ДЗ №2 Черков И. БФЗ-19-1

№1. $A = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$

1.1 $(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -i\sin\theta \\ i\sin\theta & -(\cos\theta + \lambda) \end{vmatrix} = 0$

$-(\cos - \lambda)(\cos + \lambda) + i\sin\theta \, i\sin\theta = 0$

$-\cos^2\theta + \lambda^2 - \sin^2\theta = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

при $\lambda = 1$ $\begin{cases} \cos\theta x - y i\sin\theta - \lambda x = 0 \\ x i\sin\theta - y\cos\theta - \lambda y = 0 \end{cases}$

$\left[\begin{array}{l} \begin{cases} x = i\sin\theta \\ y = \cos\theta - 1 \end{cases} \\ \begin{cases} x = \cos\theta + 1 \\ y = i\sin\theta \end{cases} \end{array}\right.$ варианты совпадают, возьмем первый

$|x|^2 + |y|^2$

$|\psi_1| = \sin^2\theta + \cos^2\theta + 2\cos\theta + 1 = 2 - 2\cos\theta$

$\vec{\psi_1} = \begin{pmatrix} \frac{i\sin\theta}{\sqrt{2 - 2\cos\theta}} \\ \frac{\cos\theta - 1}{\sqrt{2 - 2\cos\theta}} \end{pmatrix}$ ψ_2

$\lambda = -1$ $\begin{cases} \cos x - y i\sin\theta + \lambda x = 0 \\ x i\sin\theta - y\cos\theta + \lambda y = 0 \end{cases}$

$\begin{cases} x = i\sin\theta \\ y = \cos\theta - 1 \end{cases}$ или $\begin{cases} x = i\cancel{\sin\theta}\cos\theta + 1 \\ y = i\sin\theta \end{cases}$

$\sqrt{|x|^2 + |y|^2} = 2 + 2\cos\theta$

$\vec{\psi_2} = \begin{pmatrix} \frac{i\sin\theta}{\sqrt{2 + 2\cos\theta}} \\ \frac{\cos\theta + 1}{\sqrt{2 - 2\cos\theta}} \end{pmatrix}$

проверить ортогональность можем не учитывая длину векторов. т. е.

$$\langle \psi_1^* | \psi_2 \rangle = -i\sin\theta \, i\sin\theta + (\cos\theta - 1)(\cos\theta + 1) =$$
$$= \sin^2\theta + \cos^2\theta - 1 = 0 \Rightarrow$$ ~~их~~ ортогональны

1.2 $$i\alpha A = i\alpha \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$$

$$(i\alpha A)^2 = \frac{-1\alpha^2}{1!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(i\alpha A)^2 = -\frac{1 i \alpha^3}{2!} \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$$

$$(i\alpha A)^n = \begin{cases} \frac{(-1)^{\frac{n}{2}}\alpha^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{если } n - \text{четное} \\ \frac{(i\alpha)^n}{n!} \hat{A} & \text{если } n - \text{нечет} \end{cases}$$

$$e^{i\alpha\hat{A}} = \hat{E}\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + \hat{A} i \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^{2n-1}}{(2n-1)!} =$$

$$= \hat{E}\cos\alpha + \hat{A} i \sin\alpha =$$

$$= \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha \end{pmatrix} + \begin{pmatrix} i\sin\alpha\cos\theta & \sin\theta\sin\alpha \\ -\sin\theta\sin\alpha & -i\cos\theta\sin\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha + i\sin\alpha\cos\theta & \sin\theta\sin\alpha \\ -\sin\theta\sin\alpha & \cos\alpha - i\cos\theta\sin\alpha \end{pmatrix}$$

Ответ: ↑

№2. $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

2.1

$$|A - E\lambda| = \begin{vmatrix} -\lambda & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -\lambda & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{vmatrix} =$$

$$= -\lambda\left(\lambda^2 - \frac{1}{2}\right) + \frac{i}{\sqrt{2}}\left(-\frac{i\lambda}{\sqrt{2}}\right) = -\lambda^3 + \lambda = \lambda(1-\lambda^2) \Rightarrow$$

$$\Rightarrow \lambda = \left[\begin{array}{l} +1 \\ -1 \\ 0 \end{array}\right.$$

