Ch 10 linear algebra supplement: Sherman-Morrison

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Thm 10.8 in book: Sherman-Morrison Formula

If A nonsingular $n\times n$ matrix, $\vec{x},\vec{y}\in\mathbb{R}^n$ and $\vec{y}^T A^{-1}\vec{x} \neq -1$, then $A+\vec{x}\vec{y}^T$ is nonsingular and

$$(A+\vec{x}\vec{y}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{x}\vec{y}^T A^{-1}}{1+\vec{y}^T A^{-1}\vec{x}}$$

proof: just verify: if I claim B^{-1} is the inverse of B, check $B^{-1}B = I$

(and if you're square, that's enough, implies $BB^{-1}=I$ also)

So

$$\left(A^{-1} - \frac{A^{-1}xy^TA^{-1}}{1+y^TA^{-1}x}\right)\left(A+xy^T\right) =$$

OR... EXERCISE (IN CLASS)

I claim $(A+xy^T)^{-1} = A^{-1} - c\cdot A^{-1}xy^TA^{-1}$ for some scalar c. Find the value of c.

Solution:

$$I \stackrel{?}{=} \underbrace{(A+xy^T)}_{\text{"}B\text{"}}\underbrace{(A^{-1}-cA^{-1}xy^TA^{-1})}_{\text{supposed } B^{-1}} = I - c\cdot \underbrace{xy^TA^{-1}} + \underbrace{xy^TA^{-1}} - c\, xy^T\underbrace{A^{-1}x}_{d = y^TA^{-1}x}y^TA^{-1}$$

$$= I - c\cdot\underbrace{xy^TA^{-1}} + \underbrace{xy^TA^{-1}} - c\cdot d\cdot \underbrace{xy^TA^{-1}}_{\text{matrix}}$$

$$= I + \underbrace{(-c+1-c\cdot d)\cdot xy^TA^{-1}}_{\text{want this to be } 0}$$

i.e. $-c+1-cd=0,\quad c(1+d)=1,\quad c=\frac{1}{d+1} = \boxed{\frac{1}{1+y^TA^{-1}x}}$

matches the theorem!

VARIANT: aka WOODBURY MATRIX IDENTITY or MATRIX INVERSION LEMMA

Let $A\in\mathbb{R}^{n\times n}$, $U,V\in\mathbb{R}^{n\times k}$, $C\in\mathbb{R}^{k\times k}$ then

$$(A+\underbrace{U\cdot C\cdot V^T}_{n\times n \text{ inverse}})^{-1} = A^{-1} - A^{-1}\cdot U(\underbrace{C^{-1}+V^TA^{-1}U}_{k\times k \text{ inverse}})^{-1}V^TA^{-1}$$

often used when A is diagonal so A^{-1} is easy

closely related to the Schur complement