Ch 2.2: Fixed Point Iteration

Wednesday, September 17, 2025 7:12 AM

Outline:

- Definition
- Graphical Interpretation
- Theory, part 1 (existence, non-constructive)
- Theory, part 2 (constructive)
- Convergence Rate
- Pictures
- Recap
- Examples

Another variant on root-finding...
this one does extend to higher dimension
but we'll stick to scalars for now
(next week we'll extend to vectors)

Def A fixed-point* of a function g is a point p such that

$$g(p) = p$$

* unrelated to "fixed point"/"floating point" representations of number on a computer

Mathematically, equivalent to root-finding (i.e., p is a root of $f(x) := g(x) - x$)

Most examples can just as naturally be cast as finding zeros or minimization/maximization ... so why discuss?

... leads to very natural algorithm, "fixed-point iteration"

$$p_{k+1} = g(p_k).$$ (no derivatives of g needed)

or write "$p^{(k)}$" rather than "p_k" so that later when p is a vector we don't confuse with coordinates

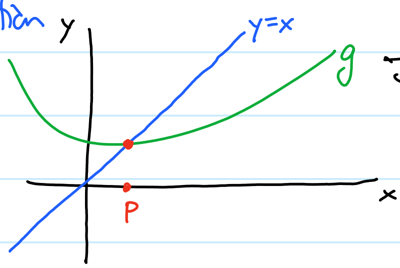
If (p_k) converges, $p_k \to p$, then (still assuming g is continuous) this limit p is a fixed point

(proof: $p = \lim_{k\to\infty} p_{k+1} = \lim_{k\to\infty} g(p_k) \overset{\text{used continuity of } g}{=} g(\lim_{k\to\infty} p_k) = g(p)$)

Graphical Interpretation

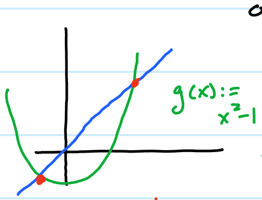
Solve $g(p) = p$

or $g(x) = x$

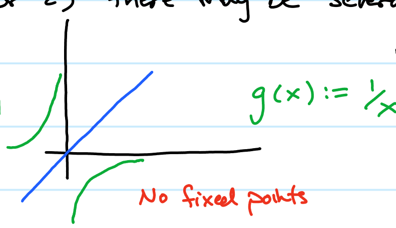


Just find the intersection of $y = f(x)$ and $y = x$

It's immediately clear 1) there may not be any fixed points
or 2) there may be several (or infinite)
ex: define $g(x) := x$



Several fixed points



No fixed points

Ch 2.2, p. 2

Wednesday, September 17, 2025 7:17 AM

Theory, part 1: existence (non-constructive)

Main tool: Intermediate Value Thm (IVT), just like for the bisection method

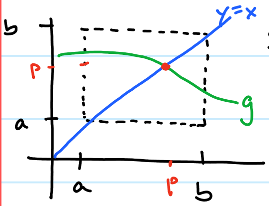
Theorem (Thm 2.3 (i) in Burden + Faires 9^{th} + 10^{th} ed.)

Let $g \in C([a,b])$, then if all its output is in $[a,b]$ as well, (i.e. $\forall x \in [a,b]$, $g(x) \in [a,b]$)

then g has a fixed point in $[a,b]$ (... may not be unique).

proof

Define $h(x) = g(x) - x$ and apply IVT to show $\exists\, x \in [a,b]$ s.t. $h(x) = 0$.



Specifically, $h(a) = g(a) - a$.

Either $g(a) = a$ (and we're done)

or $g(a) > a$ (since range is in $[a,b]$) $\Rightarrow h(a) > 0$

Similarly, $g(b) = b$ (and we're done) or $h(b) < 0$

So $h(a) \cdot h(b) < 0$, IVT applies, and $\exists\, x \in [a,b]$ w/ $h(x) = 0$ i.e. $g(x) = x$. $\square$

Theory, part 2: uniqueness, and a constructive way to find fixed pts.

Definition A function g is called **Lipschitz** with constant L on an interval $D \subseteq \mathbb{R}$ if $(\forall x, y \in D)$ $|g(x) - g(y)| \le L \cdot |x - y|$ $(*)$

(In particular, such a function is uniformly continuous, and hence continuous. Hence, sometimes we say "Lipschitz continuous")

NOT in textbook

"The" Lipschitz constant L of a Lipschitz continuous function is the smallest such L that satisfies $(*)$.

Note Suppose $g \in C^1(A)$, i.e., g' exists.

