

Example Turing Machine

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Language

$$\begin{aligned}A &= \{0, 1\} \\ A^* &= \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots\} \\ L &= \{\epsilon, 01, 0011, 000111, \dots\} \\ A^* \setminus L &= \{0, 1, 00, 10, \dots\} \\ \mathbb{N}_0 &= \{0, 1, 2, 3, \dots\} \\ L &= \{0^i 1^i \mid i \in \mathbb{N}_0\}\end{aligned}$$

Turing machine

State	Input	Write	Move	Next
q_0	—	—	R	q_a
q_0	0	—	R	q_1
q_0	1	1	R	q_f
q_1	—	—	L	q_2
q_1	0	0	R	q_1
q_1	1	1	R	q_1
q_2	—	—	R	q_f
q_2	0	0	R	q_f
q_2	1	—	L	q_3
q_3	—	—	R	q_0
q_3	0	0	L	q_3
q_3	1	1	L	q_3

Example input

$$\begin{aligned}q_0 000111 &\rightarrow q_1 00111 \rightarrow 0q_1 0111 \rightarrow 00q_1 111 \\ &\rightarrow 001q_1 11 \rightarrow 0011q_1 1 \rightarrow 00111q_1 \rightarrow 0011q_2 1 \\ &\rightarrow 001q_3 1 \rightarrow 00q_3 11 \rightarrow 0q_3 011 \rightarrow q_3 0011 \\ &\rightarrow q_3_0011 \rightarrow q_0 0011 \rightarrow q_1 011 \rightarrow 0q_1 11 \\ &\rightarrow 01q_1 1 \rightarrow 011q_1 \rightarrow 01q_2 1 \rightarrow 0q_3 1 \\ &\rightarrow q_3 01 \rightarrow q_3_01 \rightarrow q_0 01 \rightarrow q_1 1 \\ &\rightarrow 1q_1 \rightarrow q_2 1 \rightarrow q_3 \rightarrow q_1 \rightarrow q_a\end{aligned}$$

Steps

$q_0000111 \rightarrow \dots 13 \text{ steps } \dots \rightarrow q_00011$
 $\rightarrow \dots 9 \text{ steps } \dots \rightarrow q_001 \rightarrow \dots 5 \text{ steps } \dots$
 $\rightarrow q_0 \rightarrow \dots 1 \text{ step } \dots \rightarrow q_a \quad (28 \text{ total})$

Simulation

n	0	2	4	6	8	10	12	14
$f(n)$	1	6	15	28	45	66	91	120

Sequence

OEIS¹ gives sequence formula:

$$a(i) : \mathbb{N} \rightarrow \mathbb{N}_0 = i(2i - 1)$$

So, $a(1) = 1$, $a(2) = 6$, $a(3) = 15$, and so on. We index as $2\mathbb{N} = \{0, 2, 4, 6, 8, 10, \dots\}$. Transform:

$$h(n) : 2\mathbb{N}_0 \rightarrow \mathbb{N} = \frac{n}{2} + 1.$$

So, $h(0) = 1$, $h(2) = 2$, $h(4) = 3$, and so on.

$$\begin{aligned}
 f(n) : 2\mathbb{N}_0 \rightarrow \mathbb{N}_0 &= a(h(n)) \\
 &= \left(\frac{n}{2} + 1\right) \left(2\left(\frac{n}{2} + 1\right) - 1\right) \\
 &= \left(\frac{n}{2} + 1\right) (n + 2 - 1) \\
 &= \frac{1}{2} (n + 2) (n + 1) \\
 &= \frac{1}{2} (n^2 + 3n + 2)
 \end{aligned}$$

So, $f(n)$ is $O(n^2)$.

Justification

Is $f(n)$ the correct formula for the number of steps taken for an accepted input of length n ?

Each pass right and left across the j non-blank tape cells, the machine takes $j + 1$ steps right, followed by j steps left.

Start	End	Right	Left
000111	0011	7	6
0011	01	5	4
01	ϵ	3	2
ϵ	q_a	1	0

¹ OEIS Foundation Inc (2020). *The On-Line Encyclopedia of Integer Sequences*.
<https://oeis.org/A000384>.

$$\begin{aligned}
 f(n) &= (n+1) + n + \dots + 2 + 1 + 0 \\
 &= ((n+1) + 0) + ((n) + 1) + \dots \\
 &= \left(\frac{n}{2} + 1\right) (n+1)
 \end{aligned}$$

Decider

Does the Turing Machine always halt and if so, does it reject in $O(n^2)$? Is $L \in \mathbf{P}$?