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Course 625.603.86.SP19
FINAL EXAM

① $Y/x = ae^{-x/b}$

$y = axe^{-x/b}$

$\ln(y) = \ln(a) + \ln(x) + \frac{-x}{b}$

Interesting So $\log y$ is linear with x and $\log x$, we will def use $\ln a$ and b. But the $\ln(x)$ doesn't interact with the constants

So $\ln(\frac{Y}{x}) = \ln a + \frac{-1}{b}x$ let $C = \frac{-1}{b}$

Oh so $\ln(\frac{Y}{x})$ and x have a linear relationship

So we can just modify theorem 11.2.1 (S21)

$$b = \frac{n\sum_{i=1}^{n} x_i \ln(\frac{y_i}{x_i}) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \ln(\frac{y_i}{x_i})\right)}{n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i}$$

$$\ln a = \frac{\sum_{i=1}^{n} \ln \frac{y_i}{x_i} - b\sum_{i=1}^{n} x_i}{n}$$

2 $f(x) = \frac{1}{\alpha} \exp(-x/\alpha)$ $\quad 0 \le x < \infty$

$H_0: \alpha = 1$, H_0 rejected IF $x > 4$

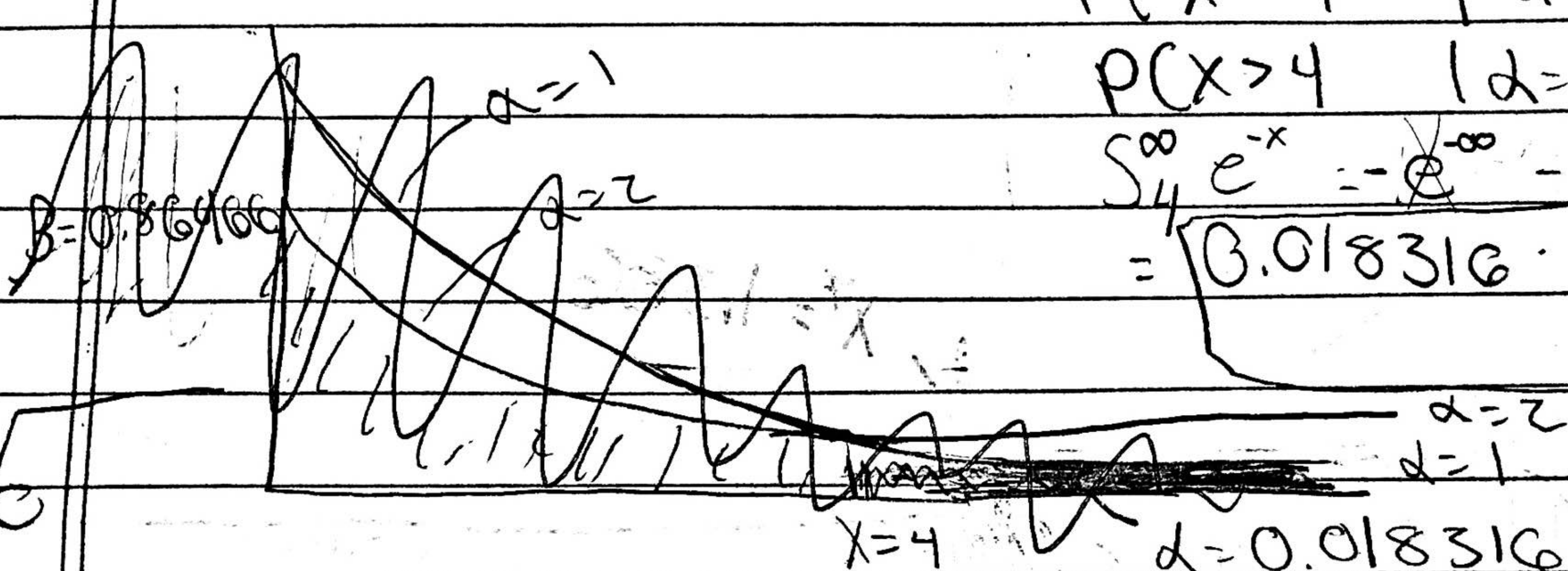
a. $\frac{1}{1} e^{-x}$ Type I error: $P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$P(X > 4 \mid \alpha = 1)$

