Deutsch's Algorithm

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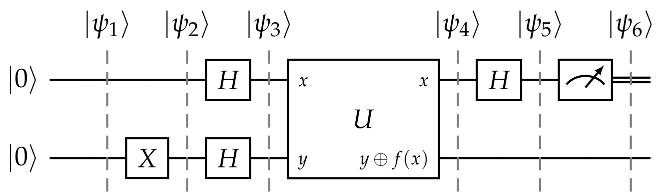


Quantum query algorithms – ibm quantum learning, 2023. URL `https://learning.quantum-computing.ibm.com/course/fundamentals-of-quantum-algorithms/quantum-query-algorithms#deutschs-algorithm`

$$|\psi_1\rangle = |0\rangle \otimes |0\rangle = |00\rangle \to \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1\\0\end{bmatrix}\\ \\ 0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$|\psi_2\rangle = (I \otimes X)\,|\psi_1\rangle = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$I \otimes X = \begin{bmatrix}1\begin{bmatrix}0&1\\1&0\end{bmatrix} & 0\begin{bmatrix}0&1\\1&0\end{bmatrix}\\ \\ 0\begin{bmatrix}0&1\\1&0\end{bmatrix} & 1\begin{bmatrix}0&1\\1&0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

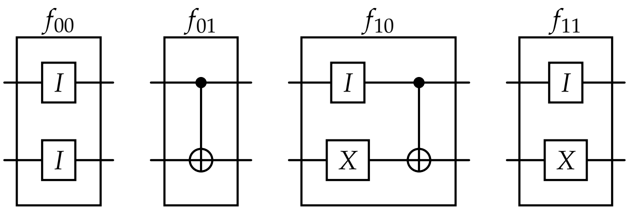
$$|\psi_3\rangle = (H \otimes H)\,|\psi_2\rangle = \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}\sqrt{2}}\begin{bmatrix}1\begin{bmatrix}1&1\\1&-1\end{bmatrix} & 1\begin{bmatrix}1&1\\1&-1\end{bmatrix}\\ \\ 1\begin{bmatrix}1&1\\1&-1\end{bmatrix} & -1\begin{bmatrix}1&1\\1&-1\end{bmatrix}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$$

U_f

x	y	$f_{00}(x)$	$y \oplus f_{00}(x)$	$f_{01}(x)$	$y \oplus f_{01}(x)$	$f_{10}(x)$	$y \oplus f_{10}(x)$	$f_{11}(x)$	$y \oplus f_{11}(x)$
0	0	0	0	0	0	1	1	1	1
0	1	0	1	0	1	1	0	1	0
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

	f_{00}	f_{01}	f_{10}	f_{11}
$\|00\rangle$	$\|00\rangle$	$\|00\rangle$	$\|01\rangle$	$\|01\rangle$
$\|01\rangle$	$\|01\rangle$	$\|01\rangle$	$\|00\rangle$	$\|00\rangle$
$\|10\rangle$	$\|10\rangle$	$\|11\rangle$	$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|11\rangle$	$\|10\rangle$	$\|11\rangle$	$\|10\rangle$



$$U_{00} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$U_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

f_{00}

$$|\psi_4\rangle = U_{00}\,|\psi_3\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$|\psi_5\rangle = (H\otimes I)\,|\psi_4\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

f_{01}

$$|\psi_4\rangle = U_{01}\,|\psi_3\rangle = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$|\psi_5\rangle = (H\otimes I)\,|\psi_4\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

f_{10}

$$|\psi_4\rangle = U_{10}\,|\psi_3\rangle = \frac{1}{2}\begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$$

$$|\psi_5\rangle = (H\otimes I)\,|\psi_4\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$$

f_{11}

$$|\psi_4\rangle = U_{11}|\psi_3\rangle = \frac{1}{2}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$$

$$|\psi_5\rangle = (H\otimes I)|\psi_4\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

$$H\otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix}1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & -1\begin{bmatrix}1&0\\0&1\end{bmatrix}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}$$

Measure

$$\begin{bmatrix}a\\b\end{bmatrix}\otimes\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}ac\\ad\\bc\\bd\end{bmatrix}$$

$$\begin{bmatrix}ac\\ad\\bc\\bd\end{bmatrix} = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

$ac = 0 \Rightarrow a = 0$ or $c = 0$

$bc = 1 \Rightarrow b \neq 0$ and $c \neq 0$

$\Rightarrow a = 0$

$|a|^2 + |b|^2 = 1$

$\Rightarrow |b|^2 = 1$