The Adaptive Accept-Reject Algorithm

625.433

In many cases, it is quite difficult to find a density, $g(x)$, such that $f(x) \leq M \cdot g(x)$ for all x in the support of both $f(x)$ and $g(x)$. In the early 1990s, some statisticians found a simple way to arrive at such a function if the density that you want to sample from is log-concave. Log-concavity simply requires that

$$\frac{\partial^2 l(x)}{\partial x^2} < 0 \;\; \forall\, x,$$

where $l(x) = \log[f(x)]$. For a density that is log-concave, the logarithm of the density looks something like this what is given in Figure 1.

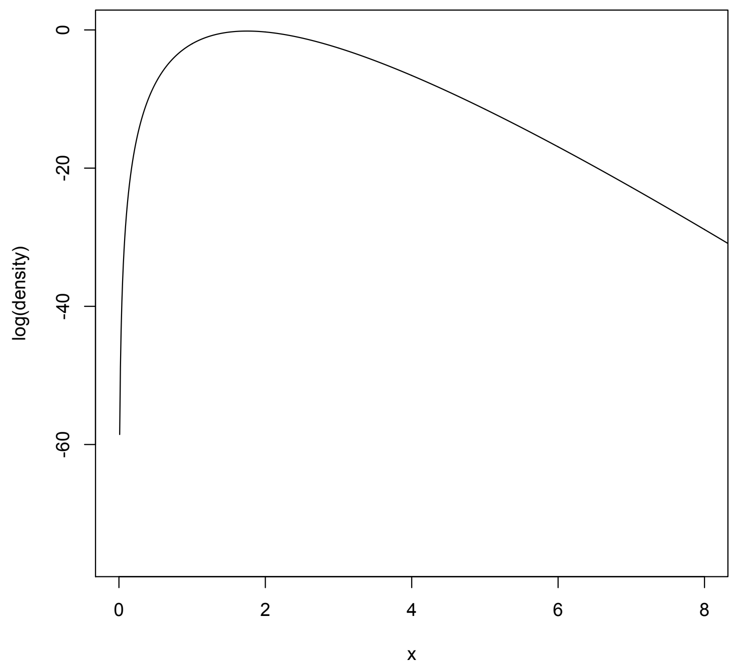

Figure 1: A plot of $\log[f_X(x)]$ vs. x

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For log-concave densities, all lines tangential to $l(x)$ are above it. This is what the adaptive accept-reject algorithm takes advantage of.

Consider a set of p points in the support of $f(x)$. We will call this set of points $\mathcal{P} = \left\{x_1^*, x_2^*, \ldots, x_p^*\right\}$, and for the purposes of this illustration, I will assume $p = 2$, making $\mathcal{P} = \{x_1^*, x_2^*\}$. Now consider the lines tangent to $l(x)$ at x_1^* and x_2^*. In Figure 2 below, $x_1^* = 0.8$ and $x_2^* = 2.5$. These lines are shown in the Figure 2, and I will refer to the one on the left as $g_{\log}^1(x)$ and the one on the right as $g_{\log}^2(x)$. The formula for these two lines are given as

$$g_{\log}^1(x) = m_1 x + b_1 \quad \text{where} \quad m_1 = \left.\frac{dl(x)}{dx}\right|_{x=x_1^*} \quad \text{and} \quad b_1 = l(x_1^*) - m_1 x_1^*$$

$$g_{\log}^2(x) = m_2 x + b_2 \quad \text{where} \quad m_2 = \left.\frac{dl(x)}{dx}\right|_{x=x_2^*} \quad \text{and} \quad b_2 = l(x_2^*) - m_2 x_2^*$$

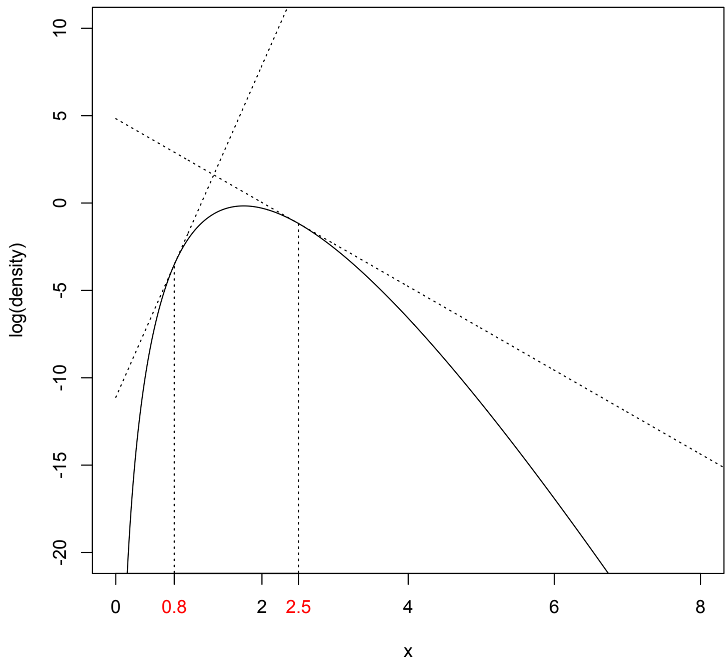

Figure 2: A plot of $\log[f_X(x)]$ and the tangential "hat" functions

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With the logarithm of the density enveloped, it is straight-forward to construct an envelope for the density $f(x)$ itself. All one needs to do is exponentiate the function and the envelope.

Within the range of $(-\infty, x_{\text{Int}})$, where x_{Int} is the $x-$coordinate where lines $g^1_{\log}(x)$ and $g^2_{\log}(x)$ intersect, the enveloping function is $g^1(x) = \exp\left[g^1_{\log}(x)\right] = \exp\left[m_1 x + b_1\right]$ andwithin the interval (x_{Int}, ∞), the enveloping function is $g^2(x) = \exp\left[g^2_{\log}(x)\right] = \exp\left[m_2 x + b_2\right].$

So to generate n values of X from $f(x)$ using the adaptive accept-reject algorithm, one needs to do the following:

1. Set `num_accepted` $= 0$
2. While `num_accepted` $<$ `n`
   (a) Sample from the enveloping function. To do this, first decide which one of the densities you will be generating from. You will generating from $g^1(x)$ with probability

   $$w_1 = \int_{-\infty}^{x_{\text{Int}}} g^1(x)dx \Big/ \left(\int_{-\infty}^{x_{\text{Int}}} g^1(x)dx + \int_{x_{\text{Int}}}^{\infty} g^2(x)dx\right)$$

   and from $g^2(x)$ with probability

   $$w_2 = \int_{-\infty}^{x_{\text{Int}}} g^2(x)dx \Big/ \left(\int_{\text{Int}}^{x_{\infty}} g^1(x)dx + \int_{x_{\text{Int}}}^{\infty} g^2(x)dx\right)$$

   (b) Once it is decided which density to sample the candidate value of X from, generate the candidate value. This is easy to do using the inverse-transform method.

(c) Now generate $U \sim \text{Unif}\left(0, g^1(x_{\text{cand}})\right)$ or $U \sim \text{Unif}\left(0, g^2(x_{\text{cand}})\right)$ (depending on which density you're sampling from).

(d) If $U \leq f(x_{\text{cand}})$, accept and set $\texttt{num}_{\texttt{a}}\texttt{ccepted} = \texttt{num}_{\texttt{a}}\texttt{ccepted} + \texttt{1}$. Otherwise, reject.