Problem Set 4 *

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Problem 3.10.6. Let $Y_1, Y_2, ..., Y_n$ be a random sample from the exponential pdf $f_y(y) = e^{-y}, y \geq 0$. What is the smallest n for which $P(Y_{min} < 0.2) > 0.9$?

Solution The evaluated n $>$ 11.513, since x must be an integer (I'm assuming since these are trials) $n \geq 12$.

$$\begin{aligned}
P(Y_{min} < 0.2) &= \int_0^{0.2} f_{Y_{min}}(y) \\
&= \int_0^{0.2} n[1 - F_Y(y)]^{n-1} f_Y(y) && \text{Theorem 3.10.1.b (pg193)} \\
&= \int_0^{0.2} n[1 - (1 - e^{-y})]^{n-1} (e^{-y}) \\
&= \int_0^{0.2} n(e^{-y})^{n-1} (e^{-y}) \\
&= \int_0^{0.2} n(e^{-ny}) \\
&= -e^{-ny}\Big|_0^{0.2} \\
&= 1 - e^{-(0.2)n} > 0.9 \\
&= log(e^{-0.2n}) < log(0.1) \\
n &> \frac{log(0.1)}{-0.2} \approx 11.513
\end{aligned}$$

*Problem list -3.10.6, 3.10.16, 3.12.6, 3.12.8

Problem 3.10.16. Suppose a device has thre independent components, all of whose lifetimes (in months are modeled by the exponential pdf, $f_y(y) = e^{-y}, y > 0$. What is the probability that all three components will fail within two months of one another?

Solution[1] Range $= Y_{max} - Y_{min} = Y_3^{`} - Y_1^{`}$. The *memoryless property of the exponential distribution*: $P(X \geq s + t | x \geq s) = P(X \geq t)$. This implies that the level of $Y_1^{`}$ is inconsequential. Thus we can assume that $Y_1^{`} = 0$. In which case we are really only interested in $P(Y_{max}^{`} < r)$. Thus we can apply theorem 3.10.1.a (pg 193) with: n=3, $f_y(y) = e^{-y}$, and $F_Y(y) = \int_0^y f_Y(y)dy = 1 - e^{-y}$.

$$
\begin{aligned}
P(Y_{max} < m) &= \int_{-\infty}^{m} n[F_Y(y)]^{n-1} f_Y(y) \\
P(Y_3^{`} < 2) &= \int_0^2 3[1 - e^{-y}]^2 e^{-y} && \text{Enter WolframAlpha} \\
&\approx 0.646.
\end{aligned}
$$

Problem 3.12.6. Find $M_Y(t)$ if Y has the pdf:

$$
f_Y(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 2 - y, & 1 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}
$$

[1] To start, I want to note that understanding of the "memoryless property of the exponential distribution" was critical to even approaching success in this problem. I studied the proof in this pdf, http://www.cs.cmu.edu/afs/cs/academic/class/15750-s19/OldScribeNotes/lecture11.pdf (pg 2). I also used wolframalpha to do some of the calculations that were a to complex for my patience level.

Solution Since X is continuous, the second part of Definition 3.12.1 (pg 206) applies, so:

$$\begin{aligned}
M_Y(t) &= E(e^{tW} = \int_{-\infty}^{\infty} e^{tw} f_W(w)dw \\
&= \int_{-\infty}^{0} e^{ty}0 + \int_{0}^{1} e^{ty}y + \int_{1}^{2} e^{ty}(2-y) + \int_{0}^{\infty} e^{ty}0 \\
&= \int_{0}^{1} e^{ty}y + \int_{1}^{2} e^{ty}(2-y) && \text{I used WolframAlpha here.} \\
&= \frac{e^{tx}(tx-1)}{t^2}\Big|_0^1 + \frac{(y-2)e^{tx}}{t}\Big|_1^2 \\
&= \frac{1-e^t}{t^2}
\end{aligned}$$

Problem 3.12.8. Let Y be a continuous random variable with $f_Y(y) = ye^{-y}, o \leq y$. Show that $M_Y(t) = \frac{1}{(1-t)^2}$

Solution Since X is continuous, the second part of Definition 3.12.1 (pg 206) applies, so:

$$\begin{aligned}
M_Y(t) &= E(e^{tW} = \int_{-\infty}^{\infty} e^{tw} f_W(w)dw \\
&= \int_{0}^{\infty} e^{ty} ye^{-y} \\
&= \int_{0}^{\infty} e^{ty-y} y && \text{The integration by parts at the end if you want to see it.} \\
&= \frac{e^{(t-1)y}((t-1)y-1)}{(t-1)^2}
\end{aligned}$$

Integration by parts

$$
\begin{aligned}
f(y) = y &\quad df = dy \\
dg(y) = e^{y(t-1)} &\quad g = \frac{e^{y(t-1)}}{t-1} \\
&= \frac{y(e^{y(t-1)}}{t-1} - \frac{1}{(t-1)} \int e^{y(t-1)} \\
&= \frac{y(e^{y(t-1)}}{t-1} - \frac{e^{y(t-1)}}{(t-1)^2}\Big|_0^\infty
\end{aligned}
$$