

Equations of motion of a bell and clapper

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1 Notation

We start by formulating the Lagrangian of a freely swinging bell and clapper. The notation is shown in fig 1. Let:

θ	=	angle of the bell from bottom dead centre
M	=	mass of the bell
b	=	distance from the axis of the bell to its centre of gravity
I'_b	=	moment of inertia of the bell about its centre of gravity
I_b	=	moment of inertia of the bell about its axis
K_b	=	friction coefficient of the bell
a	=	distance from the bell axis to the clapper axis
α	=	angle of the clapper axis from the centre line of the bell
ϕ	=	angle of the clapper from the centre line of the bell
m	=	mass of the clapper
c	=	distance from the clapper axis to the clapper centre of gravity
I'_c	=	moment of inertia of the clapper about its centre of gravity
I_c	=	moment of inertia of the clapper about its axis
K_c	=	friction coefficient of the clapper

(1)

Some useful relations follow. From the parallel axis theorem,

$$I_b = I'_b + Mb^2 \tag{2}$$

$$I_c = I'_c + mc^2 \tag{3}$$

To calculate the potential energy it is useful to know the cartesian coordinates of the bell and clapper. Taking the axis of the bell to be at $(0, 0)$, with the usual coordinates the location of the bell centre of gravity is

$$(b \sin \theta, -b \cos \theta) \tag{4}$$

The location of the clapper axis is

$$(a \sin(\theta + \alpha), -a \cos(\theta + \alpha)) \tag{5}$$

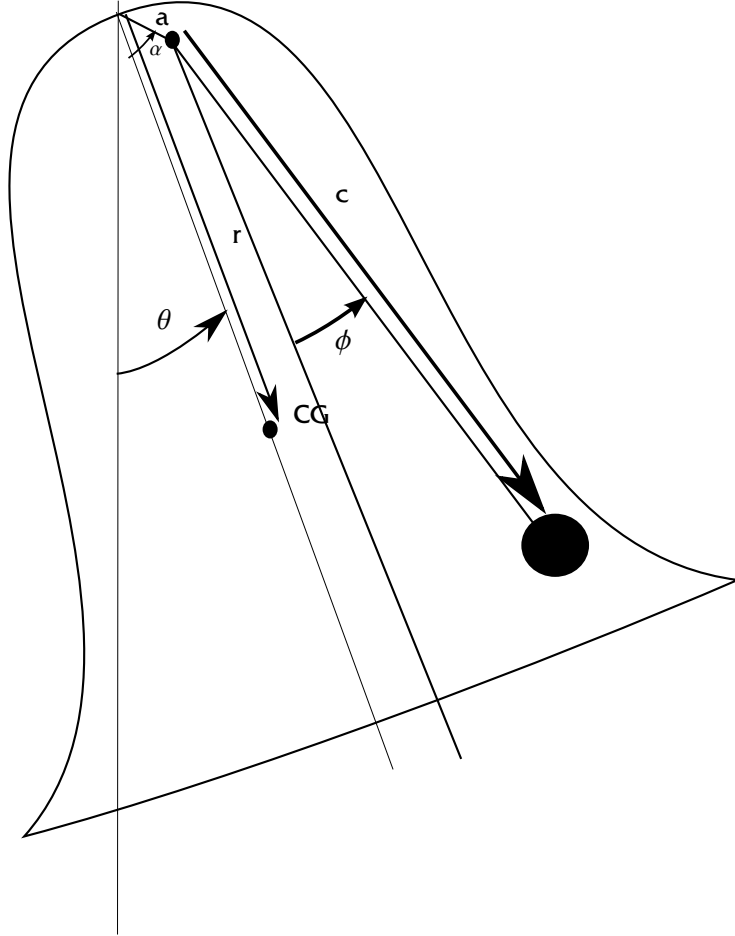


Figure 1: Notation. The bell is mass M , with centre of gravity located a distance r from the axis. θ is the angle of the bell from bottom dead-centre. The clapper has mass m , and a pivot axis located at angle α from the centre line of the bell at a distance a from the bell axis. The clapper has mass m with a centre of gravity located at a distance c from the clapper axis. The moment of inertia of the bell about its centre of gravity is I'_b , and of the clapper about its centre of gravity I'_c . The moments of inertia of the bell and clapper about their respective axes are I_b and I_c .

and the clapper centre of gravity is at

$$(a \sin(\theta + \alpha) + c \sin(\theta + \phi), -a \cos(\theta + \alpha) - c \cos(\theta + \phi)) \quad (6)$$

or, in angular coordinates,

$$\begin{aligned} r_c^2 &= a^2 + b^2 + 2ac \cos(\phi - \alpha) \\ \theta_c &= \theta + \arctan \frac{a \cos \alpha + c \cos \phi}{a \sin \alpha + c \sin \phi} \end{aligned} \quad (7)$$

