Problem Set 2 *

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1 Definitions

Def: Forward Rate Formulas (pg 79). The implied forward rate between times t_1 and t_2 is the rate of interset between those times that is consistent with a given spot rate curve. For Yearly compounding, the forward rate is:

$$f_{i,j} =[\frac{(1+s_j)^j}{(1+s_i)^i}]^{1/(j-i)} - 1$$

$$e^{s(t_2)t_2} =e^{s(t_1)t_1}e^{f_{t_1,t_2}(t_2-t_1)}$$

*Problems 1,2,3,4,5,6

Discount Factor Relation The discount facot between periods i and j is defined as

$$d_{i,j} = [\frac{1}{1 + f_{i,j}}]^{j-i}$$

These factors satisfy the compounding rule: $d_{i,k} = d_{i,j}d_{j,k}$

Def. Derivative (Ross pg 223) Let F be a real valued function defined on an open interval contained a point a. We say f is differentiable at a, or f has derivative at a if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Problem 1. If spot rates for 1 and 2 years are $s_1 = 6.3\%$ and $s_2 = 6.9\%$ what is the forward rate $f_{1,2}$

Solution Using the definition above:

$$\begin{aligned} f_{i,j} &= [\frac{(1+s_j)^j}{(1+s_i)^i}]^{1/(j-i)} - 1 \qquad\qquad \text{Def. Forward Rate (pg 79)} \\ &= [\frac{(1+s_2)^2}{(1+s_1)^1}]^{1/(2-1)} - 1 \\ &= [\frac{(1+6.9)^2}{(1+6.3)}] - 1 \end{aligned}$$

$$f_{1,2} = 7.5493\%$$

Problem 2. Given the (yearly) spot rate curve $s = (5.0, 5.3, 5.6, 5.8, 6.0, 6.1)$ find the spot rate curve for next year.

Solution The forcast spot rate is found using:

$$s`_{j-1} = f_{1,j} = [\frac{(1-s_j)^j}{1+s_1}]^{1/(j-1)} - 1$$

This gives us a forcasted $s_7 = 6.34$

	s1	s2	s3	s4	s5	s6
Current	5.0	5.3	5.6	5.8	6.0	6.1
Forecast		5.62	5.92	6.09	6.28	6.34

Problem 3. Consider two 5-year bonds: one has 9% coupon and sells for 101.00; the other has a 7% coupon and sells for 93.30 Find the price of a

5-year (zero?) coupon bond. (This question is confusing but I'm assuming it is following the structure of Example 4.3)

Solution To construct a zero coupon bond we first must make the sum of the coupons equal zero: Let x_A represent the amount of the 9% coupon bond and x_B represent the amount of the 7% coupon bond held.

$$0.09x_A + 0.07x_B$$

We also want the year 5 payout to be 100 to match the face value of 100 of a zero coupon bond.

$$109x_A + 107x_B$$

We can solve these equations for $x_A = -3.5$ and $x_B = 4.5$. To evaluate the price of the constructed zero coupon bond:

$$P_A x_A + P_B x_B = (101) * (-3.5) + (93.2) * (4.5) = 65.9$$

Problem 5. Let $s(t), 0 \leq t \leq \infty$, denote a spot rate curve; that is the present value of a dollar to be recieved at time t is $e^{-s(t)t}$. For $t_1 < t_2$ let $f(t_1, t_2)$ be the forward rate between t_1 and t_2 implied by the spot rate curve.

(a). Find an expression for f_{t_1,t_2}

(b). Let $r(t) = \lim_{t_2 \to t} f(t_1, t_2)$. We can call $r(t)$ the <u>instantaneuous interest rate</u> at time t. Show that $r(t) = s(t) + s'(t)t$

(c) x_0 is invested at $t = 0$ with instantaneous rate of interest $r(t)$ at all t (compounded). Then x(t) will satisfy $dx(t)/d = r(t)x(t)$. Find x(t).

Solution

$$
\begin{aligned}
e^{s(t_2)t_2} &= e^{s(t_1)t_1} e^{f_{t_1,t_2}(t_2-t_1)} && \text{Forward Rate Formula c (pg 79)} \\
s(t_2)t_2 &= s(t_1)t_1 + f_{t_1,t_2}(t_2-t_1) && \text{take the log} \\
f_{t_1,t_2} &= \frac{s(t_2)t_2 - s(t_1)t_1}{(t_2-t_1)} && \text{solve for f}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t_2 \to t} f_{t,t_2} &= \frac{s(t_2)t_2 - s(t)t}{(t_2-t)} \\
r(t) &= \frac{d}{dt}[s(t)t] && \text{Def. Derivative} \\
r(t) &= s(t) + s'(t)t && \text{product rule}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}x(t) &= r(t)x(t) \\
\frac{d}{dt}x(t) &= x(t) \cdot \frac{d}{dt}[s(t)t] && \text{Substitute r(t)} \\
\frac{d}{dt}x(t) \cdot \frac{1}{x(t)} &= \frac{d}{dt}[s(t)t] \\
\frac{d}{dt}ln(x(t)) &= \frac{d}{dt}[s(t)t] && \text{Note: chain rule ln(x(t))} \\
ln(x(t)) &= s(t)t && \text{Integrate} \\
x(t) &= e^{s(t)t}
\end{aligned}
$$

Problem 6. At time zero the one period discount rates are shown in column 1 and 2 below: Fine the time zero discount factors as shown in column 3 and 4 below.

Solution We can use the compounding rule here to find $d_{0,2} = d_{0,1}d_{1,2}\cdots d_{0,6} = d_{0,1}...d_{5,6}$ I preformed these calculations in an excel table and the result is shown below.

one-period discounts		time zero discounts	
d01	0.95	d01	0.95
d12	0.94	d02	0.893
d23	0.932	d03	0.832276
d34	0.925	d04	0.769855
d45	0.919	d05	0.707497
d56	0.913	d06	0.645945