Problem 1

We have F=100, C=10, $\lambda = 0.05$, n=60, m=2, the price is given by

$$\begin{aligned} P &= \frac{F}{[1+(\lambda/m)]^n} + \sum_{k=1}^{n} \frac{C/m}{[1+\lambda/m)]^k} \\ &= \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left(1 - \frac{1}{[1+(\lambda/m)]^n}\right) = 177.272 \end{aligned}$$

problem 2

Find the present value of the perpetual cash flow stream $(C_0, C_1, ..., C_n, ...)$ where the n-th year payment $C_n = n-1$ and the prevailing rate is 2%. Let $x = \dfrac{1}{1+r}$, the present value is given by

$$\begin{aligned} PV &= \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} (n-1)x^n \\ &= x^2 \sum_{n=0}^{\infty} (n-1)x^{n-2}. \end{aligned}$$

Let us simplify the quantity $\sum_{n=0}^{\infty}(n-1)x^{n-2}$. Using the change of variable $m = n-2$ we have,

$$\sum_{n=0}^{\infty}(n-1)x^{n-2} = \frac{-1}{x^2} + 0 + \sum_{m=0}^{\infty}(m+1)x^m.$$

Using the result from Lecture note 3, we know that $\sum_{m=0}^{\infty}(m+1)x^m = \dfrac{1}{(1-x)^2}$. In sum,

$$\sum_{n=0}^{\infty}(n-1)x^{n-2} = \frac{-1}{x^2} + \frac{1}{(1-x)^2}.$$

Therefore

$$PV = x^2 \sum_{n=0}^{\infty}(n-1)x^{n-2} = -1 + \frac{x^2}{(1-x)^2} = -1 + \frac{1}{r^2} = 2499.$$

Problem 3

Using the fact that the Present Value $PV = \frac{A}{r}$, with $A = 10000$ the duration is

$$D = -\frac{1+r}{PV}\frac{dPV}{dr} = -\frac{r(1+r)}{A}\frac{(-A)}{r^2} = \frac{1+r}{r} = 34.33 \text{ years.}$$

The modified duration

$$D_M = \frac{D}{1+r} = \frac{1}{r} = 33.33 \text{ years.}$$

Problem 4

Since we have a zero coupon bond the bond price is $P = \dfrac{F}{[1+(\lambda/m)]^n}$, then the convexity is given by

$$C = \frac{1}{P}\frac{d^2P}{d\lambda^2} = \frac{1}{P}\frac{n(n+1)P}{m^2\left(1+\lambda/m\right)^2} = \frac{n(n+1)}{m^2\left(1+\lambda/m\right)^2}$$

using the fact that $T = n/m$ so $n = Tm$. We have

$$C = \frac{T(T+1/m)}{\left(1+\lambda/m\right)^2}.$$

As m goes to infinity we have

$$\lim_{m\to\infty} C = \lim_{m\to\infty} \frac{T(T+1/m)}{\left(1+\lambda/m\right)^2} = T^2.$$

Problem 5

After 10 years, the payment the company needs to make if the call provision is exercise is:
$P_{10}^C = (1+0.1) \times \text{FaceValue} = 110\%$. We always assume the $\text{FaceValue} = 100\%$. Let P_{10} be the bond price after 10years, since the company assumes that exercising the call provision is advantageous, this means we should have

$$110 < P_{10} = \frac{100}{(1+\lambda)^{10}} + \frac{5}{\lambda}\left(1 - \frac{1}{(1+\lambda)^{10}}\right).$$

Solving this inequality numerically we find out that the yield to maturity is lower than 3.78%, $\lambda < 0.0378052$.