Definitions

Tree: connected graph with no cycles.

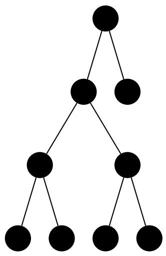
Rooted: if one node identified as root.

m**-ary:** if every parent has m children.

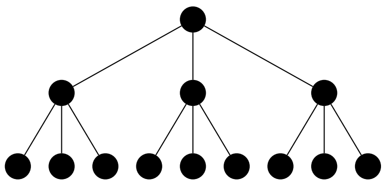
Height: maximum length of path to a leaf.

Leaf: is a node with no children.

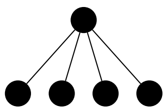
Examples



Binary (2-ary) tree of height 3.



Ternary (3-ary) tree of height 2.



4-ary tree of height 1.

Theorem

m-ary tree with l leaves has height at least $\log_m l$.

Log

$$b^a = c \Leftrightarrow \log_b c = a$$

$$10^2 = 100 \Leftrightarrow \log_{10} 100 = 2$$

Rationale

Maximum leaves is $l \leq m^h$.

Proof of theorem

$$h \geq \log_m l \;\Leftrightarrow\; m^h \geq m^{\log_m l} \Leftrightarrow m^h \geq l$$

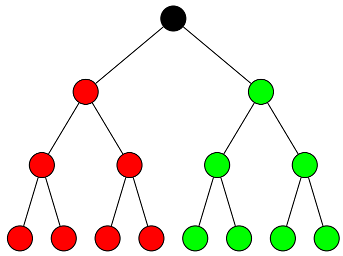
By induction on h:

$h = 0$**:** $m^0 = 1$, $l = 1$, $1 \geq 1$.

$h = n$**:** Assume true for $h = n - 1$. Removing root gives m trees with maximum m^{n-1} leaves each. In total, we get maximimum of $m(m^{n-1}) = m^n$ leaves.

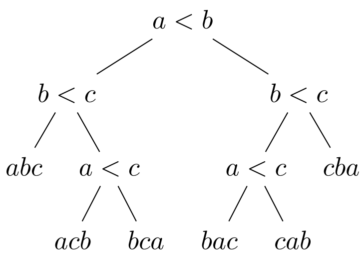
True for $h = 0$. Therefore true for $h = 0 + 1 = 1$. Therefore true for $h = 1 + 1 = 2$. Therefore true for $h = 2 + 1 = 3$. And so on.

Example



Application

To sort a list of n items we have to choose between $n!$ permutations. So, in the worst case, using binary decisions we make at least $\log_2 n!$ comparisons.



$$\log_2 3! \;=\; \log_2 6 \;>\; 2$$