1} \\ l(D^2f(Nr, \hat{I}(D_xL)^{-1}) + DfDN(r, \hat{I}(D_xL)^{-1})) \end{pmatrix} \\
&= \mathrm{rank} \begin{pmatrix} lDf \\ l(D^2f(Nr, I) + DfDN(r, I)) \end{pmatrix} \\
&= \mathrm{rank}\ C_0(z_0),
\end{aligned}$$

where the penultimate line follows from right-factoring the $n \times k$ full-rank matrix $\hat{I}(D_xL)^{-1}$ from both block rows of the original matrix. □

5 Examples

We now demonstrate Lemmas 6 and 7 in a series of three-dimensional multiple-timescale systems having a two-dimensional critical manifold. We begin with a system in standard-form in Sec. 5.1, where the fast fibers are unidirectional and the corresponding evaluation of the defining equations are trivial. We also write down the defining equations for general layer problems in standard form. In Sec. 5.2 we identify a line of contact folds and a contact cusp in a three-component biochemical oscillator model with a negative feedback loop. The contact singularities partition the critical manifold according to their attracting and repelling dynamics. We finish in Sec. 5.3 with a similar negative-feedback oscillator model: a minimal model for an embryonic cell cycle. The set of equilibria in this case consists of the union of four planes. We identify a contact fold line and a contact cusp on one of these planes.

5.1 Cusp normal form in a standard slow-fast system.

Here the straightening transformations are trivial. Consider the normal form of the singularly perturbed cusp in $\mathbb{R}^3$ in the standard case [2]:

$$\begin{aligned} x' &= \varepsilon(1 + \mathcal{O}(x, y, z, \varepsilon) \\ y' &= \varepsilon\mathcal{O}(x, y, z, \varepsilon) \\ z' &= (z^3 + yz + x) + \mathcal{O}(\varepsilon, xz, z^4). \end{aligned}$$

Here the slow variables are x, y and the fast variable is z. In terms of the Nf-splitting we have the right-hand side of the layer problem given by

$$h = Nf = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (x + yz + z^3).$$

We can check the conditions for the cusp. Note first that

$$DfN = y + 3z^2.$$

It is not hard to show that the parabola $F = \{(x, y, z) : y + 3z^2 = 0\} \cap S$ on the cusp surface $S = \{(x, y, z) : x + yz + z^3 = 0\}$ consists of contact points of at least order one. In particular we have that $l = r = 1$ for this problem and thus

$$\begin{aligned} l(D^2f(Nr, Nr) + DfDN(Nr, r) &= Hf(N, N) + DfDN(N, N) \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 6z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0 \\ &= 6z, \end{aligned}$$

whence every point on F except the point $z_0 = (0, 0, 0)$ is a fold point. At the point z_0, cusp condition (i) is satisfied.

The nondegeneracy condition (ii) reduces to checking the trivial computation

$$\begin{aligned} lD^3f(N, N, N) &= \sum_{j,k,l=1}^{3} \frac{\partial^3 f}{\partial z_j \partial z_k \partial z_m} N_j N_k N_m \\ &= 6 \neq 0. \end{aligned}$$

Finally, we check the transversality condition (iii). We have

$$\begin{aligned} C_0(z_0) &= \begin{pmatrix} Df \\ D^2fN + DfDN \end{pmatrix}\Big|_{z_0} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

as expected.

Defining equations for a contact cusp in standard slow-fast systems are given by a trivial subcase of Lemma 7. Consider the layer problem

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} g(x, y, z),$$

which is given in standard form. Then $N(x, y, z) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. Then at a test point $p_0 = (x_0, y_0, z_0)$ the (non)degeneracy conditions on the derivatives become

$$\begin{aligned} g = g_z = g_{zz} &= 0 \\ g_{zzz} &\neq 0, \end{aligned}$$

whereas the test matrix for the transversality condition is

$$\begin{aligned} C_0(z_0) &= \begin{pmatrix} Df \\ D^2fN + DfDN \end{pmatrix} \\ &= \begin{pmatrix} g_x & g_y & g_z \\ g_{xz} & g_{yz} & g_{zz} \end{pmatrix} \\ &= \begin{pmatrix} g_x & g_y & 0 \\ g_{xz} & g_{yz} & 0 \end{pmatrix}, \end{aligned}$$

giving the transversality condition

$$\begin{pmatrix} g_x \\ g_y \end{pmatrix} \cdot \begin{pmatrix} g_{yz} \\ -g_{xz} \end{pmatrix} \neq 0.$$

