mod2_mcgroarty

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Problem 1

Let X and Y have joint density f given by

$$f(x, y) = cxy \quad 0 \leq y \leq x \leq 1$$

(a). Determine the normailization constant c.

$$\begin{aligned}
f(x, y) &= \int_0^1 \int_0^x cxy \cdot dydx \\
&= \int_0^1 [cxy^2/2|_0^x] \cdot dx \\
&= \int_0^1 cx^3/2dx \\
&= cx^4/8|_0^1 \\
&= c/8 = 1 \\
c &= 8
\end{aligned}$$

(b). Determine $P(X + 2Y \leq 1)$

$$\begin{aligned}
P(X + 2Y \leq 1) &= P(X \leq 1 - 2y) \\
&= \int_0^1 \int_y^{1-2y} 8xy \cdot dxdy \\
&= \int_0^1 [4x^2y|_x^{1-2y}] \cdot dy \\
&= \int_0^1 [4(1 - 2y)^2y - 4(y)^2y \\
&= \int_0^1 12y^3 - 16y^2 + 4y) \\
&= 12y^4/4 - (12/3)y^3 + (4/2)y^2)|_0^1 \\
&= (12/4) - (12/3) + (4/2) \\
&= 1
\end{aligned}$$

(c). Find $E(X|Y=y)$

$$\begin{aligned}
f_Y(y) &= \int_y^1 8xy \cdot dx \\
&= 4x^2y|_y^1 \\
&= 4y - 4y^3 \\
f(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{8xy}{4y(1-y^2)} \\
f(x|y) &= \frac{2x}{(1-y^2)} \\
E(X|Y=y) &= \int_y^1 \frac{2x}{(1-y^2)} x \cdot dx \\
&= \frac{(2/3)x^3}{(1-y^2)}|_y^1 \\
&= \frac{(2/3)(1-y^3)}{(1-y^2)}
\end{aligned}$$

(d). Find $E(X)$

$$\begin{aligned}
f_x(x) &= \int_0^x 8xy \cdot dy \\
&= 4xy^2|_0^x \\
&= 4x^3 \\
E(X) &= \int_y^1 4x^3 \cdot x \cdot dx \\
&= x^4|_y^1 \\
&= 1 - y^4
\end{aligned}$$

Problem 2

Let $X \sim N(\mu, \Sigma)$ with $\mu^T = (2, -3, 1)$ and

$$\begin{matrix} & 1 & 1 & 1 \\ \Sigma = (& 1 & 3 & 2) \\ & 1 & 2 & 2 \end{matrix}$$

(a) Find the distribution of $Y = 3X_2 - 2X_2 + X_3$.

```
E <- matrix(c(1,1,1,1,3,2,1,2,2),nrow=3)
a <- matrix(c(3,-2,1), nrow=1)
u <- matrix(c(2,-3,1), nrow= 3)

## Expected value of Y
as.matrix(a %*% u)
```

```
[,1]
```

[1,] 13

```
## Variance of Y
as.matrix( a %*% E %*% t(a))
```

```
[,1]
```

[1,] 9

The distribution of $Y = 3X_2 - 2X_2 + X_3$ $N(13, 9)$.

(b) Find a 2 x 1 vector such that the follwoing are independent:

$$X_2, \;\; X_2 - a^T (\begin{matrix} X_1 \\ X_2 \end{matrix})$$

```
a2 <- matrix(c(0,1,1,-1,0,0), nrow=2)

## The diagonal matrix should be all zeros.
a2 %*% E %*% t(a2)
```

```
[,1] [,2]
```

[1,] 3 -2 [2,] -2 2

Problem 3

Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with pdf:

$$f(x, y) = 2x \ s.t. \ 0 \leq x \leq 1, \ 0 \leq y \leq 1$$

Find Marginal pdf:

$$\begin{aligned}
f_Y(x) &= \int_0^1 f(x, y) dx && \text{Definition} \\
&= \int_0^1 2x dx \\
&= x^2|_0^1 \\
&= 1 \\
F_Y(x) &= \int_{x^2}^{x} f_Y(x) \\
&= \int_{x}^{x^2} 1 \\
P(X^2 < Y < X) &= x - x^2
\end{aligned}$$

