Law of Large Number

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1 Introduction

The law of large numbers is one of the most important theorems in probability theory. It states that, as a probabilistic process is repeated a large number of times, the relative frequencies of its possible outcomes will get closer and closer to their respective probabilities.

For example, flipping a regular coin many times results in approximately 50% heads and 50% tails frequency, since the probabilities of those outcomes are both 0.5.

The law of large numbers demonstrates and proves the fundamental relationship between the concepts of probability and frequency. In a way, it provides the bridge between probability theory and the real world.

2 Understanding the Law of Large Numbers

The law of large numbers can refer to two different topics. First, in statistical analysis, the law of large numbers can be applied to a variety of subjects. It may not be feasible to poll every individual within a given population to collect the required amount of data, but every additional data point gathered has the potential to increase the likelihood that the outcome is a true measure of the mean.

The law of large numbers does not mean that a given sample or group of successive samples will always reflect the true population characteristics, especially for small samples. This also means that if a given sample or series of samples deviates from the true population average, the law of large numbers does not guarantee that successive samples will move the observed average toward the population mean (as suggested by the Gambler's Fallacy).

Second, the term "law of large numbers" is sometimes used in business in relation to growth rates, stated as a percentage. It suggests that, as a business expands, the percentage rate of growth becomes increasingly difficult to maintain. This is because the underlying dollar amount is actually increasing even if the growth rate as a percentage is to remain constant.

3 A formal statement

The law of large numbers was something mathematicians were aware of even around the 16th century. But it was first formally proved in the beginning of the 18th century, with significant refinements by other mathematicians throughout the following 2-3 centuries.

In words, this formulation says that when the same random process is repeated a large number of times, the relative frequency of the possible outcomes will be approximately equal to their respective probabilities.

Here's the same statement, formulated as a mathematical limit:

$$\frac{N_n(outcome)}{n} \longrightarrow P(outcome) \quad as \quad n \longrightarrow \infty$$

Where:

- n is the number of times the random process was repeated.
- $N_n(outcome)$ is the number of times a particular outcome has occurred after n repetitions.
- $P(outcome)$ is the probability of the outcome.
- The arrows are standard notation when writing limits and should be read as approaches.
- The symbol ∞ represent (positive) infinity.

4 Example

As we have seen, the law of large numbers states that as a sample size becomes larger, the sample mean gets closer to the expected value.

The most basic example of this involves flipping a coin. Each time we flip a coin, the probability that it lands on heads is 1/2. Thus, the expected proportion of heads that will appear over an infinite number of flips is 1/2 or 0.5.

However, if we flip a coin 10 times we might find that it only lands on heads 3 times. Since 10 flips is a small sample size, there's no guarantee that the proportion of heads will be close to 0.5.

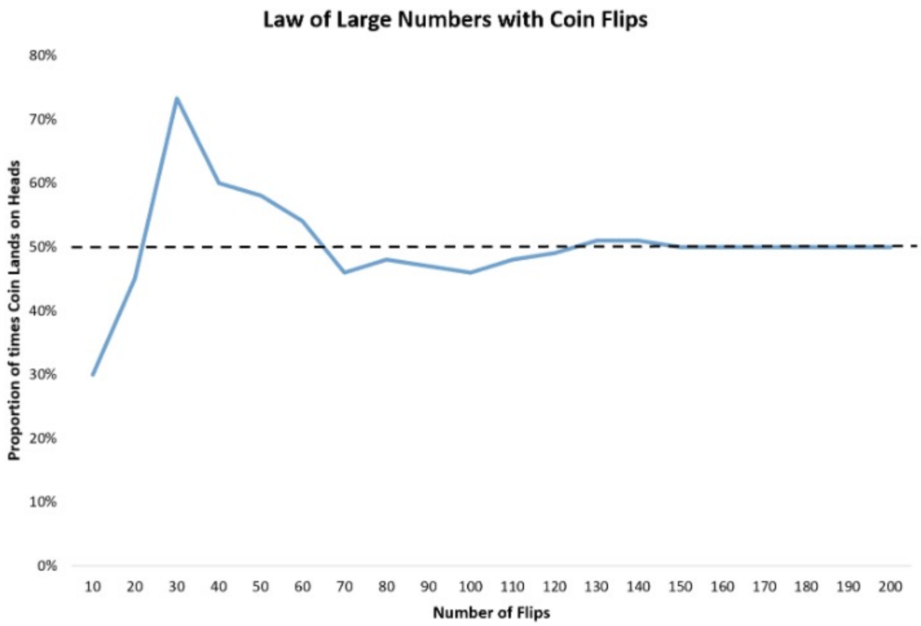

Figure 1: LLN example

If we continue flipping the coin another 10 times, we might find that it lands on heads a total of 9 times out of 20. If we flip it 10 more times, we might find that it lands on heads 22 times out of 30.

As we flip the coin more and more, the proportion of times that it lands on heads will converge to the expected proportion of 0.5.

A possible code to simulate the flipping of a coin is the following:

```
import random
random.seed(7)

def coin_flip():
    # This function simulates a coin flip: "H" for heads and "T" for tails.
    return "H" if random.random() < 0.5 else "T"

def simulate_flips(n):
```

```
    # This function simulates 'n' coin flips and returns the proportion of heads.
    heads = sum(1 for _ in range(n) if coin_flip() == "H")
    return heads / n

# Simulate coin flips for different numbers of flips:
for num_flips in [10, 100, 1000, 10000, 100000]:
    proportion_heads = simulate_flips(num_flips)
    print(f"After {num_flips} flips, the proportion of heads is: {proportion_heads:.4f}")

```

5 Weak Law of Large Numbers

There are two forms of the law of large numbers, but the differences are primarily theoretical. The weak and strong laws of large numbers both apply to a sequence of values for independent and identically distributed (i.i.d.) random variables: $X1, X2, ..., Xn$.

The weak law of large numbers states that as n increases, the sample statistic of the sequence converges in probability to the population value. Statisticians also refer to this form of the law as Khinchin's law.
Suppose you specify a nonzero difference between the theoretical value and the sample value. For example, you might define a difference between the theoretical probability for coin toss results (0.50) and the actual proportion you obtain over multiple trials. As the number of trials increases, the probability that the actual difference will be smaller than this predefined difference also increases. This probability converges on 1 as the sample size approaches infinity.

6 Strong Law of Large Numbers

The strong law of large numbers describes how a sample statistic converges on the population value as the sample size or the number of trials increases. For example, the sample mean will converge on the population mean as the sample size increases. The strong law of large numbers is also known as Kolmogorov's strong law.

Both laws apply to various characteristics, ranging from the means for continuous variables to the proportions for Bernoulli trials.