This provides the full unfolding of the cusp under the independent variation of two slow parameters. These conditions should be compared to the defining equations in [2], and in particular the transversality condition, which is a corrected version of the nondegeneracy condition (A) in their paper. We note that this does not affect the correctness of their Lemma 1, and the appropriate transversality condition is correctly identified later in equation (35), but only after a series of coordinate transformations.

5.2 Three-component negative feedback oscillator

A fundamental characteristic of biochemical oscillators is the presence of negative feedback with time delay. Novak and Tyson considered several examples of biochemical networks, including autonomous systems and delay differential equations, and argued that negative-feedback loops with at least three components are required for sustained oscillations to arise [23]. We consider the following minimal, autonomous three-component model studied in [32]:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha_1\left(\frac{1}{1+z^2}-x\right) \\ \alpha_2 x - 1 \\ \alpha_3(y-z) \end{pmatrix} y + \varepsilon \begin{pmatrix} \alpha_1\left(\frac{1}{1+z^2}-x\right) \\ \alpha_2 x \\ \alpha_3(y-z) \end{pmatrix}. \tag{14}$$

The (dimensionless) parameters are $\alpha_1, \alpha_2, \alpha_3 > 0$, and $\varepsilon \ll 1$. The layer problem of (14) is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha_1\left(\frac{1}{1+z^2}-x\right) \\ \alpha_2 x - 1 \\ \alpha_3(y-z) \end{pmatrix} y.$$

Here we have $N(x, y, z)$ the vector function on the RHS and $f(x, y, z) = y$. The (regular part of the) critical manifold is given by the plane

