Ch 10.1: example

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EXAMPLE: multi-dim. fixed pt. equation, not in book

Consider $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, find $\vec{x} = G(\vec{x})$ w/ $G(\vec{x}) = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \end{bmatrix}$

$$g_1(\vec{x}) = \tfrac{1}{2}x + y^2 \overset{\text{want}}{=} x$$
$$g_2(\vec{x}) = x^2 + \tfrac{1}{3}y^2 = y$$

Observe $\vec{x} = \vec{0}$ is a fixed point (and so is $x = (\frac{2}{9})^{1/3}$, $y = \frac{3}{2}(\frac{2}{9})^{2/3}$)

How close to $\vec{0}$ must we start for the fixed-pt. iteration to guarantee convergence to $\vec{0}$?

① Find a nice domain D (with $\vec{0}$ in its interior) such that

$\vec{x} \in D \Rightarrow G(\vec{x}) \in D$. No "correct" choice, but we want a "nice" D that we can analyze. Look for D of the form $[-a,a] \times [-b,b]$ $a, b > 0$

i.e. if $|x| \le a$, $|y| \le b$ and we define
$$x_+ = g_1(\vec{x}) = \tfrac{1}{2}x + y^2$$
$$y_+ = g_2(\vec{x}) = x^2 + \tfrac{1}{3}y^2$$

want $|x_+| \le a$ and $|y_+| \le b$ as well.

$$|x_+| = |\tfrac{1}{2}x + y^2| \le \tfrac{1}{2}|x| + |y|^2 \le \tfrac{1}{2}a + b^2 \overset{?}{\le} a \quad \text{(want)} \quad \text{i.e. want } b^2 \le \tfrac{1}{2}a$$
$$|y_+| = |x^2 + \tfrac{1}{3}y| \le |x|^2 + \tfrac{1}{3}|y| \le a^2 + \tfrac{1}{3}b \overset{?}{\le} b \quad \text{want } a^2 \le \tfrac{2}{3}b$$

combine: $a, b > 0$

$$a^4 \le (\tfrac{2}{3}b)^2 = \tfrac{4}{9}b^2 \le \tfrac{4}{9}(\tfrac{1}{2}a) \quad \text{i.e.} \quad a^3 \le 2/9, \quad a \le (\tfrac{2}{9})^{1/3}$$

so a simple choice is $a = \frac{1}{2} = (\frac{1}{8})^{1/3} \le (\frac{2}{9})^{1/3}$

and

$$b^4 \le (\tfrac{1}{2}a)^2 = \tfrac{1}{4}a^2 \le \tfrac{1}{4}(\tfrac{2}{3}b) = \tfrac{1}{6}b \quad \text{i.e.} \quad b^3 \le \tfrac{1}{6}, \quad b \le (\tfrac{1}{6})^{1/3}$$

so a simple choice is $b = \frac{1}{2} = (\frac{1}{8})^{1/3} \le (\frac{1}{6})^{1/3}$

VERIFY:
$$|x_+| \le \tfrac{1}{2}a + b^2 = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2} \checkmark$$
$$|y_+| \le a^2 + \tfrac{1}{3}b = \tfrac{1}{4} + \tfrac{1}{6} \le \tfrac{1}{2} \checkmark.$$

② Find a (possibly smaller) region where we're contractive, any norm

Option 1: Find D_1 and $L_1 < 1$ s.t. $\forall \vec{x}, \vec{x}' \in D$

' symbol unrelated to differentiation here

$$\| G(\vec{x}) - G(\vec{x}') \|_\infty \le L \cdot \| \vec{x} - \vec{x}' \|_\infty$$

Strategy: find D_1, L_1 st.
$$\| g_1(\vec{x}) - g_1(\vec{x}') \|_\infty \le L \cdot \| \vec{x} - \vec{x}' \|_\infty$$
$$\| g_2(\vec{x}) - g_2(\vec{x}') \|_\infty \le L \cdot \| \vec{x} - \vec{x}' \|_\infty$$

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Via MVT, $\exists\ \vec{\xi} \in \mathbb{R}^2$ s.t.

$$g_1(\vec{x}) - g_1(\vec{x}') = \nabla g_1(\vec{\xi})^T \cdot (\vec{x} - \vec{x}')$$

So

$$|g_1(\vec{x}) - g_1(\vec{x}')| = |\nabla g_1(\vec{\xi})^T \cdot (\vec{x} - \vec{x}')|$$

$$\le \underbrace{\|\nabla g_1(\vec{\xi})\|_1} \cdot \|\vec{x} - \vec{x}'\|_\infty \quad \text{via Hölder's Ineq. (since } \tfrac{1}{1} + \tfrac{1}{\infty} = 1\text{)}$$

so bound

$$\nabla g_1(\vec{x}) = \begin{bmatrix} \frac{1}{2} \\ 2y \end{bmatrix}, \quad \|\nabla g_1(\vec{x})\|_1 = |\tfrac{1}{2}| + |2y| = L_1$$