2 Kinetic energy

The kinetic energy of the bell is the sum of the motion of the centre of gravity plus the rotational kinetic energy.

$$KE_b = \frac{1}{2} M r^2 \dot{\theta}^2 + \frac{1}{2} I'_b \dot{\theta}^2 = \frac{1}{2} I_b \dot{\theta}^2 \quad (8)$$

To find the kinetic energy of the clapper, we first obtain the velocity in cartesian coordinates of the clapper centre of mass,

$$\vec{v}_c = (a\dot{\theta} \cos(\theta + \alpha) + c(\dot{\theta} + \dot{\phi}) \cos(\theta + \phi), a\dot{\theta} \sin(\theta + \alpha) + c(\dot{\theta} + \dot{\phi}) \sin(\theta + \phi)) \quad (9)$$

which gives the squared velocity

$$v_c^2 = a^2 \dot{\theta}^2 + c^2 (\dot{\theta} + \dot{\phi})^2 + 2ac \dot{\theta} (\dot{\theta} + \dot{\phi}) \cos(\phi - \alpha) \quad (10)$$

The kinetic energy of the clapper is the sum of the centre of mass motion plus the rotational kinetic energy about the centre of mass, $mv_c^2/2 + I'_c(\dot{\theta} + \dot{\phi})^2/2$, giving

$$KE_c = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} I_c (\dot{\theta} + \dot{\phi})^2 + m a c \dot{\theta} (\dot{\theta} + \dot{\phi}) \cos(\phi - \alpha) \quad (11)$$

where we have used the parallel axis theorem to use the moment of inertia about the clapper axis.

3 Potential energy

We've already obtained the location of the centre of mass of the bell and clapper in cartesian coordinates so finding the potential energy mgh is straightforwardly, for the bell,

$$PE_b = -Mgb \cos \theta \quad (12)$$

and

$$PE_c = -mg(a \cos(\theta + \alpha) + c \cos(\theta + \phi)) \quad (13)$$

for the clapper.

4 Lagrangian

This gives now the Lagrangian for the freely swinging bell+clapper,

$$\begin{aligned}\mathcal{L} = & I_b \dot{\theta}^2/2 + I_c (\dot{\theta} + \dot{\phi})^2/2 \\ & + ma^2 \dot{\theta}^2/2 + mac \dot{\theta} (\dot{\theta} + \dot{\phi}) \cos(\phi - \alpha) \\ & + Mgb \cos \theta + mga \cos(\theta + \alpha) + mgc \cos(\theta + \phi)\end{aligned}\quad (14)$$

5 Equations of motion

We can now obtain Lagrange's equations of motion,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad (15)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (16)$$

Firstly, for θ , we obtain

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_b \dot{\theta} + I_c (\dot{\theta} + \dot{\phi}) + ma^2 \dot{\theta} + mac(2\dot{\theta} + \dot{\phi}) \cos(\phi - \alpha) \quad (17)$$

so that

$$\begin{aligned}\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = & I_b \ddot{\theta} + I_c (\ddot{\theta} + \ddot{\phi}) + ma^2 \ddot{\theta} \\ & + mac(2\ddot{\theta} + \ddot{\phi}) \cos(\phi - \alpha) - mac\dot{\phi}(2\dot{\theta} + \dot{\phi}) \sin(\phi - \alpha)\end{aligned}\quad (18)$$

Now

$$\frac{\partial \mathcal{L}}{\partial \theta} = -Mgb \sin \theta - mga \sin(\theta + \alpha) - mgc \sin(\theta + \phi) \quad (19)$$

So our equation of motion is

$$\begin{aligned}0 = & I_b \ddot{\theta} + ma^2 \ddot{\theta} + I_c (\ddot{\theta} + \ddot{\phi}) \\ & + mac(2\ddot{\theta} + \ddot{\phi}) \cos(\phi - \alpha) - mac\dot{\phi}(2\dot{\theta} + \dot{\phi}) \sin(\phi - \alpha) \\ & + Mgb \sin \theta + mga \sin(\theta + \alpha) + mgc \sin(\theta + \phi)\end{aligned}\quad (20)$$

Secondly, for ϕ , we get

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_c (\dot{\theta} + \dot{\phi}) + mac \dot{\theta} \cos(\phi - \alpha) \quad (21)$$

so that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_c (\ddot{\theta} + \ddot{\phi}) + mac \ddot{\theta} \cos(\phi - \alpha) - mac \dot{\theta} \dot{\phi} \sin(\phi - \alpha) \quad (22)$$