$P(X > 4 \mid \alpha = 1)$

$\int_4^\infty e^{-x} = -\cancel{e^{-\infty}} - -e^{-4} = e^{-4}$

$= \boxed{0.018316}$ Prob of Type I error



I redo this graph since it sucks

b. $f_{\alpha=2}(x) = \frac{1}{2} e^{-x/2}$

P Type II error: $P(\text{Accept } H_0 \mid H_1 \text{ true})$

$P(X \le 4 \mid \alpha = 2)$

$\int_0^4 \frac{1}{2} e^{-x/2} = e^{-x/2} \Big|_0^4$

$1 - e^{-2} = \boxed{0.864166}$ Prob of type II error

In Red

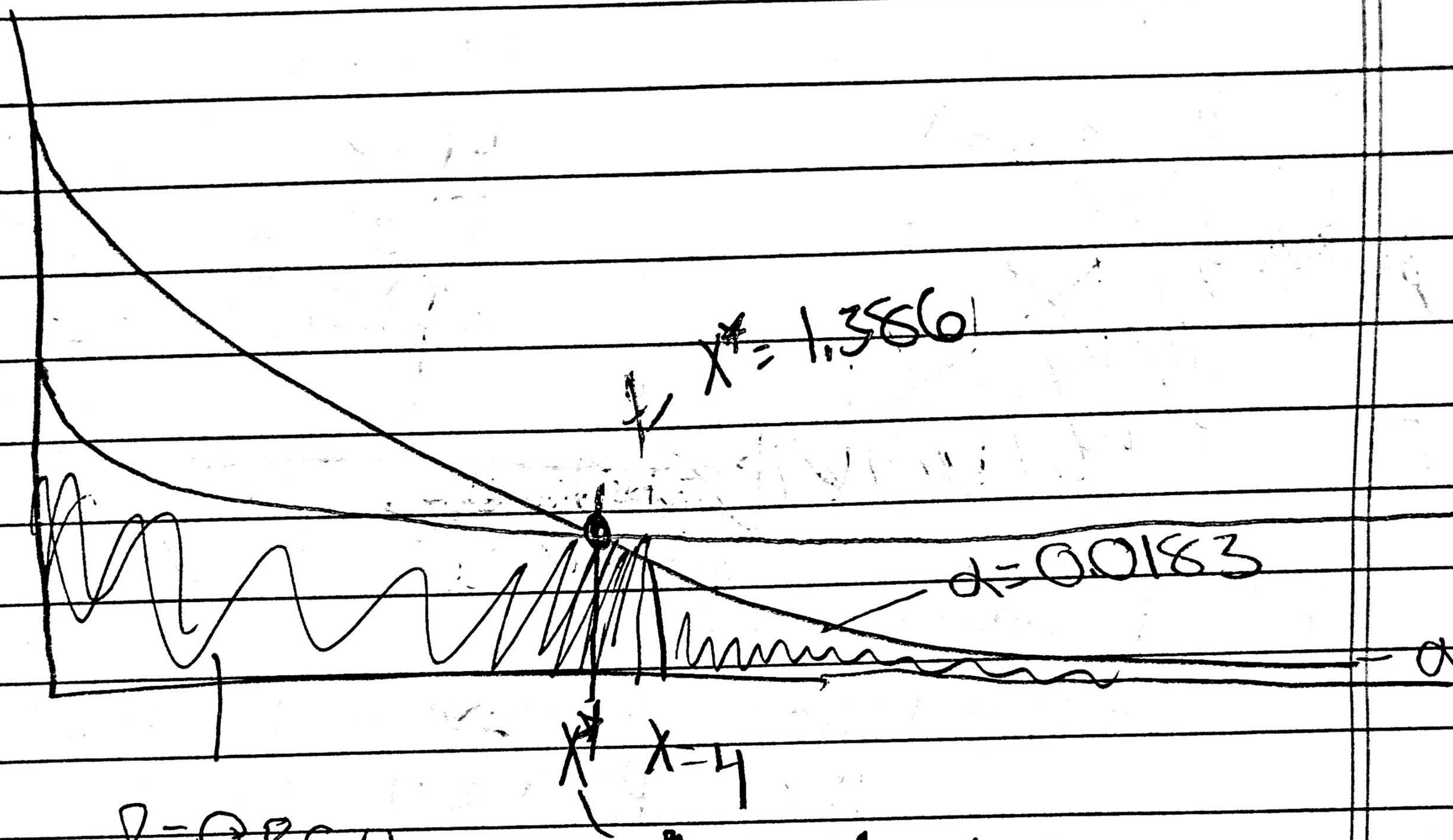
c. To minimize find the intersection of the Points (X^* on the graph)

