Poisson Process

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1 Introduction

A Poisson process is a mathematical model that describes a sequence of events occurring in continuous time.
It is named after the French mathematician Siméon Denis Poisson, who introduced it in the early 19th century. The Poisson process is widely used in various fields, including telecommunications, biology, economics, and physics, to model the random occurrence of events over time.

2 Characteristics

Key characteristics of a Poisson process are:

- **Event Occurrence:** The process represents the occurrence of events at distinct points in continuous time. These events could be arrivals at a service center, radioactive decay events, phone calls in a call center, or any other phenomena where events happen randomly.

- **Independence:** The occurrence of events is assumed to be independent. In other words, the time between events and the number of events in disjoint time intervals are not influenced by the occurrence of events in other intervals.

- **Memorylessness:** The Poisson process has the memorylessness property, meaning that the time until the next event is independent of the past. The probability distribution of the time until the next event is exponential.

- **Constant Rate:** The events occur at a constant rate λ (lambda) per unit of time. This rate represents the average number of events per unit time.

- **Stationary Increments:** The number of events in any time interval depends only on the length of the interval, not on the specific location of the interval in time.

3 Formula

The probability mass function (PMF) for the number of events in a given time interval $[t, t + \Delta t)$ is given by the Poisson distribution:

$$P(N(t + \Delta t) - N(t) = k) = \frac{e^{-\lambda \Delta t} (\lambda \Delta t)^k}{k!}$$

where:

- $N(t)$ is the number of events that have occurred by time t,
- λ is the rate of events per unit time,
- Δt is the length of the time interval,
- k is the number of events in the interval,
- e is the base of the natural logarithm.

In practice, the Poisson process is often used to model rare events happening independently over time. For example, it could be used to model the number of customer arrivals at a service center, the occurrence of accidents on a stretch of road, or the decay of radioactive particles.

The Poisson process is closely related to the exponential distribution, which describes the time between consecutive events in the process. The time between events follows an exponential distribution with parameter λ.

Mathematically, the Poisson process is often denoted as $N(t) \sim Poisson(\lambda t)$, indicating that the number of events in a time interval $[0, t]$ follows a Poisson distribution with mean λt.

4 Simulation

To simulate a Poisson process in Python, you can use the NumPy library to generate random numbers from a Poisson distribution.

```python
import numpy as np
import matplotlib.pyplot as plt

def simulate_poisson_process(rate, total_time):
    # Generate Poisson-distributed random variables for each small time interval
    time_intervals = np.linspace(0, total_time, 1000)
    poisson_values = np.random.poisson(rate * (time_intervals[1] - time_intervals[0]), len(time_intervals))

    # Calculate the cumulative sum to represent the Poisson process
    poisson_process = np.cumsum(poisson_values)

    return time_intervals, poisson_process

# Set the parameters for the Poisson process
rate = 1.5  # Average rate of events per unit time
total_time = 10  # Total simulation time

# Simulate the Poisson process
time, process = simulate_poisson_process(rate, total_time)

# Plot the results
plt.step(time, process, where='post')
plt.title('Simulated Poisson Process')
plt.xlabel('Time')
plt.ylabel('Number of Events')
plt.show()

```

In this code:

- simulate_poisson_process is a function that generates Poisson-distributed random variables for small time intervals and calculates the cumulative sum to represent the Poisson process.

- np.random.poisson is used to generate Poisson-distributed random variables.

- np.cumsum calculates the cumulative sum of the generated Poisson values, creating the Poisson process.

- The results are then plotted using matplotlib.pyplot.

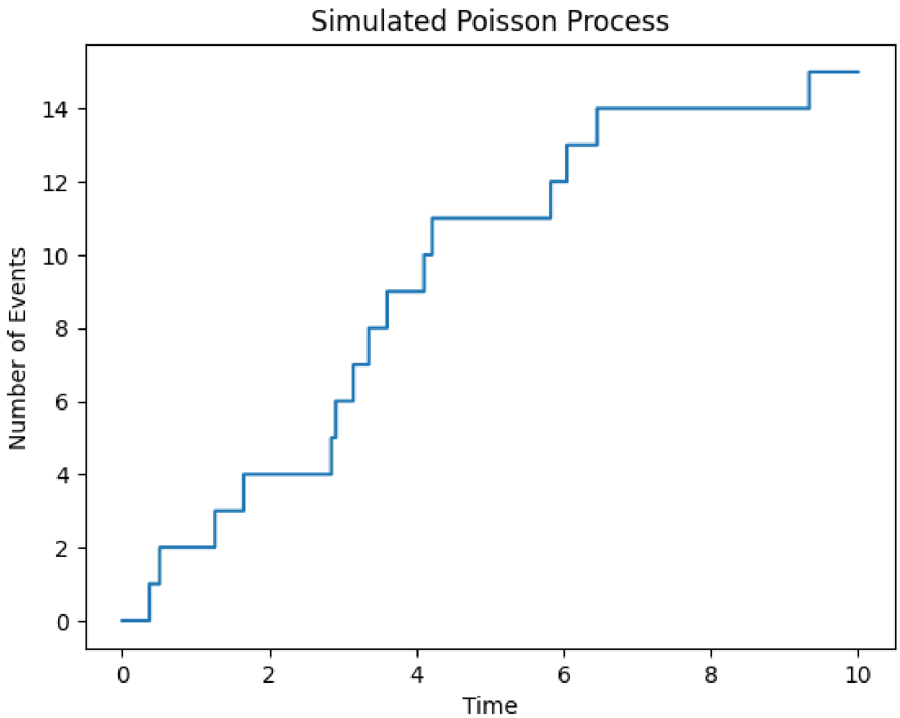

Figure 1: Poisson Process