

# Problem Set 10 \*

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Sorry in advance. Its been a week, and I didn't really have time to do this. I've included photos of the white board on which I work these out. Let me know if you have any problems. Again sorry.

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\*Problems:4,5,6,7,10

Show that 
$$W - \frac{(W - \bar{W})^2}{2\sigma^2} = -\frac{1}{2\sigma^2} [W - (\bar{W} + \sigma^2)]^2 + \bar{W} + \frac{\sigma^2}{2}$$

$$= -\frac{1}{2\sigma^2} [W - \bar{W} - \sigma^2]^2 + \bar{W} + \frac{\sigma^2}{2}$$

$$= -\frac{1}{2\sigma^2} [W^2 - 2W\bar{W} - 2W\sigma^2 + \bar{W}^2 + 2\bar{W}\sigma^2 + \sigma^2] + \bar{W} + \frac{\sigma^2}{2}$$

$$= \left( -\frac{W}{2\sigma^2} + \frac{W\bar{W}}{\sigma^2} + W - \frac{\sigma^2}{2\sigma^2} \right) - \left( \bar{W} - \frac{\bar{W}^2}{2} + \bar{W} + \frac{\bar{W}^2}{2} \right)$$

$$= -\frac{1}{2\sigma^2} [W^2 - 2W\bar{W} - 2\sigma^2 W + \bar{W}^2]$$

$$= -\frac{1}{2\sigma^2} [(W^2 - 2W\bar{W} + \bar{W}^2) - 2\sigma^2 W]$$

$$= -\frac{1}{2\sigma^2} (W - \bar{W})^2 + W$$

For complete the square later on  $a = \frac{1}{2\sigma^2}$ ,  $b = \frac{\bar{W}}{\sigma^2}$ ,  $c = \frac{\bar{W}^2}{2\sigma^2}$

$$E(W) = \int_{-\infty}^{\infty} g(w) \cdot p(w) \quad [\text{Theorem 3.53 (10)}]$$

$$p(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} \left( \frac{w - \bar{W}}{\sigma} \right)^2} \quad [\text{eqn 4.31 (10)}]$$

$$U = w$$

$$E(W) = \int_{-\infty}^{\infty} w \cdot \left( \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} \left( \frac{w - \bar{W}}{\sigma} \right)^2} \right) dw$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} w e^{-\frac{w^2}{2\sigma^2} - \frac{W\bar{W}}{\sigma^2} + W - \frac{\sigma^2}{2\sigma^2}} dw$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} w e^{-\frac{w^2}{2\sigma^2}} e^{-\frac{W\bar{W}}{\sigma^2}} e^W e^{-\frac{\sigma^2}{2\sigma^2}} dw$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{W\bar{W}}{\sigma^2}} \int_{-\infty}^{\infty} w e^{-\frac{w^2}{2\sigma^2} + W} dw$$

Complete the square  $a = \frac{1}{2\sigma^2}$ ,  $b = W$ ,  $c = \frac{W^2}{2\sigma^2}$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \left( \left( W + \frac{W\bar{W}}{\sigma^2} \right)^2 + \left( \frac{W\bar{W}}{\sigma^2} \right)^2 \right)$$

Problem 5. ss

⑤

For the log normal we have

$$E(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln w - \bar{w})^2}{2\sigma^2}} dw$$

Let  $y = \ln w - \bar{w}$

$y - \bar{w} = \ln w$   
 $w = e^{y + \bar{w}}$

$\frac{dw}{dy} = e^{y + \bar{w}}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} e^{y + \bar{w}} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2} + y + \bar{w}} dy$$

Complete the square  $a = \frac{1}{2\sigma^2}$   $b = 1$   $c = \bar{w}$

$$\frac{1}{2\sigma^2} \left( \left( y + \frac{1}{2\sigma^2} \right)^2 + \frac{1}{4\sigma^4} + \frac{1}{2\sigma^2} \right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (y + \frac{1}{2\sigma^2})^2} e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (y + \frac{1}{2\sigma^2})^2} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} u^2} du$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}} \sqrt{2\pi\sigma^2}$$

$$= e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}}$$

Var =  $[E(w^2)] - [E(w)]^2$

$$= [E(w^2)] - \left( \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{4\sigma^4} + \frac{1}{2\sigma^2}} \right)^2$$

$$= [E(w^2)] - \frac{1}{2\pi\sigma^2} e^{\frac{1}{2\sigma^4} + \frac{1}{\sigma^2}}$$

