Hadamard Gate

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Gate

H

Matrix

$$\mathsf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$|\phi_H\rangle \, \mathsf{H} \, |\phi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix}$$

Probability

$$\langle \phi_H | = \frac{1}{\sqrt{2}} \begin{bmatrix} (\alpha + \beta)^* & (\alpha - \beta)^* \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha^* + \beta^* & \alpha^* - \beta^* \end{bmatrix}$$

$$\begin{aligned} \langle \phi_H | \phi_H \rangle &= \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right) \begin{bmatrix} \alpha^* + \beta^* & \alpha^* - \beta^* \end{bmatrix} \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix} \\ &= \frac{1}{2} \left[(\alpha^* + \beta^*)(\alpha + \beta) + (\alpha^* - \beta^*)(\alpha - \beta) \right] \\ &= \frac{1}{2} \left[\alpha^* \alpha + \beta^* \alpha + \alpha^* \beta + \beta^* \beta + \alpha^* \alpha - \beta^* \alpha - \alpha^* \beta + \beta^* \beta \right] \\ &= \frac{1}{2} \left[2\alpha^* \alpha + 2\beta^* \beta \right] \\ &= \alpha^* \alpha + \beta^* \beta \end{aligned}$$

$$\langle \phi | \phi \rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha^* \alpha + \beta^* \beta = 1$$