# Final*

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**Problem 1.** Assume an asset price St follows the geometric Brownian motion, $dSt = \mu Stdt + \sigma StdZt, S0 = s > 0$ where $\mu$ and $\sigma$ are constants, r is the risk-free rate, and Zt is the Brownian motion

Part1: Using the Ito's Lemma find to the stochastic differential equation satisfied by the process $Xt = \sqrt{St}$

$$
\begin{aligned}
dX(t) =& (\frac{\partial F}{\partial S}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2)dt + \frac{\partial F}{\partial S}bdz && \text{Def: Ito's Lemma (pg. 312)}\\
\frac{\partial F}{\partial x} = \frac{1}{2S^{1/2}}|\frac{\partial^2 F}{\partial x^2} &= -\frac{1}{4S^{3/2}}|\frac{\partial F}{\partial t} = 0|a = \mu S_t|b = \sigma S_t\\
&= [\frac{1}{2S^{1/2}} \cdot \mu S_t + \frac{\sigma^2 S^2}{2} \cdot -\frac{1}{4S^{3/2}}]dt + \frac{\sigma S}{2S^{1/2}}dz_t\\
&= [\frac{\mu S_t^{1/2}}{2} - \frac{\sigma^2 S^{1/2}}{8}]dt + \frac{\sigma S_t^{1/2}}{2}dz_t\\
&= S_t^{1/2}([\frac{\mu}{2} - \frac{\sigma^2}{8}]dt + \frac{\sigma}{2}dz_t)\\
dX(t) &= X(t)([\frac{\mu}{2} - \frac{\sigma^2}{8}]dt + \frac{\sigma}{2}dz_t) && \text{Answer to Q1}\\
\frac{dX(t)}{X(t)} &= [\frac{\mu}{2} - \frac{\sigma^2}{8}]dt + \frac{\sigma}{2}dz_t && \text{Notice this is } d\ ln[X(t)]\\
d\ ln[X(t)] &= [\frac{\mu}{2} - \frac{\sigma^2}{8}]dt + \frac{\sigma}{2}dz_t\\
ln[X(t)] &= S(0) + [\frac{\mu}{2} - \frac{\sigma^2}{8}]t + \frac{\sigma}{2}Z_t && \text{Integrate}\\
X(t) &= e^{S(0)+[\frac{\mu}{2}-\frac{\sigma^2}{8}]t+\frac{\sigma}{2}Z_t} && \text{Exponentiate}\\
\text{Know that } E[e^X] = e^{\mu+1/2\sigma^2} & \text{ and that Brownian Motion} && \text{is Normal with } \mu = 0, \sigma = t\\
E[X(t)] &= S(0)e^{[\frac{\mu}{2}-\frac{\sigma^2}{8}]t+0+\frac{\sigma^2 t}{2}}\\
&= S(0)e^{[\frac{\mu}{2}-\frac{3\sigma^2}{8}]t} && \text{Answer to Q2A}\\
E[X(t)]^2 &= S(0)e^{[\mu-\frac{3\sigma^2}{4}]t}\\
E[X(t)^2] &= E[S^{1/2}] = \frac{E[S]}{S(0)^2} = e^{\mu t} && \text{See pg. 310}\\
Var(X_t) &= E[X(t)^2] - E[X(t)]^2 = S(0)^2 e^{\mu t} - e^{\mu t - \frac{3\sigma^2}{4}t}\\
&= S(0)^2 e^{\frac{3\sigma^2}{4}t} && \text{Answer to Q2B}
\end{aligned}
$$