Problem 4:

The random pair (X,Y) have the distribution:

$$\begin{array}{ccc|c} (1/12) & (2/12) & (1/12) & (4/12) \\ (2/12) & (0/12) & (2/12) & (4/12) \\ (0/12) & (4/12) & (0/12) & (4/12) \\ \hline (3/12) & (6/12) & (3/12) & 1 \end{array} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Show that X and Y are dependent: To show this we can use the fact that if $Cov(X_i, X_j) = 0$ then X_i, X_j are independent. That means that the vectors of the values of X and Y must be linerarly independent. We see that the reduced row echelon form yeilds a row of 0s. Thus, we can say that the vectors X and Y are linearly dependent and thus so are X and Y. (I don't have my liner book with me took cite the linear dependence definitions sorry).

(b) Find a matrix that is lineraly independent but has the same marginal probabilitites:

IS THERE A TRICK TO THIS??

$$\begin{array}{ccc|c} (1/12) & (2/12) & (1/12) & (4/12) \\ (2/12) & (0/12) & (2/12) & (4/12) \\ (0/12) & (4/12) & (0/12) & (4/12) \\ \hline (3/12) & (6/12) & (3/12) & 1 \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5: Suppose that $X_1...X_{20}$ are independent random variables with density function f(x) = 2x 0 < x < 1. Use the central limit theorem to approximate $P(S \leq 10)$

$$\begin{aligned}
E(S) &= \int_0^1 f(x) \cdot xdx & \text{independence} \\
&= \int_0^1 2x^2 \\
&= (2/3)x^2|_0^1 & = 2/3 \\
E(S^2) &= \int_0^1 2x^4 \\
&= 2/4 \\
V(S) &= E(S^2) - E(S)^2 \\
&= 1/18 \\
S &\sim N(20 \cdot (2/3), 20 \cdot (1/18)) \\
S &\sim N(13.3, 1.11) \\
P(S \leq 10) &= P(\frac{S - 13.33}{\sqrt{1.11}} \leq \frac{10 - 13.33}{\sqrt{1.11}}) \\
&= P(Z \leq -3.16) = 0.0008
\end{aligned}$$

```
(10 - 13.33)/(sqrt(1.11))
```

```
## [1] -3.160696
```

```
    pnorm(-3.16)
```

```
## [1] 0.0007888457
```

Problem 6: Suppose that a measurement has mean μ and variance $\sigma^2 = 25$. Let $\bar{X}$ be the average of n independent measurements. How large should n be s.t. $P(|\bar{X} - \mu| < 1) = 0.95$?

