More Accept-Reject Examples

625.433

1. Give an algorithm for generating a random variable having density function

$$f_X(x) = 2x \exp\left\{-x^2\right\} \qquad x > 0.$$

Solution: In this case, I let $g(x)$ be an exponential distribution. In other words, $g(x) = \lambda \exp\{-\lambda x\}$. Recall that I want to find an M such that $M \cdot g(x) \geq f(x)$ for all values of x, i.e.,

$$M \geq \sup_x \left\{ \frac{f_X(x)}{g(x)} \right\} \quad \forall\ x.$$

In this case,

$$\frac{f(x)}{g(x)} = \frac{2}{\lambda} \cdot x \cdot \exp\left\{-x^2 + \lambda x\right\},$$

and using basic calculus we can show that the x value at which this is optimized is $x_{\text{opt}} = \left[\lambda \pm \left(\lambda^2 + 8\right)^{\frac{1}{2}}\right]/4$. When $\lambda = 1$, $x_{\text{opt}} = 1$, making $f_X(x_{\text{opt}})/\,g(x_{\text{opt}}) = M = 2$.

See the `R` code `accReject(n)` which executes this example.

2. Let G be the distribution function with density g and suppose, for constants $a < b$, we want to generate a random variable from the distribution function

$$F(x) = \frac{G(x) - G(a)}{G(b) - G(a)} \quad a \leq x \leq b.$$

Show that the rejection method reduces in this case to generating a random variable X having distribution G (density g) and then accepting it if it lies between a and b.

Solution: In this case,

$$f(x) = \begin{cases} k \cdot g(x) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases},$$

where k is the normalizing constant given above ($k = 1/(G(b) - G(a))$). Sketch this out and prove it to yourself. To implement an accept-reject algorithm in this case, we would do the following:

   (a) Generate X_{cand} from $g(x)$. $g(x)$ in this case would be our candidate density.

   (b) Then generate $U \sim \text{Unif}(0, k \cdot g(X_{\text{cand}})) = \text{Unif}(0, f(X_{\text{cand}}))$. We can do this because we know $k \cdot g(x) \geq f(x)$ for all values of x.

   (c) If $U \leq f(x_{\text{cand}})$, accept. This will always happen in the interval (a, b) because $f(x_{\text{cand}}) = k \cdot g(x_{\text{cand}})$.

   (d) If $U \geq f(x_{\text{cand}})$, reject. This will always happen outside (a, b) since $f(x_{\text{cand}}) = 0$ outside of (a, b).

So....you draw from $g(x)$, and if you're in (a, b), you always accept. If you're outside (a, b) you always reject.