Part 2: Using the Ito's Lemma find to the stochastic differential equation satisfied by the process $Yt = S_t^2 e^{rt^2}$

$$dX(t) = \left(\frac{\partial F}{\partial S}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2\right)dt + \frac{\partial F}{\partial S}bdz \qquad \text{Def: Ito's Lemma (pg. 312)}$$

$$\frac{\partial F}{\partial S} = 2S_t e^{rt^2} | \frac{\partial^2 F}{\partial S^2} = 2e^{rt^2} | \frac{\partial F}{\partial t} = 2rS_t^2 e^{rt^2} | a = \mu S_t | b = \sigma S_t$$

$$\begin{aligned}
dY(t) &= (2S_t e^{rt^2} \cdot \mu S + 2rS_t^2 e^{rt^2} + \frac{1}{2}2e^{rt^2} \cdot \sigma^2 S_t^2)dt \\
&+ 2S_t e^{rt^2} \cdot \sigma S dz_t \\
&= (2S_t^2 e^{rt^2}\mu + 2rS_t^2 e^{rt^2} + S_t^2 e^{rt^2}\sigma^2)dt + 2S_t^2 e^{rt^2}\sigma dz_t \\
&= S_t^2 e^{rt^2}(2\mu + 2r + \sigma^2)dt + 2\sigma dz_t
\end{aligned}$$

$$dY(t) = Y(t)(2\mu + 2r + \sigma^2)dt + 2\sigma dz_t) \qquad \text{Answer to Q3}$$

$$\frac{dY(t)}{Y(t)} = (2\mu + 2r + \sigma^2)dt + 2\sigma dz_t) \qquad \text{Notice this is } d\ ln[Y(t)]$$

$$d\ ln[Y(t)] = (2\mu + 2r + \sigma^2)dt + 2\sigma dz_t)$$

$$ln[Y(t)] = S(0) + (2\mu + 2r + \sigma^2)t + 2\sigma Z_t) \qquad \text{Integrate}$$

$$Y(t) = S(0)e^{(2\mu+2r+\sigma^2)t+2\sigma Z_t)} \qquad \text{Exponentiate}$$

Know $E[e^X] = e^{\mu+1/2\sigma^2}$ and that Brownian Motion is N(0,t)

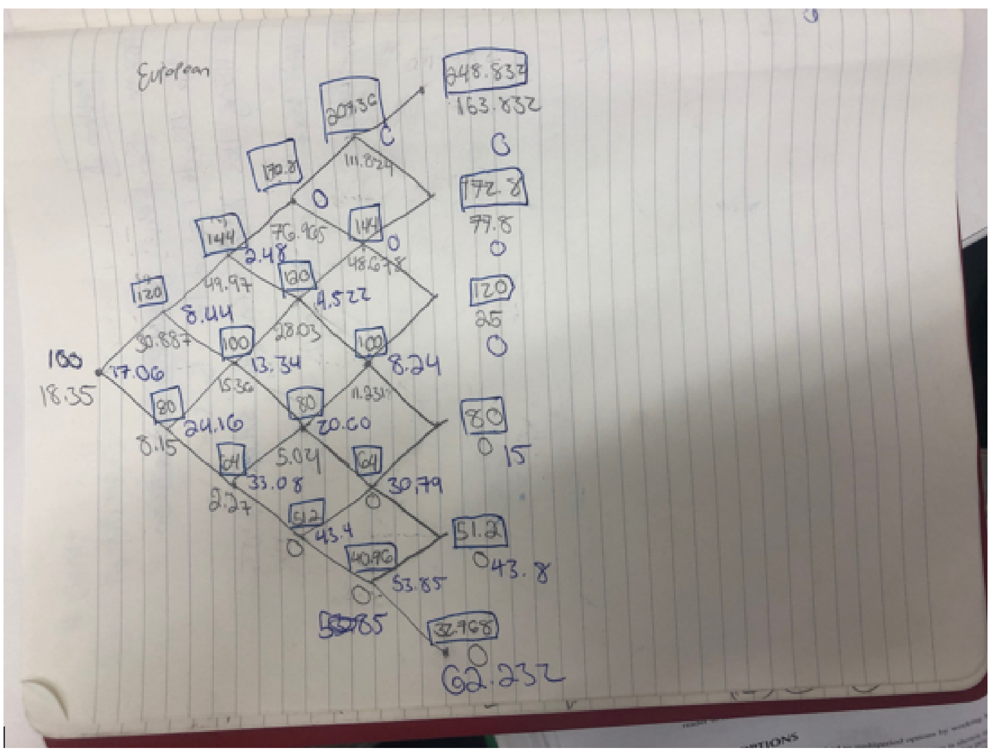
$$\begin{aligned}
E[Y(t)] &= e^{(2\mu+2r+\sigma^2)t+\sigma^2 t} \\
&= e^{2t(\mu+r+\sigma^2)} \qquad \text{Answer to Q4A} \\
E[Y(t)]^2 &= e^{4t(\mu+r+\sigma^2)} \\
E[Y(t)^2] &= E[(S(0)e^{(2\mu+2r+\sigma^2)t+2\sigma Z_t})^2 \\
&= E[S(0)^2 e^{(2\mu+2r+\sigma^2)2t+4\sigma Z_t)} \\
&= E[S(0)^2 e^{(2\mu+2r+\sigma^2)2t+2t\sigma^2)} \\
&= E[S(0)^2 e^{(\mu+r+2\sigma^2)4t} \\
Var(Y_t) &= E[X(t)^2] - E[X(t)]^2 = e^{(\mu+r+2\sigma^2)4t} - e^{4t(\mu+r+\sigma^2)} \\
&= e^{\sigma} \qquad \text{Answer to Q4B}
\end{aligned}$$

