№1.

$|\psi\rangle \qquad \psi(x) = \frac{1}{x^2+a^2}$

$$\int_{-\infty}^{+\infty} \left|\frac{1}{x^2+a^2}\right|^2 dx = 1$$

$$A^2 \int_{-\infty}^{+\infty} \left(\frac{1}{x^2+a^2}\right)^2 dx = \left| \int \frac{1}{(ax^2+b)^n} dx = \frac{2n-3}{2b(n-1)} \int_{-\infty}^{+\infty} \frac{1}{(ax^2+b)^{n-1}} + \left(\frac{x}{2b(n-1)(ax^2+b)^{n-1}}\right)\Big|_{-\infty}^{+\infty} \right| =$$

$$= \frac{A^2}{2a^2 \cdot 1} \int_{-\infty}^{+\infty} \frac{1}{x^2+a^2} dx + \frac{A^2 x}{2a^2 \cdot (x^2+a^2)}\Big|_{+\infty}^{+\infty} =$$

$$= \frac{A^2}{2a^2} \cdot \frac{1}{a} \cdot \left(\operatorname{arctg}\frac{x}{a}\right)\Big|_{-\infty}^{+\infty} + A^2(0-0) = \left|\operatorname{arctg}_{x\to\infty} x = \frac{\pi}{2}\right| =$$

$$= \frac{A^2 \pi}{2a^3}. \qquad \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

$$\frac{A^2 \pi}{2a^3} = 1 \quad \Rightarrow \quad A = \sqrt{\frac{2a^3}{\pi}} \quad \text{Ответ: } \sqrt{\frac{2a^3}{\pi}}$$

№2. $\varphi(x) = \frac{B}{x+ib}$

$$\int_{-\infty}^{+\infty} \frac{BB}{(x+ib)\cdot(x-ib)} dx = \int_{-\infty}^{+\infty} \frac{B^2}{x^2-(ib)^2} dx =$$

$$= \int_{-\infty}^{+\infty} \frac{B^2}{x^2+b^2} dx = |\text{табличн интеграл}| = \frac{B^2}{b} \operatorname{arctg}\frac{x}{b}\Big|_{-\infty}^{+\infty} =$$

$$= \frac{B^2}{b}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{B^2}{b}\pi = 1 \Rightarrow B = \sqrt{\frac{b}{\pi}}$$

Ответ: $\sqrt{\frac{b}{\pi}}$

$\langle \varphi | \psi \rangle$ №3.

$$\langle \varphi | \psi \rangle = \int_{-\infty}^{+\infty} \varphi^*(x)\, \psi(x)\, dx = 1$$

$$\int_{-\infty}^{+\infty} \frac{AB}{(x-ib)(x^2+a^2)}\, dx =$$

Далее я приведу вырезки из черновика, где раскладывал дробь на простейшие. Простите, если нужно, позже перепишу в чистовик, но сегодня не успеваю.

$$\frac{A}{x-ib} + \frac{B}{x-ia} + \frac{C}{x+ia}$$

$$A(x^2+a^2) + B(x-ib)(x+ia) + C(x-ib)(x-ia)$$

$$A(x^2+a^2) + B(x^2+xia-xib+ab) + C(x^2-xia-xib-ab$$

$$A+B+C=0$$

$$iB(a-b) + iC(+a+b) = 0$$

$$Aa^2 + abB - abC = 1$$

$$-a^2 - \frac{a^3-a^2b}{a+b} + ab - a^2b + ab^2$$

$$-a^3 - \cancel{a^2b} - a^3 + \cancel{a^2b} + \cancel{a^2b} + ab^2 - \cancel{a^2b} + ab^2$$

$$\frac{-2a^3+2ab^2}{a+b} = \frac{1}{B}$$

$$\frac{a+b}{-2a^3+2ab^2} = B$$

$$\cancel{\frac{a+b}{-2a+b^2}}$$

$$C = \frac{a-b}{-2a^3+2ab^2}$$

$$\frac{-2a}{-2a^3+2ab^2}$$

$$\frac{1}{b^2-a^2}$$

$$A+B+C=0$$

$$A + a + \cancel{b} + a - \cancel{b} = 0$$

$$A = -2a$$

Возврат к заданию 3

$$AB\left(\int_{-\infty}^{+\infty} \frac{2a}{c(x-ib)}\,dx + \int_{-\infty}^{+\infty}\frac{a+b}{c(x-ia)}\,dx + \int_{-\infty}^{+\infty}\frac{a-b}{c(x+ia)}\,dx\right) =$$

$$= \left| \text{где } c = -2a^3 + 2ab^2 \right| =$$

$$= AB\left(\frac{2a}{c} \ln(x-ib)\Big|_{-\infty}^{+\infty} + \frac{a+b}{c} \ln(x-ia)\Big|_{-\infty}^{+\infty} + \frac{a-b}{c} \ln(x+ia)\Big|_{-\infty}^{+\infty} \right) =$$

$$= AB\left(2a \ln|x-ib| + (\cancel{a+a+b\ b} \right. =$$

$$= AB\,\frac{(2a \ln(x-ib) + (a+b)\ln(x-ia) + (a-b)\ln(x+ia))}{c}\Big|_{-\infty}^{+\infty} =$$

