Module 3

1 Poisson Processes

1.1 A Generic Description and Some Examples

A Poisson Process is a stochastic process. Generally speaking, a stochastic process is a sequence of random variables $X_1, X_2, \ldots, X_t, \ldots$, typically indexed by time. This implies that X_t is the random variable that occurs at time t and X_{t-1} is the random variable that occurs at time $t-1$. There are all sorts of different kinds of stochastic processes, and we will focus on three of them, the Poisson Process, Markov processes, and Gaussian processes. For this module, we will specifically focus on the Poisson process.

We will motivate the idea of a Poisson process through an example. Consider a fast food restaurant. Let t_1 be the time that the first customer arrives. Let t_2 be the time that the second customer arrives after the first, t_3 the time that the third customer arrives after the second, and so on. From this, it follows that $T_5 = t_1 + t_2 + \cdots + t_5$ is the time that the 5^{th} customer arrives. Let us also assume that $t_1, t_2, \ldots, t_n$ are independent and identically distributed exponential random variables with parameter λ. Now define $T_n = \sum_{i=1}^{n} t_i$, with $T_0 = 0$ and $N(s) = \max\{n : T_n \leq s\}$. Intuitively, $N(s)$ is the maximum number of customers that have arrived by time s, and T_n is the time that the n^{th} customer arrives. $N(s)$ is called a "Poisson" process. It is called a Poisson process because $P(N(s) = n)$, the probability that n customers have arrived by time s, is distributed as a Poisson random variable. Specifically,

$$P(N(s) = n) = \frac{\exp(-\lambda s)\,(\lambda s)^n}{n!}.$$

Here is the proof:

$$\begin{aligned}
P(N(s) = n) &= \int_0^s P\left(n^{\text{th}} \text{ customer arrives at time } t \text{ \& } (n+1)^{\text{st}} \text{ arrives after time } s\right) dt \\
&= \int_0^s P(T_n = t \text{ \& } T_{n+1} > s)\, dt = \int_0^s P(T_n = t) P(T_{n+1} > s | T_n = t) dt \\
&= \int_0^s P(T_n = t) P(t_{n+1} > s - t) dt = \int_0^s \underbrace{\lambda \exp\{-\lambda t\}}_{=P(T_n = t)} \underbrace{\frac{(\lambda t)^{n-1}}{(n-1)!} \exp\{-\lambda(s-t)\}}_{=P(t_{n+1} > s-t)} \\
&= \frac{\lambda^n}{(n-1)!} \exp\{-\lambda s\} \int_0^s t^{n-1} dt = \frac{\lambda^n}{(n-1)!} \exp\{-\lambda s\} \left.\frac{t^n}{n}\right|_0^s = \frac{(s\lambda)^n \exp\{-\lambda s\}}{n!}
\end{aligned}$$

The second equality in the third line above is justified since T_n has a Gamma(n, λ) distribution and $P(t_{n+1} > s - t) = \exp\{-\lambda(s-t)\}$. This is proven below.

Proof: Let's start by observing that $P(T_1 = t) = \lambda \exp\{-\lambda t\}$ (this is a Gamma with $n = 1$). Now assume that it is true for n. That is, let's assume that $T_n \sim \text{Gamma}(n, \lambda)$. We will do a proof by induction and show that if it is true for n, then it must be true for $n+1$. Recall that $T_{n+1} = T_n + t_{n+1}$.

$$\begin{aligned}
P(T_{n+1} = t) &= \int_0^t P(n^{\text{th}} \text{ customer arrives at time } s \text{ \& } (n+1)^{\text{st}} \text{ customer arrives } t \text{ units afterwards}) dt \\
&= \int_0^t P(T_n = s \text{ \& } t_{n+1} = t - s)\, ds = \int_0^t P(T_n = s) P(t_{n+1} = t - s) ds \\
&= \int_0^t \underbrace{\left\{\frac{\lambda^n s^{n-1}}{(n-1)!} \exp\{-\lambda s\}\right\}}_{=P(T_n = s)} \underbrace{\lambda \exp\{-\lambda(t-s)\}}_{=P(t_{n+1} = t-s)} ds = \int_0^t \frac{\lambda^{n+1} s^{n-1}}{(n-1)!} \exp\{-\lambda t\}\, ds \\
&= \frac{\lambda^{n+1} \exp\{-\lambda t\}}{(n-1)!} \int_0^t s^{n-1} ds = \frac{\lambda (t\lambda)^n \exp\{-\lambda t\}}{n!} = \frac{\lambda^{n+1}}{\Gamma(n+1)} t^n \exp\{-t\lambda\}
\end{aligned}$$

