Example Turing Machine

ian.mcloughlin@atu.ie

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Language

$$
\begin{aligned}
A &= \{0, 1\} \\
A^* &= \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\} \\
L &= \{\epsilon, 01, 0011, 000111, \ldots\} \\
A^* \setminus L &= \{0, 1, 00, 10, \ldots\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{N}_0 &= \{0, 1, 2, 3, \ldots\} \\
L &= \{0^i 1^i | i \in \mathbb{N}_0\}
\end{aligned}
$$

Turing machine

State	Input	Write	Move	Next
q_0	_	_	R	q_a
q_0	0	_	R	q_1
q_0	1	1	R	q_f
q_1	_	_	L	q_2
q_1	0	0	R	q_1
q_1	1	1	R	q_1
q_2	_	_	R	q_f
q_2	0	0	R	q_f
q_2	1	_	L	q_3
q_3	_	_	R	q_0
q_3	0	0	L	q_3
q_3	1	1	L	q_3

Example input

$$
\begin{aligned}
&q_0000111 \rightarrow q_100111 \rightarrow 0q_10111 \rightarrow 00q_1111 \\
&\rightarrow 001q_111 \rightarrow 0011q_11 \rightarrow 00111q_1 \rightarrow 0011q_21 \\
&\rightarrow 001q_31 \rightarrow 00q_311 \rightarrow 0q_3011 \rightarrow q_30011 \\
&\rightarrow q_3\_0011 \rightarrow q_00011 \rightarrow q_1011 \rightarrow 0q_111 \\
&\rightarrow 01q_11 \rightarrow 011q_1 \rightarrow 01q_21 \rightarrow 0q_31 \\
&\rightarrow q_301 \rightarrow q_3\_01 \rightarrow q_001 \rightarrow q_11 \\
&\rightarrow 1q_1 \rightarrow q_21 \rightarrow q_3 \rightarrow q_1 \rightarrow q_a
\end{aligned}
$$

Steps

$$\begin{aligned} &q_0000111 \to \ldots 13 \text{ steps} \ldots \to q_00011 \\ &\to \ldots 9 \text{ steps} \ldots \to q_001 \to \ldots 5 \text{ steps} \ldots \\ &\to q_0 \to \ldots 1 \text{ step} \ldots \to q_a \qquad (28 \text{ total}) \end{aligned}$$

Simulation

n	0	2	4	6	8	10	12	14
$f(n)$	1	6	15	28	45	66	91	120

Sequence

OEIS[1] gives sequence formula:

[1] OEIS Foundation Inc (2020). *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org/A000384.

$$a(i) : \mathbb{N} \to \mathbb{N}_0 = i(2i-1)$$

So, $a(1) = 1, a(2) = 6, a(3) = 15$, and so on. We index as $2\mathbb{N} = \{0, 2, 4, 6, 8, 10, \ldots\}$. Transform:

$$h(n) : 2\mathbb{N}_0 \to \mathbb{N} = \frac{n}{2} + 1.$$

So, $h(0) = 1, h(2) = 2, h(4) = 3$, and so on.

$$\begin{aligned} f(n) : 2\mathbb{N}_0 \to \mathbb{N}_0 &= a(h(n)) \\ &= \left(\frac{n}{2}+1\right)\left(2\left(\frac{n}{2}+1\right)-1\right) \\ &= \left(\frac{n}{2}+1\right)(n+2-1) \\ &= \frac{1}{2}(n+2)(n+1) \\ &= \frac{1}{2}\left(n^2+3n+2\right) \end{aligned}$$

So, $f(n)$ is $O(n^2)$.

Justification

Is $f(n)$ the correct formula for the number of steps taken for an accepted input of length n?

Each pass right and left across the j non-blank tape cells, the machine takes $j+1$ steps right, followed by j steps left.

Start	**End**	**Right**	**Left**
000111	0011	7	6
0011	01	5	4
01	ϵ	3	2
ϵ	q_a	1	0

$$\begin{aligned} f(n) &= (n+1) + n + \ldots + 2 + 1 + 0 \\ &= ((n+1) + 0) + ((n) + 1) + \ldots \\ &= \left(\frac{n}{2} + 1\right)(n+1) \end{aligned}$$

Decider

Does the Turing Machine always halt and if so, does it reject in $O(n^2)$? Is $L \in \mathbf{P}$?