$$S = \{y = 0\}.$$

The nontrivial eigenvalues are encoded in the (in this case *scalar*) function

$$DfN|_S = \alpha_2 x - 1.$$

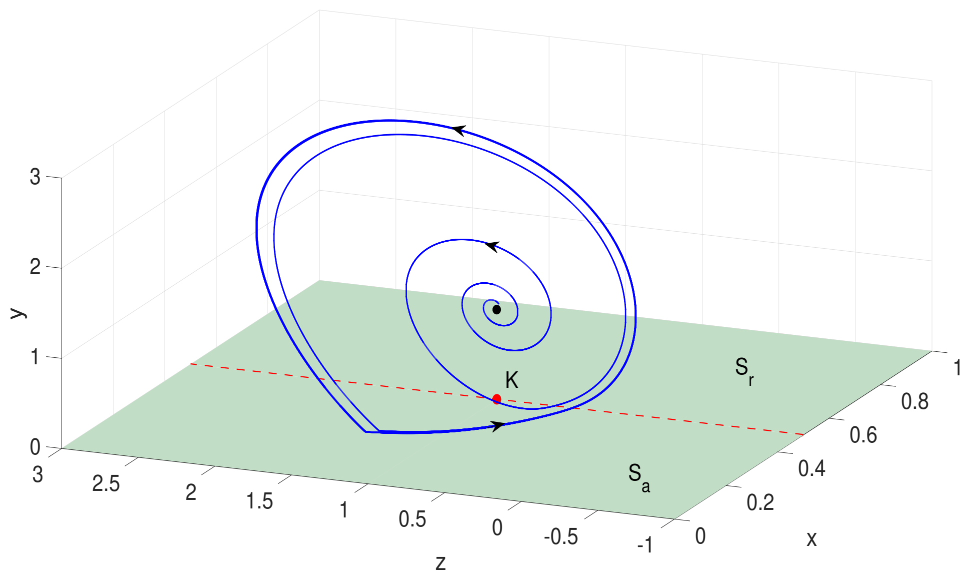
The fold curve

$$\tilde{F} = \{(x, y, z) \in S : x = 1/\alpha_2\}$$

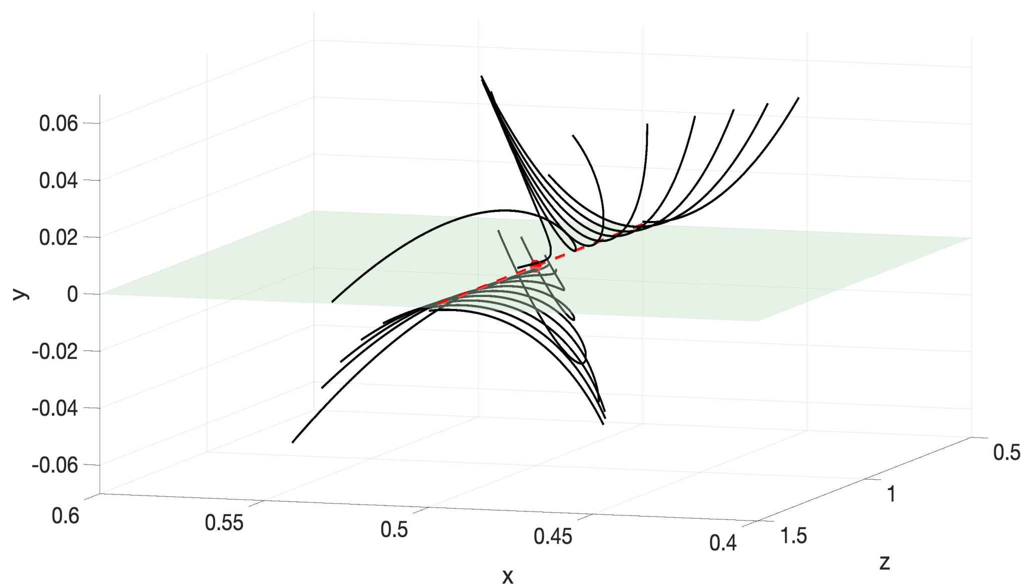
therefore divides S into an attracting and repelling branch. The system admits an isolated saddle-focus equilibrium point away from S. A stable relaxation oscillation can be computed for the parameter values $\varepsilon = 0.0005$, $\alpha_1 = 0.2$, $\alpha_2 = 2$, $\alpha_3 = 0.2$ (see Fig. 2). This periodic orbit can be decomposed into a slow segment near S which crosses $\tilde{F}$, and a fast global reinjection arising from intersections between the two-dimensional unstable manifold of the saddle-focus and the fast fiber bundle near S. See [32] for further details on the global dynamics.

Contact folds. Most of the points on $\tilde{F}$ are fold points. We observe this by checking the defining and nondegeneracy conditions:

$$\begin{aligned} \operatorname{rank}(Df)|_{\tilde{F}} &= n - k = 1 \\ DfDNN|_{\tilde{F}} &= \alpha_1 \frac{\alpha_2 - (1+z^2)}{1+z^2}. \end{aligned}$$



(a)



(b)

Figure 2: Sample trajectory (blue curve) of the three-component negative feedback oscillator (14) with initial condition $(x, y, z) \approx (0.5198, 1.0205, 1.0205)$ and parameter set $(\varepsilon, \alpha_1, \alpha_2, \alpha_3) = (0.0005, 0.2, 2, 0.2)$. An isolated contact point K (red point) of order 2 ((15)) lies on the fold line (red dashed line). A saddle-focus equilibrium point (black point) of (14) is also shown. (b) Local geometry of the fast fibers near the line of folds (note that this figure is rotated to better show the curvature).

(observe how simple the nondegeneracy condition becomes in the codimension-one case: we have $\mathrm{adj}(DfN) = 1$).