Now

$$\frac{\partial \mathcal{L}}{\partial \phi} = mac \dot{\theta} (\dot{\theta} + \dot{\phi}) \sin(\phi - \alpha) - mgc \sin(\theta + \phi) \quad (23)$$

so that our equation of motion is

$$0 = I_c(\ddot{\theta} + \ddot{\phi}) + mac\ddot{\theta}\cos(\phi - \alpha) - mac\dot{\theta}\dot{\phi}\sin(\phi - \alpha) + mac\dot{\theta}(\dot{\theta} + \dot{\phi})\sin(\phi - \alpha) + mgc\sin(\theta + \phi) \quad (24)$$

which simplifies to

$$0 = I_c(\ddot{\theta} + \ddot{\phi}) + mac\ddot{\theta}\cos(\phi - \alpha) + mac\dot{\theta}^2\sin(\phi - \alpha) + mgc\sin(\theta + \phi) \quad (25)$$

This is the equation of motion for the clapper. It will be valid only when the clapper is in free motion, ie when

$$\phi_1 \leq \phi \leq \phi_2 \quad (26)$$

where ϕ_1, ϕ_2 are the boundary angles at which the clapper strikes the bell.

Our first equation Eq. (20) includes several terms in this equation. Hence we can simplify Eq. (20) into

$$0 = (I_b + ma^2)\ddot{\theta} + mac(\ddot{\theta} + \ddot{\phi})\cos(\phi - \alpha) - mac(\dot{\theta} + \dot{\phi})^2\sin(\phi - \alpha) + Mgb\sin\theta + mga\sin(\theta + \alpha) \quad (27)$$

We take Eq. (25) and Eq. (27) as our equations of motion.

We will work at simplifying these equations, but firstly lets verify the correctness. In the special case where $\alpha = 0, a = b$ (ie, when the clapper axis is located at the centre of gravity of the bell) this system reduces to a double pendulum. Comparing with the Mathworld equations [1], we set the notation

$$\begin{aligned} m_1 &= M \\ l_1 &= b \\ \theta_1 &= \theta \\ m_2 &= m \\ l_2 &= c \\ \theta_2 &= \theta + \phi \end{aligned} \quad (28)$$

and $I_b = m_1 l_1^2, I_c = m_2 l_2^2$. Substituting into the equation of motion Eq. (25) we obtain

$$0 = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + m_2 g l_2 \sin \theta_2 \quad (29)$$

which is identical to equation (18) of the double pendulum equations at the Wolfram website. Substituting into the equation of motion Eq. (27) we obtain

$$0 = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + (m_1 + m_2) l_1 g \sin \theta_1 \quad (30)$$

which coincides with equation (13) of the same reference.

6 Friction

We can incorporate friction in the clapper and the bell by modifying Lagrange's equations to

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta \quad (31)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = Q_\phi \quad (32)$$

where the friction forces are

$$Q_\theta = -K_b \dot{\theta} + f(t) \quad (33)$$

$$Q_\phi = -K_c \dot{\phi} \quad (34)$$

and we have also added an external force on the bell which will be exerted by the ringer. This modifies the equations of motion to

$$0 = K_c \dot{\phi} + I_c (\ddot{\theta} + \ddot{\phi}) + mac \ddot{\theta} \cos(\phi - \alpha) + mac \dot{\theta}^2 \sin(\phi - \alpha) + mgc \sin(\theta + \phi) \quad (35)$$

and, recalling that we derived Eq. (27) by subtracting Eq. (25) from Eq. (20), we obtain

$$\begin{aligned} f(t) = & K_b \dot{\theta} - K_c \dot{\phi} + (I_b + ma^2) \ddot{\theta} \\ & + mac(\ddot{\theta} + \ddot{\phi}) \cos(\phi - \alpha) - mac(\dot{\theta} + \dot{\phi})^2 \sin(\phi - \alpha) \\ & + Mgb \sin \theta + mga \sin(\theta + \alpha) \end{aligned} \quad (36)$$

We note that Eq. (35) is identical (up to a small change of notation) to that obtained by F.H. King [2]. Our equation of motion for the bell Eq. (36) is quite different however, which is due to the fact that we have allowed the motion of the clapper to influence the bell. In most bells where the clapper is insignificant to the motion of the bell, which amounts to $mac \ll I_b$, and remembering that K_c will be proportional to the mass of the clapper so that is also negligible, then Eq. (36) simplifies quite considerably, to

$$f(t) = K_b \dot{\theta} + I_b \ddot{\theta} + Mgb \sin \theta \quad (37)$$

which is the same as the equation listed in [2]. However in situations where the motion of the clapper does significantly affect the bell, then we must retain all terms in Eq. (36), and also determine what happens when the clapper impacts with the bell.