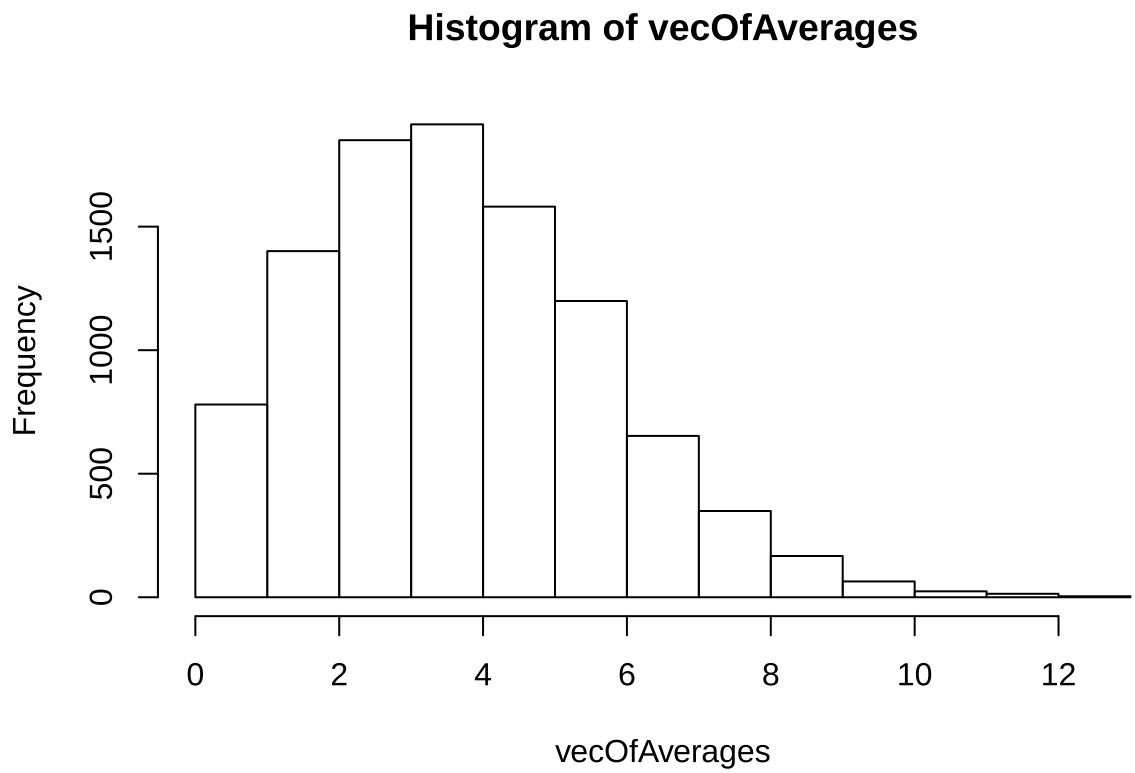
$$
\begin{aligned}
P(|\bar{X} - \mu| < 1) &= 0.95 \\
P(\frac{|\bar{X} - \mu|}{\sigma/\sqrt{n}} < \frac{1}{\sigma/\sqrt{n}}) & & \text{divide} \\
P(Z < \frac{1}{\sigma/\sqrt{n}}) \\
P(Z < \frac{1}{5/\sqrt{n}}) & & = 0.95 \text{Z score of } 1.96 \\
\frac{1}{5/\sqrt{n}} &= 1.96 n & = 96.04 trials
\end{aligned}
$$

```
(5*1.96)^2
```

```
## [1] 96.04
```

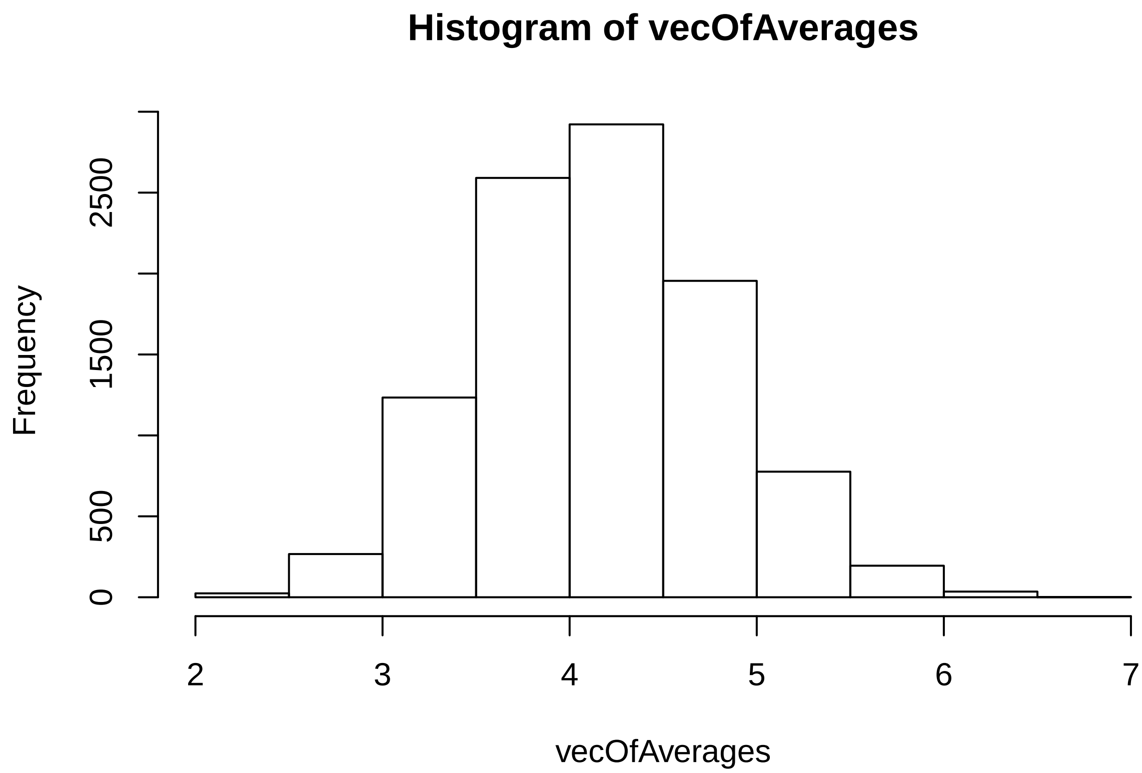
Problem 7: Not sure how to show this but I'll do it for a few iterations?

```
limitTheorem = function(n,epsilon) {

    vecOfAverages = c(rep(0,10000))
    for (i in 1:10000) {
        smpl = rpois(n,4.2)
        avg = mean(smpl);
        vecOfAverages[i] = avg;
    }
    hist(vecOfAverages);
    vecOfDifferences = vecOfAverages - 4.2*c(rep(1,10000));
    nmbCloseToTruth = length(vecOfDifferences[abs(vecOfDifferences) <= epsilon]);
    print(nmbCloseToTruth/10000);
}

limitTheorem(1,1)
```



