Trees

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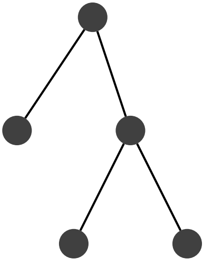
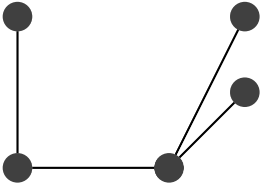
Definition

Tree

A *tree* is a graph where every pair of vertices has a path between them, and there are no cycles.





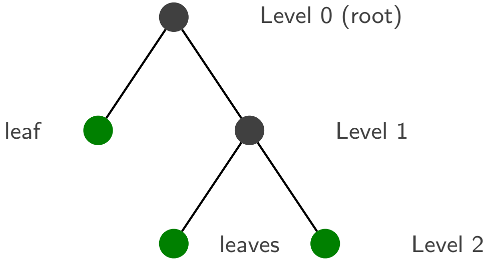
Rooted trees

Any vertex of a tree can be called its root.

Levels Root is at level 0, neighbours of the root are at level 1, their other neighbours at level 2, and so on.

Height of a tree is h, where there's vertex at level h but not at level $h+1$.

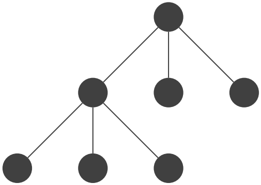
Leaf Vertex at level i not connected to a vertex at level $i+1$.

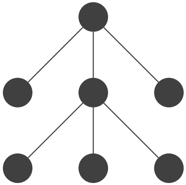


Isomorphic Rooted Trees

Definition

Two rooted trees are said to be *isomorphic* if there is a graph isomorphism between them which takes the root of one tree to the root of the other.

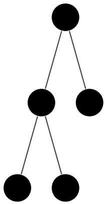


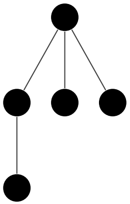


Example: non-isomorphic rooted trees

Note

Trees can be isomorphic as graphs and not as rooted trees.

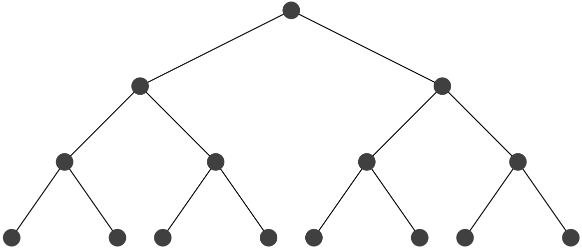




m-ary Rooted Tree

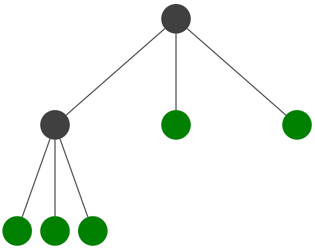
Definition

When a vertex at level i is connected to a vertex at level $i + 1$ it's common to call the former the *parent* and the latter the *child*. A rooted tree is m-ary if every parent has the same number of children. A 2-ary rooted tree is called a *binary tree*.

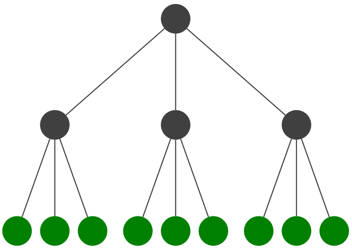


Examples of heights and leaves of m-ary trees

$m = 3$, $h = 2$, $l = 5$



$m = 3$, $h = 2$, $l = 9$



Logarithms

We define $\log$ in the following way:

$$m^h = l \Leftrightarrow \log_m l = h$$

What does log mean?

Suppose we have two numbers m and h and we ask the question "what is m to the power of h?" Let's call the answer l, so $l = m^h$.

The $\log$ function asks the inverse question: "what do we need to raise m to the power of to get l?" The answer is h.

For example, $10^2 = 100$ so $\log_{10} 100 = 2$. The subscript 10 is called the *base*.

Heights and leaves of m-ary rooted trees

Theorem

The height h of an m-ary rooted tree with l leaves is at least $\log_m l$. That is: $h \geq \log_m l$.

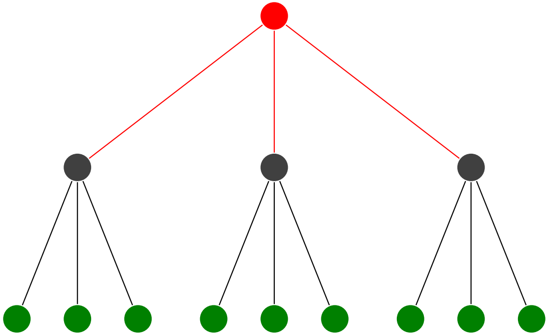
Proof.

Note that $h \geq log_m l \Leftrightarrow m^h \geq m^{log_m l} \Leftrightarrow m^h \geq l$. So, just show that l is at most m^h.

For a tree of height 0, $m^0 = 1$ and $l = 0$ giving $m^h = l$. Next, assume trees of height $i-1$ have at most m^{i-1} leaves. From a tree of height i, we can create m trees of height at most $i-1$ by deleting the root. Each of these smaller trees has at most m^{i-1} leaves. So, the big tree has at most $m \times m^{i-1} = m^i$ leaves. $\square$

Deleting the root of an m-ary tree

m smaller trees of height $h-1$



Spanning trees

Subgraph

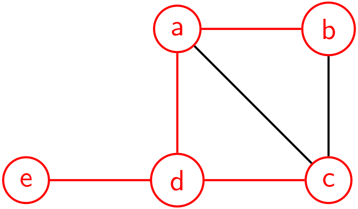
A *subgraph* $H = (V_H, E_H)$ of a graph $G = (V, E)$ is a graph such that V_H is a subset of V, E_H is a subset of E, and no edge in E_H contains a vertex not in V_H.

Spanning Tree

A *spanning tree* T of a connected graph G is a subgraph of G such that:

- the vertex set of T is the vertex set of G and
- T is a tree.

Spanning tree example



Spanning tree in red.