Problem Set 5

Ian McGroarty

01MAR2020

Problem 1

Apply Inverse Transform Method to Laplace distribution

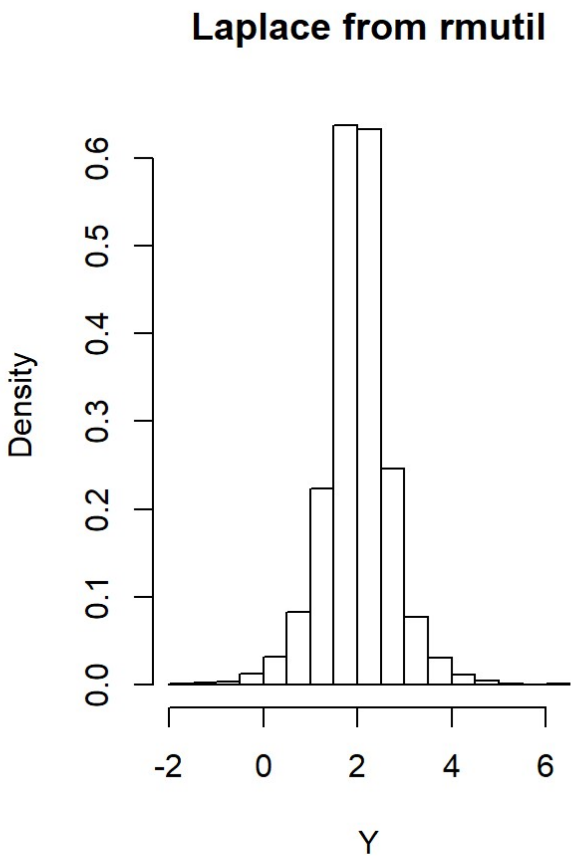
$$
\begin{aligned}
f(x) &= \frac{\lambda}{2} e^{-\lambda|x-\theta} \\
&= e^{-2|x-2} && (\lambda = 2, \theta = 2) \\
&\text{Break into cases for absolute} \\
f(x) &= e^{-2(x-2)} && x > 2 \\
f(x) &= e^{2(x-2)} && x < 2 \\
&\text{Integrate to find F(x)} \\
f(x) &= 1 - \frac{1}{2} \cdot e^{-2x+4)} && x > 2 \\
f(x) &= \frac{1}{2} \cdot e^{2x-4)} && x < 2 \\
&\text{Inverte by setting = to U} \\
x &= -\frac{1}{2}[log(-2u+2) - 4] \\
x &= \frac{1}{2}[log(2u) + 4]
\end{aligned}
$$

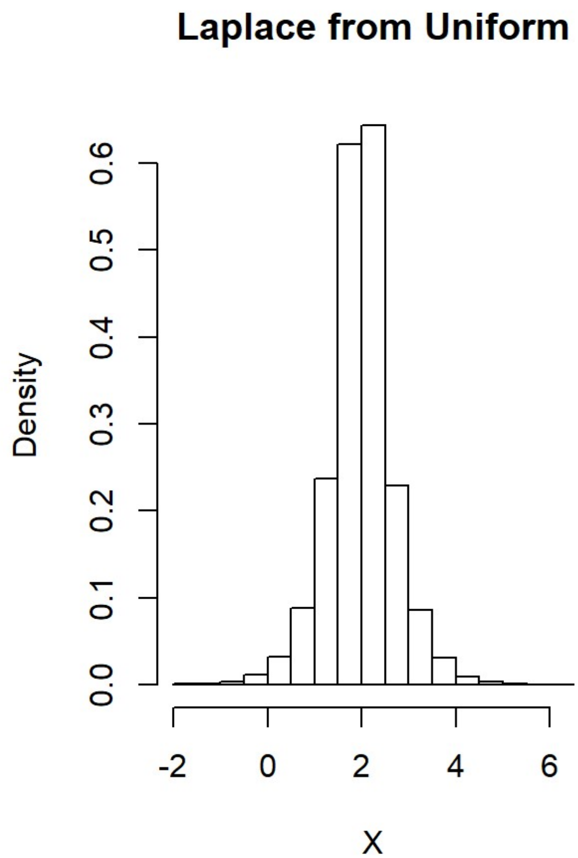
We see that the two values of X are equally likely so we integrate from 0 to 0.5 and 0.5 to 1 over U. This is demonstrated with R code below. I've also compared the derived distribution to the Laplace distribution generated by the R function *rlaplace* in package *rmutil*

```
Nsim <- 10^4
U <- runif(Nsim , min=0,max=1)

## From the actual Lapace distribution through rmutil package
  # m = theta = 2 & s = 1/lambda = (1/2)
Y <- rlaplace(Nsim, m=2,s=(1/2))

## Generate from the inverse transform method
X <- ifelse(U>.5,(log(-2*(U) + 2)-4)/(-2),
     ifelse(U<.5,(log(2*U)+4)/(2),2))
par(mfrow = c(1,2))
hist(Y,freq=F, main= "Laplace from rmutil")
hist(X,freq=F, main= "Laplace from Uniform")
```





