Problem Set 6

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08MAR2020

Problem 1

Assume that $X \sim N(0, \sigma^2)$ ### (a) Prove that $E(exp(-X^2)) = \frac{1}{\sqrt{2\sigma^2+1}}$

$$
\begin{aligned}
let g(x) &= e^{-X^2} = e^{(\mu+\sigma\cdot x)^2} = e^{\sigma^2\cdot x^2} && \text{Normalize} \\
let f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{(1/2)\cdot(x^2/2)} \\
E[g(x)] &= \int_{-\infty}^{\infty} g(x)\cdot f(x)dx && \text{Theorem 3.5.3 (148) Larsen \& Marx} \\
&= \int_{-\infty}^{\infty} e^{\sigma^2\cdot x^2}\cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{(1/2)\cdot(x^2/2)} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp(\sigma^2 x^2 - \frac{x^2}{2}) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp((x^2/2)\cdot(2\sigma+1)) \\
&= \frac{2\sigma+1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\sigma^2/2} dx \\
&= \frac{2\sigma+1}{\sqrt{2\pi\sigma^2}}
\end{aligned}
$$

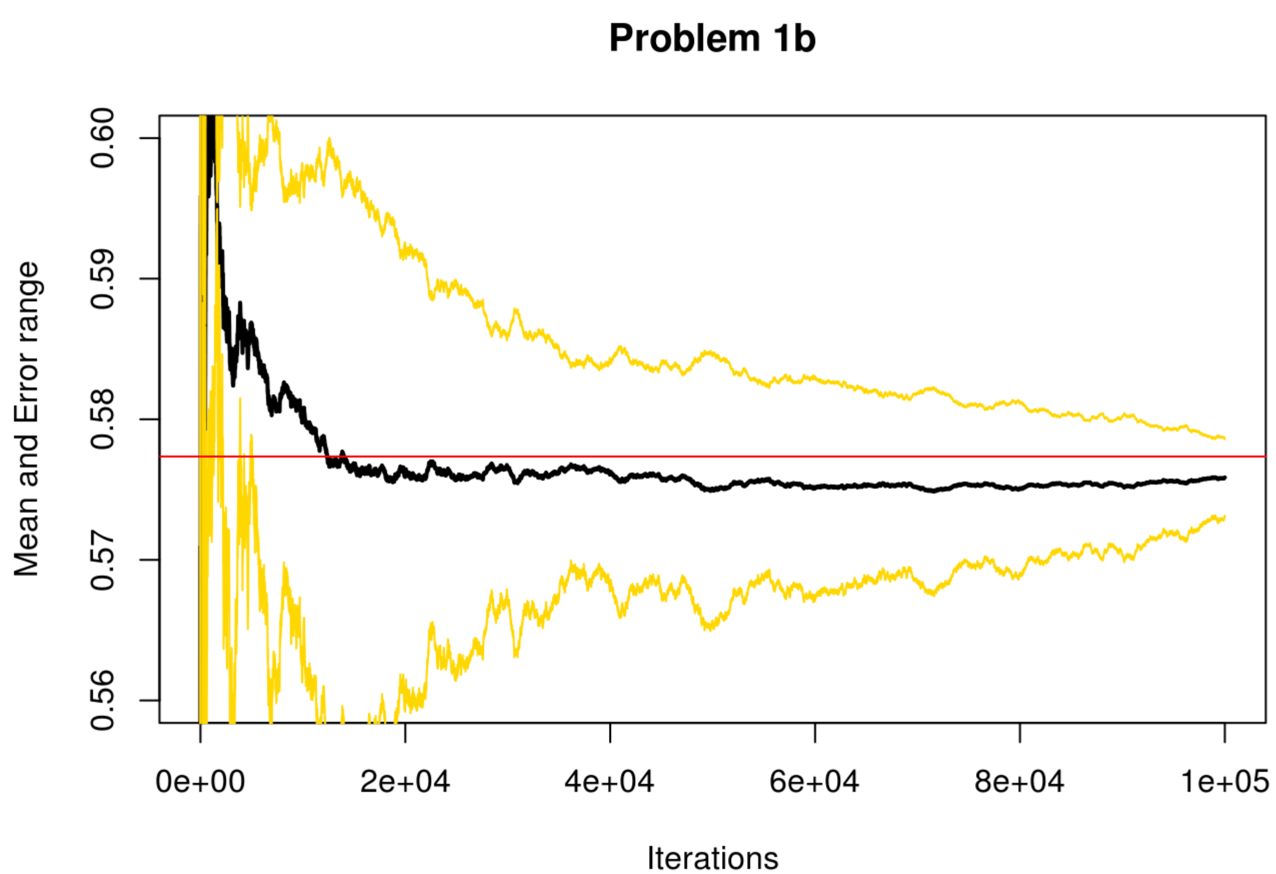
I couldn't get it all the way but I got close. I suspect I am doing something wrong with the normal distribution but I can't tell exactly what.