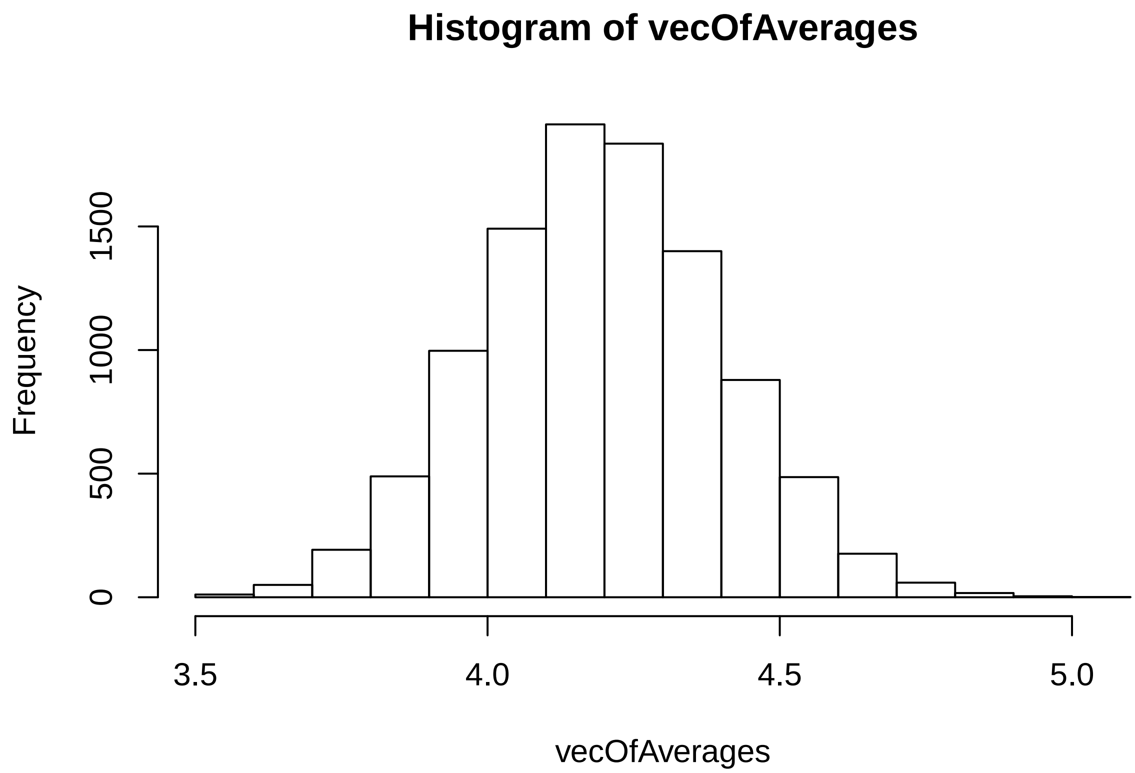

```
## [1] 0.3495
```

```r
limitTheorem(10,0.1)
```