$\ln \left\{ e^{-x} = \frac{1}{2} e^{-x/2} \right\}$ $\quad -x = \ln(1/2) - \frac{x}{2}$

$-\frac{1}{2}x = \ln(1/2)$

$x = -2\ln(1/2)$

$X^* = 1.386$



x^* is the technical best x for minimizing errors

3) $f(x) = 2x \qquad 0 < x < 1 \qquad n = 40$

$15/40 \qquad 0 < x < 0.5$

under normal conditions

$$\int_0^{0.5} 2x = \quad x^2 \Big|_0^{0.5} \quad (0.5)^2 = \boxed{0.25} - 0$$

Using theorem 5.3.1 (299)

$$\frac{k}{n} \pm z_{\alpha/2} \sqrt{\frac{(k/n)(1-k/n)}{n}} \qquad \text{s.t. } k = 15$$

$$n = 40$$

$$z_{0.05/2} = 1.96$$

$$\frac{15}{40} \pm (1.96) \sqrt{\frac{(15/40)(1 - 15/40)}{40}}$$

$$0.375 \pm (1.96) \sqrt{(0.375)(0.625) \cdot \frac{1}{40}}$$

$$0.375 \pm (1.96)(0.0765) =$$

$$= [0.2251, 0.525]$$

Since the true probability of 0.25 lies within this range it is a good fit.

(4) Coin toss

Prior $g(\theta) = \frac{\pi}{2} \sin(\pi\theta)$

Well since it is a coin toss the probability model for X - outcome of the toss is given by

$$P_X(K|\theta) = \binom{n}{K} \theta^K (1-\theta)^{n-K}$$

By definition 5.8.1 on (331) the posterior distribution is given by

$$\frac{g(\theta)\, P_X(K|\theta)}{\int_{-\infty}^{\infty} g(\theta)\, P(\theta)\, d\theta}$$

Which gives

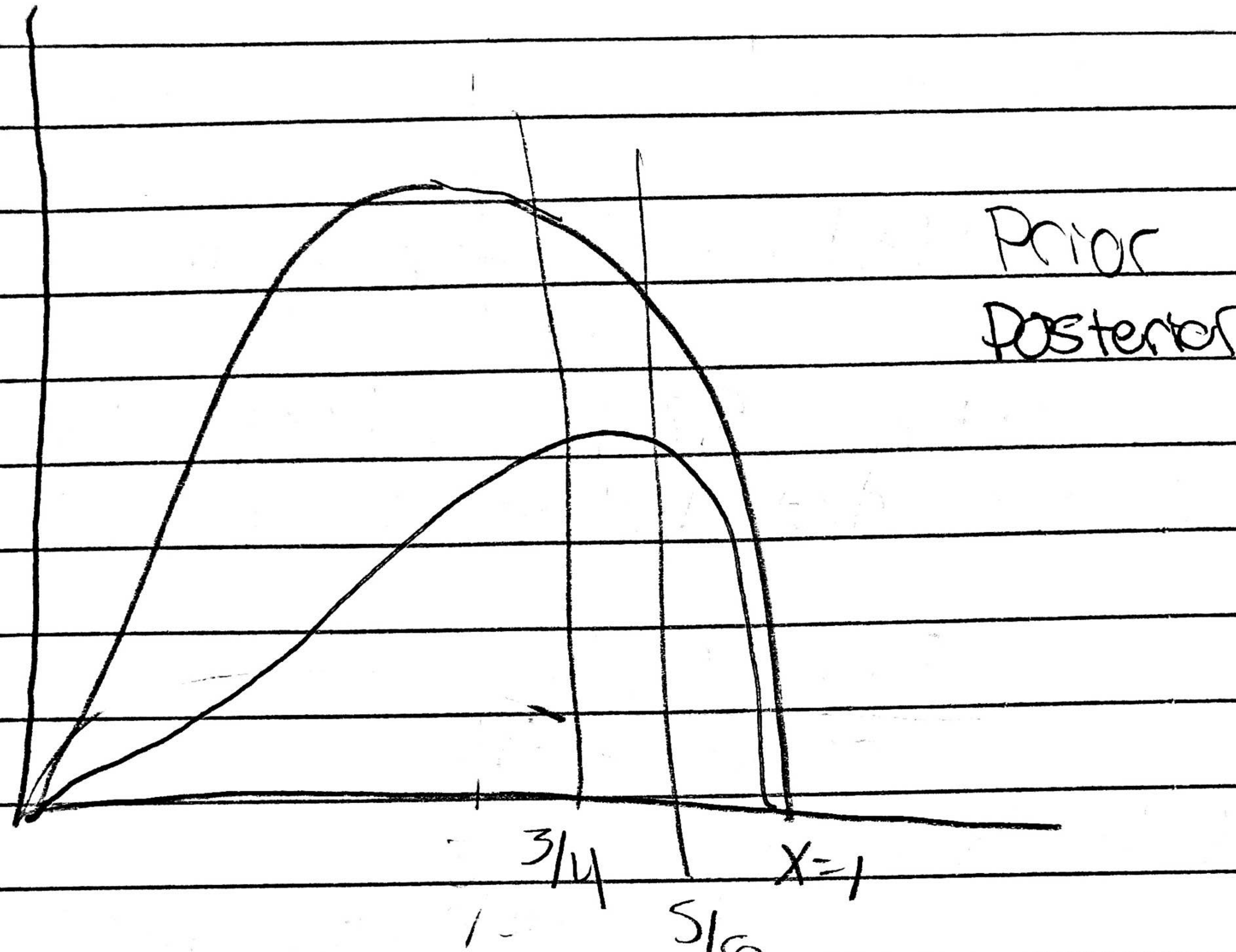
$$\frac{\frac{\pi}{2}\sin(\pi\theta) \cdot \binom{n}{K}\theta^K(1-\theta)^{n-K}}{\int_{-\infty}^{\infty} \frac{\pi}{2}\sin(\pi\theta) \cdot \binom{n}{K}\theta^K(1-\theta)^{n-K}}$$