Set equal

So... $|y| = \frac{1}{2}(L_1 - \frac{1}{2})$ works.

i.e. if $b < \frac{1}{2}$ and $|y| \le b$ then $L_1 < 1$ as desired

Similarly for g_2:

$$\nabla g_2(\vec{x}) = \begin{bmatrix} 2x \\ \frac{1}{3} \end{bmatrix}, \quad \|\nabla g_2(\vec{x})\|_1 = |2x| + |\tfrac{1}{3}| = L_1$$

So... $|x| = \frac{1}{2}(L_1 - \frac{1}{3})$ works

i.e. if $a < \frac{1}{3}$ and $|x| \le a$ then $L_1 < 1$ as desired

Putting it together:

let $0 < a < \frac{1}{3}$, $0 < b < \frac{1}{2}$, $D_1 = [-a, a] \times [-b, b]$, then

G is L_1 Lipschitz on D_1 (and $L_1 < 1$)

Since

$$\|G(\vec{x}) - G(\vec{x}')\|_\infty := \max\{|g_1(\vec{x}) - g_1(\vec{x}')|, |g_2(\vec{x}) - g_2(\vec{x}')|\}$$

$$\le \max\{L_1 \cdot \|\vec{x} - \vec{x}'\|_\infty, L_1 \cdot \|\vec{x} - \vec{x}'\|_\infty\}$$

$$= L_1 \cdot \|\vec{x} - \vec{x}'\|_\infty$$

Option 2 Find a region D_2 and a constant $\boxed{L_2 < \frac{1}{\sqrt{n}}}$ i.e. $L_2 < \frac{1}{\sqrt{2}}$

both g_1 and g_2 are L_2-Lipschitz on D_2 with respect to $\|\cdot\|_2$ norm

Similar to before:

$$|g_1(\vec{x}) - g_1(\vec{x}')| \le \|\nabla g_1(\vec{\xi})\|_2 \cdot \|\vec{x} - \vec{x}'\|_2$$

via Cauchy-Schwarz (special case of Hölder)

$$\|\nabla g_1(\vec{x})\|_2 = \sqrt{(\tfrac{1}{2})^2 + (2y)^2} = L_2 < \tfrac{1}{\sqrt{2}}$$

want

i.e. $\frac{1}{4} + 4y^2 = L_2^2$, $y^2 = \frac{1}{4}(L_2^2 - \frac{1}{4})$, so if $|y| \le b < \frac{1}{4}$ then $L_2 < \frac{1}{\sqrt{2}}$

$$\|\nabla g_2(\vec{x})\|_2 = \sqrt{(2x)^2 + (\tfrac{1}{3})^2} = L_2 < \tfrac{1}{\sqrt{2}}$$

want

i.e. $4x^2 + \frac{1}{9} = L_2^2$, $x^2 = \frac{1}{4}(L_2^2 - \frac{1}{9})$, so if $|x| \le a < \frac{1}{6}\sqrt{\frac{7}{2}}$ then $L_2 < 1/\sqrt{2}$

Example, p. 3

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and put it together:

$$\|G(\vec{x})-G(\vec{x}')\|_2 = \sqrt{(g_1(\vec{x})-g_1(\vec{x}'))^2 + (g_2(\vec{x})-g_2(\vec{x}'))^2}$$

$$\le \sqrt{L_2^2 \cdot \|\vec{x}-\vec{x}'\|_2^2 + L_2^2 \|\vec{x}-\vec{x}'\|_2^2}$$

$$= \underbrace{\sqrt{2}\cdot L_2}_{<1} \cdot \|\vec{x}-\vec{x}'\|_2$$

Option 3 Suppose each g_i is L_3 Lipshitz with respect to the ℓ^1 norm,

i.e. bound $\|\nabla g_i(\vec{x})\|_\infty$ Not a typo

with $L_3 < \frac{1}{n}$ (i.e. $\frac{1}{2}$ in this example)

$$\|G(\vec{x})-G(\vec{x}')\|_1 = |g_1(\vec{x})-g_1(\vec{x}')| + |g_2(\vec{x})-g_2(\vec{x}')|$$

$$\le L_3\|\vec{x}-\vec{x}'\|_1 + L_3 \cdot \|\vec{x}-\vec{x}'\|_1$$

$$= \underbrace{2\cdot L_3}_{<1} \cdot \|\vec{x}-\vec{x}'\|_1$$

etc.

Conclusion:

If we start in $D = [-a,a] \times [-b,b]$

with $a < 1/3$ ← smallest of "a" from step 1 and "a" from step 2

$b < 1/2$

then CMT guarantees we converge to $\vec{0}$

Recall: $a \le 1/2, b \le 1/2$ guarantees $G: D \to D$

use either
- $a < 1/3$, $b < 1/2$ guarantees G contractive in ℓ^∞ norm
- $a < \frac{1}{6}\sqrt{\frac{7}{2}}$, $b < \frac{1}{4}$ guarantees G contractive in ℓ^2 norm
- ...