If $|g'(x)| \le L$ $(\forall x \in A)$ then g is L-Lipschitz

(proof: F.T.C., $|g(x) - g(y)| = \left| \int_y^x g'(s)\,ds \right|$

$\le \int_y^x |g'(s)|\,ds \le L \cdot |x-y|$)

or ... MVT $\frac{g(x)-g(y)}{x-y} = g'(\xi)$

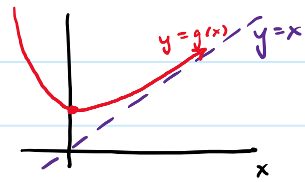
or ... Taylor remainder thm

So often we just show $|g'| \le L$. However, you can have a Lipschitz function that isn't differentiable (ex: $g(x) = |x|$ is $L=1$ Lipschitz)

Definition A function g is a **contraction** on $D \subseteq \mathbb{R}$ (domain) if it is Lipschitz with $L < 1$ on D

⚠ Contraction means $\forall x, y \quad |g(x) - g(y)| \le L \cdot |x - y|$ for some $L < 1$

this is not quite the same as $\forall x, y \quad |g(x) - g(y)| < |x - y|$



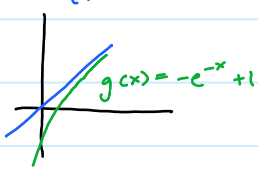
DO AS IN-CLASS EXERCISE:
1) $D = \mathbb{R}$
2) $D = [0, \infty)$
3) $D = [0, 100]$
... but do later.

Ex: $g(x) = e^{-x} + x$ so $g'(x) = -e^{-x} + 1$

on $D = [0, 100]$, g is a contraction since $|g'(x)| < 1 - e^{-100}$ on D ... but doesn't map D into D i.e. $g(100) > 100$

on $D = [0, \infty)$, g is not a contraction since $\lim_{x \to \infty} |g'(x)| = 1$

(in neither case is there a fixed point: the issue)



(asymptotes to $y = x$ but doesn't cross)

Theorem (Banach Fixed point theorem aka Contraction Mapping Theorem) or "Banach-Picard", specialized to 1D

domain D is either $[a,b]$ or $\mathbb{R}$ or $[a, \infty)$ or $(-\infty, b]$. Really, any closed set.

Let g be a contraction (1) on D and $(\forall x \in [a,b])\ g(x) \in [a,b]$ (2)

then

(1) there is a unique fixed point of g inside D, and call the limit p
(existence and uniqueness)

(2) defining the fixed point iteration by $p_{k+1} = g(p_k)$, $p_0 \in D$ arbitrary

then $\lim_{k \to \infty} p_k = p$ and at rate $|p_k - p| \le L^k \cdot |p_0 - p|$ Recall this is "linear convergence"

(construction... i.e., how to find it)

⚠ The theorem gives conditions that guarantee existence and uniqueness, but they are not necessary (in particular, D might be too big)

Proof First uniqueness (part I), then existence + construction (part II)

(I) Suppose there is a fixed point $p \in [a,b]$ with $g(p) = p$, and let $q \in [a,b]$ with $g(q) = q$ also. Then

$$|p - q| \overset{\text{def. fixed pt.}}{=} |g(p) - g(q)| \le L \cdot |p - q| \quad \text{for } L < 1$$

So if $|p - q| \neq 0$, divide equation by it, to get $1 \le L$. Contradiction

Hence $|p - q| = 0$, ie $p = q$, meaning there cannot be distinct fixed pts.

Wednesday, September 17, 2025 7:20 AM

(II) To show existence, we show (p_k) is a Cauchy sequence. This isn't in the scope of our course, so don't worry about it. (or use the IVT) i.e. use previous theorem if $D=[a,b]$

To show contraction, i.e. $|p_k - p| \le L^k \cdot |p_0 - p|$ and rate

just need to calculate: $|p_k - p| = |g(p_{k-1}) - g(p)|$ ← by def'n of fixed pt.

$\le L \cdot |p_{k-1} - p|$ since it is a contraction $(L<1)$

$\le L^k |p_0 - p|$.

$0<L<1 \Rightarrow L^k \to 0$

so $p_k \to p$ $\square$

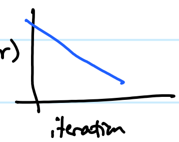
Convergence rate

$\underbrace{|p_k - p|}_{e_k} \le L^k \cdot |p_0 - p|$, $L<1$ is linear convergence

e_k, e for "error"

$e_k \le L^k \cdot e_0$ i.e., $\overbrace{\log(e_k)}^{y} \le k \cdot \overbrace{\log(L)}^{m} + \overbrace{\log(e_0)}^{b}$

i.e., log(error)