(b) Use Monte Carlo Methods to prove your result from (a)

```
## Set the function
  h <- function(x){
    exp(-x^2)
  }

## Integrate - find the true mean in this case given by problem
  truemean <- function(s){
    1/(sqrt(2*s^2+1))
  }
  truemean(1)
```

```
## [1] 0.5773503
```

```
## Sampling
  # Set Nsim
    Nsim <- 10^5
  # Generate uniform
    x <- rnorm(Nsim)
  # Evaluate h(x)
    hx <- h(x)
  # Average of hx
    estint <- cumsum(hx)/(1:Nsim)
  # Get the variation from the true mean for everyvalue of hx
    esterr <- sqrt(cumsum(hx-estint)^2)/(1:Nsim)

## Graph
#par(mar=c(2,2,2,1),mfrow=c(1,2))
#curve(h,xlab="Function",ylab="",lwd=2)
plot(estint, type='l',lwd=2,
     xlab="Iterations",ylab="Mean and Error range", main="Problem 1b",
     ylim=c(0.56,0.6))

lines(estint+3*esterr,col="gold",lwd=1)
lines(estint-3*esterr,col="gold",lwd=1)
abline(h=truemean(1),col = "red")
```



Problem 2

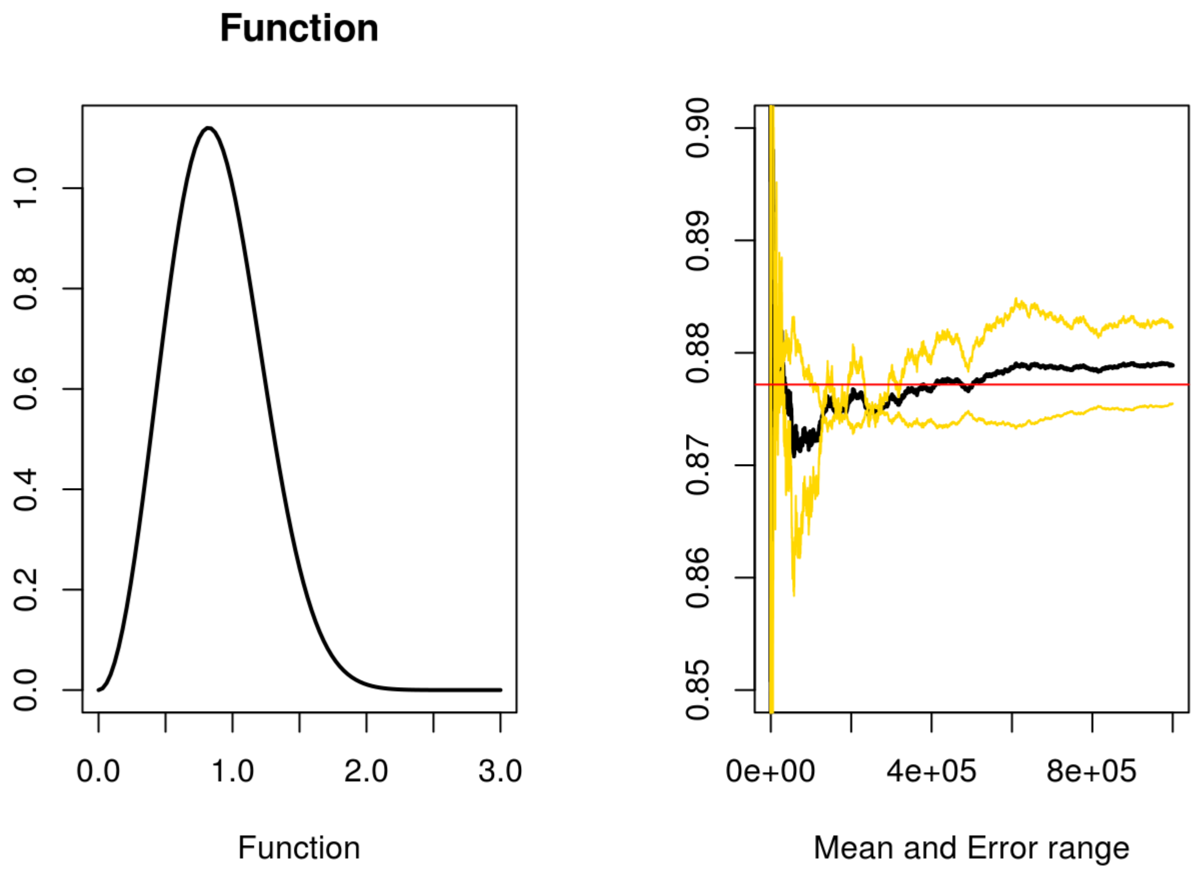
Estimate the mean and variance of this distribution using Monte Carlo methods.

