Law of Large Number

Chiara Iannicelli

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1 Introduction

The law of large numbers is one of the most important theorems in probability theory. It states that, as a probabilistic process is repeated a large number of times, the relative frequencies of its possible outcomes will get closer and closer to their respective probabilities.

For example, flipping a regular coin many times results in approximately 50% heads and 50% tails frequency, since the probabilities of those outcomes are both 0.5.

The law of large numbers demonstrates and proves the fundamental relationship between the concepts of probability and frequency. In a way, it provides the bridge between probability theory and the real world.

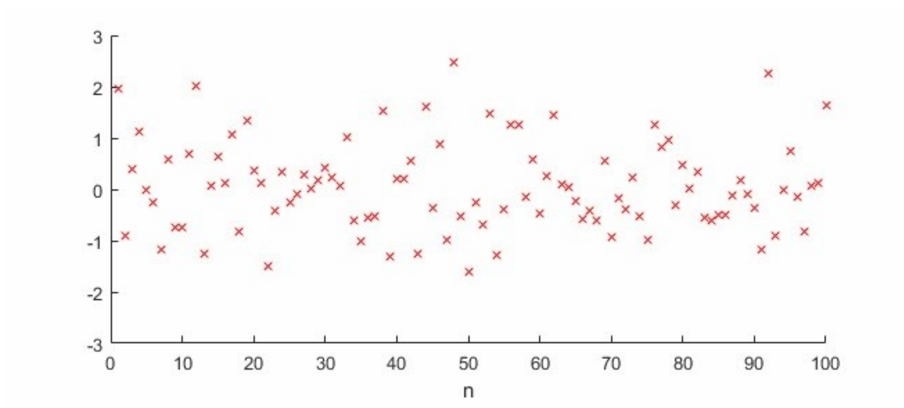

Figure 1: Sequence of random variables

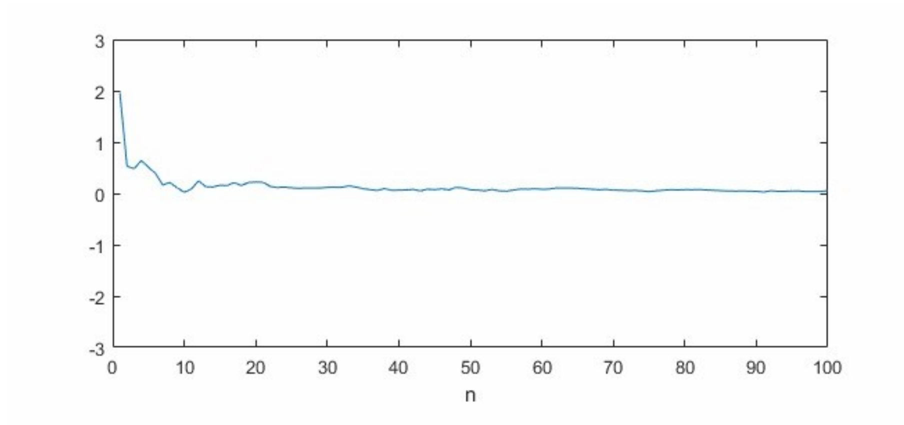

Figure 2: Sample mean of the first n element of the sequence

2 A formal statement

The law of large numbers was something mathematicians were aware of even around the 16th century. But it was first formally proved in the beginning of the 18th century, with significant refinements by other mathematicians throughout the following 2-3 centuries.

In words, this formulation says that when the same random process is repeated a large number of times, the relative frequency of the possible outcomes will be approximately equal to their respective probabilities.

Here's the same statement, formulated as a mathematical limit:

$$\frac{N_n(outcome)}{n} \longrightarrow P(outcome) \quad as \quad n \longrightarrow \infty$$

Here n is the number of times the random process was repeated.
$N_n(outcome)$ is the number of times a particular outcome has occurred after n repetitions.
$P(outcome)$ is the probability of the outcome.
The arrows are standard notation when writing limits and should be read as *approaches*.
The symbol represent (positive) infinity.

For any given n, $N_n(outcome)/n$ is equal to the relative frequency of that outcome after n repetitions.

3 Weak Law of Large Numbers

There are two forms of the law of large numbers, but the differences are primarily theoretical. The weak and strong laws of large numbers both apply to a sequence of values for independent and identically distributed (i.i.d.) random variables: $X1, X2, \ldots, Xn$.

The weak law of large numbers states that as n increases, the sample statistic of the sequence converges in probability to the population value. Statisticians also refer to this form of the law as Khinchin's law.

Here's what that means. Suppose you specify a nonzero difference between the theoretical value and the sample value. For example, you might define a difference between the theoretical probability for coin toss results (0.50) and the actual proportion you obtain over multiple trials. As the number of trials increases, the probability that the actual difference will be smaller than this predefined difference also increases. This probability converges on 1 as the sample size approaches infinity.

4 Strong Law of Large Numbers

The strong law of large numbers describes how a sample statistic converges on the population value as the sample size or the number of trials increases. For example, the sample mean will converge on the population mean as the sample size increases. The strong law of large numbers is also known as Kolmogorov's strong law.

Both laws apply to various characteristics, ranging from the means for continuous variables to the proportions for Bernoulli trials.