

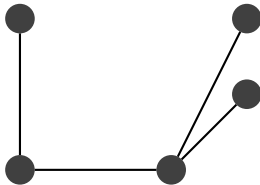
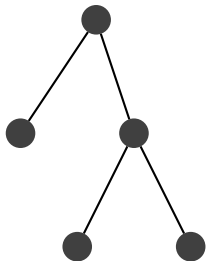
# Trees

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## Definition

### Tree

A *tree* is a graph where every pair of vertices has a path between them, and there are no cycles.



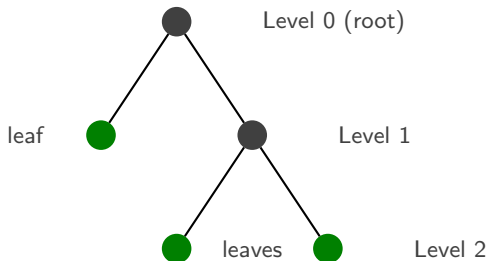
## Rooted trees

**Any vertex** of a tree can be called its root.

**Levels** Root is at level 0, neighbours of the root are at level 1, their other neighbours at level 2, and so on.

**Height** of a tree is  $h$ , where there's vertex at level  $h$  but not at level  $h + 1$ .

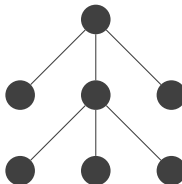
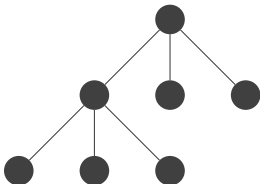
**Leaf** Vertex at level  $i$  not connected to a vertex at level  $i + 1$ .



# Isomorphic Rooted Trees

## Definition

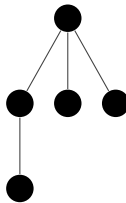
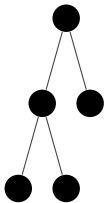
Two rooted trees are said to be *isomorphic* if there is a graph isomorphism between them which takes the root of one tree to the root of the other.



## Example: non-isomorphic rooted trees

### Note

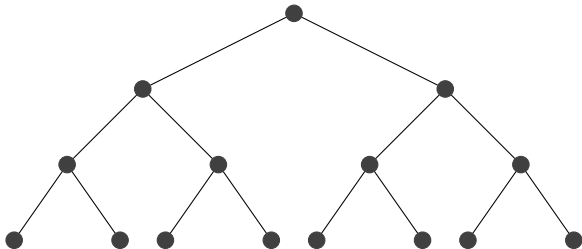
Trees can be isomorphic as graphs and not as rooted trees.



## $m$ -ary Rooted Tree

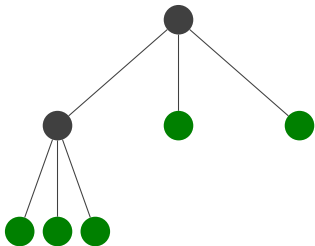
### Definition

When a vertex at level  $i$  is connected to a vertex at level  $i + 1$  it's common to call the former the *parent* and the latter the *child*. A rooted tree is  $m$ -ary if every parent has the same number of children. A 2-ary rooted tree is called a *binary tree*.

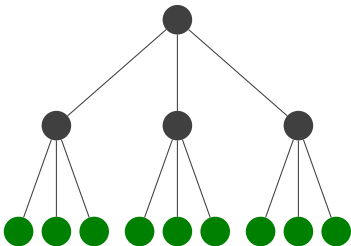


## Examples of heights and leaves of $m$ -ary trees

$$m = 3, h = 2, l = 5$$



$$m = 3, h = 2, l = 9$$



# Logarithms

We define  $\log$  in the following way:

$$m^h = l \Leftrightarrow \log_m l = h$$

## What does *log* mean?

Suppose we have two numbers  $m$  and  $h$  and we ask the question “what is  $m$  to the power of  $h$ ?” Let’s call the answer  $l$ , so  $l = m^h$ .

The  $\log$  function asks the inverse question: “what do we need to raise  $m$  to the power of to get  $l$ ?” The answer is  $h$ .

For example,  $10^2 = 100$  so  $\log_{10} 100 = 2$ . The subscript 10 is called the *base*.



## Heights and leaves of $m$ -ary rooted trees

### Theorem

*The height  $h$  of an  $m$ -ary rooted tree with  $l$  leaves is at least  $\log_m l$ . That is:  $h \geq \log_m l$ .*

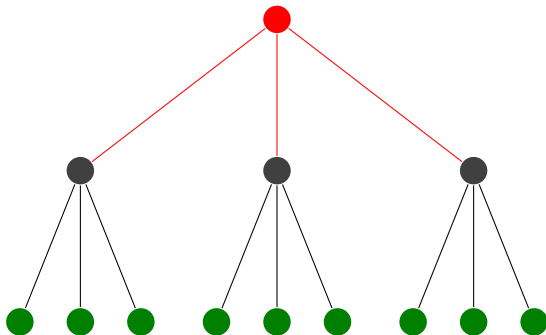
### Proof.

Note that  $h \geq \log_m l \Leftrightarrow m^h \geq m^{\log_m l} \Leftrightarrow m^h \geq l$ . So, just show that  $l$  is at most  $m^h$ .

For a tree of height 0,  $m^0 = 1$  and  $l = 1$  giving  $m^h = l$ . Next, assume trees of height  $i - 1$  have at most  $m^{i-1}$  leaves. From a tree of height  $i$ , we can create  $m$  trees of height at most  $i - 1$  by deleting the root. Each of these smaller trees has at most  $m^{i-1}$  leaves. So, the big tree has at most  $m \times m^{i-1} = m^i$  leaves.  $\square$

## Deleting the root of an $m$ -ary tree

$m$  smaller trees of height  $h - 1$



# Spanning trees

## Subgraph

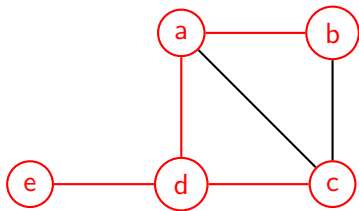
A *subgraph*  $H = (V_H, E_H)$  of a graph  $G = (V, E)$  is a graph such that  $V_H$  is a subset of  $V$ ,  $E_H$  is a subset of  $E$ , and no edge in  $E_H$  contains a vertex not in  $V_H$ .

## Spanning Tree

A *spanning tree*  $T$  of a connected graph  $G$  is a subgraph of  $G$  such that:

- the vertex set of  $T$  is the vertex set of  $G$  and
- $T$  is a tree.

## Spanning tree example



Spanning tree in red.