Unitary Error Basis

ian.mcloughlin@atu.ie

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Pauli Matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Inner Product

A, B: $d \times d$ matrices.

$\langle A, B \rangle = \mathrm{tr}(A^* B)/d$

$\mathrm{tr}(A) = \sum_i a_{ii}$

Andreas Klappenecker and Martin Rötteler. Unitary error bases: Constructions, equivalence, and applications. In Marc Fossorier, Tom Høholdt, and Alain Poli, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 139–149, Berlin, Heidelberg, 2003. Springer Berlin Heidelberg. ISBN 978-3-540-44828-0

Andreas Klappenecker and Martin Roetteler. On the monomiality of nice error bases, 2003

Conjugate Transpose

$A^* = A^{\mathsf{H}} = A^{\dagger} = \bar{A}^{\mathsf{T}}$

$$\begin{bmatrix} i & 2+i \\ 1 & 3-2i \end{bmatrix}^* = \begin{bmatrix} -i & 1 \\ 2-i & 3+2i \end{bmatrix}$$

Trace

$\mathrm{tr}(A) = \sum_i a_{ii}$

$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$

$\mathrm{tr}(cA) = c\mathrm{tr}(A)$

$\mathrm{tr}(A) = \sum_i \lambda_i$

Orthogonality

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$A^*B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$\mathrm{tr}(A^*B) = 0 + 0 = 0$

$d = 2$

$\mathrm{tr}(A^*B)/2 = 0/2 = 0$

Normality

$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad A^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$A^*A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix}$

$\mathrm{tr}(A^*A) = -i^2 + -i^2 = 2$

$d = 2$

$\mathrm{tr}(A^*B)/2 = 2/2 = 1$

Equivalence

Unitary matrix: $U^*U = I$.

$\mathcal{U}(d)$*:* group of all $d \times d$ unitary matrices.

$\mathcal{E}, \mathcal{E}'$*:* $d \times d$ Unitary Error Bases.

$$\mathcal{E} \equiv \mathcal{E}' \Leftrightarrow \exists A, B \in \mathcal{U}(d), f : \mathcal{E} \to \mathbb{C} : \mathcal{E}' = \{f(E)AEB \mid E \in \mathcal{E}\}$$

Pauli Basis is Unique

Lemma 1. *Up to equivalence, the Pauli basis is unique in dimension 2.*

Proof. Let $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ be a 2×2 unitary error basis.
Then $E_1^*\mathcal{E} = \{I, E_1^*E_2, E_1^*E_3, E_1^*E_4\}$ is also a unitary error basis.
As is $E_1^*\mathcal{E} = \{I, E_1^*E_2, E_1^*E_3, E_1^*E_4\}$ is also a unitary error basis. □

Characterization

U*:* $d \times d$ unitary matrix.

$E_i(U)$*:* $\sqrt{d}(U_{i,nd+m})_{n,m=0,\ldots,d-1}$

$$U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$E_0(U) = \sqrt{2} \begin{bmatrix} U_{0,0} & U_{0,1} \\ U_{0,2} & U_{0,3} \end{bmatrix}$$

TODO: Fix above

Knill

E. Knill. Group representations, error bases and quantum codes, 1996

Reinhard F. Werner. All teleportation and dense coding schemes. *Journal of Physics A*, 34:7081–7094, 2000. URL `https://api.semanticscholar.org/CorpusID:9684671`