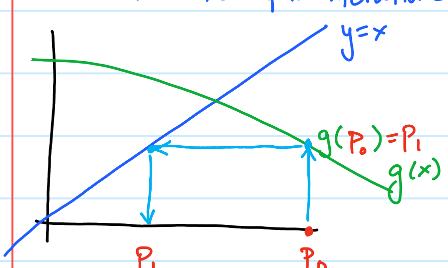


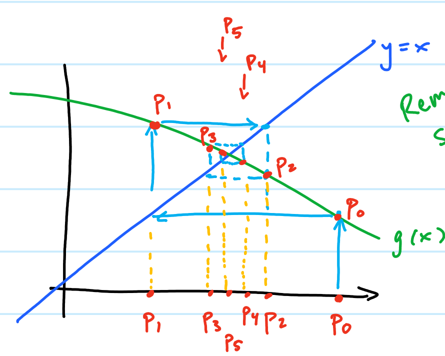
You can estimate slopes by running a least-squares fit to $\log(e_k)$, eg polyfit in Matlab

(in practice, often exclude some of the small k points if visually they don't appear to be following the trend yet)

See demos, labs for cobweb plots

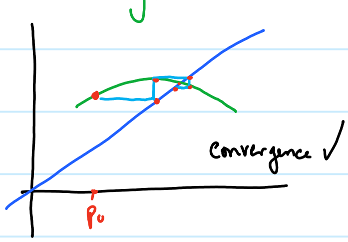
Pictures of fixed-pt. iterations

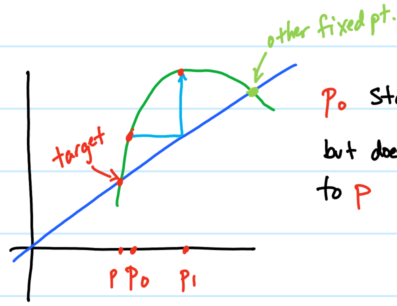




Remind you of stable equilibrium you learned about in APPM 2360?

⚠ Things don't always work! $g(x) = x^2 - 4x + 3.5$





p_0 starts near p but doesn't converge to p

Ch 2.2, p. 5

Wednesday, September 17, 2025 7:22 AM

Recap,

or on $\mathbb{R}$ or $[a,\infty)$ or $(-\infty,b]$ ...

$|g'(x)| \le L < 1$ on $[a,b]$, and $g(x) \in [a,b]$ $\forall x \in [a,b]$,

is sufficient for guaranteeing fixed-pt. iterations will find the unique fixed point on $[a,b]$.

(linear convergence)

Error converges to 0 at least as fast as $|e_n| \lesssim \text{const} \cdot L^n$

As $p_n \to p$, we can shrink $[a,b]$, and find a tighter bound on $|g'(x)|$

i.e., $|g'(p)|$ will determine final convergence rate

If $g'(p) = 0$, we get superlinear convergence!

Note: $f'(p)=0$ is bad for rootfinding, but no contradiction since $f(x) = g(x) - x$ so $g'(p)=0 \Rightarrow f'(p)=1$

$\kappa^{abs}(p) = 1$

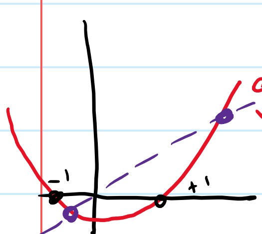
Examples

1. Show $g(x) = \frac{(x^2-1)}{3}$ has a unique fixed pt. on $[-1,1]$

First, check that $g(x) \in [-1,1]$ for $x \in [-1,1]$

(why do we do this step? If $g(x)$ is outside the domain, how can we have $x = g(x)$?)

out of domain ↗ x ; in domain ↖ $g(x)$



So, what is $\max_{x\in[-1,1]} g(x)$ and $\min_{x\in[-1,1]} g(x)$?

- check endpts, $g(-1) = 0$ and $g(1) = 0$
- check critical pts, ie., where $g'(x) = 0$. $g'(x) = \frac{2x}{3}$ so $g'(x) = 0 \Rightarrow x = 0$

so $g(0) = -1/3$

So $g(x) \in [-1/3, 0] \subseteq [-1,1]$ for $x \in [-1,1]$ ✓

Second, check if it's contractive. Since g' exists, just check $|g'(x)|$ on $x \in [-1,1]$. Since $g'(x) = 2/3\, x$

then $|g'(x)| \le 2/3$, so it's a contraction ($2/3 < 1$). ... so we would be able to find the other fixed pt

Not a contraction if $|x| \geq 3/2$

(2) Show $g(x) = 3^{-x}$ has a fixed pt. on $[0,1]$

Use $3^{-x} = e^{-\ln(3)\cdot x}$

Note $g'(x) = -3^{-x}\cdot \ln(3)$, so $g'(x) < 0$, so g is decreasing.

