Problem Set 11 *

Ian McGroarty
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*Problems:13.1, 13.5, 13.6, 13.11, 13.25

Problem 15.4. Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. This means that $S_0 = 50, K = 50, r = 0.1, \sigma = 0.3, T = 0.25$

$$
\begin{aligned}
d_1 &= \frac{ln(S_0/K) + (r - \sigma^2/2)(T)}{\sigma\sqrt{T-t}} && \text{(pg 334)} \\
&= \frac{ln(50/50) + (0.1 - (0.3)^2/2)(0.25)}{0.3\sqrt{0.25}} \\
d_1 &= 0.24167 \\
d_2 &= d_1 - \sigma\sqrt{T} && \text{Luenberger (pg. 334)} \\
&= 0.24167 - 0.3\sqrt{0.25} \\
d_2 &= 0.091667 \\
Ke^{-rT} &= 50 \cdot e^{-0.025} = 48.765 && \text{Eqn. 15.19 (pg. 333)} \\
p &= Ke^{-rt} \cdot N(-d_2) - S_0 \cdot N(-d_1) && \text{Eqn 15.21 (pg. 333)} \\
&= 48.765 \cdot N(-0.091667) - 50 \cdot N(-0.24167) \\
&= 48.765 \cdot (0.4634) - 50 \cdot (0.4045) \\
p &= 2.3
\end{aligned}
$$

Problem 15.5. What difference does it make to you calculations in Problem 15.4 if a dividend of \$1.50 is expected in two months.

Well the present value of the dividend is $1.50 \cdot e^{-0.025} = 1.496$. We now need to take this out of our S_0, I don't fully understand why?? so $S_0 - 50 - 1.496 = 48.504$. We see below that the price of the put option increases with the dividend.

$$
\begin{aligned}
d_1 &= \frac{ln(48.504/50) + (0.1 - (0.3)^2/2)(0.25)}{0.3\sqrt{0.25}} = 0.03915 \\
d_2 &= 0.03915 - 0.3\sqrt{0.25} = -0.111 \\
p &= 48.765 \cdot N(0.111) - 48.504 \cdot N(-0.03915) \\
p &= 3.04
\end{aligned}
$$

Problem 15.11. I'm not entirely sure how to use the risk neutral valuation in it's purest form here. But here we go. Following Section 15.7 (pg 332-333): First, assume that the expected return from the underlying asset is the risk free rate.

$$\mu = r$$

Second calculate the expected payoff:

$$
\begin{aligned}
lnS_T \ &\sim \Phi[lnS_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T] && \text{Eqn 14.19/15.3 (pg 313/320)} \\
&\implies E(lnS_T) = lnS_0 + (\mu - \frac{\sigma^2}{2})T \\
&= \implies \hat{E}(lnS_T) = lnS_0 + (r - \frac{\sigma^2}{2})T && \text{Risk Neutral} \\
f &= e^{-rT}\hat{E} && \text{Discounted to today} \\
f &= e^{-rT}(lnS_0 + (r - \frac{\sigma^2}{2})T)
\end{aligned}
$$

Confirm that this satisfies 15.16

$$
\begin{aligned}
rf &= \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} && \text{Eqn. 15.16 (pg 330)} \\
\frac{\partial f}{\partial t} &= -r(e^{-rT}(lnS_0 + (r - \frac{\sigma^2}{2})T)) - (e^{-rT}(r - \frac{\sigma^2}{2}) \\
\frac{\partial f}{\partial S} &= \frac{e^{-rT}}{S} \\
\frac{\partial^2 f}{\partial S^2} &= -\frac{e^{-rT}}{S^2} \\
rf &= -r(e^{-rT}(lnS_0 + (r - \frac{\sigma^2}{2})T)) \\
&\quad - (e^{-rT}(r - \frac{\sigma^2}{2}) + rS\frac{e^{-rT}}{S} - \frac{1}{2}\sigma^2 S^2 \frac{e^{-rT}}{S^2} \\
rf &= -re^{-rT}(lnS_0 + (r - \frac{\sigma^2}{2})T) \\
&\quad - e^{-rT}[(r - \frac{\sigma^2}{2}) + (r - \frac{\sigma^2}{2})] \\
rf &= -re^{-rT}(lnS_0 + (r - \frac{\sigma^2}{2})T) && \text{We are satisfied}
\end{aligned}
$$

Problem 15.13. Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$52, the risk-free interest rate is 12% per annum, and the volatility is 30% per annum. This means that $S_0 = 52, K = 50, r = 0.12, \sigma = 0.3, T = 0.25$ Since this is essentially the same format of 15.4.

