Problem Set 4 *

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*Problem list 3.10.6, 3.10.16, 3.12.6, 3.12.8

Problem 3.10.6. Let $Y_1, Y_2, ..., Y_n$ be a random sample from the exponential pdf $f_y(y) = e^{-y}, y \geq 0$. What is the smallest n for which $P(Y_{min} < 0.2) > 0.9$?

Solution The evaluated n $>$ 11.513, since x must be an integer (I'm assuming since these are trials) $n \geq 12$.

$$\begin{aligned}
P(Y_{min} < 0.2) &= \int_0^{0.2} f_{Y_{min}}(y) \\
&= \int_0^{0.2} n[1 - F_Y(y)]^{n-1} f_Y(y) \qquad \text{Theorem 3.10.1.b (pg193)} \\
&= \int_0^{0.2} n[1 - (1 - e^{-y})]^{n-1} (e^{-y}) \\
&= \int_0^{0.2} n(e^{-y})^{n-1} (e^{-y}) \\
&= \int_0^{0.2} n(e^{-ny}) \\
&= -e^{-ny}\Big|_0^{0.2} \\
&= -e^{-(0.2)n} - (-1) > 0.9 \\
&= log(e^{-0.2n}) < log(0.1) \\
n &> \frac{log(0.1)}{-0.2} \approx 11.513
\end{aligned}$$

Problem 3.10.16. Suppose a device has three independent components, all of whose lifetimes (in months) are modeled by the exponential pdf, $f_y(y) = e^{-y}, y > 0$. What is the probability that all three components will fail within two months of one another?

Solution[1] Range = $Y_{max} - Y_{min} = Y_3^{‘} - Y_1^{‘}$. $P(R < r) = 0.646$

The *memoryless property of the exponential distribution*:

$$P(X \geq s + t | x \geq s) = P(X \geq t)$$

This implies that the level of $Y_1^{‘}$ is inconsequential. Thus, we can assume that $Y_1^{‘} = 0$. In which case we are really only interested in $P(Y_{max}^{‘} < r)$. Thus, we can apply Theorem 3.10.1.a (pg 193) with: n=3, $f_y(y) = e^{-y}$, and $F_Y(y) = \int_0^y f_Y(y)dy = 1 - e^{-y}$.

$$\begin{aligned} P(Y_{max} < m) &= \int_{-\infty}^{m} n[F_Y(y)]^{n-1} f_Y(y) \\ P(Y_3^{‘} < 2) &= \int_0^2 3[1 - e^{-y}]^2 e^{-y} && \text{Enter WolframAlpha} \\ &\approx 0.646. \end{aligned}$$

Problem 3.12.6. Find $M_Y(t)$ if Y has the pdf:

$$f_Y(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 2 - y, & 1 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Solution $M_Y(t) = \frac{1-e^t}{t^2}$

[1] To start, I want to note that understanding of the "memoryless property of the exponential distribution" was critical to even approaching success in this problem. I studied the proof in this pdf, http://www.cs.cmu.edu/afs/cs/academic/class/15750-s19/OldScribeNotes/lecture11.pdf (pg 2). I also used wolframalpha to do some of the calculations that were a to complex for my patience level.

Since X is continuous, the second part of Definition 3.12.1 (pg 206) applies:

$$\begin{aligned}
M_Y(t) = E(e^{tW}) &= \int_{-\infty}^{\infty} e^{tw} f_W(w) dw \\
&= \int_{-\infty}^{0} e^{ty} 0 + \int_{0}^{1} e^{ty} y + \int_{1}^{2} e^{ty}(2-y) + \int_{0}^{\infty} e^{ty} 0 \\
&= \int_{0}^{1} e^{ty} y + \int_{1}^{2} e^{ty}(2-y) && \text{I used WolframAlpha here.} \\
&= (\frac{1}{t}y - \frac{1}{t^2})e^{ty}\Big|_0^1 + \frac{2}{t}e^{ty}\Big|_1^2 - (\frac{1}{t}y - \frac{1}{t^2})e^{ty}\Big|_1^2 \\
&= (\frac{1}{t} - \frac{1}{t^2})e^{t} + (\frac{1}{t^2}) + \frac{2}{t}e^{2t} - \frac{2}{t}e^{t} - (\frac{2}{t} - \frac{1}{t^2})e^{2t} + (\frac{1}{t} - \frac{1}{t^2})e^{t} \\
&= \frac{1}{t^2} + \frac{1}{t^2}e^{2t} - \frac{2}{t^2}e^{t} \\
&= \frac{1}{t^2}(e^t - 1)
\end{aligned}$$

