Problem Set 8 *

Ian McGroarty

Course Number: 625.641

July 22, 2019

*Problems:3,4,5,6,9

Problem 3 (risk aversion). Suppose $U(x)$ is a utility function with arrow-pratt risk aversion coefficient $a(x)$. Let $V(x) = a + bU(x)$. Find the risk aversion coefficient for $V(x)$

Solution

$$a(x) = \frac{U''(x)}{U(x)} \text{ Definition of Risk Aversion Coefficient (pg 233)}$$

So $V'(x) = \frac{d}{dx}a + b(U(x)) = b \cdot U'(x)$ and $V''(x) = \frac{d}{dx}b \cdot U'(x) = b \cdot U''(x)$
Thus $a_V(x) = \frac{V''(x)}{V(x)} = \frac{bU''(x)}{bU(x)} = a(x)$

Problem 4 Relative Risk Aversion. The Arrow-Pratt relative risk aversion coefficient is

$$\mu(x) = \frac{xU''(x)}{U'(x)}$$

Show that the utility function $U(x) = lnx$ and $U(x) = \gamma x^\gamma$ have constant relative risk aversion coefficients.

Solution

$$\frac{d}{dx}ln(x) = 1/x \;\&\; \frac{d}{dx}1/x = -1/x^2 \implies \mu(x) = \frac{-1/x^2 \cdot x}{1/x} = \frac{-1/x}{1/x} = -1$$

$$\frac{d}{dx}yx^y = y^2x^{y-1} \;\&\; \frac{d}{dx}y^2x^{y-1} = y^2(y-1)x^{y-2} \implies \mu(x) = \frac{y^2(y-1)x^{y-2} \cdot x}{y^2x^{y-1}} =$$

$$= \frac{y^2(y-1)x^{y-1}}{y^2x^{y-1}} = (y-1)$$

Problem (5) Equivalence. Utility function $U(x)$ over $A \le x \le B$. $U(A) = A$ and $U(B) = B$. Equivalent utility function $V(x)$ over $A' \le x \le B'$.

$V(A') = A'$ and $V(B') = B'$. $V(x) = aU(x) + b$ Find a, b.

Solution

$$
\begin{aligned}
V(x) = aU(x) + b &\implies V(A') = aU(A') + b \\
&\implies V(A') - aU(A') = A' - aU(A') = b = B' - aU(B') \\
&\implies a = \frac{A' - B'}{U(A') - U(B')} \\
&\implies V(x) = \frac{A' - B'}{U(A') - U(B')} U(x) + b \\
&\implies V(A') - \frac{A' - B'}{U(A') - U(B')} \cdot U(A') = b \\
&\implies b = A' - \frac{A' - B'}{U(A') - U(B')} \cdot U(A')
\end{aligned}
$$

Problem 6 (HARA). The HARA class of utility functions is defined by:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}$$

Show how the parameters γ, a, b can be chosen to represent:

(a) Linear: Let $b \to 0, \gamma \to 1, a = 1$:

$$U(x) = \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma} + b)^{\gamma} = \frac{1-\gamma}{\gamma}(\frac{x}{1-\gamma})^{\gamma}$$
$$lim_{\gamma\to 1}x^{\gamma} = x$$
$$\implies lim_{\gamma\to 1}U(x) = \frac{1-\gamma}{\gamma}(\frac{x}{1-\gamma}) = \frac{x}{\gamma} = x$$

(b) Quadratic: Let $\gamma = 2, a = 2, b = 1$:

$$U(x) = \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma} + b)^{\gamma} = \frac{1-2}{2}(\frac{2x}{1-2})^{2}$$
$$= -1/2[-x+1]^2 = -1/2[x^2 - 2x + 1] = -1/2x^2 + x - 1/2$$