Problem 2

Let Randomn variable X has pdf:

$$f(x) = \begin{cases} (1/4) & 0 < x < 1 \\ x - (3/4) & 1 \leq x \leq 2 \end{cases}$$

(a) Gen a r.v. using inverse transform method 1. First we integrate to find F(x)

$$
\begin{aligned}
0 < x < 1 \qquad F(x) &= \int_{\infty}^{x} \\
&= \int_{\infty}^{0} dt + \int_{0}^{x} \frac{1}{4} \\
&= \frac{t}{4}|_0^x \\
&= \frac{x}{4} \\
1 \leq x \leq 2 \qquad F(X) &= \int_{\infty}^{x} \\
&= \int_{\infty}^{0} dt + \int_{0}^{1} \frac{1}{4} dt + \int_{1}^{x} (t - (3/4) dt) \\
&= \frac{1}{4} + [\frac{1}{2}t^2 - \frac{3}{4}t]_1^x \\
&= \frac{1}{4} + [\frac{1}{2}x^2 - \frac{3}{4}x] - [\frac{1}{2} - \frac{3}{4}] \\
&= \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{2}
\end{aligned}
$$

$$
F(x) = \begin{cases} \frac{1}{4}x & 0 < x < 1 \\ \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{2} & 1 \leq x \leq 2 \end{cases}
$$

2. Find $F^{-}(x)$ by setting equal to U and solving for x:

$$
\begin{aligned}
U &= \frac{1}{4} & &\implies x = 4U|_0^{0.25} \\
U &= \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{2} & &\implies x = \frac{1}{4}(3 \mp \sqrt{32U - 7}|_{0.25}^{1})
\end{aligned}
$$

3. Using R and a uniform distribution from 0,1 with each x having equal probability for U: (R code and plot provided at the end of the problem). See that the histogram follows the line generated from plotting f(x) very well.

(b) Gen a r.v. using the accept reject method

We don't really need to do any fancy calculations for the max, so observe that:

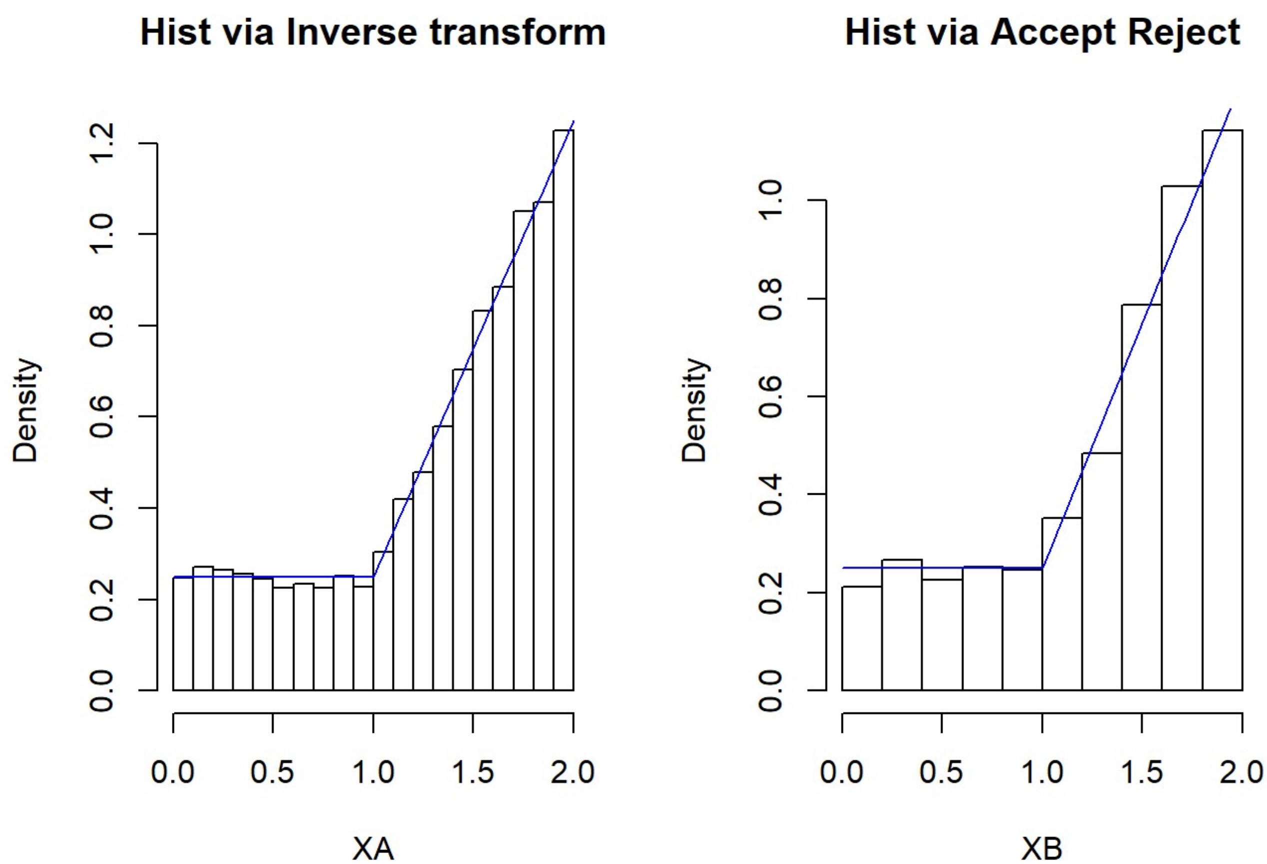
$$
sup(f(x)) = x - (3/4)|x = 2 \implies x_{opt} = (5/4) \rightarrow M = \frac{f((5/4))}{g((5/4))} = 2
$$

The rest is demonstrated in the R code below under PART (b)

```
## PART (a) USING THE INVERSE TRANSFORM METHOD
# Set the number of simulations
  Nsim <- 10^4
# Generate the uniform values to plug into the inverse
  U <- runif(Nsim , min=0,max=1)
# Evaluate the inverse for the Us
  XA <- ifelse(U<(1/4),4*U,ifelse(U>(1/4),(1/4)*(3 + sqrt(32*U -7)),NA))


## PART (b) USING ACCEPT REJECT
# Set the max
  M <- (2)
# Set the function
  fx <- function(x){
    ifelse((x<1),(1/4),ifelse((1<=x & x<= 2),x-(3/4),NA))
  }
# Generate x candidates over the range under condiseration
  Xcand <- runif(Nsim,min=0,max=2)
# Generate y candidates up to M
  Ycand <- runif(Nsim, min=0, max=M)
# Keep the x candidates for which the Y candidates are viable solutions
  XB <- Xcand[Ycand < fx(Xcand)]

## Plots for (a) and (b)
par(mfrow = c(1,2))
hist(XA,freq=F,main="Hist via Inverse transform")
plot(fx,xlim=c(0,2.5),ylim=c(0,2.5),col = "blue",add=TRUE)
hist(XB,freq=F,main="Hist via Accept Reject")
plot(fx,xlim=c(0,2.5),ylim=c(0,2.5),col = "blue",add=TRUE)
```