$$f(x) = 3.852985 \cdot exp(-x^2\sqrt{x})[sin(x)]^2$$

```
###
  # Set the function
    f <- function(x){
    3.852985*exp((-x^2*sqrt(x)))*(sin(x)^2)
    }

  # Set sim
    Nsim <- 10^6
  # Generate g(x) - use exponential because of the previous exercise? (3.13)
    x <- rexp(Nsim)
  # Evaluate h(x)
    fx <- f(x)
  # Get the density weight [f(x)/g(x)]
    w <- f(x)/dexp(x)
  # Compute the h function
    h <- w * x
    #max(h)

  # Average of hx
    estint <- cumsum(h)/(1:Nsim)
  # Get the variation from the true mean for everyvalue of hx
    esterr <- sqrt(cumsum(h-estint)^2)/(1:Nsim)
  # Mean & Variance
      print(paste0("The function has a mean of ",mean(estint)," and a variance of ",var(estin
t)))
```

```
## [1] "The function has a mean of 0.877178381365843 and a variance of 2.40488090721826e-05"
```

```
  ## Graph
    par(mfrow=c(1,2))
    curve(f,xlab="Function",ylab="",main="Function", xlim=c(0,3),lwd=2)
    plot(estint, xlab="Mean and Error range",ylab="",type='l',lwd=2,
         ylim=c(0.85,0.9))
    lines(estint+2*esterr,col="gold",lwd=1)
    lines(estint-2*esterr,col="gold",lwd=1)
    abline(h=mean(estint), col = "red")
```



Problem 3

For the density in the previous problem find the $P(X > 3)$

```
## P(X>3)
    sum(as.numeric(h>3))/Nsim
```

```
## [1] 0
```

Problem 4 (exercise 3.14)

When a cdf F(x) has a tail power of $$ (i.e., when $1 - F(x) \propto x^{-\alpha}$). Show that $E[X|X > K] = K\alpha/(\alpha - 1)$

