Homework 5
APPM 4600 Numerical Analysis, Fall 2025

Due date: Friday, September 26, before midnight, via Gradescope. **Instructor**: Prof. Becker
Revision date: 9/25/2025

Theme: contraction mapping theorem, sparse matrices, linear algebra and least-squares review.

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with. Please also follow our AI policy.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary (and you can use any language you want). You may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code). For nicely exporting code to a PDF, see the APPM 4600 HW submission guide FAQ.

Reading You may want to brush up on your linear algebra before starting this. If you took APPM 3310 Matrix Methods recently, you likely used the textbook "Applied Linear Algebra" (2nd ed) by Olver and Shakiban. In that textbook, relevant sections to brush up on include §3.3 norms, §4.3 orthogonal matrices, §5.4 least squares, §8.2 eigenvalues and eigenvectors, §8.3 eigenvector bases, §8.5 eigenvectors of symmetric matrices, and §8.7 singular values.

Problem 1: Which of the following iterations will definitely converge to the indicated fixed point p (provided x_0 is sufficiently close to p)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, p is the larger of the two fixed points (so you have to find it! You can do this algebraically. *Hint: convert the fixed point problem to a root-finding problem with a quadratic polynomial.*)

b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, p is the unique fixed point (again, you have to find it).

c) $x_{n+1} = \frac{12}{1+x_n}$, p is the larger of the two fixed points (again, you have to find it).

Problem 2: Linear algebra review Let $\mathbf{A}$ be a $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$, and write $\mathbf{I}$ to denote the $n \times n$ identity matrix. Note that throughout this homework we assume $\sigma_n > 0$ (which is not generally true).

a) Is $\mathbf{A}^\top \mathbf{A}$ diagonalizable? (i.e., is it definitely diagononalizable? or possibly diagonalizable? or never diagonalizable?) Why or why not? If it is diagonalizable, can we find an *orthonormal* eigenvector basis?

b) In terms of the singular values σ_i, what are the eigenvalues λ_i of $\mathbf{A}^\top \mathbf{A}$?

c) Let $\eta \in \mathbb{R}$ be a scalar. In terms of η and the eigenvalues λ_i of $\mathbf{A}^\top \mathbf{A}$, what are all the eigenvalues of $\mathbf{I} - \eta \mathbf{A}^\top \mathbf{A}$?

d) Let $\boldsymbol{b} \in \mathbb{R}^n$ be any fixed vector, and define $G(\boldsymbol{x}) = \boldsymbol{x} - \eta \mathbf{A}^\top (\mathbf{A}\boldsymbol{x} - \boldsymbol{b})$. In terms of the eigenvalues λ_i of $\mathbf{A}^\top \mathbf{A}$, for which values of η (if any) is G a contraction with respect to the Euclidean norm? Explain your work. *Hint: use the previous parts of this problem. This problem may take some time, depending on your linear algebra background.*

Problem 3: We'll now specialize to a particular $\mathbf{A}$. Define $\mathbf{A} = \mathbf{I} + \boldsymbol{u}\boldsymbol{v}^\top$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ where $\boldsymbol{v} = \boldsymbol{e}_1$ is the unit vector $\boldsymbol{v} = (1, 0, \ldots, 0)^\top$, and $\boldsymbol{u} = \frac{1}{\sqrt{n-1}}(0, 1, 1, \ldots, 1)^\top$.

a) Find all the unique eigenvalues of $\mathbf{A}^\top\mathbf{A}$ (i.e., a formula). Because of the special construction of $\mathbf{A}$, this should *not* depend on n. *Hint: check your answer numerically! Also, you should find that all λ_i are strictly positive.*

b) Using your work from the previous problem, find a stepsize η for which $\mathbf{I} - \eta\mathbf{A}^\top\mathbf{A}$ is a contraction.

Problem 4: **Computation** Let $\mathbf{A}$ be the same $n \times n$ matrix defined in the previous problem, and we'll explore ways to solve least-squares problem $\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_2$. This can be done via "direct" methods (e.g., the QR factorization), but we can do better than direct methods when $\mathbf{A}$ is sparse or very structured.

a) Create $\mathbf{A}$ on the computer in a way that you can compute that matrix-vector product ("mat-vec") $\mathbf{A}\boldsymbol{x}$ in $\mathcal{O}(n)$ flops, rather than $\mathcal{O}(n^2)$ flops that a naive implementation would require. You can do this in several ways, such as writing a custom mat-vec function, or using sparse matrices. Turn in (a) your code to construct this object, and (b) a plot showing the time for a mat-vec as a function of n, for a variety of n in the range between 10^3 and 10^4. *How should you scale each of your axes for this plot — logarithmic or linear? The plot should easily convince the grader that your method really is $\mathcal{O}(n)$.*

b) Let $\boldsymbol{p}$ be the all ones vector, and set $\boldsymbol{b} = \mathbf{A}\boldsymbol{p}$. Now let's suppose we didn't know $\boldsymbol{p}$ and we'll try to find it by solving the least-squares problem $\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_2$. We know that the solution satisfies the normal equations, $\mathbf{A}^\top\mathbf{A}\boldsymbol{p} = \mathbf{A}^\top\boldsymbol{b}$. Convert this into a fixed-point equation, $\boldsymbol{p} = G(\boldsymbol{p})$ with $G(\boldsymbol{p}) = \boldsymbol{p} - \eta\mathbf{A}^\top(\mathbf{A}\boldsymbol{p} - \boldsymbol{b})$, and solve with the fixed-point iteration, $\boldsymbol{x}^{(k+1)} = G(\boldsymbol{x}^{(k)})$. Choose η so that you are guaranteed convergence (use your work from the previous problems!). For $n = 10^6$ and $\boldsymbol{x}^{(0)} = \mathbf{0}$, run your code, and plot the error $\|\boldsymbol{x}^{(k)} - \boldsymbol{p}\|_2$ as a function of k. Turn in your code and the plot. *How should you scale each of your axes for this plot — logarithmic or linear? Tip: If you run your code for about 100 iteration, it should take a handful of seconds. If it's significanlty slower, check your implementation! Also, if you were unable to find a stepsize η that guarantees convergence, then find a stepsize by trial-and-error.*

c) According to theory we discussed in class, for your particular stepsize, what kind of convergence rate do you expect? (i.e., sublinear, linear, superlinear, quadratic?) If it's linear, what's the rate?

d) According to your numerical experiment, what kind of rate did you observe? Does this match your theory?

e) **Comment — no work required** We're working with matrices because they simplify things, but it's worth pointing out that for linear least squares, there are more specialized methods based on the idea of Krylov subspaces (e.g., conjugate gradient, LSQR. If you're curious to try these out, look into `scipy.sparse.linalg.lsqr`). Furthermore, the contraction mapping theorem for linear operators can be strengthened by using the concept of the "spectral radius" which means you don't have to choose a norm for the contraction. We've also required $\mathbf{A}$ to be square, in order to make the problem easier, but our theory would work for any rectangular $\mathbf{A}$ that has full column rank. If you have a matrix $\mathbf{A}$ that doesn't have full column rank, the fixed point iteration still converges (for the same values of η) albeit more slowly and with some subtlety (since the solution isn't unique!), and you can't use the contraction mapping theorem to prove it but have to rely on other tools.