1) $\lambda = \pm 1$
$$\begin{cases} -i/\sqrt{2}\cdot\psi_2 = \pm\psi_1 \\ i\sqrt{2}\cdot\psi_1 - i/\sqrt{2}\,\psi_3 = \pm\psi_2 \\ i/\sqrt{2}\,\psi_2 = \pm\psi_3 \end{cases}$$

$$\begin{cases} -i\,\psi_2 = \pm\sqrt{2}\,\psi_1 \\ i\,\psi_2 = \pm\sqrt{2}\,\psi_3 \end{cases} \Rightarrow \begin{cases} \psi_1 = \mp i \\ \psi_2 = \sqrt{2} \\ \psi_3 = \pm i \end{cases}$$

$|u_1\rangle = \frac{1}{a}\begin{pmatrix} -i \\ \sqrt{2} \\ +i \end{pmatrix}$ $\quad |u_2\rangle = \frac{1}{a}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$

$\lambda = 0$ $\quad -i/\sqrt{2}\,\psi_2 = 0$

$i/\sqrt{2}\,\psi_1 - i\sqrt{2}\,\psi_3 = 0$ $\quad |u_3\rangle = b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$a = \sqrt{1+2+1} = 2$ $\quad b = \sqrt{2}$

$|u_1\rangle = \frac{1}{2}\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix}$ $\quad |u_2\rangle = \frac{1}{2}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$ $\quad |u_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\langle u_1|u_2\rangle = i\,i + \sqrt{2}\cdot\sqrt{2} - i(-i) = -1+2-1 = 0$

$\langle u_2|u_3\rangle = -i + i = 0$ $\quad \langle u_3|u_1\rangle = -i + i = 0$

2.2

$$A^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \hat{B}$$

$$A^3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\cancel{2}i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$A^4 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \hat{B}$$

$$e^{i\theta A} = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n} B}{(2n)!} + \sum_{n=1}^{\infty} \frac{(i\theta)^{2n-1} A}{(2n-1)!} =$$

$$= B\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + Ai\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\theta^{(2n-1)}}{(2n-1)!} =$$

$$= B\cos\theta + Ai\sin\theta =$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{2}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} i\sin\theta =$$

$$= \begin{pmatrix} \frac{1}{2}\cos\theta & \frac{\sin\theta}{\sqrt{2}} & -\frac{\cos\theta}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{2} \end{pmatrix}$$

2.3 $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x|u_1\rangle + y|u_2\rangle + z|u_3\rangle$

$$\begin{cases} -\frac{xi}{2} + \frac{iy}{2} + \frac{z}{\sqrt{2}} = 1 \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0 \qquad x = -y \\ \frac{ix}{2} - \frac{yi}{2} + \frac{z}{\sqrt{2}} = 0 \end{cases}$$

$$\begin{cases} \frac{iy}{2} + \frac{iy}{2} + \frac{z}{\sqrt{2}} = 1 \\ \frac{-yi}{2} - \frac{yi}{2} + \frac{z}{\sqrt{2}} = 0 \end{cases} +$$

$$\frac{\sqrt{2}z}{\sqrt{2}} = 1$$

$$\begin{cases} z = \frac{1}{\sqrt{2}} \\ x = \frac{i}{2} \\ y = \frac{-i}{2} \end{cases}$$

$y = \frac{-i}{2}$

Ответ:

$|\psi\rangle = \frac{i}{2}|u_1\rangle - \frac{i}{2}|u_2\rangle + \frac{1}{\sqrt{2}}|u_3\rangle$

№3. $\hat{T}_a \psi(x) = \psi(x+a)$

1. $\langle \varphi(x) | \hat{T}_a \psi(x) = \int_a^b \varphi^*(x) T_a \psi(x) dx =$

$= \int_a^b \varphi^*(x) \psi(x+a) dx = \begin{vmatrix} x+a=y \\ y-a=x \\ dx=dy \end{vmatrix} = \int^{d_y} \varphi^*(y-a) \psi(y) dy$

$= \int_{c_y}^{d_y} \underbrace{T_{-a}}_{= \hat{T}_a^+ \text{ (пункт 2)}} \varphi^*(y) \psi(y) dy \neq \int_{c_y}^{d_y} \hat{T}_a \varphi'(y)$

$\neq \langle \hat{T}_a \varphi(y) | \psi(y) \rangle$ $\hat{T}_a$ – не эрмитов

3.2. $\hat{T}_{-a} = \hat{T}_a^+$

3.3. $\hat{T}_a^+ \hat{T}_a$

$\hat{T}_a \psi(x) = \psi(x+a)$

$\hat{T}_{-a} \psi(x+a) = \psi(x) = \hat{T}_{-a} \hat{T}_{+a} \psi(x) \Rightarrow \hat{T}_{-a} \hat{T}_{+a} = 1$

