Johns Hopkins Engineering

Mathematical Finance

Module 1	Lecture 2

JOHNS HOPKINS
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Present Value

- Definition

The present value of an amount *A* that will be received in one year is

$$PV = \frac{A}{1+r}$$

where *r* is the interest rate.

- The quantity $d_1 = \frac{1}{1+r}$ is called the one-year discount factor.

- So we have

$$PV = d_1 A = \left(\frac{1}{1+r}\right) A$$

Present Value

•The present value of a future amount *A* which will be received after *k* periods of compounding is

$$PV = \frac{A}{\left(1 + \frac{r}{m}\right)^k}$$

the discount factor is

$$d_k = \frac{1}{\left(1 + \frac{r}{m}\right)^k}$$

Present and Future Value of Streams

- Consider a cash stream $(x_0, x_1, \ldots, x_n)$ such that at the end of each period *k* the amount x_k is invested in an account with rate *r* , we have: x_0 $x_0(1+r)^n$

- x_1 will grow to $x_1(1+r)^{n-1}$
- x_2 will grow to $x_2(1+r)^{n-2}$
- x_{n-1} will grow to $x_{n-1}(1+r)^1$
- x_n will grow to x_n
- will grow to

- Thus the Future Value of the entire cash stream is

$$FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \cdots + x_n$$

- Likewise the Present Value of this cash stream is

$$PV = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

Present and Future Value of Streams

•We have the following relationship between the present and future values

$$PV = \frac{FV}{(1+r)^n}$$

• Example

Find the future and present values of the cash flow stream (1,1,1,2) when the periods are years and the annual rate is r = 10%.

The future value is

$$FV = 1(1+0.1)^3 + 1(1+0.1)^2 + 1(1+0.1)^1 + 2(1+0.1)^0 = 5.641$$

The present value is $$PV = \frac{FV}{(1+0.1)^3} = 4.2382$$

Present and Future Value of Streams

Remark

•If the interest is compounded m times in a year, then the present value of the cash flow $(x_0, x_1, \cdots, x_n)$ is

$$PV = \sum_{k=0}^{n} \frac{x_k}{\left(1 + \frac{r}{m}\right)^k}$$

•If the interest is compounded continuously and the cash flow investments occur at times $(t_0, t_1, \cdots, t_n)$ the present value is

$$PV = \sum_{k=0}^{n} x(t_k) e^{-r t_k}$$

Present and Future Value of Streams

Theorem and Definition

Two cash flow streams $(x_0, x_1, \cdots, x_n)$ and $(y_0, y_1, \cdots, y_n)$ are equivalent for a constant rate r if and only if the present values of the two streams evaluated at the rate r are equal.

Example

Consider the cash flow streams $x = (1,1,1,4)$ and $y = (2,1,1,2)$ at rate *r* =5%.

Are they equivalent? If not find the *r* such that they are equivalent.

Solution

$$PV(x) = \sum_{k=0}^{n} \frac{x_k}{(1+0.05)^k} = 6.31476 \qquad PV(y) = \sum_{k=0}^{n} \frac{y_k}{(1+0.05)^k} = 5.58709$$

So the streams *x* and *y* are not equivalent. In order to the value of *r*, we solve

$$PV(x) = \sum_{k=0}^{n} \frac{x_k}{(1+r)^k} = \sum_{k=0}^{n} \frac{y_k}{(1+r)^k} = PV(y)$$

and find *r = 0.258.*

Internal Rate of Return

Definition

Given a cash flow $(x_0, x_1, \cdots, x_n)$, the internal rate of return of this cash flow stream is a the number *r* satisfying the equation

$$\sum_{k=0}^{n} \frac{x_k}{(1+r)^k} = 0$$

In other terms, the internal rate of return is the number *r* such that the present value of the stream is zero.

Example

Find the internal rate of return of the sequence (3/2,-11/4,1/4,1).

Solution

We use the change of variable *1/(1+r)=c* the new equation becomes

0=(c-3/4)(c-1)(c+2) so c=3/4 or c =1 or c = -2, and r = 1/c-1.

Therefore the only meaningful value of *r* is *r* = *1/3.*