

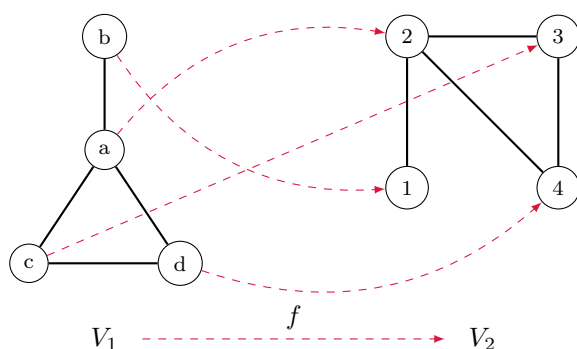
Graph

Simple graph: $G = (V, E)$; V a set; E a set of two-subsets of V .

Isomorphism

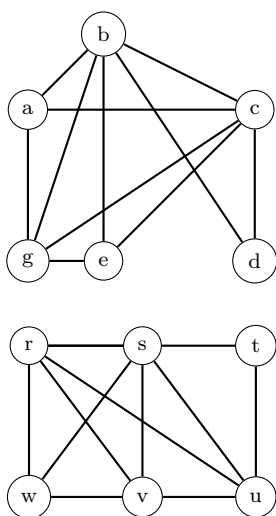
Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Bijection $f : V_1 \rightarrow V_2$ such that $f(E_1) = E_2$ where $f(E_1) = \{\{f(v_1), f(v_2)\} | \{v_1, v_2\} \in E_1\}$.

Example



$$\begin{aligned} f(E_1) &= \{\{f(a), f(b)\}, \{f(a), f(c)\}, \\ &\quad \{f(a), f(d)\}, \{f(c), f(d)\}\} \\ &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\} = E_2 \end{aligned}$$

Non-isomorphism



No of maps

$f(a) \rightarrow 6$ choices; $f(b) \rightarrow 5$ choices; $f(c) \rightarrow 4$ choices; etc. So, $n!$ maps between the vertex sets of two graphs with n vertices.

Some invariants

- Degrees.
- Paths.
- Connection.

Adjacency matrix

Fix a listing of V . $[a_{ij}]$ where a_{ij} is 1 if $\{v_i, v_j\} \in E$ else 0.

Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Permutation matrix

Isomorphic $\leftrightarrow \exists P$ such that $A = PBP^T$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Binary encoding

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$\rightarrow 0110011011111101110110000110011111010$

or **111011011011101**

Decision problem

$f(110101, 101011) \rightarrow \text{Yes}$

$f(111011011011101, 1011111101110011) \rightarrow \text{No}$

$f : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\} = 1$ iff isomorphic

$$\text{GRAPHISO} = \{(G_1, G_2) | f(G_1, G_2) = 1\}$$