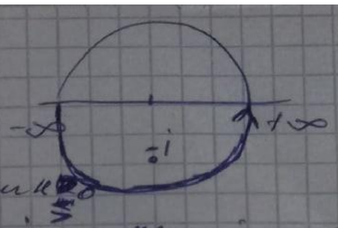
№1. $\int\limits_{-\infty}^{+\infty} \frac{e^{-kix}\,dx}{(x+i)^3}$



$f(x) = \frac{e^{-ikx}}{(x+i)^3}$ $\quad x=-i$ третий порядок

при $k>0$ $\quad$ при ~~$k<0 \Rightarrow -k > 0 \Rightarrow$ жордана уместна~~

$$\oint = \int\limits_{-\infty}^{+\infty} + \int\limits_{CR}^{\to 0} \ominus \;\Big|\; res\, f(x) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}} f(x)(x-x_0)^n\Big|_{x\to x_0} =$$

$$= \frac{1}{2}\frac{d^2}{dx^2}\frac{e^{-kix}}{\cancel{(x+i)^3}}\cancel{(x+i)^3} = \frac{1}{2}(-ki)(-ki)\,e^{-kix} =$$

$$= -\frac{1}{2}k^2 e^{-kix}\Big|_{x\to -i} = -\frac{1}{2}k^2 e^{-k}\;\Big|\;\ominus$$

$$\ominus\; -2\pi i \cdot -\frac{1}{2}k^2 e^{k} = +\pi i k^2 e^{-k}$$

при $k<0$

$$\oint = \int\limits_{-\infty}^{+\infty} + \int\limits_{CR}^{\to 0} = +2\pi i \sum res\, f(x)$$

т.к. ф-ция ~~$f(x)$~~ $f(x)$ аналитична в некоторой замкнутой односвязной обл-сти без особых точек $\Rightarrow res\, f(x) = 0$

$\Rightarrow$ Ответ: $\begin{cases} +\pi i k^2 e^{-k} & k>0 \\ 0 & k<0 \end{cases}$

В №2.

$$U(x) = \frac{1}{x^4+1}$$

$$f(x) = \int_0^{+\infty} [a(\lambda) \cos\lambda x + b(\lambda) \sin\lambda x]\, d\lambda$$

$$a(\lambda) = \frac{1}{\pi} \int^{+\infty} f(\xi) \cos \lambda\xi \, d\xi$$

$$b(\lambda) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\xi) \sin \lambda\xi \, d\xi$$

$\frac{1}{x^4+1}$ — четная функция $\Rightarrow$

$$a = \frac{2}{\pi} \int_0^{+\infty} f(\xi)\, \lambda\xi \, d\xi$$

$$b(\lambda) = 0$$

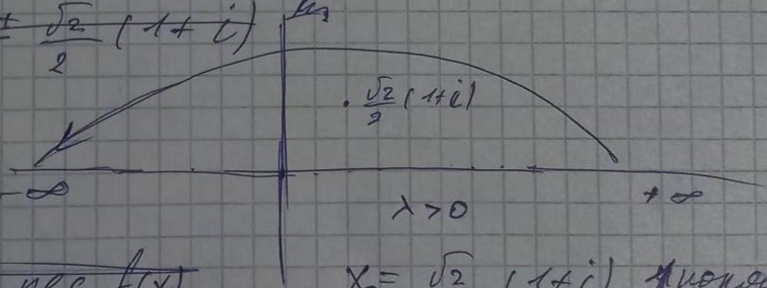
$$f(x) = \frac{2}{\pi} \int_0^{+\infty} \cos \lambda x \, d\lambda \int_0^{+\infty} f(\xi) \cos \lambda\xi \, d\xi$$

$$a(\lambda) = \int_0^{+\infty} \frac{1}{x^4+1} \cos(\lambda x)\, dx = \int_0 \frac{1}{x^4+1} \left(\frac{e^{i\lambda x} + e^{-i\lambda x}}{2}\right) dx =$$

$$= \frac{1}{2} \int_0^{+\infty} \frac{e^{i\lambda x}}{x^4+1} dx + \frac{1}{2} \int_0^{+\infty} \frac{e^{-i\lambda x}}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{x^4+1} dx$$