Thus, the parabolic coefficient is nontrivial everywhere on the contact set except where $z = \sqrt{\alpha_2 - 1}$. We call this distinguished point

$$K \;=\; \{(x, y, z) \in \tilde{F} : z = \sqrt{\alpha_2 - 1}\} = \{1/\alpha_2, 0, \sqrt{\alpha_2 - 1}\}. \tag{15}$$

Contact cusps. We check whether K is a contact cusp. The identities $D^2 f = 0$ and $D^3 f = 0$ greatly simplify the calculations; we need only evaluate $Df D^2 N(N, N, 1)$ and $Df DN DN N$. The second derivative in the first term admits the following simple formula in the codimension-one case:

$$Df D^2(N, N, 1) \;=\; Df \begin{pmatrix} N^T H N_1 N \\ N^T H N_2 N \\ \vdots \\ N^T H N_n N \end{pmatrix},$$

where HN_i denotes the Hessian of the scalar function $N_i(z)$. This term evaluates to 0, which can be read off from the fact that $N_2(z)$ is only linear, whereas $D_x f$ and $D_z f$ are both zero. On the other hand, the last term is nontrivial:

$$Df DN DN N \;=\; \frac{2\alpha_1 \alpha_3 (\alpha_2 - 1)}{\alpha_2}.$$

As long as $\alpha_2 > 1$, K has contact-order of 2 (note that K exists for $\alpha_2 \geq 1$).

We now check the remaining transversality condition. As before, we write down

$$C_0(K) \;=\; \left.\begin{pmatrix} Df \\ D^2 f N + Df DN \end{pmatrix}\right|_{z_0}.$$

at the contact point K, we have

$$\begin{aligned} Df &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ DN &= \begin{pmatrix} \alpha_1 & 0 & -2(\alpha_1/\alpha_2)\sqrt{\alpha_2 - 1} \\ \alpha_2 & 0 & 0 \\ 0 & \alpha_3 & -\alpha_3 \end{pmatrix}, \end{aligned}$$

so

$$C_0(K) \;=\; \begin{pmatrix} 0 & 1 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix},$$

which has the maximal rank of 2 as long as $\alpha_2 \neq 0$. The three-component feedback oscillator therefore exhibits a contact cusp at the point K (see (15)).

5.3 Mitotic oscillator

We demonstrate the existence of a cusp in Goldbeter's minimal model for the embryonic cell cycle [7]. The original formulation contains terms of Michaelis-Menten type to study the existence of sustained oscillations due to negative feedback loops. An analysis from the GSPT point of view is provided in [13], where an isolated, strongly attracting limit cycle is proven to exist for sufficiently small values of a singular perturbation parameter (see Fig. 3). Following their formulation, consider the system

$$\begin{aligned}
\frac{dX}{dt} &= \left(M(1-X)(\varepsilon+X)-\frac{7}{10}X(\varepsilon+1-X)\right)F_\varepsilon(M)\\
\frac{dM}{dt} &= \left(\frac{6C}{1+2C}(1-M)(\varepsilon+M)-\frac{3}{2}M(\varepsilon+1-M)\right)F_\varepsilon(X)\\
\frac{dC}{dt} &= \frac{1}{4}(1-X-C)F_\varepsilon(X,M),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
F_\varepsilon(X,M) &= F_\varepsilon(X)F_\varepsilon(M)\\
F_\varepsilon(X) &= (\varepsilon+1-X)(\varepsilon+X)\\
F_\varepsilon(M) &= (\varepsilon+1-M)(\varepsilon+M).
\end{aligned}$$