Problem 3

Let Randomn variable X has pdf:

$$f(x) = \begin{cases} (1/2)x & 0 < x < 1 \\ (1/2) & 1 \leq x \leq \frac{5}{2} \end{cases}$$

1. First we integrate to find F(x)

$$
\begin{aligned}
0 < x < 1 \qquad F(x) &= \int_{\infty}^{x} \\
&= \int_{\infty}^{0} dt + \int_{0}^{x} \frac{x}{2} \\
&= \frac{x^2}{4}|_0^x \\
&= \frac{x^2}{4} \\
1 \le x \le 2 \qquad F(X) &= \int_{\infty}^{x} \\
&= \int_{\infty}^{0} dt + \int_{0}^{1} \frac{x}{2} dt + \int_{1}^{x} \frac{1}{2} dt) \\
&= \frac{1}{4} + \frac{t}{2}|_1^x \\
&= \frac{1}{2} + \frac{x}{2} - \frac{1}{2} \\
&= \frac{x}{2} - \frac{1}{4}
\end{aligned}
$$

$$
F(x) = \begin{cases} \frac{1}{4}x^2 & 0 < x < 1 \\ \frac{1}{2}x - \frac{1}{4} & 1 \le x \le \frac{5}{2} \end{cases}
$$

2. Find $F^{-}(x)$ by setting equal to U and solving for x:

$$
\begin{aligned}
U &= \frac{1}{4}x^2 && \implies x = \sqrt{4U}|_0^{0.25} \\
U &= \frac{1}{2}x - \frac{1}{4} && \implies x = 2U + \frac{1}{2}{}_{0.25}^{1}
\end{aligned}
$$

3. Using R and a uniform distribution from 0,1 with each x having equal probability for U: (R code and plot provided at the end of the problem). See that the histogram follows the line generated from plotting f(x) very well.