$$\cancel{x = \pm\sqrt{i} = \pm \frac{\sqrt{2}}{2}(1+i)}$$

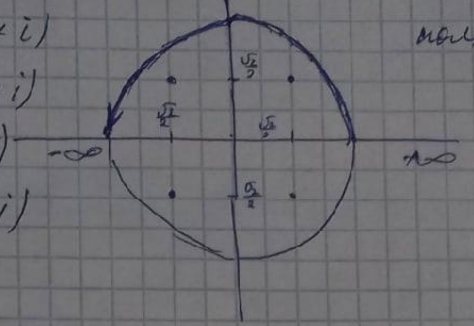
$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{x^4+1} dx$$



$$\cancel{\oint = \int_{-\infty}^{+\infty} + \int_{CR} = \sum res\, f(x)}$$

$x_0 = \frac{\sqrt{2}}{2}(1+i)$ 1 порядок

$$\left[\begin{array}{l} x = +\sqrt{i} = \frac{\sqrt2}{2}(1+i)\\ x = -\sqrt{i} = -\frac{\sqrt2}{2}(1+i)\\ x=+i\sqrt{i} = \frac{\sqrt2}{2}(i-1)\\ x = -i\sqrt{i} = \frac{\sqrt2}{2}(1-i)\end{array}\right.$$



полюса 1 порядка

$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R} = 2\pi i \sum \operatorname{res} f(x)$$

res $x_0 = +\sqrt{i}$ $\quad \frac{1}{1}\,\dfrac{e^{i\lambda x}\cancel{(x-\sqrt{i})}}{(x^2+i)\cancel{(x-\sqrt{i})}(x+\sqrt{i})}\Bigg|_{x\to x_0(\sqrt{i})}$

res $x_0 = +i\sqrt{i}$ $\quad \frac{1}{1}\,\dfrac{e^{i\lambda x}\cancel{(x-\sqrt{-i})}}{\cancel{(x-\sqrt{-i})}(x+\sqrt{-i})\underbrace{(x+\sqrt{i})(x-\sqrt{i})}_{(x^2-i)}}\Bigg|_{x\to i\sqrt{i}}$

$$\sum \operatorname{res} = \frac{e^{\frac{\sqrt2}{2}(1+i)i\lambda}}{(i+i)(\sqrt2(1+i))} + \frac{e^{\frac{\sqrt2}{2}(i-1)i\lambda}}{\sqrt2(i-1)(-i-i)} =$$

$$= \frac{e^{\frac{\sqrt2}{2}(i-1)\lambda}}{2\sqrt2 i(1+i)} + \frac{e^{\frac{\sqrt2}{2}(-1-i)\lambda}}{-2\sqrt2 i(i-1)} =$$

$$= \frac{e^{\frac{\sqrt2}{2}(i-1)\lambda}}{2\sqrt2(i-1)} + \frac{e^{-\frac{\sqrt2}{2}(-1+i)\lambda}}{2\sqrt2(1+i)} \quad (\text{для } \lambda>0)$$

для $x = -\sqrt{i}$, $x = i\sqrt{i}$ будет аналогично, но другие знаки

$$\oint = -2\pi i \sum \operatorname{res} f(x) =$$

$$\frac{e^{\frac{\sqrt2}{2}(1-i)\lambda}}{2\sqrt2(1-i)} + \frac{e^{\frac{\sqrt2}{2}(-1+i)}}{-2\sqrt2(-1+i)} \quad (\text{для } \lambda<0)$$

$$\frac{e^{\frac{\sqrt{2}}{2}\lambda(i-1)}}{2\sqrt{2}(i-1)}\cdot(i+1) + \frac{e^{-\frac{\sqrt{2}}{2}(i+1)\lambda}}{2\sqrt{2}(1+i)}(1-i) = \left| \begin{array}{l} \frac{\sqrt{2}}{2}\lambda = a \\ 2\sqrt{2} = b \end{array} \right.$$

$$= \frac{ie^{ai}\cdot e^{-a} + e^{ai}e^{-a} + e^{-ai}\cdot e^{-a} - ie^{-ai}\cdot e^{-a}}{b\cdot 2} =$$