Another thing to be aware of is if $N(s)$ is a Poisson process, then

1. $N(0) = 0$ (i.e., nobody arrives before time 0).
2. $N(t+s) - N(s) \sim \text{Poi}(\lambda t)$ (the number of people arriving between time s to $t-s$ has a Poisson distribution with parameter λt.)
3. $N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ N(t_3) - N(t_2)$ are all independent. (i.e., the number of people that arrive in the time increment $t_1 - t_0, t_2 - t_1, t_3 - t_2, \ldots,$ are independent).

The first example given below illustrates some results regarding the exponential distribution, so read it carefully.

Example 1 (from Durrett) Alice and Betty enter a beauty parlor simultaneously, Alice to get a manicure and Betty to get a haircut. Suppose the time for a manicure (haircut) is exponentially distributed with mean 20 (30) minutes.

1. What is the probability Alice gets done first? Ans: Let $A =$ the time that Alice gets done, and $B =$ the time that Betty gets done. To make notation simple at this point, let $A \sim \exp(\lambda_A)$ and $B \sim \exp(\lambda_B)$, where $\lambda_A = \frac{1}{20}$ and $\lambda_B = \frac{1}{30}$. In this case,

$$\begin{aligned} P(A < B) &= \int_0^\infty P(A = s)P(B > s | A = s) ds = \int_0^\infty \lambda_A \exp\{-\lambda_A s\} \exp\{-\lambda_B s\}\, ds \\ &= \int_0^\infty \lambda_A \exp\{-(\lambda_A + \lambda_B)s\}\, ds = \frac{\lambda_A}{\lambda_A + \lambda_B} \underbrace{\int_0^\infty (\lambda_A + \lambda_B) \exp\{-(\lambda_A + \lambda_B)s\}\, ds}_{=1} \\ &= \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{3}{5}. \end{aligned}$$

This probability makes sense, right? The probability that one finishes first should be proportional to its rate parameter as is shown above. This can be generalized to many exponentials. If $X_1, X_2, \ldots, X_n$ are such that $X_i \sim \text{Exp}(\lambda_i)$, then $P(X_i = \min(X_1, X_2, \ldots, X_n)) = \lambda_i / \sum_{j=1}^n \lambda_j$.

2. What is the expected amount of time Alice and Betty both get done? Ans: there are two ways to do this problem a straight-forward (yet tedious) way to do it, and an elegant (yet tricky) way.

Straight-forward way: Let $Z = \max(A, B)$. We know the density of Z can be found by

$$f_Z(z) = \frac{d}{dz} P(Z \le z) = \frac{d}{dz} \{P(A \le z)P(B \le z)\} = \frac{d}{dz} \{(1 - \exp(-\lambda_A z))(1 - \exp(-\lambda_B z))\}.$$

Elegant way: Observe that $E(A+B) = E\{\min(A,B) + \max(A,B)\}$. It turns out that finding the density of $C = \min(A,B)$ is reasonably straight-forward. Observe that

$$P(C \ge c) = P(A \ge c)P(B \ge c) = \exp(-\lambda_A c) \exp(-\lambda_B c) = \exp(-(\lambda_A + \lambda_B) c),$$

implying that $C \sim \text{Exp}(\lambda_A + \lambda_B)$, and with this, we get $E(C) = \frac{1}{\lambda_A + \lambda_B}$. With this, we can find $E(\max(A,B))$. Since $E\{\min(A,B) + \max(A,B)\} = E(A+B) = 50\text{min}$ it follows that $E(\max(A,B)) = 50 - E(\min(A,B)) = 50 - 1/\left(\frac{1}{20} + \frac{1}{30}\right) = 38$. One should also note that the result stated above can be generalized for n exponential random variables. If $X_1, \ldots, X_n$ are independent such that $X_i \sim \text{Exp}(\lambda_i)$, then $\min(X_1, \ldots, X_n) \sim \text{Exp}\left(\sum_{j=1}^n \lambda_j\right)$.

Example 2 : Suppose the number of calls/hour is distributed as a Poisson random variable with $\lambda = 4$. Knowing this, answer the following.

1. What is the probability that fewer than two calls come in the first hour?
$P(N(1) < 2) = P(N(1) = 0) + P(N(1) = 1) = \exp\{-\lambda\} + \lambda \exp\{-\lambda\} = .0915.$
2. Suppose that six calls arrive in the first hour. What is the probability that there will be less than two in the second hour? Because of independence, the answer is the same as above. .0915.