The layer problem is given by

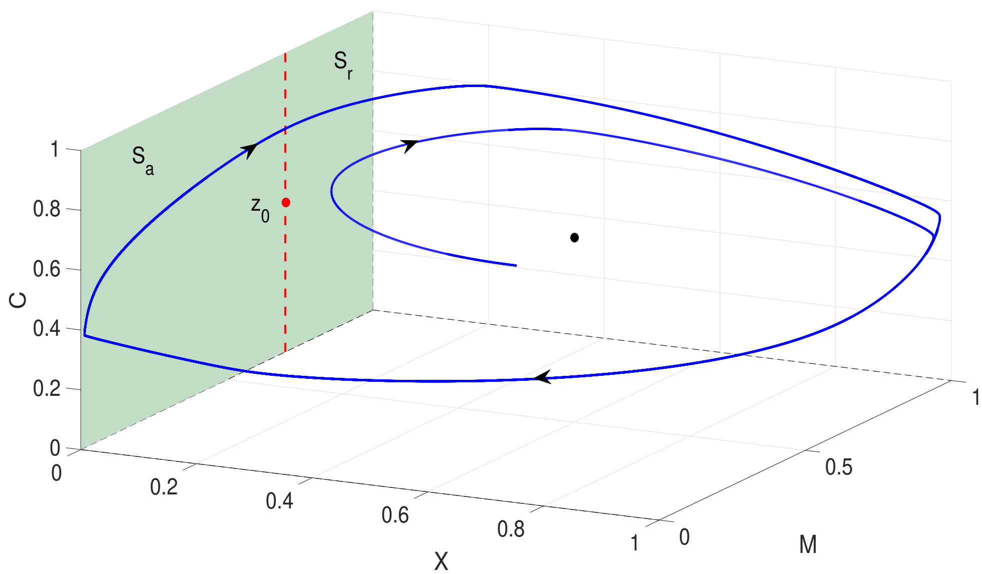
$$\begin{aligned}
\frac{dX}{dt} &= \left(M-\frac{7}{10}\right)F_0(X,M,C)\\
\frac{dM}{dt} &= \left(\frac{6C}{1+2C}-\frac{3}{2}\right)F_0(X,M,C)\\
\frac{dC}{dt} &= \frac{1}{4}(1-X-C)F_0(X,M,C),
\end{aligned}$$

for

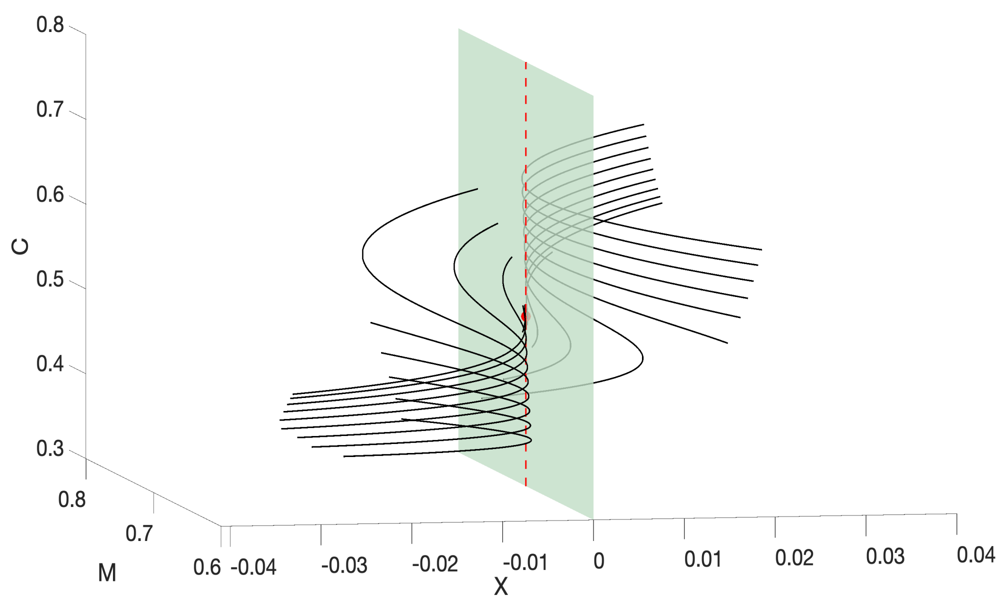
$$F_0(X,M,C) = XM(1-X)(1-M).$$

The critical manifold S is given by regular two-dimensional subsets of the zero set $\{F_0(X,M,C)=0\}$; in particular $S=\{X=0\}\cup\{X=1\}\cup\{M=0\}\cup\{M=1\}$. The one-dimensional linear fast fibers are spanned by the vector

$$N(X,M,C) = \begin{pmatrix} M-\frac{7}{10}\\ \frac{6C}{1+2C}-\frac{3}{2}\\ \frac{1}{4}(1-X-C)\end{pmatrix}$$