**Problem 2.** Suppose you have \$1,000,000 invested evenly in the following stocks IBM, GE, MSFT, XOM on August 14, 2019. Use daily adjusted closing prices from August 13, 2018 to August 13, 2019 as historical data. We assume that the daily rates of return rIBM, rGE, rMSFT, rXOM of these stocks follow the normal distribution.

<u>Part 1-3</u>. See Excel File:

<u>Part 4</u> The weights are all equal by problem construction. So by sub additivity of Value at Risk $VaR_h(X+Y) \leq VaR_h(X) + VaR_h(y)$. See Module 7 Problem 5 for proof. In the excel file we see that the sum of the values at risk (without diversification) is 0.3702.

**Problem 3.** We want to price options using the binomial lattice. The current stock price is 100 and the strike price is 95. Assume that the stock up-trend rate is $u = 1.2$ with probability p=0.45 and the down-trend rate is d = 0.8 with probability 1-p=0.55 The annual risk-free rate is r=0.02. Assume that the length of a period is one month. See Photo below and excel sheet. The stock prices are in pencil with a blue boarder in node 1, the value of the calls are in pencil (no boarder) in node 3, and the value of the put are in blue pen in node 2. All the numbers are also in the excel file if you have trouble reading it!



**Problem 4.** Consider Apple Inc. as the underlying asset, use its daily adjusted closing prices from August 13, 2018 to August 13, 2019 as historical data.

Part 1,2 See Excel Sheet.

Part 3 $\sigma = 0.314,\ r = 0.02, T = 12,\ S_0 = 204.75,\ K_1 = 180,\ K_2 = 225.$

Def: Black Scholes Option Formulas

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(d_1)$$
$$d_1 = \frac{ln(s/k) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 = \sigma\sqrt{T-t}$$
$$\Delta_{call} = N(d_1)$$
$$\Gamma_{call} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$
$$\Delta_{put} = -N(-d_1)$$

We have 12 months so T-t is 1. I've coded all of this into R. So I will not be demonstrating the formulas step by step. For the call option: $d_1 = 0.5997,\ d_2 = 0.2857$ these give: $N(d_1) = 0.72565,\ N(d_2) = 0.6124483$.And hence, $C(S,t) = 39.067$. With $\Delta = 0.72565$ and $\Gamma = 0.0926$

For the Put option: $d_1 = -0.11092,\ d_2 = -0.42491$ these give: $N(d_1) = 0.54415,\ N(d_2) = 0.6645$. And thus, $P(S,t) = 36.23531$. With $\Delta = -0.54415$ and $\Gamma = 0.08811015$