

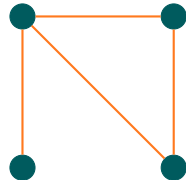
Graph Isomorphism Problem

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January 12, 2023

Graphs

A graph¹ is a set of nodes² V connected by a set of edges E .



The set V can be anything but the set E must be a set of 2-subsets of V . For example, V might be the set of cities in Ireland and E might represent motorways connecting cities. A motorway in this case would be defined by the two cities it connects.

Set Notation

A set is a collection of objects, usually denoted using curly braces.³ For example, the set A below contains the three objects 1, 2, and 3. The objects are usually called elements of the set.

$$A = \{1, 2, 3\}$$

Sets can be infinite, in which case the elements can be identified by an algorithm or property. In this case we usually assume the infinite set of counting numbers⁴ $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ is a given⁵.

In the below, the set T of even positive natural numbers is given by an algorithm. The algorithm says start with a natural number and multiply it by two. The set P is given by the property that each element is prime.

$$T = \{2n \mid n \in \mathbb{N}\}$$

$$P = \{p \in \mathbb{N} \mid p \text{ is prime}\}$$

Two important properties of sets are that sets are unordered and that each element is distinct. Note there is no mention of order in the definition *collection of objects*⁶. Likewise, the idea of an *object* is that it is unique — we did not say an instance of an object or anything like that.

We say B is a subset of A if all of the elements on B are also in A . When B has k elements, we sometimes say B is a k -subset of A . Under this definition, the empty set and A itself are always subsets of a set A .

Note that a set B is an object itself, and so might be an element of a set A . In this case, we are not saying that the elements of B

¹ Michael Sipser. *Introduction to the Theory of Computation*. Third. Boston, MA: Course Technology, 2013. ISBN: 113318779X, p. 10.

² Nodes are sometimes also called vertices.

³ Sipser, *Introduction to the Theory of Computation*, p. 3.

⁴ The numbers are usually called the natural numbers and \mathbb{N}_0 is the set of natural numbers including zero.

⁵ Sometimes it is convenient to not include the element 0, in which case we denote the set \mathbb{N} .

⁶ We can create an order or ordering of a set if we wish but that is something we must treat alongside the set.

are individually in A , although that could also be the case. The distinction is important⁷.

Tuple Notation

When order matters, we use tuples. Tuples are basically the same as lists of arrays in programming languages.

A tuple is a finite sequence.⁸ A sequence is a list of objects, usually stipulated to come from a set or sets. A tuple of length k is sometimes called a k -tuple, although a 2-tuple is usually just called a pair.

The word *list* implies an order — we can talk about the first thing on a list, if it exists.

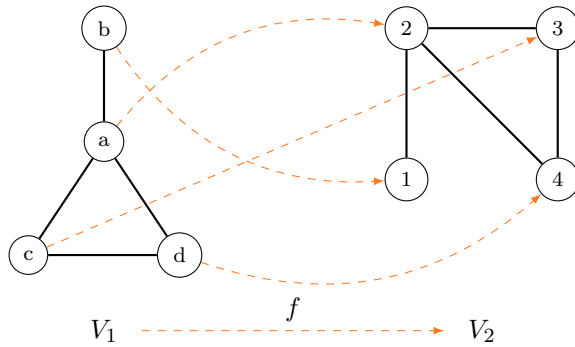
Graph Notation

A simple graph G is a pair $G = (V, E)$ where V is a set and E is a set of 2-subsets of V .

Isomorphism

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Bijection $f : V_1 \rightarrow V_2$ such that $f(E_1) = E_2$ where $f(E_1) = \{\{f(v_1), f(v_2)\} \mid \{v_1, v_2\} \in E_1\}$.

Example

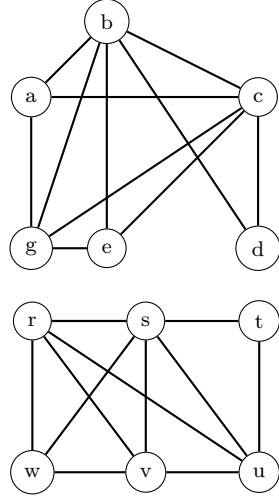


$$\begin{aligned}
 f(E_1) &= \{\{f(a), f(b)\}, \{f(a), f(c)\}, \\
 &\quad \{f(a), f(d)\}, \{f(c), f(d)\}\} \\
 &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\} = E_2
 \end{aligned}$$

⁷ Bertrand Russel is known for Russell's paradox about a set R which is the set of all sets that do not contain themselves. A set seemingly may contain itself — consider the set of all sets. Does R contain itself?

⁸ Sipser, *Introduction to the Theory of Computation*, p. 6.

Non-isomorphism



No of maps

$f(a) \rightarrow 6$ choices; $f(b) \rightarrow 5$ choices; $f(c) \rightarrow 4$ choices; etc. So, $n!$ maps between the vertex sets of two graphs with n vertices.

Some invariants

- Degrees.
- Paths.
- Connection.

Adjacency matrix

Fix a listing of V . $[a_{ij}]$ where a_{ij} is 1 if $\{v_i, v_j\} \in E$ else 0.

Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Permutation matrix

Isomorphic $\leftrightarrow \exists P$ such that $A = BP^T$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Binary encoding

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$\rightarrow 011001101111110111011000011001111010$

or **111011011011101**

Decision problem

$f(110101, 101011) \rightarrow \text{Yes}$

$f(111011011011101, 1011111101110011) \rightarrow \text{No}$

$f : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\} = 1$ iff isomorphic

$$\mathbf{GRAPHISO} = \{(G_1, G_2) | f(G_1, G_2) = 1\}$$