$E(w^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln w - \bar{w})^2}{2\sigma^2}} w^2 dw$

This is basically the same except  $\frac{dw}{dy} = w$  so  $\frac{dw}{dy} = e^{y + \bar{w}}$  and  $w = e^{y + \bar{w}}$  so  $\frac{dw}{dy} = e^{y + \bar{w}}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} e^{y + \bar{w}} w^2 dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2} + y + \bar{w}} w^2 dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2} + y + \bar{w}} e^{2(y + \bar{w})} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2} + 3y + 3\bar{w}} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{9}{2\sigma^2} + 3\bar{w}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2} + 3y} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{9}{2\sigma^2} + 3\bar{w}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (y - 3\sigma^2)^2} e^{\frac{9\sigma^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{9}{2\sigma^2} + 3\bar{w}} e^{\frac{9\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (y - 3\sigma^2)^2} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{9}{2\sigma^2} + 3\bar{w}} e^{\frac{9\sigma^2}{2}} \sqrt{2\pi\sigma^2}$$

$$= e^{\frac{9}{2\sigma^2} + 3\bar{w} + \frac{9\sigma^2}{2}}$$

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Problem 6. ss

⑥  $\mu = 0.20$   
 $\sigma = 0.40$   
 $S(0) = 1$

$E[\ln S(t)] =$

Relations for geometric  
 Brownian motion,  $S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$   
 $E[\ln(S(t)/S(0))] = \mu t$   
 $\text{Stdev}[\ln(S(t)/S(0))] = \sigma\sqrt{t}$   
 $E\left[\frac{S(t)}{S(0)}\right] = e^{\mu t}$   
 $\text{Stdev}\left[\frac{S(t)}{S(0)}\right] = e^{\mu t}(e^{\sigma^2 t} - 1)^{1/2}$   
 (310)

$E[\ln S(t)] = \ln S(0) + \left(\mu + \frac{\sigma^2}{2}\right)t + E\left[\sigma W_t\right]$   
 $\ln(1) + (0.2 + \frac{0.4^2}{2})1$

$E[\ln S(t)] = 0.28$

$\text{Stdev} = \ln[S(t)] + \frac{\sigma}{\sqrt{t}}$   
 $= \frac{0.4}{\sqrt{1}}$

$\text{Stdev}[\ln S(t)] = 0.4$

$E[S(t)] = S(0) + e^{0.20}$

$E[S(t)] = 1.2214$

$\text{Stdev}[S(t)] = S(0) + e^{\mu t}(e^{\sigma^2 t} - 1)^{1/2}$   
 $1 + 1.2214(e^{0.16} - 1)^{1/2}$

$\text{Stdev}[S(t)] = 0.5087$

Problem 7. ss

⑦ Stock  $S$  governed by  
 $dS = aSdt + bSdz$   
 Find the Process that governs  
 $G(t) = S^{1/2}t$   
 $dG = \left(\frac{1}{2}a - \frac{1}{8}b^2\right)Gdt + \frac{1}{2}bGdz$

$$\frac{\partial G}{\partial S} = \frac{1}{2S^{1/2}}$$

$$\frac{\partial^2 G}{\partial S^2} = \frac{1}{4}S^{-3/2}$$

$$dS^{1/2} = \left(\frac{a}{2}S^{-1/2} - \frac{b^2}{8}S^{-3/2}\right)dt + \frac{b}{2}S^{-1/2}dz$$

Problem 10. ss

let  $w = e^{\sigma z - \frac{1}{2}t}$   
 Find PDE governing  $w$

$$w = e^{\sigma z} \cdot e^{-\frac{1}{2}t}$$

$$\frac{\partial w}{\partial t} = e^{\sigma z} \cdot e^{-\frac{1}{2}t} \cdot -\frac{1}{2}$$

$$\frac{\partial^2 w}{\partial z^2} = e^{\sigma z} \cdot e^{-\frac{1}{2}t} \cdot \sigma^2$$

$$= e^{\sigma z} \cdot e^{-\frac{1}{2}t} (\sigma^2 - \frac{1}{2})$$

Ho's Lemma  
 $dG(w) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}b dz$

$$\left(-\frac{1}{2}e^{\sigma z} \cdot e^{-\frac{1}{2}t} + \frac{b^2}{2}(\sigma^2 - \frac{1}{2})e^{\sigma z} \cdot e^{-\frac{1}{2}t}\right)dt + \sigma e^{\sigma z} \cdot e^{-\frac{1}{2}t} dz$$