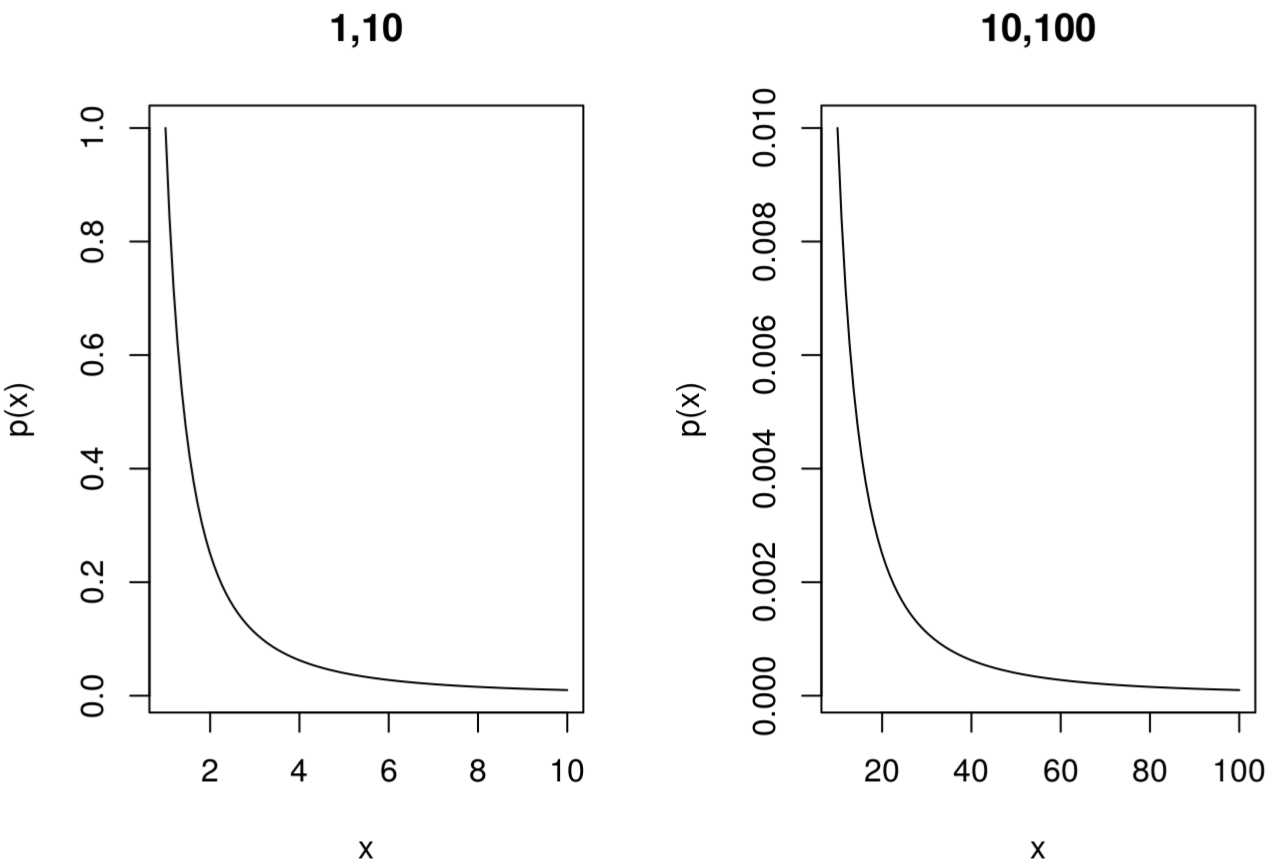
$$cdf : 1 - F(X) = (1 - x^{-\alpha})$$
$$pdf : \frac{d}{dx}(1 - x^{-\alpha}) = \alpha x^{-\alpha-1}$$
$$\begin{aligned}
E[X|X > K] = E[X > K] &= \int_K^\infty x \cdot f(x) \\
&= \int_K^\infty x \cdot \alpha x^{-\alpha-1} dx \\
&= \alpha \int_K^\infty x^{-\alpha} dx \\
&= \alpha[\frac{x^{-\alpha+1}}{-\alpha+1}|_K^\infty] \\
&= 0 - \alpha \frac{K^{-\alpha+1}}{-\alpha+1} \\
&=? \frac{K}{1-\alpha} \cdot \alpha
\end{aligned}$$

(b) Derive this?

Okay I’m not sure what to do here but I found it realy useful to play with this. So I’ll talk about that.