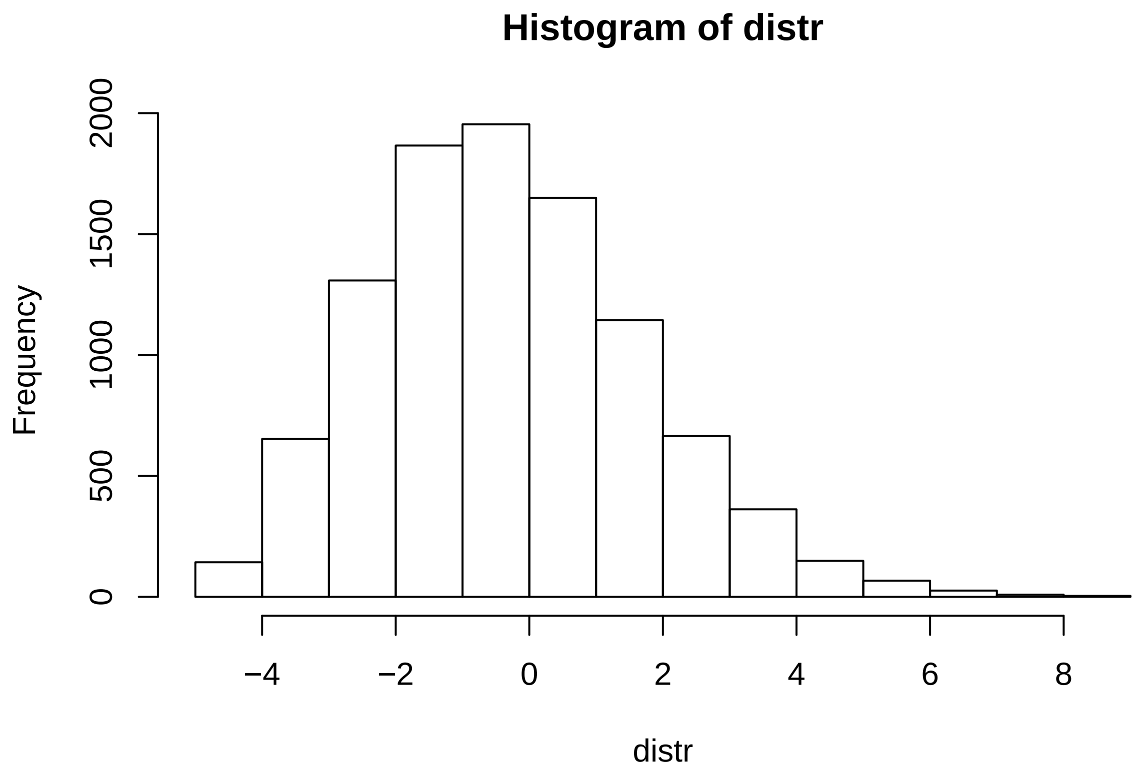

```
## [1] 0.1174
```

```
limitTheorem(100,0.01)
```