So $\max\limits_{x\in[0,1]} g(x) = g(0) = 1$ and $\min\limits_{x\in[0,1]} g(x) = g(1) = 1/3$

So $g(x)\in[0,1]$ on $x\in[0,1]$. Since g is continuous, our first theorem guarantees there is a fixed pt.

Note we can't use the contraction mapping to prove uniqueness, since g isn't a contraction on $[0,1]$ since $g'(0) = -\ln(3) = -1.0986$

$|g'(0)| > 1$

(3) Revisit $e^{-x}+x$ on $[0,\infty)$ example

(4) Root-finding $\Rightarrow$ Fixed pt.

From §2.2 in Burden and Faires

Solve for a root of $x^3+4x^2-10=0$ in $[1,2]$ $(p\approx 1.36)$

Contraction on $[1,2]$?

* This is done in a DEMO and LAB #5

didn't present since did Lab5 instead

A) $\underbrace{x^3+4x^2-10+x}_{g_1(x)} = x$ So solve $x=g_1(x)$

In the real world, use a specialized polynomial method, see ch. 2.6

No, doesn't map $[1,2]$ to $[1,2]$

So may fail

in demo/lab, switched order

B) $4x^2 = 10-x^3$

$x = \pm\frac{1}{2}\sqrt{10-x^3}$

So solve $x = \underbrace{\frac{1}{2}\sqrt{10-x^3}}_{g_2(x)}$

Not on $[1,2]$

$g'(2) = 2.12 > 1$

though $[1,1.5]$ works

Doesn't quite map $[1,2]$ into $[1,2]$

Fixed pt. iter at $x_0=1$ or 2 actually works

C) $x^3+4x^2-10=0$ ($x=0$ isn't a root)

$x^3 = 10-4x^2$ $/x$, $x^2 = \frac{10}{x}-4x$, So solve $x = \underbrace{\sqrt{10/x-4x}}_{g_3(x)}$

No, doesn't map $[1,2]$ to $[1,2]$

and $g'(p) = 3.4 > 1$

"REPULSIVE FIXED PT."!!

Definitely won't work

D) $x^3+4x^2 = 10$

$x^2(x+4) = 10$, $x^2 = \frac{10}{x+4}$ So solve $x = \underbrace{\sqrt{\frac{10}{x+4}}}_{g_4(x)}$

Yes, $|g'(x)| < 0.15$

Fast convergence!

and yes, maps $[1,2]$ to $1<\sqrt{\frac{10}{5}}$, $\sqrt{\frac{10}{6}} < 2$ ✓

E) $-\frac{x^3+4x^2-10}{3x^2+8x} + x = x$, so solve $x = \underbrace{x - \frac{x^3+4x^2-10}{3x^2+8x}}_{g_5(x)}$

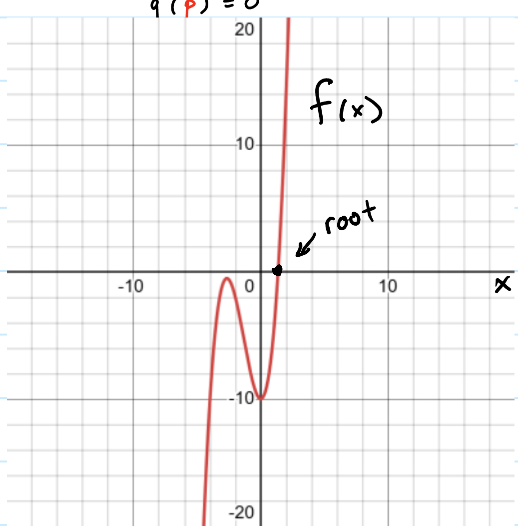
$\to$ nonzero at a root

(we know this since a simple root and denominator is f')

Yes, $|g'(x)| < .5$

and $g'(p) = 0$

and yes, maps $[1,2]$ into $[1,2]$



Ch 2.2, p. 7 (not covered)

Wednesday, September 17, 2025 7:23 AM

= Advanced, optional topic =

For fun... we discuss solving $x = g(x)$, $x \in \mathbb{R}$, a 1D vector space

This contraction-mapping idea extends to vector spaces $\mathbb{R}^n$ and even infinite-dimensional vector spaces! For example, you can think of a function f as a "point" or "vector" in a function space.

we'll cover in ch. 10

From the exam for 1st year Applied Math PhD students in Aug '19:

let h be a continuous function on $[0,1]$. Show there exists a unique continuous function f on $[0,1]$ satisfying

$$f(x) = h(x) + \underbrace{\int_0^x f(x-t)\, e^{-t^2}\, dt}_{G(f)}$$

proof sketch

Rewrite as $f = G(f)$ and show G is a contraction, so apply Banach fixed pt. theorem.