$$= \frac{e^{-a}\left(ie^{ai} - ie^{-ai} + e^{ai} + e^{-ai}\right)}{2b} =$$

$$= \frac{e^{-a}}{b}\left[\underbrace{\frac{ii\left(e^{ai} - e^{-ai}\right)}{2i}}_{\sin a} + \frac{\cancel{4}\, e^{ai} + e^{-ai}}{2\cancel{4}}\right] =$$

$$= \frac{e^{-a}}{b}\left[-\sin a + \cos a\right] = \frac{e^{-\frac{\sqrt{2}}{2}\lambda}}{2\sqrt{2}}\left[\cos\frac{\sqrt{2}}{2}\lambda - \sin\frac{\sqrt{2}}{2}\lambda\right]$$

при $\lambda > 0$

$$\frac{e^{\frac{\sqrt{2}}{2}\lambda(1-i)}}{2\sqrt{2}(1-i)} + \frac{e^{\frac{\sqrt{2}}{2}\lambda(1+i)}}{-2\sqrt{2}(1+i)} = \left| \begin{matrix} a=\frac{\sqrt{2}}{2}\lambda \\ b=2\sqrt{2} \end{matrix} \right| =$$

$$= \frac{e^{a(1-i)}(1+i) - e^{a(1+i)}(1-i)}{2\cdot b} =$$

$$= \frac{e^{a}\cdot e^{-ia} + i e^{a}\cdot e^{-ia} - e^{a}e^{ai} + i e^{a}e^{ai}}{2b} =$$

$$= \frac{e^{a}}{b}\,\frac{\left(i\left(e^{-ia}+e^{ai}\right) + e^{-ia} - e^{ia}\right)}{2} =$$

$$= \frac{e^{a}}{b}\left[\underbrace{\frac{i\left(e^{-ia}+e^{ia}\right)}{2}}_{\cos a} - i\underbrace{\frac{\left(-e^{-ia}+e^{ia}\right)}{2i}}_{\sin a}\right] =$$

$$= \frac{e^{a}}{b}\, i\left(\cos a - \sin a\right) =$$

$$= \frac{e^{\frac{\sqrt{2}}{2}\lambda}}{2\sqrt{2}}\, i\left(\cos\frac{\sqrt{2}}{2}\lambda - \sin\frac{\sqrt{2}}{2}\lambda\right) \quad \text{для } \lambda<0$$

т.е.

$$f(x) = \begin{cases} \frac{2}{\pi}\int_0^{+\infty} \cos\lambda x\, d\lambda \cdot \frac{e^{-\frac{\sqrt{2}}{2}\lambda}}{\sqrt{2}\cdot 2}\left[\cos\frac{\sqrt{2}}{2}\lambda - \sin\frac{\sqrt{2}}{2}\lambda\right] & \text{для } \lambda>0 \\ \frac{2}{\pi}\int_0^{+\infty} \cos\lambda x\, d\lambda\, \frac{e^{\frac{\sqrt{2}}{2}\lambda}}{2\sqrt{2}}\, i\left(\cos\frac{\sqrt{2}}{2}\lambda - \sin\frac{\sqrt{2}}{2}\lambda\right) & \lambda<0 \end{cases}$$

т.к. пределы от 0 до $+\infty$ то для $\lambda<0$ — не рассматривается

Ответ:

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \cos\lambda x\, d\lambda\, \frac{e^{-\frac{\sqrt{2}}{2}\lambda}}{2\sqrt{2}}\left(\cos\frac{\sqrt{2}}{2}\lambda - \sin\frac{\sqrt{2}}{2}\lambda\right)$$

№ 8.

$$\int_0^{+\infty} e^{-pt}\, t^{z-1}\, dt$$

$$\int_0^{+\infty} e^{-pt}\, dt = -\frac{1}{p} e^{-pt}\Big|_0^{+\infty} = -\frac{1}{p}(0-1) = \frac{1}{p}$$

