Problem Set 5 *

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*Problem list 5.2.12, 5.3.10, 5.3.14, 5.4.20, 5.6.6

Problem 5.2.12. A random sample of size n is taken from the pdf:

$$\begin{aligned} f_y(y;\theta) &= \frac{2y}{\theta^2},\ 0 \le y \le \theta \\ L(\theta) &= \prod_{i=1}^{n} f_y(y_i;\theta) && \text{Def. 5.2.1 (281)} \\ &= \theta^{-2n}(2y)^n \end{aligned}$$

It is easy to see that $L(\theta)$ is decreasing in θ. To maximize $L(\theta)$ we must minimize θ. Since $y \le \theta$, $\hat{\theta} = Y_{max}$.

Problem 5.3.10. Babe Ruth batted 0.356 with 192 hits in 540 at-bats. Construct a 95% confidence interval.

Solution To construct a 95% confidence interval, we can apply Theorem 5.3.1 (299)with k = 192, n = 540, and $z_{\alpha/2} = 1.96$.[1]

$$\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}, \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}$$
$$\frac{192}{540} - 1.96\sqrt{\frac{(192/540)(1-192/540)}{540}}, \frac{192}{540} + 1.96\sqrt{\frac{(192/540)(1-192/540)}{540}}$$
$$(0.315, 0.396)$$

This is his batting average, so the 95% confidence interval for hits is $540 * (0.315, 0.396) =$ (170,214).

[1]From Table A.1 (675)

Problem 5.3.14. If (0.57,0.63) is a 50% confindence interval for p, find $\frac{k}{n}$ and n.

Solution The margin of error can be found by using the average distance from the mean, which is the difference in the bounds. $\frac{0.63-0.57}{2} = 0.03 = d = \frac{k}{n}$. To find how many observations were taken we use Definition 5.3.1 (301) $d = \frac{z_{\alpha/2}}{2\sqrt{n}} \implies n = (\frac{z_{\alpha/2}}{2d})^2 = (\frac{0.675}{2*(0.03)})^2 = 126.6 \approx 127$ observations.

Problem 5.4.20. Calculate the relative efficiency (r.e.) of $\hat{\lambda}_1 = X_1$ and $\hat{\lambda}_2 = \bar{X}_2$.

Proof. I mean they are both "relatively efficient"[2]. □

Solution Since it is a Poisson distribution:

$$\begin{aligned} Var_1(X) &= \lambda && \text{Theorem 4.2.4 (224)} \\ Var_2(\bar{X}) &= Var(\frac{X_1}{n} + \cdots + \frac{X_n}{n}) \\ &= \frac{Var(x)}{n} = \frac{\lambda}{n} \end{aligned}$$

The relative efficiency is defined as:

$$\begin{aligned} r.e.(\hat{\lambda}_1) &= \frac{Var(\hat{\lambda}_2)}{Var(\hat{\lambda}_1)} && \text{Def. 5.4.2 (314)} \\ &= \frac{\lambda/n}{\lambda} = \frac{1}{n} \end{aligned}$$

[2]Einstein's Theory of relativity

Problem 5.6.6. Is $W = \prod_{i=1}^{n} Y_i$ a sufficient statistic for θ.

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\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} f_Y(y;\theta) && \text{Def. 5.6.1 (322)} \\
&= \prod_{i=1}^{n} \theta y^{\theta-1} \\
&= \theta^n (\prod_{i=1}^{n} y_i)^{\theta-1}
\end{aligned}
$$

We can see that $L(\theta)$ depends on θ but only depends on y through the values of $\prod_{i=1}^{n} y_i$. So define $h(Y) = \prod_{i=1}^{n} y_i$ and $g(h(x);\theta) = \theta^n(\prod_{i=1}^{n} y_i)^{\theta-1}$ and $b(k_1, \cdots, k_n) = 1$. By theorem 5.6.1 (324) $W = \prod_{i=1}^{n} Y_i$ is sufficient for θ