7 Solving the simplified equations

We now examine what is required to solve the equations Eq. (35) and Eq. (37). Following [2], let us define

$$l_b = \frac{I_b}{Mb} \quad (38)$$

$$k_b = \frac{K_b}{Mb} \quad (39)$$

$$l_c = \frac{I_c}{mc} \quad (40)$$

$$k_c = \frac{K_c}{mc} \quad (41)$$

$$F(t) = \frac{f(t)}{I_b} \quad (42)$$

the equations become

$$F(t) = \ddot{\theta} + \frac{k_b}{l_b} \dot{\theta} + \frac{g}{l_b} \sin \theta \quad (43)$$

and

$$0 = (\ddot{\theta} + \ddot{\phi}) + \frac{k_c}{l_c} \dot{\phi} + \frac{c}{l_c} \ddot{\theta} \cos(\phi - \alpha) + \frac{c}{l_c} \dot{\theta}^2 \sin(\phi - \alpha) + \frac{g}{l_c} \sin(\theta + \phi) \quad (44)$$

To solve these, we convert into set of coupled first-order equations by introducing a set of auxiliary variables v_θ and v_ϕ . We can solve θ independently,

$$\dot{\theta} = v_\theta \quad (45)$$

$$\dot{v}_\theta = F(t) - \frac{k_b}{l_b} v_\theta - \frac{g}{l_b} \sin \theta \quad (46)$$

which can be used in an ODE solver. We can also solve for the clapper position,

$$\dot{\phi} = v_\phi \quad (47)$$

$$\dot{v}_\phi = \dot{v}_\theta \left(1 + \frac{c}{l_c} \cos(\phi - \alpha) \right) - \frac{k_c}{l_c} v_\phi + \frac{c}{l_c} v_\theta^2 \sin(\phi - \alpha) + \frac{g}{l_c} \sin(\theta + \phi) \quad (48)$$

Strictly speaking we shouldn't have \dot{v}_θ on the right hand side, but this is defined in Eq. (46) anyway.

8 Pull-ometer

In the case where the motion of the clapper is insignificant to the motion of the bell, we need to solve only the pair of first order ODE's Eq. (45,46). If the accelerometer is located at a distance r from the bell axis, at an angle β from bottom dead centre, orientated such that the x axis is tangential to the wheel

and the y axis is pointing inwards, then the x axis will measure the angular acceleration plus a component due to gravity,

$$\ddot{x} = r\ddot{\theta} + g \sin(\theta + \beta) \quad (49)$$

and the y axis will measure the centripetal acceleration plus gravity,

$$\ddot{y} = -r\dot{\theta}^2 - g \cos(\theta + \beta) \quad (50)$$

Inserting the equations of motion,

$$\ddot{x} = r \left(F(t) - \frac{k_b}{l_b} v_\theta - \frac{g}{l_b} \sin \theta \right) + g \sin(\theta + \beta) \quad (51)$$

and

$$\ddot{y} = -rv_\theta^2 - g \cos(\theta + \beta) \quad (52)$$

The idea will be that we solve the equations of motion, which give the instantaneous θ and v_θ . To find the time derivatives, we need to know what $F(t)$ is. We can calculate that from knowing \ddot{x} . But we also know \ddot{y} , so we ideally want to use this to improve the fit. Effectively, we have two equations (for \ddot{x} and \ddot{y}), with three unknowns, θ , v_θ and $F(t)$.

8.1 Free case

We firstly consider the case where there is no external force on the bell. We then have

$$\ddot{x} = -\frac{r}{l_b} (k_b v_\theta + g \sin \theta) + g \sin(\theta + \beta) \quad (53)$$

$$\ddot{y} = -rv_\theta^2 - g \cos(\theta + \beta) \quad (54)$$

If we have all the constants calibrated then these two equations determine θ and v_θ . However we have calibration constants to determine, r , l_b , k_b , and β . Ideally we want to tune the position of the sensor such that $r = l_b$ and $\beta = 0$. However this may be difficult, so we need at minimum a way to determine these values empirically. From small oscillations of the bell, we obtain

$$\ddot{\theta} = -\frac{g}{l_b} \theta \quad (55)$$

which has the solution $\theta = A \sin(\omega t + \theta_0)$ where

$$\omega = \sqrt{\frac{g}{l_b}} \quad (56)$$

which gives

$$\ddot{x} = -r \frac{g}{l_b} A \sin(\omega t + \theta_0) + g \sin \beta + g A \cos \beta \sin(\omega t + \theta_0) \quad (57)$$

or

$$\ddot{x} = gA \left(\cos \beta - \frac{r}{l_b} \right) \sin(\omega t + \theta_0) + g \sin \beta \quad (58)$$