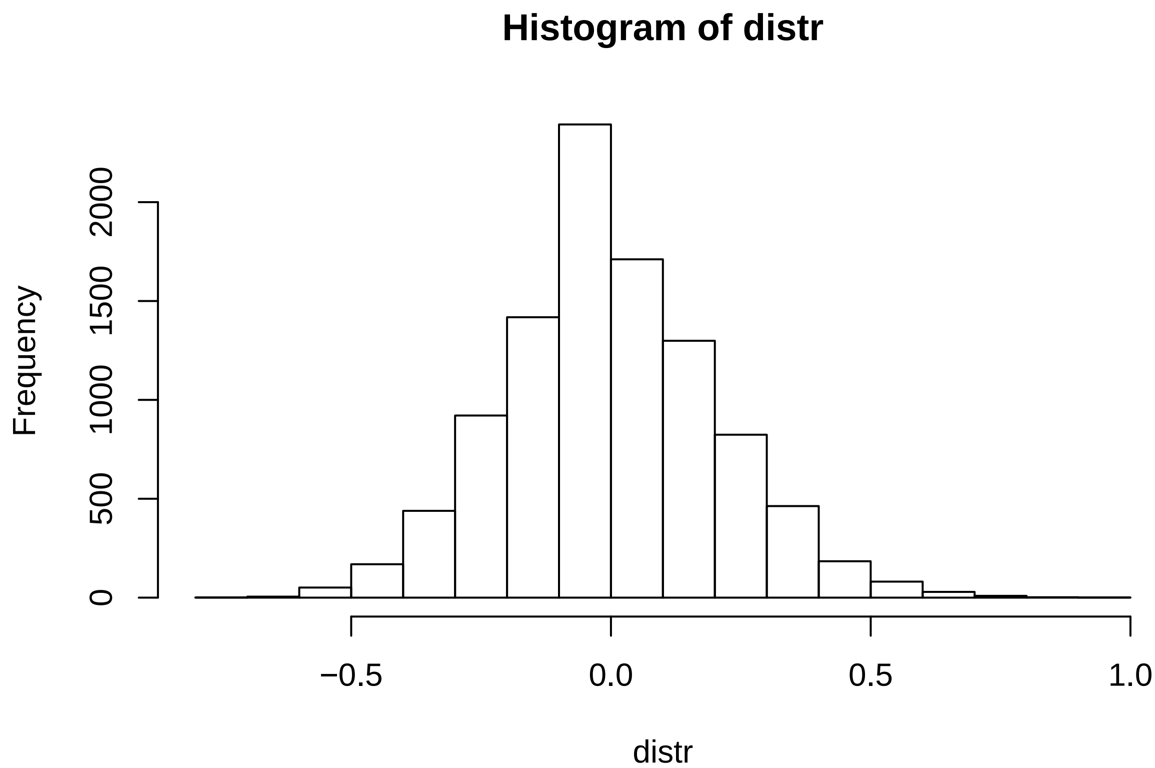

```
## [1] 0.0604
```

```r
### Adjust to prove Central limit Theorm
    limitTheorem2 = function(n,epsilon) {

      vecOfAverages = c(rep(0,10000))
      distr = c(rep(0,10000))
      for (i in 1:10000) {
        smpl = rpois(n,4.2)
        avg = mean(smpl);
        normdis <- ((avg) - 4.2)/(n/sqrt(n))
        vecOfAverages[i] = avg;
        distr[i] <- normdis
      }
      hist(distr);
        vecOfDifferences = vecOfAverages - 4.2*c(rep(1,10000));
      nmbCloseToTruth = length(vecOfDifferences[abs(vecOfDifferences) <= epsilon]);
      print(nmbCloseToTruth/10000);
    }

limitTheorem2(1,1)
```