пусть $z-1=n$

$$\int_0^{+\infty} e^{-pt}\, t^{n}\, dt = \Big| \text{для } n=2 \quad \int e^{-pt}\, t^{2}\, dt =$$

$$= \Big| \cancel{\begin{matrix} u = e^{-pt} & du = -p\, e^{-pt}\, dt \\ dv = t^2\, dt & v = \end{matrix}}$$

$$= \Big| \begin{matrix} u = t^2 & du = 2t\, dt \\ dv = e^{-pt}\, dt & v = \dfrac{e^{-pt}}{-p} \end{matrix} \Big| = \underbrace{\frac{t^2 e^{-pt}}{-p}\Big|_0^{+\infty}}_{=0} + \underbrace{\int \frac{2t}{+p}\, e^{-pt}\, dt}_{I}$$

$$I = \int \frac{2t\, e^{-pt}}{p}\, dt = \frac{2}{p} \Big| \begin{matrix} u = t & du = dt \\ dv = e^{-pt}\, dt & v = \dfrac{e^{-pt}}{-p} \end{matrix} \Big| =$$

$$= \frac{2}{p}\left(\underbrace{\frac{t\, e^{-pt}}{-p}\Big|_0^{+\infty}}_{=0} + \underbrace{\int_0^{+\infty} \frac{e^{-pt}}{+p}\, dt}_{=\frac{1}{p^2}} \right) = \frac{2}{p^3}$$

для $n = 3$

$$\int_0^{+\infty} e^{-pt} t^3\, dt = \left| \begin{matrix} u = t^3 & du = 3t^2 dt \\ dv = e^{-pt} dt & v = \frac{e^{-pt}}{-p} \end{matrix} \right| =$$

$$= \underbrace{t^3 \frac{e^{-pt}}{-p} \Bigg|_0^{+\infty}}_{=0} + \int \frac{3t^2 e^{-pt}}{+p}\, dt =$$

$$= \left| \begin{matrix} t^2 = u & du = 2t\, dt \\ dv = e^{-pt} dt & v = \frac{e^{-pt}}{-p} \end{matrix} \right| = \frac{3}{p}\Bigg(\underbrace{\frac{t^2 e^{-pt}}{-p} \Bigg|_0^{+\infty}}_{=0} + \int \frac{2t\, e^{-pt}}{+p}\, dt =$$

$$= \left| \begin{matrix} u = t & du = dt \\ dv = e^{-pt} dt & v = \frac{e^{-pt}}{-p} \end{matrix} \right| = \frac{3}{p} \cdot \frac{2}{p} \Bigg(\underbrace{\frac{t\, e^{-pt}}{-p} \Bigg|_0^{+\infty}}_{=0} + \underbrace{\int \frac{e^{-pt}}{+p}\, dt}_{=\frac{1}{p^2}} =$$

$$= \frac{3}{p} \cdot \frac{2}{p} \cdot \frac{1}{p^2} = \frac{6}{p^4}$$

для $n = 0$ $\quad I = \frac{1}{p}$

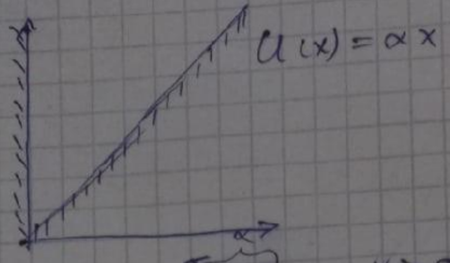
для $n = 2$ $\quad I = \frac{2}{p^3}$

для $n = 3$ $\quad I = \frac{6}{p^4}$ $\quad$ т.е. $\quad I = \frac{n!}{p^{n+1}}$ $\quad n = z - 1$

тогда $\displaystyle\int_0^{+\infty} e^{-pt} t^{z-1}\, dt = \frac{(z-1)!}{p^z}$

Ответ: $\dfrac{(z-1)!}{p^z}$

№3 $U(x) = \alpha x$ $\qquad$ $E(p) = c|p|$



$$U(x) = \begin{cases} mg\,x, & x > 0 \\ \infty, & x = 0 \end{cases}$$

$$-\frac{\hbar^2}{2m}\psi'' + \underbrace{U(x)}_{\alpha x}\psi(x) = E\cdot\psi(x)$$

$$\psi(0) = 0$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$H = \frac{\hat{p}^2}{2m} + \alpha\hat{x}$$