(a)



(b)

Figure 3: Sample trajectory (black curve) of the system (16) with initial condition $(X, M, C) = (0.5, 0.5, 0.5)$ and $\varepsilon = 0.0021$. The green plane $X = 0$ lies in the set of equilibria of (16). An isolated contact point K (red point) of order 2 ((17)) lies on the fold line $F = \{X = 0\} \cap \{M = 7/10\}$ (red dashed line). An isolated saddle-focus equilibrium of (16) is also depicted (black point). (b) Local geometry of the fast fibers near the fold line. Note that we extend the fast fibers into the unphysical regime $X < 0$.

at points $(X, M, C) \in S$. In the sequel we denote $f = F_0$ so that we can read off the defining equations classifying the singularities along the contact set.

Let us record the following derivatives:

$$\begin{aligned}
Df &= \begin{pmatrix} M(1-M)(1-2X) & X(1-X)(1-2M) & 0 \end{pmatrix} \\
D^2f &= \begin{pmatrix} 2M(M-1) & (1-2X)(1-2M) & 0 \\ (1-2X)(1-2M) & 2X(X-1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
DN &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{6}{(2C+1)^2} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \end{pmatrix}
\end{aligned}$$

For the remainder of the problem we restrict ourselves to the subset $S = \{X = 0\}$ of the critical manifold (see Fig. 3).
We have

$$DfN|_{S'} = M(1-M)(M-7/10).$$

We read off that S loses normal hyperbolicity along the lines $M = 0$, $M = 7/10$, and $M = 1$, and that S is attracting on the subset $S_a = S \cap \{0 < M < 7/10\}$ and repelling on the subset $S_r = S \cap \{7/10 < M < 1\}$. We will not consider the degenerate lines $M = 0$ and $M = 1$. These define the 'corners' of S; blow-up is used to analyze the dynamics nearby (see [13]). We focus on the fold line $F = S \cap \{M = 7/10\}$. Note that the critical manifold remains locally two-dimensional along this line.

The matrix DfN drops rank along F, giving us the trivial left- and right- nullvectors $l = r = 1$ of DfN.

Contact folds. We test the nondegeneracy condition:

$$\begin{aligned}
l(D^2f(Nr, Nr) + DfDN(Nr, r)) &= N^T D^2 f N + DfDNN \\
&= 0 + \frac{63}{200}\frac{2C-1}{2C+1}
\end{aligned}$$

along F. Thus, a line of fold points separates S_a from S_r, but there is a distinguished point

$$z_0 = F \cap \{C = 1/2\} = \{(X, M, C) = (0, 7/10, 1/2)\}, \tag{17}$$

which has higher contact order.

Contact cusps. We test the nondegeneracy condition:

$$
\begin{aligned}
& l\cdot(D^3f(Nr,Nr,Nr)+3D^2f(DN(Nr,r),Nr)+DfD^2N(Nr,Nr,r)+DfDN(DN(Nr,r),r)) \\
= \;& D^3f(N,N,N)+3D^2f(DNN,N)+DfD^2N(N,N,1)+DfDNDNN \\
= \;& 0+0+0+\frac{63}{1000}
\end{aligned}
$$

when evaluated at z_0.

We test the transversality condition:

$$
C_0(z_0) \;=\; \begin{pmatrix} Df \\ N^T D^2f + DfDN \end{pmatrix}\Bigg|_{z_0}.
$$

At z_0 we have

$$
\begin{aligned}
Df &= \begin{pmatrix} 21/100 & 0 & 0 \end{pmatrix} \\
D^2f &= \begin{pmatrix} -21/50 & -2/5 & 0 \\ -2/5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
N &= \begin{pmatrix} 0 \\ 0 \\ 1/8 \end{pmatrix} \\
DN &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3/2 \\ -1/4 & 0 & -1/4 \end{pmatrix},
\end{aligned}
$$

and so altogether we have

$$
C_0(z_0) \;=\; \begin{pmatrix} 21/100 & 0 & 0 \\ 0 & 21/100 & 0 \end{pmatrix}.
$$

The mitotic oscillator therefore exhibits a contact cusp at the point z_0 (see (17)).