(c) Exponential $Let \gamma \to -\infty, b = 0$:

$$\begin{aligned}
U(x) &= \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma} + b)^{\gamma} \\
&= \frac{1+\infty}{\infty}[\frac{ax}{1-\gamma}]^{-\infty} \\
&= \frac{\infty}{\infty}[\frac{ax}{1-\gamma}]^{-\infty}
\end{aligned}$$

$$\begin{aligned}
&= [\frac{(ax)^{\gamma}}{(1-\gamma)^{\gamma}} \\
ln[U(x)] &= \gamma \cdot ln[ax] - \gamma \cdot ln[1-\gamma] \\
&= \gamma(ln[ax] - ln[1-y]) \\
&= -e^{-ax}
\end{aligned}$$

(d) Power: $letb \to 0$:

$$\begin{aligned}
U(x) &= \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma} + b)^{\gamma} = \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma})^{\gamma} \\
&= \frac{1-\gamma}{\gamma}(\frac{a^{\gamma}x^{\gamma}}{(1-\gamma)^{\gamma}}) \\
\text{Let } c &= \frac{1-\gamma}{\gamma}(\frac{a^{\gamma}}{(1-\gamma)^{\gamma}} \implies U(x) = cx^{\gamma}
\end{aligned}$$

(e) Logarithmic Let $a = 1, b \to 0, \gamma \to 0$

$$U(x) = \frac{1-\gamma}{\gamma}(\frac{ax}{1-\gamma} + b)^{\gamma} = \frac{1-\gamma}{\gamma}(\frac{x}{1-\gamma})^{\gamma}$$
$$= \frac{1-\gamma}{\gamma}(\frac{x^{\gamma}}{(1-\gamma)^{\gamma}})$$
$$lim_{\gamma\to 0}\frac{1-\gamma}{\gamma}(\frac{x^{\gamma}}{(1-\gamma)^{\gamma}}) = lim_{\gamma\to 0}\frac{1-\gamma}{\gamma} \cdot lim_{\gamma\to 0}(\frac{x^{\gamma}}{(1-\gamma)^{\gamma}})$$

By Theorem 9.4 (Ross pg 47) $\quad lim(s_n t_n) = (lim s_n)(lim t_n)$

$$\text{First: } lim_{\gamma\to 0}\frac{1-\gamma}{\gamma} = \frac{1}{\gamma}$$
$$\text{Second: } lim_{\gamma\to 0}(\frac{x^{\gamma}}{(1-\gamma)^{\gamma}}) = (\frac{x^{\gamma}}{(1)^{\gamma}}) = x^{\gamma}$$
$$U(x) = \frac{1}{\gamma}x^{\gamma}$$
$$\frac{d}{dx}U(x) = x^{\gamma-1} \implies lim_{\gamma\to 0}\frac{d}{dx}U(x) = \frac{1}{x}$$
$$\int \frac{d}{dx}U(x) = U(x) = \int \frac{1}{x} = ln[x]$$
$$U(x) = ln[x]$$

Risk Aversion Coefficient

Problem 9 Quadratic mean variance. An investor with unit wealth maximizes the expected value of the utility function $U(x) = ax - bx^2/2$ and obtains a mean-variance efficient portolio. A friend of his with wealth W and the same utility function does the same calculation but gets a different portfolio return. However, changing b to b' does the trick.

Solution

Well investor 1 will maximize

$U(x) = ax - bx^2/2$ **(1a)**

subject to $P_x \cdot x = 1.$ **(2a)**

To maximize, take the derivative and set to zero:

$\frac{d}{dx}ax - bx^2/2 = a - bx = 0$

So $x = a/b$ **(3a)**

Well investor 2 will maximize

$U(x) = ax' - b'x'^2/2$ **(1a)**

subject to $P_{x'} \cdot x' = W.$ **(2b)**

To maximize, take the derivative and set to zero:

$\frac{d}{dx}ax' - b'x'^2/2 = a - bx' = 0$

So $x' = a/b'$ **(3b)**

Now solve (2b) for P_x to get $P_x = W/x'$. Plug into (2a) to get $w/x' \cdot x = 1$ Then plug in (3a) to get $w/x' \cdot a/b \implies b/w = a/x'$ Finally, plug in (3b) to yield: $b/w = a/(a/b') \implies b' = b/w$