(b) Gen a r.v. using the accept reject method

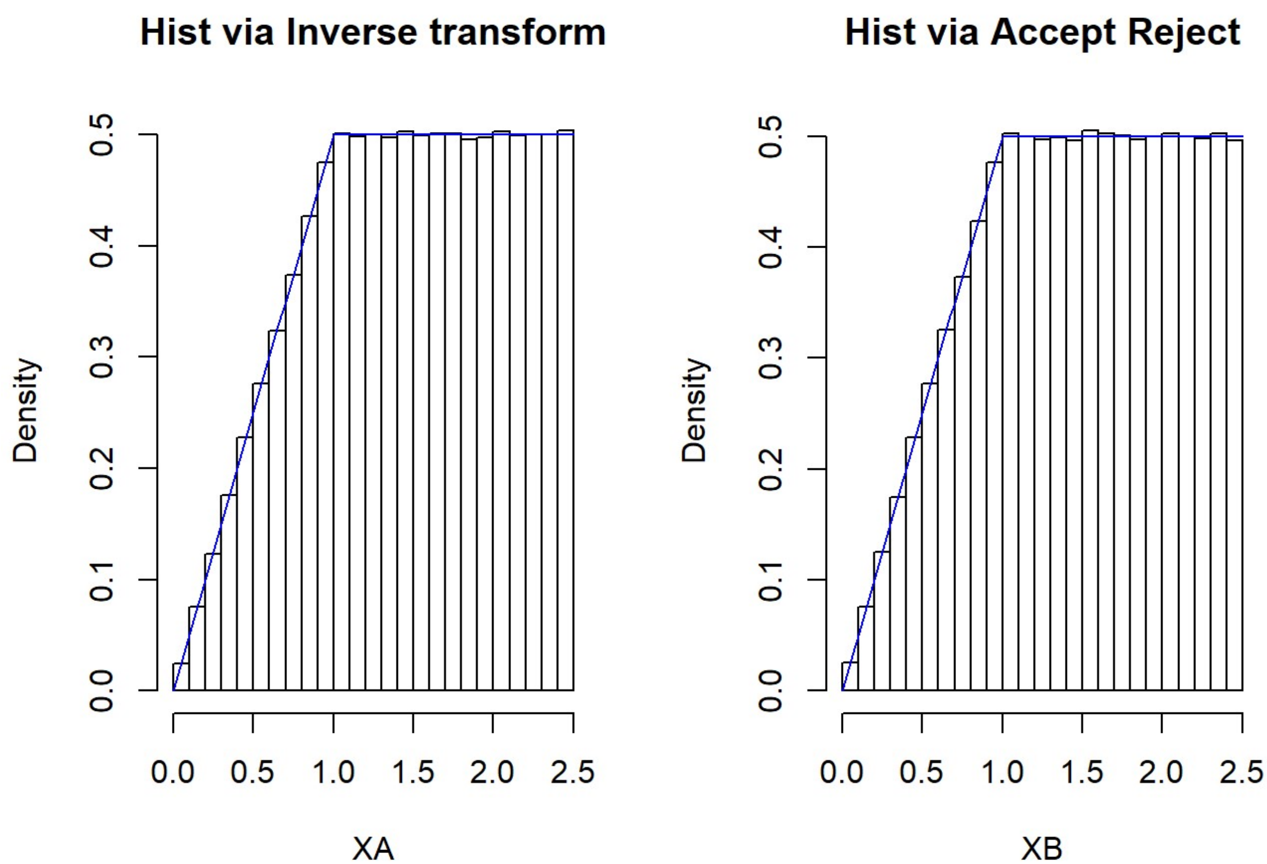
$$
sup(f(x)) = (1/2)|x = 1 \implies x_{opt} = \frac{16}{25} \rightarrow M = \frac{f(1/2)}{g(1/2)}
$$

```r
## PART (a) USING THE INVERSE TRANSFORM METHOD
# Set the number of simulations
  Nsim <- 10^6
# Generate the uniform values to plug into the inverse
  U <- runif(Nsim , min=0,max=(1))
# Evaluate the inverse for the Us
  XA <- ifelse(U<(1/4),sqrt(4*U),ifelse(U>(1/4),2*U+(1/2),NA))



## PART (b) USING ACCEPT REJECT
# Set the max
  M <- ((1/2))
# Set the function
  fx <- function(x){
    ifelse((x<1),(x/2),ifelse((1<=x & x<= 2.5),(1/2),NA))
  }
# Generate x candidates over the range under condiseration
  Xcand <- runif(Nsim,min=0,max=2.5)
# Generate y candidates up to M
  Ycand <- runif(Nsim, min=0, max=M)
# Keep the x candidates for which the Y candidates are viable solutions
  XB <- Xcand[Ycand < fx(Xcand)]

## Plots for (a) and (b)
par(mfrow = c(1,2))
hist(XA,freq=F,main="Hist via Inverse transform")
plot(fx,xlim=c(0,2.5),ylim=c(0,2.5),col = "blue",add=TRUE)
hist(XB,freq=F,main="Hist via Accept Reject")
plot(fx,xlim=c(0,2.5),ylim=c(0,2.5),col = "blue",add=TRUE)
```