We can easily calculate ω , which determines l_b . But we can't get much further because A is unknown, and we can't easily determine it accurately. From the equation for \ddot{y} , we get

$$\ddot{y} = -\frac{rgA^2}{2l_b}(1 + \cos(2\omega t + 2\theta_0)) - g \cos \beta + gA \sin(\omega t + \theta_0) \sin \beta \quad (59)$$

At this point, we should remember to allow for a mixing angle between \ddot{x} and \ddot{y} , which is the error in the angle of the accelerometer axes. Let this angle be γ , so that our measured \ddot{x}' and \ddot{y}' are

$$\begin{aligned} \ddot{x}' &= \ddot{x} \cos \gamma + \ddot{y} \sin \gamma \\ \ddot{y}' &= -\ddot{x} \sin \gamma + \ddot{y} \cos \gamma \end{aligned} \quad (60)$$

Firstly, supposing we had $\gamma = 0$ and $\beta = 0$. Then the peak-to-peak amplitude of the \ddot{x} oscillation is $2gA(1 - \frac{r}{l_b})$. Whereas for \ddot{y} it is $\frac{rgA^2}{l_b}$. This gives

$$\frac{A_x^2}{A_y} = \frac{4g(l_b^2 + r^2 - 2rl_b)}{rl_b} = \frac{4g(l_b - r)^2}{rl_b} \quad (61)$$

which determines r . Unfortunately this is a quadratic with two solutions, so we don't know if r is too big or too small by this method.

Alternatively, if we subtract off the constant offset in \ddot{x} and \ddot{y} , then the oscillating parts are

$$\ddot{x}_1 = gA(1 - \frac{r}{l_b}) \sin(\omega t + \theta_0) \quad (62)$$

and

$$\ddot{y}_1 = -\frac{rgA^2}{2l_b} \cos(2\omega t + 2\theta_0) \quad (63)$$

Thus, if we move the bell out to an angle θ_0 and let it go, the sign of the oscillation in \ddot{x} tells us whether r is larger or smaller than l_b . If $r > l_b$, meaning the sensor is too far away from the axis, then \ddot{x} will have the same sign as $\ddot{\theta}$. If $r < l_b$, then the sensor is too close to the axis and \ddot{x} will have the opposite sign to $\ddot{\theta}$.

Unfortunately, if we have $\beta \neq 0$, then this will confuse the situation. We then have

$$\ddot{x}_1 = gA(\cos \beta - \frac{r}{l_b}) \sin(\omega t + \theta_0) \quad (64)$$

so by following the above procedure we would mistakenly end up finding $r = l_b \cos \beta$.

Lets be a bit more systematic. We note that \ddot{y} contains terms that oscillate at both ω and 2ω . If we have tuned $\beta = 0$ then the ω oscillation drops out but we always have a 2ω component. On the other hand, \ddot{x} only has ω oscillation

components. Therefore, we can tune the angle γ separately first, such that we eliminate any 2ω oscillation from \ddot{x} .

Then, we can tune β to eliminate any ω oscillation from \ddot{y} .

Finally, we can tune r to eliminate the ω oscillation from \ddot{x} .

As long as we keep the other parameters constant when adjusting γ , then β , then r , this should systematically get the correct position of the sensor.

Another calibration option is to put a constant force on the rope and solve for the steady state. Let $F(t) = F$ be a constant. Then in the steady state,

$$F = \frac{g}{l_b} \sin \theta \quad (65)$$

Perhaps a good approach is to obtain l_b and initially ignore friction, but solve for $F(t)$. Then the contribution of the friction term can be obtained later.

9 Clapper-Bell Interaction

10 Gyro-based pullometer

We can use a gyro as a pullometer, which is based around the equation for the external force on the bell,

$$F(t) = \ddot{\theta} + (g/l) \sin \theta \quad (66)$$

where g is the acceleration due to gravity, and l is the pendulum length (0.640m for the 5th at St Johns).

The output of the gyroscope will be proportional to the velocity $\dot{\theta}$, say

$$\dot{\theta} = kv \quad (67)$$

where v is the raw gyro output and k is some calibration constant. This gives

$$\ddot{\theta} = k\dot{v} \quad (68)$$

and

$$\theta = k \int_0^t v(v') dt' = kx \quad (69)$$

where we assume that $\theta(0) = 0$. With respect to v , our equation of motion is therefore

$$F(t) = k\dot{v} + (g/l) \sin kx \quad (70)$$

For small oscillations with no external forces we therefore have

$$0 = \dot{v} + (g/l)x \quad (71)$$

which allows us to calibrate g/l . Once this constant is calibrated, larger period oscillations with no external force allows us to calibrate k , by fixing k such that

$$\frac{\sin kx}{k} = -(l/g)\dot{v} \quad (72)$$

11 The rope

Lets consider how the rope influences the equations of motion. To simplify, we assume that the section of the rope that goes around the wheel is massless, that is, we model the rope as a mass on the end of a thin piece of string. This isn't strictly true – the section of rope that wraps around the wheel will reduce the mass of the rope that is pulling downwards on the bell, and the rope around the wheel will also have a small effect on the centre of gravity and moment of inertia of the bell, but we neglect these effects for the time being.