$$\begin{aligned}
d_1 &= \frac{ln(S_0/K) + (r + \sigma^2/2)(T)}{\sigma\sqrt{T-t}} && \text{(pg 334)} \\
&= \frac{ln(52/50) + (0.12 - (0.3)^2/2)(0.25)}{0.3\sqrt{0.25}} \\
d_1 &= 1.934 \\
d_2 &= d_1 - \sigma\sqrt{T} && \text{Luenberger (pg. 334)} \\
&= 1.934 - 0.3\sqrt{0.25} \\
d_2 &= 1.874 \\
c &= S_0 \cdot N(d_1) - K \cdot e^{-rt} \cdot N(d_2) && \text{Eqn 15.20 (pg. 333)} \\
&= 52 \cdot N(1.934_50 \cdot e^{-0.12*0.25} \cdot N(1.874) \\
c &= 3.575
\end{aligned}$$

Problem 15.15. Consider an American call option on a stock. $S_0 = 70, r = 0.10, K = 65, \sigma = 0.32, T = 0.666$ Dividend after 3 and 6 months. It is never optimal to exercise before 3 months if:

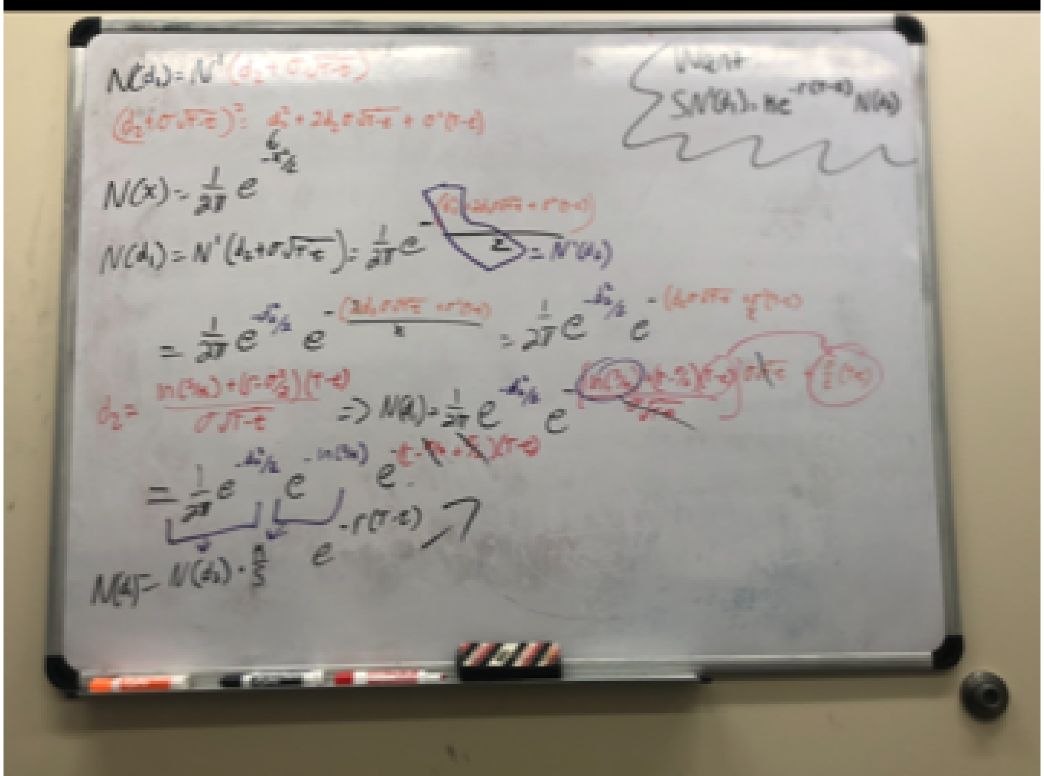
$$\begin{aligned}
D_i &\leq K[1 - e^{-r(t_{i+1}-t_i)}] && \text{Eqn. 15.25 (pg 343)} \\
D_1 &\leq 65 \cdot [1 - e^{-0.1((6/12)-(3/12))}] \\
\$1 &\leq 1.604 && \text{15.25 holds} \\
D_2 &\leq 65 \cdot [1 - e^{-0.1((8/12)-(6/12))}] \\
\$1 &\leq 1.074 && \text{15.25 holds}
\end{aligned}$$

Since 15.25 holds for both times corresponding to all dividends, it is never optimal to exercise the call and thus it can be treated as a European call option.