3. Given that exactly six calls arrive in the first two hours, what is the conditional probability that exactly two arrive in the first hour and exactly four arrived in the second hour? The trick here is to split this up into independent intervals $(0,1)$ and (1,2).

$$P\left(N(1)=2 \;\&\; N(2)-N(1)=4|N(2)=6\right)=\frac{\exp\{-4\}\,4^2}{2!}\times\frac{\exp\{-4\}\,4^4}{4!}=.02862.$$

4. Suppose that an operator gets to take a break after she has answered ten phone calls. How long are her average work periods? Remember that $T_{10}=t_1+t_2+\cdots+t_{10}$ and $T_{10}\sim\text{Gamma}(10,4)$. The expected value of a gamma distribution is $\frac{n}{\lambda}$, so the expected weight time is 2.5 hours.

Example 3 : Suppose $N(t)$ is a Poisson process with rate 2. Compute the following conditional probabilities:

1. $P(N(3)=4|N(1)=1)$. In this case, we split the intervals up into *independent* intervals. So that

$$\begin{aligned}P(N(3)=4|N(1)=1) &= \frac{P\left(N(3)-N(1)=3 \;\&\; N(1)=1\right)}{P(N(1)=1)}=\frac{P(N(3)-N(1)=3)P(N(1)=1)}{P(N(1)=1)}\\ &= \exp\left(-4\right)4^3\big/\,3!\end{aligned}$$

2. $P(N(1)=1|N(3)=4)$. We split it up as before, but here things are a little different.

$$\begin{aligned}P(N(1)=1|N(3)=4) &= \frac{P(N(3)-N(1)=3\;\&N(1)=1)}{P(N(3)=4)}=\frac{P(N(3)-N(1)=3)P(N(1)=1)}{P(N(3)=4}\\ &= \frac{\exp(-4)4^3\big/\,3!\,\exp(-2)2^1\big/\,1!}{\exp(-6)6^4\big/\,4!}=.395.\end{aligned}$$

1.2 Relaxing the Assumptions

Something to note, of course, is that in the Poisson processes we've discussed so far, we have made two (critical) assumptions. They are given below.

1. All indivudals are assumed to have the same probability of arriving
2. The arrival rate of the indiviauls is constant throughout the time period considered.

In this subsection, we address what happens when these two assumptions are relaxed.

1.2.1 Relaxing Assumption # 1

With regard to assumption #1, it turns out that as long as the probability of arrival is small enough for each individual, the total number of arrivals can still be approximated with a Poisson distribution. Theorem 1 and Coroloary 1 make this clear.

Theorem 1 *For each n, let $X_{n,m}$ be independent random variables with $P(X_{n,m}=1)=p_{n,m}$ and $P\left(X_{n,m}=0\right)=1-p_{n,m}$. Let*

$$S_n=\text{total number of arrivals of the } n \text{ individuals}=\sum_{j=1}^{n}X_{n,j},\quad\text{and}\quad\lambda_n=E\left(S_n\right)=\sum_{j=1}^{m}p_{n,j}.$$

Let $Z_n\sim\text{Poisson}\left(\lambda_n\right)$. Then for any set $\mathcal{A}$,

$$\left|P\left(S_n\in\mathcal{A}\right)-P\left(Z_n\in\mathcal{A}\right)\right|\leq\sum_{j=1}^{n}p_{n,j}^2.$$

This theorem states that if we have a total of n indivuduals, all with different probabilities of arriving, then the probability that the total number of arrivals will be within a certain set of values, $\mathcal{A}$, can be approximated (within $\sum_{j=1}^{n} p_{n,j}^2$) by a Poisson random variable.

The following corollary gives a stronger result, but only on the condition that the probabilities of arrival are very small.

Corollary 1 *Suppose* $\lambda_n \longrightarrow \lambda$ *and* $\max_k (p_{n,k}) \longrightarrow 0$*, then*

$$\max_{\mathcal{A}} |P(S_n \in \mathcal{A}) - P(Z_n \in \mathcal{A})| \longrightarrow 0.$$

Below is a proof of this corollary.
Proof: We know $p_{n,m}^2 \leq p_{n,m} \cdot \max_k (p_{n,k}) \Longrightarrow \sum_{j=1}^{n} p_{n,j}^2 \leq \sum_{j=1}^{n} [\max_j (p_{n,j})] p_{n,j} = \max_j(p_{n,j}) \sum_j p_{n,j}$. The right term, by assumption, goes to $\lim_{n\to\infty} [\max_j (p_{n,j})] \cdot \lim_{n\to\infty} \left[\sum_j p_{n,j}\right] = 0 \cdot \lambda = 0$.

