Problem Set 7 *

Ian McGroarty

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1 Theorems

Theorem 3.9.2 (Larsen Marx (2018) page 184): Let X and Y be any two random variables, and let a and b be constants. Then:

$$E(aX + bY) = aE(X) + bE(Y)$$

Theorem 3.9.5 (Larsen, Marx (2018) page 188) Suppose X and Y are random variables with finite variances, and a and b are constants. Then

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

*Problems: 2,5,6,8,9

Proposition 1 (notes module 7) Suppose X follows a normal distribution with mean μ and standard deviation σ. Then

$$VaR_h(X) = -\sigma F_N^{-1}(h) - \mu$$

Definition: Conditional Value at Risk (notes module 7) of a position X is the conditional expectation values of the associates loss of X given that the losses are at least equal to its value at risk:

$$CVaR_h(X)CVaR_h(X) = E[-X|X \leq -VaR_h(X)]$$

Definition: Expected Value (Larsen, Marx (2018) page 138) Let Y be a continuous random variable with pdf $f_Y(y)$, $E(Y) = \mu = \mu_Y = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy$

Problem 2. Suppose X is a normal with zero mean and standard deviation of \$10 million:

(a) Find the value at risk for X for the risk tolerances $h = 0.01, 0.02, 0.05, 0.10, 0.50, 0.60,$ and 0.95. **Solution:** See figure 1.

(b) Is there a relation between VaR for values of $h \leq 0.50$ and values for $h \geq 0.50$

Solution Well it seems that 0.5 is the 0 point. Values $h \leq 0.5$ are associated with a positive value at risk and values $g \geq 0.5$ are associated with negative values at risk. It also appears that they are invereses of eachother which makes sense since it is $(1 - h)$. Since X is normal, there is symmetry about the mean.

Figure 1: Value at Risk for various Risk Tolerances

Mean	0	
Standard Deviation	10	
Risk Tolarance h	FN(h)	VaR
0.01	2.326348	23.26348
0.02	2.053749	20.53749
0.05	1.644854	16.44854
0.1	1.281552	12.81552
0.5	0	0
0.6	-0.25335	-2.53347
0.95	-1.64485	-16.4485

Problem 5. Suppose X_1 and X_2 are jointly normal positions with parameters $\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}$. Show that: $VaR_h(X_1+X_2) \leq VaR_h(X_1)+VaR_h(X_2)$

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\begin{aligned}
VaR_h(X_1 + X_2) &\leq VaR_h(X_1) + VaR_h(X_2) & \\
-\sigma_{12}F_N^{-1}(h) - \mu_{12} &\leq -\sigma_1 F_N^{-1}(h) - \mu_1 + -\sigma_2 F_N^{-1}(h) - \mu_2 & \text{Proposition 1 in notes} \\
\mu_{12} &= \mu_1 + \mu_2 & \text{By Theorem 3.9.2. Thus:} \\
-\sigma_{12}F_N^{-1}(h) &\leq -\sigma_1 F_N^{-1}(h) + -\sigma_2 F_N^{-1}(h) & \\
-\sigma_{12} &\leq \sigma_1 + \sigma_2 & \\
-\sigma_{12}^2 &\leq (\sigma_1 + \sigma_2)^2 & \text{Square both sides} \\
Var(X) + Var(Y) + 2Cov(X,Y) &\leq \sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2 & \text{Theorem 3.9.5} \\
2Cov(X,Y) &\leq 2\sigma_1\sigma_2 & \\
\rho\sigma_1\sigma_2 \leq \sigma_1\sigma_2 & &
\end{aligned}
$$

Problem 6. ind $AVaR_h(X)$ for the X of Exercise 3. (Also see Exercise 9.)

Solution We know that the density of X is $f(x) = \frac{1}{60--40} = \frac{1}{100}$ for $x \in (-40, 60)$ because it is uniform. The distribution function is $F_X(x) = \frac{x+40}{100}$ for $x \in (-40, 60)$. So we have $F_X^{-1}(h) = 100h - 40$. Therefore the Value at Risk is

$$VaR_h = -F_X^{-1}(h) = 40 - 100h$$

The Average value at Risk is

$$\begin{aligned} AVaR_h(X) &= \frac{1}{h}\int_0^h VaR_u(X)du \\ &= \frac{1}{h}\int_0^h 40 - 100h \\ &= 40 - 50h \end{aligned}$$

Problem 8. Let X be a position with a probability distribution F that is strictly increasing and smooth. Let $f(x) = F'(x)$ be the associated probability density.

(a) Verify that $CVaR_h(X) = -\frac{1}{h}\int_{-\infty}^{-VaR_h(X)} xf(x)dx$

Solution By Def CVaR: $CVaR_h(X) = E[-X|X \leq -VaR_h(X)]$. This is equivalent to: $E[-X]$ $s.t.$ $x \in (-\infty, -VaR_h(X)]$. So by Def. Expected Value

$$E[-X] = \int_{-\infty}^{-VaR_h(X)} f_X(x) \cdot x$$

Since VaR is based on the h quantile we multiply by $1/h$?

(b) For any $u \in (0,1)$ let $x = F^{-1}(u)$ be the value oof X that defines the u-quantile of X. Conversely, for any specific value x of X, we have $u = F(x)$ as the quantile value associates with x Using the change of variable $u = F(x)$ in the equation (of part a) show that: $CVaR_h(X) = -\frac{1}{h}\int_0^h F^{-1}(u)du$

$$\begin{aligned}
\frac{1}{h}\int_{-\infty}^{-VaR_h(X)} f_X(x) \cdot x &= \frac{1}{h}\int_{-\infty}^{-VaR_h(F^{-}(u))} f_X(F^{-}(u)) \cdot F^{-}(u) \\
&= \frac{1}{h}\int_{-\infty}^{-VaR_h(F^{-}(u))} F^{-}(u)du
\end{aligned}$$

To determine $-VaR_h(F^{-}(u))$: $h = P(-X > V) = P(-F^{-}(u) > V) = P(F^{-}(u) \leq V) = P(u \leq F(V)) = F(F^{-}(V)) = V \implies VaR(x) = h$? Thus, $CVaR_h(X) = -\frac{1}{h}\int_0^h F^{-1}(u)du$

(c) Interpret the right hand side of equation (in part b) to obtain:

$$AVaR(X) = -\frac{1}{h}\int_0^h F^{-1}(u)du$$

and hence conclude that $CVaR_h(X) = AVaR_h(X)$. **Solution** Does anything need to be done here?

Problem 9. ind $CVaR_h(X)$ for the linear case of exercise 3 (also see exercise 6).

Solution We know that the density of X is $f(x) = \frac{1}{60--40} = \frac{1}{100}$ for $x \in (-40, 60)$ because it is uniform. The distribution function is $F_X(x) = \frac{x+40}{100}$ for $x \in (-40, 60)$. So we have $F_X^{-1}(h) = 100h - 40$. Therefore the Value at Risk is

$$VaR_h = -F_X^{-1}(h) = 40 - 100h$$

For the conditional VaR we have $CVaR_h(X) = E[-X|X \leq -VaR_h(X)]$ and $P(X \leq -VaR_h(X)) = h$. So the conditional VaR is

$$CVaR_h(X) = \int_{-40}^{100h-40} \frac{-x}{100h} dx = 40 - 50h$$