Problem 15.17. $N`(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the probability density function for a standard normal distribution (pg 19.2). For part (b),(d),(f) see the photos. I knew if I typed it I'd miss something so I had to write it out.

$$\begin{aligned}
\text{(c): } d_1 &= \frac{ln(S_0/K) + (r+\sigma^2/2)(T)}{\sigma\sqrt{T-t}} \quad \text{(pg 334)} \\
\text{Let } a &= \frac{(r+\sigma^2/2)(T)}{\sigma\sqrt{T-t}} \\
\frac{\partial d_1}{\partial S} &= \frac{d}{dS}ln(S/K) \cdot \frac{1}{\sigma\sqrt{T-t}} + a \\
&= \frac{1}{S \cdot \sigma\sqrt{T-t}} \\
&= \frac{\partial d_2}{\partial S} && \text{pretty clearly they will be the same} \\
\text{(e): } c &= SN(d_1) - Ke^{-r(T-t)N(d_2)} \\
\frac{\partial c}{\partial S} &= N(d_1) && \text{Trivial Solution}
\end{aligned}$$

(g): As $t \to T \implies \sqrt{T-t} \to 0 \implies (d_1, d_2) \to \pm\infty \implies N(d) \to (0,1)$. If 0 then c=0. If 1 then c= S-K.



$$C = SN'(d_1) - Ke^{-r(T-t)} N'(d_2)$$

$$\frac{dC}{dt} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)} N'(d_2) - Ke^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial t}$$

$$C = SN'(d_1) - Ke^{-r(T-t)} N'(d_2)$$

$$\frac{dC}{dt} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)} N'(d_2) - \underbrace{Ke^{-r(T-t)} N'(d_2)}_{SN'(d_1)\frac{\partial d_2}{\partial t}}\frac{\partial d_2}{\partial t}$$

$$SN'(d_1)\boxed{\left[\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right]}$$ Just need this.

want $\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\sigma}{2\sqrt{T-t}}$

$$d_1 - d_2 = \int \frac{\sigma}{2\sqrt{T-t}}$$

$$\sigma\sqrt{T-t} = \int \frac{\sigma}{2\sqrt{T-t}}$$

$$\frac{d}{dt}\left[\sigma\sqrt{T-t}\right] = \boxed{\frac{\sigma}{2\sqrt{T-t}}} \checkmark$$

$$\frac{dC}{dt} = SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)} N'(d_2) \checkmark$$

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial c^2}{\partial S^2}$$

This term is done

$$\to \boxed{-rKe^{-r(T-t)}N(d_2)} - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rS\cdot N(d_1) + \frac{\frac{1}{2}\sigma^2 S^2}{S\sigma\sqrt{T-t}}$$

$$a = -rKe^{-r(T-t)}N(d_2)$$

done

$$a - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + \boxed{rSN(d_1)} + \frac{\sigma S}{2\sqrt{T-t}}$$

$$b = rSN(d_1)$$

$$a + b - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + \frac{\sigma S}{2\sqrt{T-t}}$$

$$a + b + \boxed{\frac{\sigma S - S\sigma N'(d_1)}{2\sqrt{T-t}}}$$ Somehow this is 0

and we have

$$a + b = rc = -rKe^{r(T-t)}N(d_2) + rSN(d_1)$$

Problem 15.28. See Photo.

Week	Price	S_n/S_(n-1)	ln(C)	D^2			
1	30.2					SUM	0.094708
2	32	1.059602649	0.057893978	0.003352		AVG	0.011586
3	31.3	0.978125	-0.022117805	0.000489		STD	0.029016
4	30.1	0.961661342	-0.039092926	0.001528		Vol per Annum	0.209239
5	30.2	1.003322259	0.003316753	1.1E-05		Std. Error	0.039542
6	30.3	1.003311258	0.003305788	1.09E-05			
7	30.6	1.00990099	0.009852296	9.71E-05			
8	33	1.078431373	0.075507553	0.005701			
9	32.9	0.996969697	-0.003034904	9.21E-06			
10	33	1.003039514	0.003034904	9.21E-06			
11	33.5	1.015151515	0.015037877	0.000226			
12	33.5	1	0	0			
13	33.7	1.005970149	0.005952399	3.54E-05			
14	33.5	0.994065282	-0.005952399	3.54E-05			
15	33.2	0.991044776	-0.008995563	8.09E-05			