Let the radius of the wheel be l_w . For simplicity we will assume that the pulley is located flush with the horizontal and vertical edges of the wheel, and the garter hole is located at the angular position γ on the wheel. For a typical bell, $\gamma \simeq 3\pi/4$. At $\theta + \gamma = 0$ and $\theta + \gamma = \pi/2$ the distance to the pulley is l_w . At $\theta + \gamma = \pi/4$, distance is $(\sqrt{2} - 1)l_w$, which is the minimum height of the rope. The vertical height of the rope, relative to the location of the pulley, is therefore

$$\frac{y_r}{l_w} = \begin{cases} \theta + \gamma - \pi/2 + 1, & \theta + \gamma > \pi/2 \\ \sqrt{3 - 2\sqrt{2}\cos(\theta + \gamma - \pi/4)}, & 0 < \theta + \gamma < \pi/2 \\ 1 - (\theta + \gamma), & \theta + \gamma < 0 \end{cases} \quad (73)$$

This is a rather ugly function in the region $0 < \theta + \gamma < \pi/2$. Physically, eg for $\gamma = 3\pi/4$, this region is is when the bell is at angles from $-3\pi/4 < \theta < \pi/4$, which is when the bell is in free movement. We can try a quadratic approximation in this region, where we match y_r and $\frac{dy_r}{d\theta}$ at the boundaries ($\theta + \gamma - \pi/4 = \pm\pi/4$),

$$\begin{aligned} y_r/l_w &= \frac{2}{\pi}(\theta + \gamma - \pi/4)^2 + 1 - \pi/8 \\ &= \frac{2}{\pi}(\theta + \gamma)^2 - (\theta + \gamma) + 1 \end{aligned} \quad (74)$$

This should be a quite reasonable approximation. The largest error will be when $\theta + \gamma = \pi/4$, where the length of the rope is $y_r/l_w = 1 - \pi/8 \simeq 0.393$, rather than the exact value $\sqrt{2} - 1 \simeq 0.414$.

If we let M_r be the mass of the rope, then we get a potential energy term of

$$PE_r = M_r g y_r \quad (75)$$

To get the kinetic energy, we need the rope velocity as a function of θ and $\dot{\theta}$. The only complicated part is when $0 < \theta + \gamma < \pi/2$, when the velocity depends explicitly on θ . We can determine $\dot{y}_r = \frac{\partial y_r}{\partial \theta} \dot{\theta}$. This gives,

$$\frac{v_r}{l_w} = \begin{cases} \dot{\theta}, & \theta + \gamma > \pi/2 \\ \dot{\theta} \frac{4}{\pi}(\theta + \gamma - \frac{\pi}{4}), & 0 < \theta + \gamma < \pi/2 \\ -\dot{\theta}, & \theta + \gamma < 0 \end{cases} \quad (76)$$

Aside: the exact expression is

$$\frac{v_r}{l_w} = \begin{cases} \dot{\theta}, & \theta + \gamma > \pi/2 \\ \frac{2\sqrt{2}\dot{\theta}\sin(\theta+\gamma-\pi/4)}{2y_r/l_w}, & 0 < \theta + \gamma < \pi/2 \\ -\dot{\theta}, & \theta + \gamma < 0 \end{cases} \quad (77)$$

So, the kinetic energy of the rope (approximate expression) is

$$KE_r = M_r l_w^2 \dot{\theta}^2 / 2 \times \begin{cases} 1, & \theta + \gamma > \pi/2 \\ \frac{16}{\pi^2} (\theta + \gamma - \frac{\pi}{4})^2, & 0 < \theta + \gamma < \pi/2 \\ 1, & \theta + \gamma < 0 \end{cases} \quad (78)$$

We can now construct the Lagrangian of the bell + wheel. For the time being, we neglect the clapper.

$$\mathcal{L} = I_b \dot{\theta}^2 / 2 + Mgb \cos \theta + KE_r - PE_r \quad (79)$$

The equations of motions in the 3 sectors:

1. $\theta + \gamma > \pi/2$ In this region the Lagrangian is

$$\mathcal{L} = (I_b + M_r l_w^2) \dot{\theta}^2 / 2 + Mgb \cos \theta - M_r g l_w (\theta + \gamma - \pi/2 + 1) \quad (80)$$

For the equations of motion, we calculate

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (I_b + M_r l_w^2) \dot{\theta} \quad (81)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = -Mgb \sin \theta - M_r g l_w \quad (82)$$

which gives the equation of motion

$$(I_b + M_r l_w^2) \ddot{\theta} = -Mgb \sin \theta - M_r g l_w \quad (83)$$

This regime corresponds to the backstroke, where the rope is getting lifted up. So we see that the effect of the rope is a uniform force, in the direction of $-\theta$.