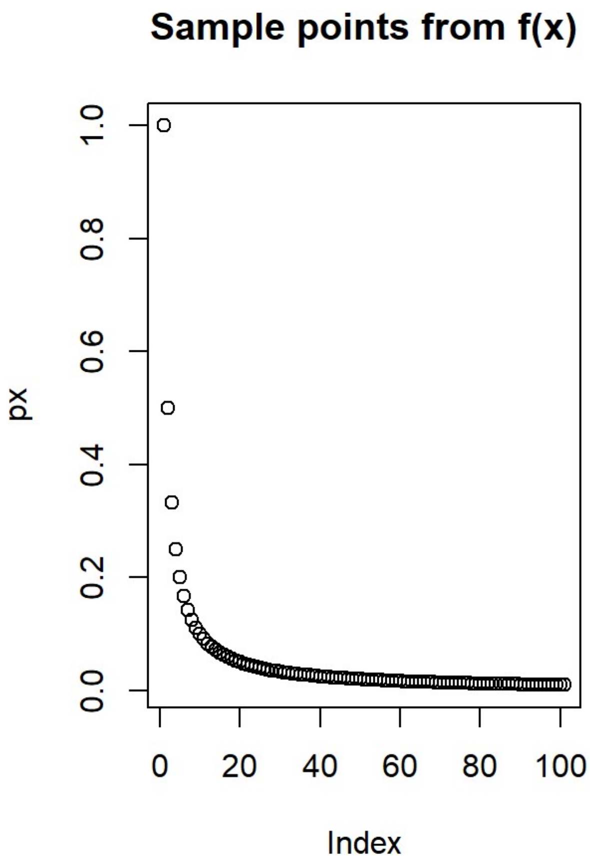
Problem 4

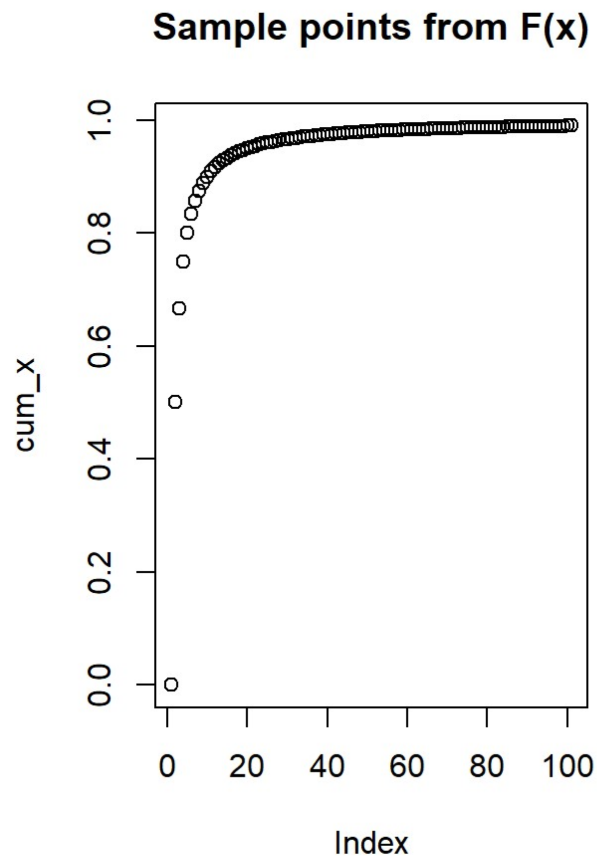
Apply the inverse-transform method to generate a random variable from the discrete uniform distribution with pdf

$$f(x) = \begin{cases} \dfrac{1}{n+1} & x = 0, 1 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

First I wanted to get an idea of what this might look like. In doing this I computed $F(X) = \frac{x}{n+1}$ which makes sense when you look at the cumulative normal distribution plot:

```
n <- seq(0,100)
px <- (1/(n+1))
cum_x <- (n/(n+1))
par(mfrow = c(1,2))
plot(px,main = "Sample points from f(x)")
plot(cum_x, main= "Sample points from F(x)")
```



