Decision problems

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Example: PRIMES

$$\mathrm{PRIMES} = \{i \; : \; j \nmid i \; \forall \; j < i \; ; \; 1 < i, j \in \mathbb{N}\}$$

PRIMES is a subset of the natural numbers.

Decision problem: map f from $\mathbb{N}$ to $\{0, 1\}$.

Indicates whether $i \in \mathrm{PRIMES}$ $\big(f(i) = 1\big)$ or not $\big(f(i) = 0\big)$.

Stipulate the elements of PRIMES are written in binary, e.g. 7 is 111.

Then PRIMES is a language over $\{0, 1\}$.

Decision problem

$$f : S \to T \text{ where } |T| = 2$$

A decision problem is a map to a set with two elements. Usually $T = \{0, 1\}$ and S is a language over $\{0, 1\}$.

Example

$$f : \{0, 1\}^* \to \{0, 1\}$$

$$f(s) = 0 \Leftrightarrow |s| \equiv_2 0$$

Another example

$$f : \{0, 1\}^* \to \{0, 1\}$$

$$f(s) = 0 \Leftrightarrow wt(s) \equiv_2 0$$

Set: collection of objects

- Denoted by capital letters: A, B, X
- Objects in a set are called elements.
- Elements are denoted by lower case letters: a, b, x
- Curly braces around elements: $A = \{a_0, a_1, a_2\}$

Examples

$$A = \{1, 2, 3\}$$
$$B = \{p \mid p \text{ is a prime number}\}$$

No order and no count

A set doesn't maintain an order of its elements:

$$\{1,2,3\} \;=\; \{1,3,2\} \;=\; \{2,1,3\} \;=\; \{2,3,1\}$$
$$=\; \{3,1,2\} \;=\; \{3,2,1\}$$

An object is either in the set or not:

$$\{1,2,2,3\} \;=\; \{1,2,3\}$$

Sets containing sets

Subsets

A is a subset of B if all the elements of A are in B.

$$A = \{1, 2, 3, 4\} \qquad B = \{2, 3\} \qquad B \subset A$$

Powersets

Some sets contain other sets as elements. The powerset of a set is the set containing all subsets of it:

$$A = \{1, 2, 3\}$$
$$\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Note A contains 3 elements and $\mathcal{P}(A)$ contains $2^3 = 8$.

Famous sets

$\mathbb{N}$ – the natural numbers $\{1, 2, 3, \ldots\}$.

$\mathbb{N}_0$ – the natural numbers with zero $\{0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}$ – the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{Q}$ – the rational numbers $\{\frac{m}{n} \mid m, n \in \mathbb{Z}\}$.

$\mathbb{R}$ – the real numbers.

$\mathbb{C}$ – the complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$.

Tuples: finite list of elements taken from sets

$$t = (2, 1, 1) \qquad t \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \qquad |t| = 3$$

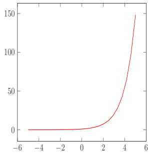
- Round brackets denote tuples, and t is a 3-tuple or a triple.
- Tuples have order, and can repeat elements.
- Sometimes we omit the brackets and commas: $t = 211$.
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is sometimes shortened to $\mathbb{N}^3$.
- The first $\mathbb{N}$ means the first element comes from $\mathbb{N}$.
- The second $\mathbb{N}$ means the second element comes from $\mathbb{N}$, etc.
- Note that there is a single empty tuple: $()$.

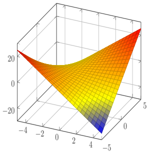
Cartesian products of sets

$$A = \{1, 2, 3\} \qquad B = \{x, y\}$$

$$A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$$

- $A \times B$ is called the cartesian product of A and B – the set of tuples with first element from A and second from B.
- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the usual 2D plane where we draw plots.
- Can extend to any length of tuple: $\mathbb{R}^3$ is the 3D plane.





Maps

Definition of map

A map from a set A to a set B is a subset M of $A \times B$ where each element of A appears as the first element of a tuple in M exactly once.

$$A = \{a, b, c\} \qquad B = \{x, y, z\}$$

Maps

- $\{(a, x), (b, x), (c, x)\}$
- $\{(a, x), (b, y), (c, z)\}$

Not maps

- $\{(a, x), (a, y), (b, x), (c, x)\}$
- $\{(a, x), (b, y)\}$

Languages

Alphabet: finite set of symbols, denoted Σ.

String: tuple w over Σ.

Star: all strings over Σ, denoted Σ^*.

Language: subset L of Σ^*.

Length: of a string, denoted $|w|$.

Deciding PRIMES

Is there an algorithm that decides if an arbitrary natural number is a prime number?

Yes — there are many algorithms such as trial division.

```
for i in range(2, n):
  if n % i == 0:
    return False
return True
```

Agrawal, Kayal and Saxena 2002 showed that PRIMES is in P.

An undecidable language

- Encode all Turing machines as strings over $\{0, 1\}$.
- Consider the subset of Turing machines that don't ever get stuck in an infinite loop irrespective of the input.
- This set is undecidable.

SAT

Example propositional formula: $(A \vee B) \wedge (\neg A \vee \neg C)$.

Variables: $A, B, C, \ldots$ – boolean.

Operations: AND ($\wedge$), OR ($\vee$), NOT ($\neg$).

Brackets: $()$.

A formula is satisfiable if there is any values for the variables that makes the formula True.

Boolean Satisfiability Problem (SAT): $\{w : w \text{ is satisfiable}\}$.

Turing machines recap

For a given input a Turing machine does one of three things:

Accepts the input string by finishing in the accept state in a finite number of steps.

Rejects the input string by finishing in the reject/fail state in a finite number of steps.

Continues indefinitely in some sort of infinite loop.

Remember there are a finite number of states and tape symbols.

Deciders

$$f : \Sigma^* \to \{q_f, q_a\}$$

Decider: a Turing machine that always finishes in a finite number of steps.

Decides: decides the language it accepts.

Decidable: a language is called decidable if any Turing machine decides it.

Important question: are all languages decidable?