Problem Set 1 *

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*Problems:1,2,3,4,5

Problem 1. Verify the Following:

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

See the Venn diagram below with the labled sections:

$$
\begin{aligned}
P(A) &= z + y + w + v \\
P(B) &= x + u + y + w \\
P(C) &= t + u + w + v \\
P(A \cap B) &= y + w \\
P(A \cap C) &= v + w \\
P(B \cap C) &= u + w \\
P(A \cap B \cap C) &= w \\
P(A) + P(B) + P(C) & \\
-P(A \cap B) - P(A \cap C) - & \\
P(B \cap C) + P(A \cap B \cap C) &= z + y + w \\
&+ vx + u + y + w + x \\
&+ u + y + w + t + u + w + v \\
&- (y + w + v + w + \\
& u + w) + w \\
&= z + t + 2y + 4w + 2v + 2u - y - 3w - v - u \\
&= z + t + x + y + w + v + u \\
&= P(A \cup B \cup C)
\end{aligned}
$$

For the Formal proof:

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) \cup P(A \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

Problem 2. A player throws darts ar a target. On each trial, independently of tthe other trials he this the bullseys wit probabilty 0.05 How many times should he throw it so that his probability ofh itting the bullseys at least once is 0.5.

Soultion: The probability of hitting one bullseye is the same as the probability of not hitting the bullseye every time except once: $P(A) = 1 - P(A^c)$. Since the throws are independent: $P(A \cap B) = P(A) \cdot P(B)$. Since the probability is the same every throw: P(A)=P(B). Thus, the probability of not hitting the bullseye in n throws is $P(A^c)^n = 0.95^n \geq 0.5 \implies n = \frac{log(2)}{log(20/19)} \approx 13.5$. Therefore, our boi should take 14 throws (or 13 if he is a gambling man).

Problem 3. Probability mass function of the basketball attempts.

*(*Solution: The probability that A wins on the first basket is p_1 the probability that B wins on the second basket is $(1 - p_1) \cdot p_2$. The probability that A wins on the third basket is $p_1^2(1-p_2)$. We can see that this will continue such that the probability that A/B will win on the nth basket is $p_1^{n-2}(1 - p_2)^{n-2}$

Problem 4. A line segment of length 1 is cut once at random. What is the probability that the longer piece is more than twice the length of the shorter piece.

Solution: This one is more intuitive. Let l_1, l_2 be the lengths of the two line segments. If $l_1 = 2 \cdot l_2$ it must be the case that $l_2 + 2 \cdot l_2 = 1$. Therefore, the smaller piece has to be no more than %33 of the total length. So the probability is 1/3.

Problem 5. Find the expected value and variance of a binomial random variable.

Solution. Intuitively, if we think of a binomial random variable we can think of n independent trials with probability of success, P(S). Since the trials are independent and we know that $P(A \cap B) = P(A) \cdot P(B)$, we can see that the probabilities of each trial are multiplicative, thus for n independent trials the expected value is np. This matches the formal proof below. I didn't see it in the notes but we also know that the variance of independent trails are additive (Larsen & Marx (2018) Corollary to Theorem 3.9.5 pg 189). So if we take one trial (n=1), $E(X) = p$ and $E(X^2) = 1^2 \cdot p = p$. Using the formula presented on page 6 of the notes we can say $Var(x) = p - p^2 = p(1-p)$. I feel like this is a substantial enough proof and the proof for variance seems much more involved so I'm going to leave it at that.

$$\begin{aligned}
E(X) &= \Sigma\binom{n}{x} \cdot p^x(1-p)^{n-x} \\
&= \Sigma\frac{x \cdot n!}{x(n-x)!}p^x(1-p)^{n-x} \\
&= \Sigma\frac{n!}{(k-1)!(n-k)!}p^x(1-p)^{n-x} \\
&= np\Sigma\frac{(n-1)!}{(k-1)!(n-k)!}p^{k-1}(1-p)^{n-k} \\
&= np\Sigma\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}
\end{aligned}$$

This sum is equal to 1 because of Result 3

$$E(X) = np$$

Problem 6. Consider a random experiment where we raw uniformly and independently n numbers fro the interval [0,1]. Let $X_{(1)}$ be the smallest of the n numbers. Determine the pdf of $X_{(1)}$.

Solution. So before we start lets think about where we are about to go. It is clear that we are going to have to go from a CDF to a PDF. Which means that we are going to have $P(F_{X_{min}} < x)$. And we know that the events are independent so we are going to multiply these values for n trials. With this let us start the proof:

$$
\begin{aligned}
F_{X_{(1)}} &= P(X_{(1)} \leq x) && \text{base} \\
&= 1 - P(X_{(1)} > x) && \text{Result 5 Mod 1} \\
&= 1 - P(min(x_1 ... x_n) > x) && \text{Def. min} \\
&= 1 - [[1 - P(x_1 \leq x)] \cdots [1 - P(x_n \leq x)]] && \text{Independence} \\
&= 1 - [[1 - F_{X_1}] \cdots [1 - F_{X_n}]] && \text{Def. CDF} \\
&= 1 - [1 - F_X(x))^n && \text{collapse} \\
F_{X_{(1)}} &= 1 - (1 - x)^n && \text{E(X) | X U(0,1)} \\
f_{X_{(1)}} &= \frac{d}{dx} F_{X_{(1)}} && \text{FTC} \\
&= \frac{d}{dx} 1 - (1 - x)^n \\
f_{X_{(1)}} &= -n(1 - x)^{n-1}
\end{aligned}
$$

Problem 7. Let $Y = e^X | X \ N(0,1)$ Here I'm just following section 2.4 Transformations (Mod 1).

$$
\begin{aligned}
f_Y(y) = \frac{d}{dx} F_Y(y) &= \frac{d}{dx} P(Y \leq y) \\
&= \frac{d}{dx} P(e^X \leq y) \\
&= \frac{d}{dx} P(X \leq ln(y)) \\
&= \frac{d}{dx} \int_{-\infty}^{ln(y)} f_X(d) dx \\
&= \frac{1}{y} \cdot f_X(ln(y))
\end{aligned}
$$