Look at the `R` program `DifferentArrivalProbs`. This function does a simulation to illustrate how the distribution of S_n and Z_n compare. There are three arguments to the function:

1. $n =$ the total number of people who may or may not arrive.
2. UppLimit = the upper limit of the arrival probabilities for the n people. Make this small.
3. nmbSimulations = the total number of simulations.

The output of this function is the simulated distribution of S_n and the distribution of $Z_n \sim \text{Poi}(\lambda_n)$.

1.2.2 Relaxing Assumption #2

With regard to assumption #2 (the part about the non-constant rates), we have to think about non-homogenous Poisson Process. A non-homogenous Process is defined below.

Non-homogenous Poisson Process (as defined by Durrett): we say $\{N(s), s \geq 0\}$ is a non-homogenous Poisson process with rate $\lambda(r)$ if

1. $N(0) = 0$ (no arrivals before time 0, obviously)
2. $N(t)$ has independent increments (recall this implies that $N(t_1) - N(0)$, $N(t_2) - N(t_1)$, $N(t_3) - N(t_2), \cdots$ are indepenent.)
3. $N(t+s) - N(s)$ is Poisson with rate $\displaystyle\int_s^t \lambda(r)dr$.

The first thing to observe about a non-homogenous Poisson process is that the underlying distribution which generates the process is not exponential (as was the case with the regular Poisson process). The proof of this is given below:

$$\begin{aligned}
P(t_1 \geq t) = P(N(t) = 0) &= \exp\left\{-\int_0^t \lambda(s)ds\right\} \\
\Longrightarrow P(t_1 = t) &= \frac{d}{dt} P(t_1 \leq t) = \frac{d}{dt}[1 - P(t_1 \geq t)] \\
&= -\frac{d}{dt} P(t_1 \geq t) = \lambda(t) \exp\left\{-\int_0^t \lambda(s)ds\right\}.
\end{aligned}$$

The last equation is, of course, not an exponential density.

In a non-homogenous Poisson Process, it is also the case that the arrival times are not independent. The proof of this is given below:

$$P(T_1 = u, T_2 = v) = P(T_2 = v | T_1 = u) P(T_1 = u) = \lambda(v) \exp\{-(\mu(v) - \mu(u))\} \times \lambda(u) \exp\{-\mu(u)\}.$$

Changing variables to the incremental arrival times (and recalling that $T_1 = t_1$ and $T_2 = t_1 + t_2$), we get

$$P(t_1 = s, t_2 = t) = \lambda(s) \exp\{-\mu(s)\} \lambda(s+t) \exp\{-\mu(t+s) - \mu(s)\}, \tag{1}$$

and Equation (1) is not the product of two independent densities. The point of all of this: with non-homogenous Poisson processes, things get complicated. Example 4, given below, shows how to apply non-homogenous Poisson processes.

Example 4 (borrowed from Durrett's book). There's a tailor shop in Annapolis (called Naval Tailor). Suppose the arrival rate starts at 0 at 10:00, increases to 4 at 12:00, to 6 by 2:00, drops to 2 by 4:00 and decreases to 0 by the time the store closes at 6:00, and that the arrival rates are linear in between these time points.

1. If the tailor decides to close the store at 5:30 instead of 6:00, what is the expected number of customers that are lost?

   Answer: Recall that the rate is linear and that the number of customers in between time periods is Poisson with rate $\int_s^t \lambda(r)dr$. In this case, $N(6:00) - N(5:30)$ is Poisson with parameter $\int_{5:30}^{6:00} \lambda(r)dr = .25/2.$. Draw the linear rates and prove this to yourself. Since $X \sim \text{Poisson}(\lambda) \Longrightarrow E(X) = \lambda$, it follows that the expected number of customers that are lost is $\frac{1}{8}$.

2. What is the probability that at least one customer arrives to find the store closed (if the tailor decides to close the store at 5:30)?

   Answer: We calculated the probability of arrivals in the first part of this question. To answer this question, we need to calculate $P(\text{at least 1 customer}) = 1 - P(0) = 1 - \exp\{-\lambda\} \lambda^0 / 0! = 1 - \exp\left\{-\frac{1}{8}\right\}$.