$$\frac{p^2}{2m}a(p) + \alpha i\hbar\frac{d}{dp}a(p) = c|p|\cdot a(p)$$

$$\psi(x) = \frac{1}{\sqrt{L}}\int e^{ipx/\hbar}a(p)\frac{dp\,L}{2\pi\hbar}$$

$$\psi(0) = \frac{1}{\sqrt{L}}\int a(p)\,dp\frac{L}{2\pi\hbar} = 0$$

$$\int_{-\infty}^{+\infty} a(p)\,dp = 0$$

$$\frac{da}{dp} = \left(E - \frac{p^2}{2m}\right)\frac{a(p)}{\alpha i\hbar}$$

$$\int_{-\infty}^{+\infty}\frac{da}{dp} = \int -\frac{i}{\alpha\hbar}\left(E - \frac{p^2}{2m}\right)dp$$

$$E = \cancel{c|p|} \quad \frac{ck^2\sqrt{p^2}}{2m}$$

$$\int \frac{-i}{\alpha\hbar}\left(\frac{c\sqrt{p^2}\,k^2}{2m} - \frac{p^2}{2m}\right) dp = \Big|$$

$$= \left| \begin{array}{l} p^2 = t \qquad p = \sqrt{t} \\ dp = \dfrac{dt}{2p} = \dfrac{dt}{2\sqrt{t}} \end{array} \right| = \int \frac{-i}{\alpha\hbar}\left(\frac{c\sqrt{t}\,k^2}{2m} - \frac{t}{2m}\right)\frac{dt}{2\sqrt{t}} =$$

$$= \int \frac{-i}{2\alpha\hbar}\left(\frac{c\,\cancel{\sqrt{t}}\,k^2}{2m} - \frac{\sqrt{t}}{2m}\right) dt \qquad \frac{2}{3}t^{\frac{3}{2}}$$

$$\cancel{\int} \frac{-i}{2\alpha\hbar}\left(t\,\frac{ck^2}{2m} - \frac{\cancel{2}}{3}\,\frac{t^{\frac{3}{2}}}{\cancel{2}m}\right) \cancel{dt} \qquad t^{\frac{3}{2}} = \frac{3}{2}t^{\frac{1}{2}}$$

$$\frac{-i}{2\alpha\hbar}\left(\frac{p^2ck^2}{2m} - \frac{p^3}{3m}\right)$$

$$a(p) = C\,e^{\frac{-i}{2\alpha\hbar}\left(\frac{p^2ck^2}{2m} - \frac{p^3}{3m}\right)}$$

$$\int_{-\infty}^{+\infty} a(p)\,dp = 0 = C\int e^{-\frac{i}{2\alpha\hbar}\left(\frac{3p^2ck^2}{6m} - \frac{2p^3}{6m}\right)} dp = 0$$

$$\cancel{3p^2ck^2 - 2p^3}$$

$3k^4t^2ck^2 - 2k^6t^3 \quad (p = k^2 t)$

$$\int e^{-\frac{k^6}{2\alpha\hbar m} i \left(\frac{\cancel{3}t^2 c}{2} - \frac{\cancel{2}t^3}{3} \right)}$$

$$S(t) = \frac{t^3}{3} - \frac{t^2 c}{2}$$

$$S'(t) = t^2 - tc$$

$$t(t-c)$$

$$\left[\begin{array}{l} t = 0 \\ t = c \end{array}\right.$$

$$S''(t) = 2t - c$$

$$S''(0) = -c \qquad < 0$$

$$S''(c) = c \qquad > 0$$

$$S(0) = 0$$

$$S(c) = \frac{c^3}{3} - \frac{c^2 \cdot c}{2} = -\frac{c^3}{6}$$

$$\sqrt{\frac{2\pi}{\lambda c}} \left\{ e^{-\frac{c^3}{6} i\lambda + \frac{i\pi}{4}} \Big|_{t=c} - e^{0 i\lambda - \frac{i\pi}{4}} \Big|_{t=0} \right\} = 0$$

когда $\frac{-c^3}{6} i\lambda + i\frac{\pi}{4} = -\frac{\pi i}{4}$

$$\frac{c^3}{6} \cancel{i}\lambda = \frac{\cancel{i}\pi}{2} \qquad \frac{k^6}{2m\alpha\hbar} = \frac{3\pi}{c^3}$$

$$\lambda = \frac{3\pi}{c^3} \qquad k = \sqrt[6]{\frac{3\pi}{c^3} 2m\alpha\hbar}$$