3.4 $\hat{T} \psi(x) = \int \hat{T}(x, x') \psi(x') dx' = \psi(x+a)$

из свойства $\delta(x)$

$\int \varphi(x') \delta(x' - x) dx' = \varphi(x)$ при подстановке $\varphi(x')$ получим

т.е. $\int \varphi(x') \delta(x' - (x+a)) dx' = \int \hat{T}(x, x') \psi(x') dx'$

откуда $T(x, x') = \delta(x' - (x+a))$

№4 $\hat{S} = \alpha\left(x^2 \frac{d}{dx} - \frac{d}{dx} x^2\right)$

4.1 для любого $\psi(x)$ $\hat{S}\psi(x) = \alpha\left(x^2 \frac{d(\psi(x))}{dx} - x^2 \frac{d\psi(x)}{dx} - 2x\psi(x)\right) = -2\alpha x \psi(x)$ т.е. $\hat{S} = -2\alpha x$

$\langle \psi^* | \hat{S} | \psi \rangle = \int \psi^*(x)(-2\alpha x)\psi(x)\,dx =$
$= \int -2\alpha x \psi^*(x)\psi(x)\,dx = \int (\hat{S}^\dagger \psi)^* \psi\,dx \Rightarrow$
$\Rightarrow S^\dagger = -2\alpha^* x$

4.2 $S^\dagger = S \Leftrightarrow \alpha^* = \alpha$ Ответ: да

4.3
$\hat{S}\psi(x) = -2\alpha x\psi(x) = \int S(x, x')\psi(x')\,dx'$

аналогично с номером 3 распишем св-во
$\delta(x)$: $\int \psi(x')\delta(x'-x)\,dx' = \psi(x)$

домножим на $(-2\alpha x)$ и внесем его под интеграл т.к. $-2\alpha x \neq x'$

$-2\alpha x\psi(x) = \int -2\alpha x\,\delta(x'-x)\psi(x')\,dx' =$
$= \int S(x, x')\psi(x')\,dx'$

откуда $S(x, x') = -2\alpha x\,\delta(x'-x)$

Ответ: ↑

№5. $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$B^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$B \cdot B^{-1} = E$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ab}{ad-bc} \\ \frac{dc-dc}{ad-bc} & \frac{-bc+da}{ad-bc} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ч.т.д.

$O(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

5.2 $O^{+}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$O^{-1}(\theta) = \frac{1}{\cos^2\theta - (-\sin\theta)\sin\theta} \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =$

$= \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = O^{+}$

№6. ~~пусть $(\hat{A}\hat{B}) =$~~

1) $\langle \varphi | \hat{A}\hat{B}\psi \rangle = |$ пусть $\hat{B}\psi = f | =$

$= \langle \varphi | \hat{A} f \rangle = \langle \hat{A}^{+}\varphi | f \rangle = | \; A^{+}\varphi = g \quad f = \hat{B}\psi$

$= \langle g | \hat{B}\psi \rangle = \langle B^{+} g | \psi \rangle = | \; g = A^{+}\varphi \; | =$

$= \langle B^{+} A^{+} \varphi | \psi \rangle$

2) пусть $\hat{A}\hat{B} = \hat{C}$ $\quad \langle \varphi | \hat{C}\psi \rangle = \langle \hat{C}^{+}\varphi | \psi \rangle =$

$= \langle (\hat{A}\hat{B})^{+} \varphi | \psi \rangle = \langle \varphi | \hat{A}\hat{B}\psi \rangle = \langle A^{+} B^{+} \varphi | \psi \rangle$

т.е. $(AB)^{+} = A^{+}B^{+}$ ч.т.д.

№7. $\hat{C}^{+}\hat{C}$ $\quad \langle \psi | \hat{C}^{+}\hat{C}\psi \rangle =$

$= \left| \hat{C}\psi = \varphi \right| = \langle \psi | \hat{C}^{+}\varphi \rangle = \langle (\hat{C}^{+})^{+}\psi | \varphi \rangle =$

$= \langle \hat{C}\psi | \varphi \rangle = \langle \varphi | \varphi \rangle = \left[\begin{array}{l} 1 \text{ при } |\varphi\rangle \neq |0\rangle \\ 0 \text{ при } |\varphi\rangle = |0\rangle \end{array} \right. \Bigg| \Rightarrow$

$\Rightarrow \langle \varphi | \varphi \rangle \geqslant 0$ т.е. $\langle \psi | \hat{C}^{+}\hat{C}\psi \rangle \geqslant 0$ ч.т.д.