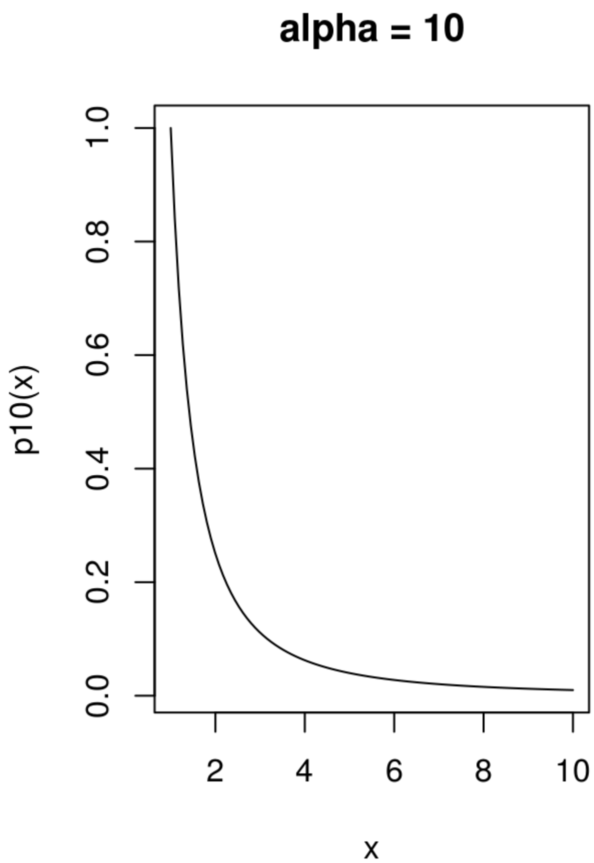
First note that $\alpha > 1$ since $1-$ can not be zero and must be positive. we can see that in the graph that the line does not change much regardless of the x range:

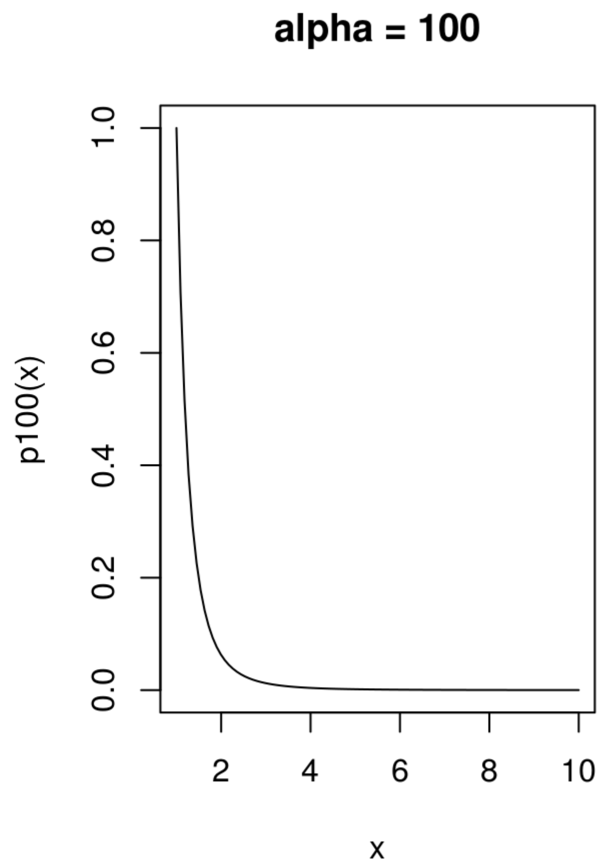
```
p <- function(x){
  x^(-2)
}

par(mfrow=c(1,2))
curve(p,xlim=c(1,10),main="1,10")
curve(p,xlim=c(10,100),main="10,100")
```



Now also notice how the curve gets “sharper?” as $$ increases.

```
p10 <- function(x){
  x^(-2)
}

p100 <- function(x){
  x^(-4)
}


par(mfrow=c(1,2))
curve(p10,xlim=c(1,10),main="alpha = 10")
curve(p100,xlim=c(1,10),main="alpha = 100")
```





If you really look at it this looks pretty similar to a geometic probability distribution. Which makes sense since there are going to be pretty small changes between values of x in the pdf so you can almost think of each increase in x as an independent event. But the $$ is going to directly determine both how high you start at K and how gradual the descent.