-NoValue-

*Unitary Error Basis*

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## *Pauli Matrices*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

## *Inner Product*

$A, B$: $d \times d$ matrices.

$\langle A, B \rangle = \mathsf{tr}(A^* B)/d$

$\mathsf{tr}(A) = \sum_i a_{ii}$

## *Conjugate Transpose*

$A^* = A^\mathsf{H} = A^\dagger = \bar{A}^\mathsf{T}$

$$\begin{bmatrix} i & 2+i \\ 1 & 3-2i \end{bmatrix}^* = \begin{bmatrix} -i & 1 \\ 2-i & 3+2i \end{bmatrix}$$

## *Trace*

$\mathsf{tr}(A) = \sum_i a_{ii}$

$\mathsf{tr}(A \otimes B) = \mathsf{tr}(A)\mathsf{tr}(B)$

$\mathsf{tr}(cA) = c\mathsf{tr}(A)$

$\mathsf{tr}(A) = \sum_i \lambda_i$

## *Orthogonality*

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$A^* B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$\mathsf{tr}(A^* B) = 0 + 0 = 0$

$d = 2$

$\mathsf{tr}(A^* B)/2 = 0/2 = 0$

## *Normality*

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad A^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix}$$

$\mathrm{tr}(A^*A) = -i^2 + -i^2 = 2$

$d = 2$

$\mathrm{tr}(A^*B)/2 = 2/2 = 1$

## *Equivalence*

*Unitary matrix:* $U^*U = I$.

$\mathcal{U}(d)$*:* group of all $d \times d$ unitary matrices.

$\mathcal{E}, \mathcal{E}'$*:* $d \times d$ Unitary Error Bases.

$$\mathcal{E} \equiv \mathcal{E}' \Leftrightarrow \exists A, B \in \mathcal{U}(d), f : \mathcal{E} \to \mathbb{C} : \mathcal{E}' = \{f(E)AEB \mid E \in \mathcal{E}\}$$

## *Pauli Basis is Unique*

**Lemma 1.** *Up to equivalence, the Pauli basis is unique in dimension 2.*

*Proof.* Let $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ be a $2 \times 2$ unitary error basis.
Then $E_1^*\mathcal{E} = \{I, E_1^*E_2, E_1^*E_3, E_1^*E_4\}$ is also a unitary error basis.
As is $E_1^*\mathcal{E} = \{I, E_1^*E_2, E_1^*E_3, E_1^*E_4\}$ is also a unitary error basis. □

## *Characterization*

$U$*:* $d \times d$ unitary matrix.

$E_i(U)$*:* $\sqrt{d}(U_{i,nd+m})_{n,m=0,\ldots,d-1}$

$$U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$E_0(U) = \sqrt{2} \begin{bmatrix} U_{0,0} & U_{0,1} \\ U_{0,2} & U_{0,3} \end{bmatrix}$$

TODO: Fix above

## *Knill*