Language

$$\begin{aligned} A &= \{0,1\} \\ A^* &= \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\} \\ L &= \{\epsilon, 01, 0011, 000111, \ldots\} \\ A^* \setminus L &= \{0, 1, 00, 10, \ldots\} \end{aligned}$$

$$\begin{aligned} \mathbb{N}_0 &= \{0,1,2,3,\ldots\} \\ L &= \{0^i 1^i | i \in \mathbb{N}_0\} \end{aligned}$$

Turing machine

State	Input	Write	Move	Next
q_0	$\sqcup$	$\sqcup$	R	q_a
q_0	0	$\sqcup$	R	q_1
q_0	1	1	R	q_f
q_1	$\sqcup$	$\sqcup$	L	q_2
q_1	0	0	R	q_1
q_1	1	1	R	q_1
q_2	$\sqcup$	$\sqcup$	R	q_f
q_2	0	0	R	q_f
q_2	1	$\sqcup$	L	q_3
q_3	$\sqcup$	$\sqcup$	R	q_0
q_3	0	0	L	q_3
q_3	1	1	L	q_3

Example input

$$\begin{aligned} & q_0 000111 \to q_1 00111 \to 0 q_1 0111 \to 00 q_1 111 \\ & \to 001 q_1 11 \to 0011 q_1 1 \to 00111 q_1 \to 0011 q_2 1 \\ & \to 001 q_3 1 \to 00 q_3 11 \to 0 q_3 011 \to q_3 0011 \\ & \to q_3 \sqcup 0011 \to q_0 0011 \to q_1 011 \to 0 q_1 11 \\ & \to 01 q_1 1 \to 011 q_1 \to 01 q_2 1 \to 0 q_3 1 \\ & \to q_3 01 \to q_3 \sqcup 01 \to q_0 01 \to q_1 1 \\ & \to 1 q_1 \to q_2 1 \to q_3 \to q_1 \to q_a \end{aligned}$$

Steps

$$\begin{aligned} & q_0 000111 \to \ldots 13 \text{ steps} \ldots \to q_0 0011 \\ & \to \ldots 9 \text{ steps} \ldots \to q_0 01 \to \ldots 5 \text{ steps} \ldots \\ & \to q_0 \to \ldots 1 \text{ step} \ldots \to q_a \qquad (28 \text{ total}) \end{aligned}$$

Simulation

n	0	2	4	6	8	10	12	14
$f(n)$	1	6	15	28	45	66	91	120

Sequence

OEIS [1] gives sequence formula:

$$a(i) : \mathbb{N} \to \mathbb{N}_0 = i(2i-1)$$

So, $a(1) = 1$, $a(2) = 6$, $a(3) = 15$, and so on. We index as $2\mathbb{N} = \{0, 2, 4, 6, 8, 10, \ldots\}$. Transform:

$$h(n) : 2\mathbb{N}_0 \to \mathbb{N} = \frac{n}{2} + 1.$$

So, $h(0) = 1$, $h(2) = 2$, $h(4) = 3$, and so on.

$$\begin{aligned} f(n) : 2\mathbb{N}_0 \to \mathbb{N}_0 &= a(h(n)) \\ &= \left(\frac{n}{2} + 1\right)\left(2\left(\frac{n}{2} + 1\right) - 1\right) \\ &= \left(\frac{n}{2} + 1\right)(n + 2 - 1) \\ &= \frac{1}{2}(n+2)(n+1) \\ &= \frac{1}{2}\left(n^2 + 3n + 2\right) \end{aligned}$$

So, $f(n)$ is $O(n^2)$.

Justification

Is $f(n)$ the correct formula for the number of steps taken for an accepted input of length n?
Each pass right and left across the j non-blank tape cells, the machine takes $j+1$ steps right, followed by j steps left.

Start	**End**	**Right**	**Left**
000111	0011	7	6
0011	01	5	4
01	ϵ	3	2
ϵ	q_a	1	0

$$\begin{aligned} f(n) &= (n+1) + n + \ldots + 2 + 1 + 0 \\ &= ((n+1) + 0) + ((n) + 1) + \ldots \\ &= \left(\frac{n}{2} + 1\right)(n+1) \end{aligned}$$

Decider

Does the Turing Machine always halt and if so, does it reject in $O(n^2)$? Is $L \in \mathbf{P}$?

References

[1] OEIS Foundation Inc (2020). The on-line encyclopedia of integer sequences. `https://oeis.org/A000384`.