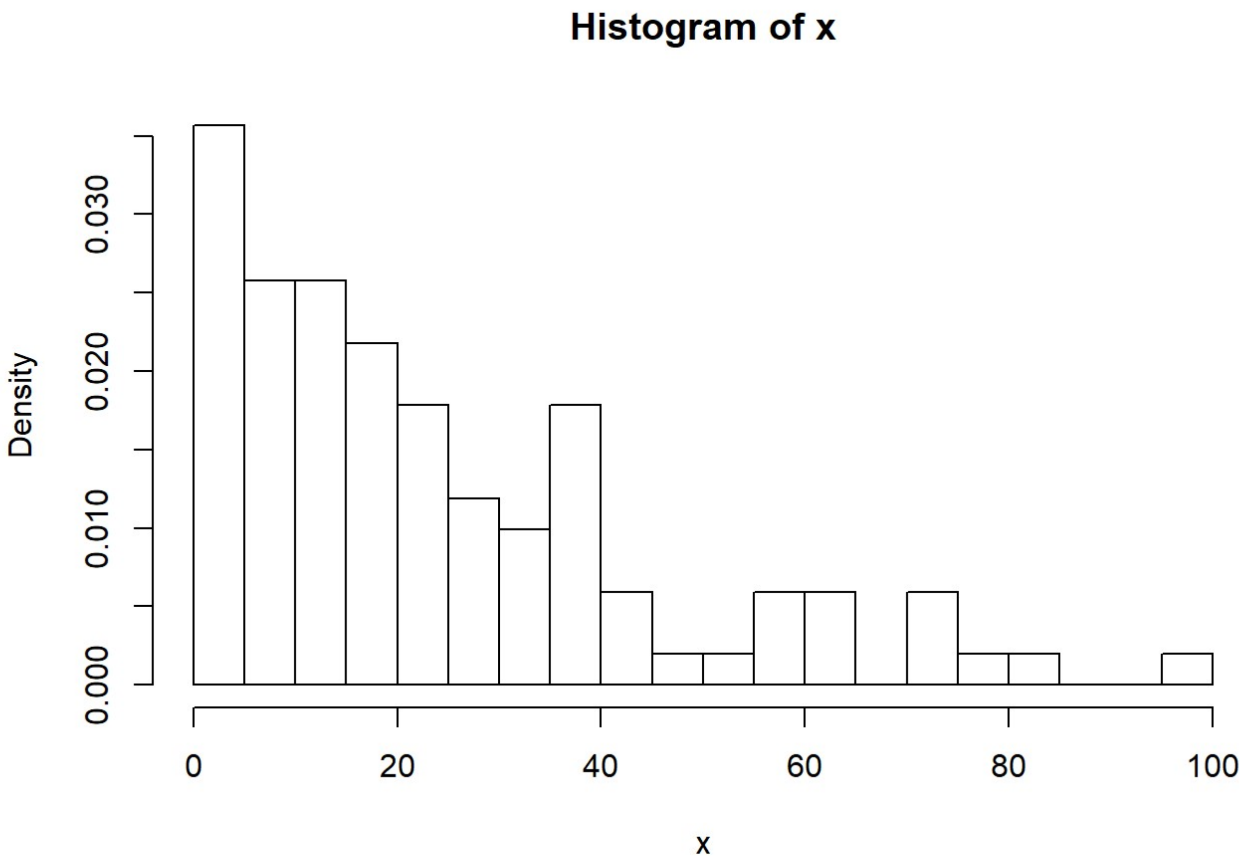


Using the inverse transform method it is clear that $X = U(n+1)$ where $U \sim unif(0,1)$ With this I made the following histogram. It doesn't look perfect but It gets the general idea.

```
U <- runif(100,min=0,max=1)
x <- U*(n+1)
```

```
## Warning in U * (n + 1): longer object length is not a multiple of shorter
## object length
```

```
hist(x,freq=F,breaks=seq(0,100,5))
```


Histogram of x
Density
0.000
0.010
0.020
0.030
0
20
40
60
80
100
x