2. The region $\theta + \gamma < 0$. This region is the handstroke.

$$\mathcal{L} = (I_b + M_r l_w^2) \dot{\theta}^2 / 2 + Mgb \cos \theta - M_r g l_w (1 - (\theta + \gamma)) \quad (84)$$

Up to irrelevant constants, this is the same as the backstroke Lagrangian, except that the sign of the potential term $M_r g l_w \theta$ is reversed. The equation of motion is

$$(I_b + M_r l_w^2) \ddot{\theta} = -Mgb \sin \theta + M_r g l_w \quad (85)$$

and we have a uniform force now in the $+\theta$ direction.

3. Now the region $0 < \theta + \gamma < \pi/2$

In this region the Lagrangian is

$$\mathcal{L} = \left(I_b + M_r l_w^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right)^2 \right) \dot{\theta}^2 / 2 + Mgb \cos \theta - M_r g l_w \left(\frac{2}{\pi} (\theta + \gamma)^2 - (\theta + \gamma) + 1 \right) \quad (86)$$

Firstly,

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \left(I_b + M_r l_w^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right)^2 \right) \dot{\theta} \quad (87)$$

so that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \ddot{\theta} \left(I_b + M_r l_w^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right)^2 \right) + M_r l_w^2 \frac{32}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right) \dot{\theta}^2 \quad (88)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = M_r l_w^2 \dot{\theta}^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right) - Mgb \sin \theta - M_r g l_w \frac{4}{\pi} \left(\theta + \gamma - \frac{\pi}{4} \right) \quad (89)$$

which leads to the equation of motion

$$\ddot{\theta} = \frac{-M_r l_w^2 \dot{\theta}^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right) - Mgb \sin \theta - M_r g l_w \frac{4}{\pi} \left(\theta + \gamma - \frac{\pi}{4} \right)}{I_b + M_r l_w^2 \frac{16}{\pi^2} \left(\theta + \gamma - \frac{\pi}{4} \right)^2} \quad (90)$$

11.1 Small oscillations

Small oscillations of the bell will be in the regime $\theta + \gamma > \pi/2$. Note that the rest position of the bell is shifted from $\theta = 0$, and will be at $\theta = -\theta_0$ (note we choose a minus sign here to make $\theta_0 > 0$,

$$Mgb \sin \theta_0 = M_r g l_w \quad (91)$$

Assuming θ_0 is small, we have

$$\theta_0 = \frac{M_r l_w}{Mb} \quad (92)$$

For small oscillations, let $\theta = -\theta_0 + \epsilon$. Then

$$\ddot{\epsilon} = -\frac{Mgb \sin(-\theta_0 + \epsilon) + M_r g l_w}{I_b + M_r l_w^2} = -\frac{Mbg}{I_b + M_r l_w^2} (\sin(-\theta_0 + \epsilon) + \theta_0) \quad (93)$$

Hence

$$\ddot{\epsilon} = -\frac{Mbg}{I_b + M_r l_w^2} \epsilon \quad (94)$$

which corresponds to oscillation with frequency

$$\omega_0 = \sqrt{\frac{Mbg}{I_b + M_r l_w^2}} \quad (95)$$

11.2 parameterization

Lets try and improve the notation a bit. Extending our existing definitions,

$$l_b = \frac{I_b}{Mb} \quad (96)$$

$$l_r = \frac{M_r l_w^2}{Mb} \quad (97)$$

$$\theta_0 = \frac{M_r l_w}{Mb} = l_r / l_w \quad (98)$$

Note that l_r is a small parameter. This gives

$$\omega_0 = \sqrt{\frac{g}{l_b + l_r}} \quad (99)$$

The equations of motions in the 3 sectors:

1. $\theta + \gamma > \pi/2$

$$\ddot{\theta} = -\frac{g}{l_b + l_r} ((\sin \theta) + \theta_0) \quad (100)$$

2. $\theta + \gamma < 0$

$$\ddot{\theta} = -\frac{g}{l_b + l_r} ((\sin \theta) - \theta_0) \quad (101)$$

3. $0 < \theta + \gamma < \pi/2$

$$\text{Let } n(\theta) = \frac{4}{\pi}(\theta + \gamma - \frac{\pi}{4}).$$

$$\ddot{\theta} = \frac{-l_r \frac{4}{\pi} n(\theta) \dot{\theta}^2 - g(\sin \theta + \theta_0 n(\theta))}{l_b + l_r n(\theta)^2} \quad (102)$$

11.3 Modified potential

Going beyond the assumption that the rope is a thin string with a mass on the end, we now consider it to have a mass per unit length. As the rope wraps around the wheel, the potential energy will be modified. In principle the moment of inertia and the centre of mass change as well, but we will ignore this for the time being, and just model the rope gravitational potential energy. (In fact I think this is probably sufficient; the change in moment is already taken care of with the kinetic and potential energy of the rope). To do this, we calculate the height h of the rope at a distance y along the rope above the pulley. Thus $y = 0$ is the pulley, and $y > 0$ measures along the rope. Recall that the amount of rope above the pulley is y_r . Again this divides into 3 regimes.

For $\theta + \gamma > \pi/2$, the rope goes straight up to the wheel for a distance l_w , and then rotates around the wheel for the remainder of the length. Thus we have

$$h(y) = \begin{cases} y, & y < l_w \\ l_w \left(1 + \sin \frac{y-l_w}{l_w}\right), & y > l_w \end{cases} \quad (103)$$

For $0 < \theta + \gamma < \pi/2$, the rope travels in a straight line to the wheel. We will use the quadratic approximation. The height in this case is,

$$h(y) = y \frac{l_r (1 - \cos(\theta + \gamma))}{y_r(\theta)} \quad (104)$$

For $\theta + \gamma < 0$, at the handstroke, the rope travels horizontally to distance l_r and then wraps around the wheel. The height is

$$h(y) = \begin{cases} 0, & y < l_w \\ l_w \left(1 - \cos \frac{y-l_w}{l_w}\right), & y > l_w \end{cases} \quad (105)$$

To get the gravitational potential, we need to integrate these equations from 0 to y_r . In the first case, $\theta + \gamma > \pi/2$, we have

$$\frac{V^+}{g\rho} = \int_0^{y_r} h(y) dy = \begin{cases} y_r^2/2, & y_r < l_w \\ l_w^2/2 + l_w y_r - l_w^2 \cos \frac{y_r-l_w}{l_w}, & y_r > l_w \end{cases} \quad (106)$$

We can ignore the first case, because when $\theta + \gamma > \pi/2$, we always have $y_r \geq l_w$.

In the second case, $0 < \theta + \gamma < \pi/2$,

$$\frac{V^+}{g\rho} = \int_0^{y_r} h(y) dy = y_r^2 \frac{(1 - \cos(\theta + \gamma))}{2y_r} \quad (107)$$

In the third case, $\theta + \gamma < 0$,

$$\frac{V^+}{g\rho} = \int_0^{y_r} h(y) dy = \begin{cases} 0, & y < l_w \\ l_w y_r - l_w^2 \sin \frac{y_r-l_w}{l_w} - l_w^2, & y > l_w \end{cases} \quad (108)$$

Again we can ignore the first case, because when $\theta + \gamma < 0$, we have $y \geq l_w$.

The final potential energy is the sum of the potential above the rope, V^+ , and the potential below, which is $V^- = g\rho (L_r - y_r)^2/2$, where L_r is the total length of the rope (not to be confused with l_r , which is the change in equivalent pendulum length induced by the rope). Note that the total mass of the rope is $M_r = L_r \rho$.

Hence our final expression for the potential energy is

$$\frac{PE_r}{g\rho} = \frac{(L_r - y_r)^2}{2} + \frac{V^+}{g\rho} \quad (109)$$

where

$$\frac{V^+}{g\rho} = \begin{cases} l_w^2/2 + l_w^2(\theta + \gamma - \pi/2 + 1) - l_w^2 \cos(\theta + \gamma - \pi/2), & \theta + \gamma > \pi/2 \\ \frac{1}{2} \left(\frac{2}{\pi} (\theta + \gamma)^2 - (\theta + \gamma) + 1 \right) (1 - \cos(\theta + \gamma)), & 0 < \theta + \gamma < \pi/2 \\ l_w^2 (\sin(\theta + \gamma) - (\theta + \gamma)), & \theta + \gamma < 0 \end{cases} \quad (110)$$

References

- [1] <http://scienceworld.wolfram.com/physics/DoublePendulum.html>
- [2] F.H. King, *Equations of Motion of a Free Bell and Clapper*,
<http://www.cl.cam.ac.uk/~fhk1/equations.pdf>