Problem 3.12.8. Let Y be a continuous random variable with $f_Y(y) = ye^{-y}, o \leq y$. Show that $M_Y(t) = \frac{1}{(1-t)^2}$

Solution Since X is continuous, the second part of Definition 3.12.1 (pg 206) applies:

$$
\begin{aligned}
M_Y(t) = E(e^{tW}) &= \int_{-\infty}^{\infty} e^{tw} f_W(w)dw \\
&= \int_0^{\infty} e^{ty} y e^{-y} \\
&= \int_0^{\infty} e^{ty-y} y \\
&= \frac{e^{(t-1)y}((t-1)y-1)}{(t-1)^2}\Big|_0^{\infty} \\
&= lim_{y\to\infty} \frac{e^{(t-1)y}((t-1)y-1)}{(t-1)^2} - lim_{y\to 0} \frac{e^{(t-1)y}((t-1)y-1)}{(t-1)^2} \\
&= lim_{y\to\infty} \frac{e^{(t-1)y}((t-1)y-1)}{(t-1)^2} - \frac{e^{(t-1)0}((t-1)0-1)}{(t-1)^2} \\
&= lim_{y\to\infty} \frac{e^{U}(U-1)}{(t-1)^2} - \frac{(-1)}{(t-1)^2} && \text{Distribute, let U =(yt-y)} \\
&= \frac{(lim_{y\to\infty} e^{yt-y})(lim_{y\to\infty}(yt-y-1))}{(t-1)^2} - \frac{(-1)}{(t-1)^2} && \text{Theorem 20.4 (Ross pg 156)}
\end{aligned}
$$

It will suffice to show that if $lim_{y\to\infty} e^{yt-y}$ converges to 0 then $M_Y(t) \to \frac{1}{(t-1)^2}$. Since $0 * (lim_{y\to\infty}(yt-y-1)) = 0$ and $\frac{0}{(t-1)^2} - \frac{(-1)}{(t-1)^2} = \frac{1}{(t-1)^2}$.

Proof. Let $\epsilon > 0$ and $\delta = \frac{log(\epsilon)}{(t-1)}$. Note that, $\forall\ t$ such that $0 < t < 1,\ \delta > 0$. So if $0 < |x| < \delta$ then $|x| < \frac{log(\epsilon)}{(t-1)} \implies (t-1)|x| = log(\epsilon) \implies e^{xt-x} < \epsilon$ Thus, by Theorem 20.6 (Ross pg 159[2]), $\lim_{x\to\infty} e^{xt-x} = 0$. $\square$

[2]Ross, Kenneth: Elementary Analysis the Theory of Calculus. Undergraduate Texts in Mathematics, 2nd edn. Springer, New York/Heidelberg/Berlin (2013)

Problem Assignment. The Inverse Transform Sampling procedure requires $F_Y^{-}(u)$ where $F_Y^{-}(u)$ is the inverse of the cumulative distribution function, $F_Y(y)$. We saw in problem 3.10.6 that $F_Y(y) = (1 - e^{-}y)$. Solving for $x = -log(1 - u)$ is our inverse CDF function. Now to be honest with you, I couldn't really follow your R code so I wrote my own. Now I'm not really sure what you are looking for to by way of the solution. But I've included my code below, and I am getting very close to 0.9 so I believe it is correct.

```
# inverse transfrom sampling


# This is the number of times you run the process,
positions <- 1000


# This will be a vector of minimums
samples <- c()


# You add one number to sample each process
# and you want to run the process position times.
while( length(samples) < positions )
{
#This is the number computed in the problem.
#The n needed to produce the minimum
num.samples <-  12
```

```
# The uniform distribution from 0 to 1
U              <-  runif(num.samples,0,1)


# The pdf
Y <- exp(-U)


# The inverse cdf function
X              <- -log(1-U)


# take the minimum
sortedx <- sort(X)
minx = min(sortedx)
samples <- append(samples,minx)
}


# Get the ratio of samples <0.2 to total samples.
# This is your probability.
less <- sum(samples <0.2)
ratio <- less/positions



# plot
hist(samples, breaks=30, freq=F, xlab='X', main='Generating Exponential
```

```r
curve(dexp(X, rate=2) , 0, 3, lwd=2, xlab = "", ylab = "", add = T)
```