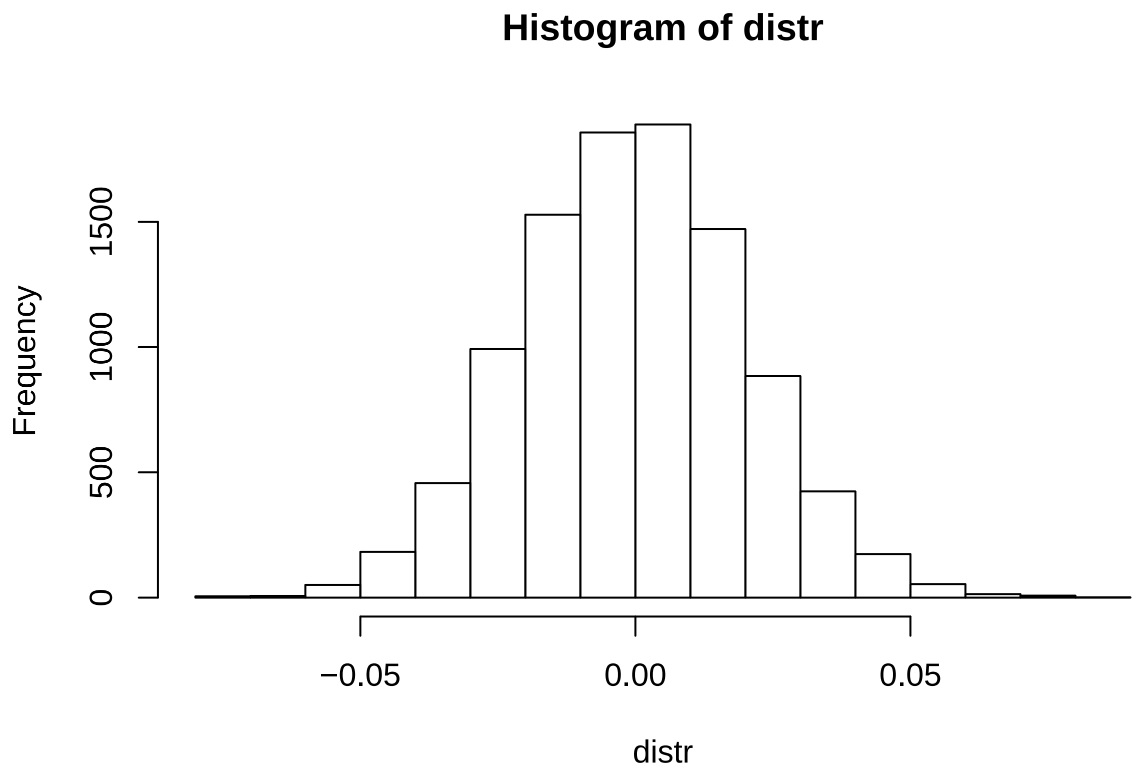

```
## [1] 0.3604
```

```
limitTheorem2(10,0.1)
```




```
## [1] 0.1237
```

```r
limitTheorem2(100,0.01)
```


Histogram of distr
Frequency
0
500
1000
1500
−0.05
0.00
0.05
distr

```
## [1] 0.0573
```

Problem 8

The independent rndom variables have the common distribution function:

$$P(X_i \leq x|\alpha,\beta) = \begin{array}{cc} 0 & x < 0 \\ (x/\beta)^\alpha & 0 \leq x \leq \beta \\ 1 & x > \beta \end{array}$$

$$\begin{aligned}
f(\alpha\beta) &= \frac{d}{dx}\frac{x}{\beta}^{\alpha} \\
&= \alpha \cdot \frac{x}{\beta}^{\alpha-1} \\
L(\alpha,\beta) &= \prod_{j=1}^{n} \alpha \cdot \frac{x}{\beta}^{\alpha-1} \\
&= \alpha^(n+1) \cdot \frac{1}{\beta}^{n(\alpha)} \cdot \prod x_j^\alpha \\
&= (n) \cdot ln(\alpha) - (n-1)\alpha \cdot ln(1/\beta) + (n-1)\alpha ln(x_j) \\
& \beta \text{is strictly decreaseing, so max L is at min} \beta = X_n \\
L(\alpha\beta)_{min} &= \frac{\partial\alpha,\beta}{\partial\alpha}(n) \cdot ln(\alpha)(n-1)\alpha \cdot ln(1/X_n) + (n-1)\alpha ln(x_j) \\
0 &= \frac{n}{\alpha} - (n-1) \cdot ln(1/X_n) + (n-1)ln(x_j) \\
\alpha &= \frac{n}{(n-1) \cdot ln(1/X_n) + (n-1)ln(x_j)}
\end{aligned}$$

For the data $\beta = 25$ since this is the max(X) and $\alpha =$

```
#(n-1)\cdotln(1/X_n) + (n-1)ln(x_j)
14/((13*log(1/25))+13*log(20.9))
```

```
## [1] -6.012076
```

```
14/((13*log(1/25))+13*log(23.9))
```

```
## [1] -23.93302
```

```
14/((13*log(1/25))+13*log(24))
```

```
## [1] -26.38095
```