If $n=1$ and $K=1$ then this becomes

a. $$\boxed{\frac{\frac{\pi}{2}\sin(\pi\theta)\cdot\theta}{\int_{-\infty}^{\infty} \pi/2 \sin(\pi\theta)\cdot\theta}}$$

b.

→



$L\left(\frac{3}{4}\right) = \frac{\pi}{2}\sin\left(\pi\frac{3}{4}\right)\cdot\frac{3}{4} \approx 0.83$

$= \frac{\pi}{2}\sin\left(\pi\frac{5}{6}\right)\cdot\frac{5}{6} \approx 0.65$

So 3/4 has a higher posterior probability

⑤ $f(x|\theta) = \dfrac{\theta}{2\sqrt{x}} \exp\left(-\dfrac{\theta}{\sqrt{x}}\right)$ $\quad x \geq 0$, $\theta \geq 0$

Def 5.2.1 (281) $L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2\sqrt{x}} e^{-\theta/\sqrt{x}}$

$L(\theta) = \left(\dfrac{\theta}{2\sqrt{x}}\right)^n e^{-n\theta/\sqrt{x}} = \dfrac{\theta^n}{2^n} \dfrac{1}{\prod \sqrt{x_i}} e^{-\theta\left(\sum \frac{1}{\sqrt{x_i}}\right)}$

$\ln L(\theta) = n\ln\left(\frac{\theta}{2}\right) + \ln\left[\prod \sqrt{x_i}\right]^{-1} + -\theta\left(\sum \frac{1}{\sqrt{x_i}}\right)$

$\dfrac{d \ln L(\theta)}{d(\theta)} = n\theta + -\frac{1}{2}\theta^2 \sum \frac{1}{\sqrt{x_i}}$

Set derivative $= 0$

$n\theta + -\frac{1}{2}\theta^2 \sum \frac{1}{\sqrt{x}} = 0$

$n\theta = \frac{1}{2}\theta^2 \sum \frac{1}{\sqrt{x_i}}$

$\dfrac{2n}{\sum \frac{1}{\sqrt{x_i}}} = \theta$

6. $p = 0.52$ for heads

Def 5.3.1 (301) $d = \frac{z_{\alpha/2}}{2\sqrt{n}}$ $\Big\}$ for 99% conf $\alpha/2 = 0.005 \Rightarrow z = 2.345$

$$n = \left(\frac{z}{2d}\right)^2 = \left[\frac{2.345}{2d}\right]^2$$

We need a d.

$d = \frac{k}{n}$ s.t. $np = k$ (# of heads)

$n(0.52) = k$

$$d = \frac{n(0.52)}{n} = 0.52$$

$$n = \left(\frac{z}{2d}\right)^2 = \left(\frac{2.345}{2(0.52)}\right)^2 = 5.084,$$ So between 5/6 tosses

$\boxed{\text{6 to be sure}}$