6 Concluding remarks

In this paper we discussed singularities of the contact set

$$
F \;=\; \{z\in S : \operatorname{rank}(DfN) = n-k-1\}
$$

for multiple-timescale dynamical systems along points of the critical manifold S. We expect that the analysis of unfoldings of 'rank drops larger than one' (i.e. for the set $\{z \in S :$

rank$(DfN) < n - k - 1\}$ quickly become more complicated. For example, nilpotent cusp-type flows on the center manifold of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha x^2 + \beta xy \end{pmatrix},$$

where $\alpha \neq 0$, may occur for rank-two drops. This vector field admits a three-parameter unfolding [3]. The list of nilpotent cases continues to grow in length and complexity for larger rank drops, but in principle the unfolding scenarios can be systematically classified as in the present paper.

We may also consider subsets where complex-conjugate pairs of eigenvalues cross the imaginary axis. This set is related to the existence of delayed Hopf and singular Hopf bifurcations. These may produce novel types of relaxation oscillations, arising from the interactions between these local bifurcations which produce small-amplitude oscillations, together with a nonstandard global return mechanisms due to curving of the fast fibers. Analysis of these complex eigenvalue crossings is a topic of future work.

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A Appendix

We write down basic definitions for jet spaces and the contact group of diffeomorphisms (see for eg. [8, 10] for a full treatment of the standard singularity theory).

Definition 11. The k-jet space $J^k(n,m)$ of smooth germs $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$J^k(n,m) \;=\; \mathcal{M}_n \cdot \mathcal{E}(n,m)/\mathcal{M}_n^{k+1} \cdot \mathcal{E}(n,m),$$

where

$$\mathcal{E}(n,m) \;=\; (\mathcal{E}_n)^m$$

is the direct product of m copies of the set $\mathcal{E}_n$ of smooth germs from $\mathbb{R}^n$ to $\mathbb{R}$,

$$\mathcal{M}_n \;=\; \mathcal{E}_n \cdot \{x_1, \cdots, x_n\}$$

is the unique maximal ideal of germs vanishing at the origin, and

$$\mathcal{M}_n^k \;=\; \mathcal{E}_n \cdot \{x_1^{i_1}, \cdots, x_n^{i_n}, i_1 + \cdots + i_n = k\}$$

is the set of germs with vanishing partial derivatives of order less than or equal to $k-1$ at the origin.

Remark 8. The set $J^k(n,m)$ may be identified with the set of polynomials of total degree less than or equal to k.

The definition of contact classes used in the paper is due to Mather:

Definition 12. The *contact group* $\mathcal{K}$ is the set of germs of diffeomorphisms of $(\mathbb{R}^n \times \mathbb{R}^m, (0,0))$ which can be written in the form

$$H(x,y) \quad = \quad (h(x), H_1(x,y)),$$

where h acts on the right (i.e. $h \cdot f = f \circ h^{-1}$) and $H_1(x,0) = 0$ for x near 0. We say that f is $\mathcal{K}$-*equivalent* to g if g lies in the group orbit of f. We refer to this as the contact class of f.

Remark 9. Suppose $f, g \in \mathcal{M}_n \cdot \mathcal{E}(n,m)$ and $k = (h, H) \in \mathcal{K}$. Then $g = k \cdot f$ if and only if

$$(x, g(x)) \quad = \quad H(h^{-1}(x), f(h^{-1}(x))).$$

Observe that H sends the graph of f to the graph of g near 0 (i.e. the zero sets of $\mathcal{K}$-equivalent germs are diffeomorphic).