$$= AB\left(\frac{2a}{c} \ln(x-ib) + \frac{a}{c}\ln(x-ia) + \frac{a}{c}\ln(x+ia) + \frac{b}{c}(\ln(x-ia) - \frac{b}{c}(\ln(x+ia) \right)\Big|_{-\infty}^{+\infty} =$$

$$= AB\left(\frac{2a}{c} \ln(x-ib) + \frac{a}{c}\ln(x^2+a^2) + \frac{b}{c}\cancel{\frac{\ln(x-ia)}{\ln(x+ia)}} \right)\Big|_{-\infty}^{+\infty}$$

$$= AB\left(\frac{2a}{c} \underbrace{\ln(x-ib)}_{=i\pi} + \frac{a}{c}\cdot 0 + \frac{b}{c}\overbrace{\ln(x-ia)}^{i\pi} - \frac{b}{c}\underbrace{\ln(x+ia)}_{-i\pi} \right) =$$

$$= AB\left(\frac{2a}{c} i\pi + \frac{b}{c} i\pi + \frac{b}{c} i\pi \right) =$$

$$= AB\left(\frac{2a - 2b}{-2a^3+2ab^2} \right) i\pi =$$

$$= AB\left(\frac{a-b}{a(a^2-b^2)} \right) i\pi = \sqrt{\frac{2a^3}{\pi}} \cdot \sqrt{\frac{b}{\pi}}\, \frac{i\pi}{(a+b)\cdot a} =$$

$$= \frac{\sqrt{2}\, a^{\frac{3}{2}} \sqrt{b}\; i}{(a+b)\, a} = \frac{\sqrt{2ab}}{a+b}\, i$$

№4. $\delta(f(x)) = \delta(\sum f'(x_i)\cdot(x-x_i)) =$

т.к. $\delta(ax) = \frac{1}{|a|}\delta(x) \Rightarrow = \sum \frac{\delta(x-x_i)}{|f'(x_i)|}$

ч.т.д.

№5. $f_1(x) = \alpha_1 e^{i\pi x/a}$ $\quad f_2(x) = \alpha_2 e^{-i\pi x/a}$

1. $\langle f_1|f_2\rangle = \int_0^a f_1^*(x) f_2(x)\,dx = \int_0^a \alpha_1 e^{-i\pi x/a}\alpha_2 e^{-i\pi x/a}\,dx =$

$= \alpha_1\alpha_2 \int_0^a e^{\frac{-i\pi x - i\pi x}{a}}\,dx = \alpha_1\alpha_2\int_0^a e^{\frac{-2i\pi x}{a}}\,dx =$

$= \alpha_1\alpha_2\int_0^a \left(\cos\frac{2\pi x}{a} - i\sin\frac{2\pi x}{a}\right)dx = \alpha_1\alpha_2\frac{a}{2\pi}\cdot$

$\cdot\left(\cancel{\sin 2\pi\ \cos}\ \sin\frac{2\pi x}{a} + i\cos\frac{2\pi x}{a}\right)\Big|_0^a =$

$= \frac{\alpha_1\alpha_2 a}{2\pi}(\underbrace{\sin 2\pi}_{=0} - \underbrace{\sin 0}_{=0} + i\underbrace{\cos 2\pi}_{=1} - i\underbrace{\cos 0}_{=1}) = \frac{\alpha_1\alpha_2 a}{2\pi}\cdot 0 = 0$

2. $\cancel{\int_0^a |\alpha_1 e^{\frac{i\pi x}{a}}|^2\,dx} = \alpha_1^2\int_0^a e^{\frac{-i\pi x}{a}}\cdot e^{\frac{i\pi x}{a}}\,dx =$ верно

$= \alpha_1^2\int_0^a e^{\frac{\pi x i}{a} - \frac{i\pi x}{a}}\,dx = \alpha_1^2\int_0^a dx = \alpha_1^2(a-0) = 1$

$\alpha_1^2 a = 1 \Rightarrow \alpha_1 = \sqrt{\frac{1}{a}}$

аналогично $\alpha_2 = \sqrt{\frac{1}{a}}$

3. $\psi(x) = \sqrt{\frac{2}{a}}\sin(\pi x/a)$ $\quad |\psi\rangle = c_1|f_1\rangle + c_2|f_2\rangle$

$\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{x a}\right) = \frac{c_1}{\sqrt{a}}\left(\cos\frac{\pi x}{a} + i\sin(\pi x/a)\right) + c_2/\sqrt{a}\left(\cos\frac{\pi x}{a} - i\sin(\pi x/a)\right)$

$\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \frac{c_1 + c_2}{\sqrt{a}}\cos\frac{\pi x}{a} + i\frac{(c_1 - c_2)}{\sqrt{a}}\sin\frac{\pi x}{a}$

$$\begin{cases} \dfrac{C_1 + C_2}{\sqrt{a}} = 0 \\ \dfrac{i\,(C_1 - C_2)}{\sqrt{a}} = \sqrt{\dfrac{2}{a}} \end{cases} \qquad \begin{cases} C_2 = -C_1 \\ i\,(C_1 + C_2) = \sqrt{2} \end{cases} \Rightarrow C_1 = -\frac{\sqrt{2}}{2}\,i \qquad C_2 = +\frac{\sqrt{2}}{2}\,i$$

Ответ: $|\psi\rangle = -i\,\frac{\sqrt{2}}{2}\,|f_1\rangle + i\,\frac{\sqrt{2}}{2}\,|f_2\rangle$