Midterm: APMA 2550, Oct. 31, 12pm
The exam is open book and notes. However, you cannot talk to anyone about the exam except me. Use Latex to write your solutions. Turn in paper copy in my dropbox.

1. Problem #1

Consider the following equation

$$\begin{aligned} u_t(x,t) =& (-1)^{m+1}\partial_x^{(2m)}u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where f is 2π periodic. Here, m is a positive integer.
Consider the following method for this problem

$$\begin{aligned} v_j^{n+1} =& v_j^n + k(-1)^{m+1}(D_+D_-)^m v_j^n, \\ v_j^0 =& f_j. \end{aligned}$$

a) Calculate $\hat{Q}(\omega h)$.

First, let us find the form of $(D_+D_-)^m$:

$$\begin{aligned} (D_+D_-)^m &= \left(\frac{E - 2E^0 + E^{-1}}{h^2}\right)^m \\ &= \frac{(E - 2E^0 + E^{-1})^m}{h^{2m}} \\ &= \frac{((E - E^0)(E^0 - E^{-1}))^m}{h^{2m}} \\ &= \frac{((E - E^0)(E - E^0)E^{-1})^m}{h^{2m}} \\ &= \frac{(E - E^0)^{2m}E^{-m}}{h^{2m}} \\ &= \frac{E^{-m}}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}E^k(E^0)^{2m-k} \\ &= \frac{E^{-m}}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}E^k \\ (D_+D_-)^m &= \frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k}E^{k-m} \end{aligned}$$

Consider $v_j^n = \frac{1}{\sqrt{2\pi}}e^{i\omega x_j}\hat{v}^n(\omega)$

$$v_j^{n+1} = v_j^n + k(-1)^{m+1}(D_+D_-)^m v_j^n \tag{1.1}$$

$$\frac{1}{\sqrt{2\pi}} e^{i\omega x_j}\hat{v}^{n+1}(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j}\hat{v}^{n}(\omega) + k(-1)^{m+1}(D_+D_-)^m \frac{1}{\sqrt{2\pi}} e^{i\omega x_j}\hat{v}^{n}(\omega) \tag{1.2}$$

$$e^{i\omega x_j}\hat{v}^{n+1}(\omega) = e^{i\omega x_j}\hat{v}^{n}(\omega) + k(-1)^{m+1}(D_+D_-)^m e^{i\omega x_j}\hat{v}^{n}(\omega) \tag{1.3}$$

$$e^{i\omega x_j}\hat{v}^{n+1}(\omega) = e^{i\omega x_j}\hat{v}^{n}(\omega) + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} E^{k-m} e^{i\omega x_j}\hat{v}^{n}(\omega) \tag{1.4}$$

$$e^{i\omega x_j}\hat{v}^{n+1}(\omega) = e^{i\omega x_j}\hat{v}^{n}(\omega) + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{i\omega(x_j+(k-m)h)}\hat{v}^{n}(\omega) \tag{1.5}$$

$$\hat{v}^{n+1}(\omega) = \hat{v}^{n}(\omega) + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{(k-m)\omega hi}\hat{v}^{n}(\omega) \tag{1.6}$$

$$\hat{v}^{n+1}(\omega) = \hat{v}(\omega)\left(1 + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{(k-m)\omega hi}\right) \tag{1.7}$$

Thus

$$\hat{Q}(\omega h) = 1 + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{(k-m)\omega hi} \tag{1.8}$$

b) Is there a condition on k, h so that $|\hat{Q}| \le 1$?

$$|\hat{Q}| = \left|1 + k(-1)^{m+1}\frac{1}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{(k-m)\omega hi}\right| \tag{1.9}$$

$$= \left|1 + k(-1)^{m+1}\frac{e^{-m\omega hi}}{h^{2m}}\sum_{k=0}^{2m}\binom{2m}{k} e^{k\omega hi}\right| \tag{1.10}$$

$$= \left|1 + k(-1)^{m+1}\frac{e^{-m\omega hi}}{h^{2m}}(e^{\omega hi}+1)^{2m}\right| \tag{1.11}$$

$$= \left|1 + k(-1)^{m+1}\frac{1}{h^{2m}}(1+e^{-\omega hi})^m(e^{\omega hi}+1)^{m}\right| \tag{1.12}$$

$$= \left|1 + k(-1)^{m+1}\frac{1}{h^{2m}}(e^{-\omega hi}+2+e^{\omega hi})^{m}\right| \tag{1.13}$$

$$= \left|1 + k(-1)^{m+1}\frac{1}{h^{2m}}(2+2\cos(\omega h))^{m}\right| \tag{1.14}$$

$$= \left|1 + k(-1)^{m+1}\frac{2^m}{h^{2m}}(2\cos^2(\omega h/2))^{m}\right| \tag{1.15}$$

$$= \left|1 + k(-1)^{m+1}\left(\frac{2\cos(\omega h/2)}{h}\right)^{2m}\right| \tag{1.16}$$

If m is odd, then

$$|\hat{Q}| = \left|1 + k\left(\frac{2\cos(\omega h/2)}{h}\right)^{2m}\right| \tag{1.17}$$

and thus $|\hat{Q}| > 1$ for any k, h. In the case where m is even, then

$$|\hat{Q}| = \left|1 - k\left(\frac{2\cos(\omega h/2)}{h}\right)^{2m}\right| \tag{1.18}$$

Thus $|\hat{Q}| \leq 1$ if

$$k\left(\frac{2\cos(\omega h/2)}{h}\right)^{2m} \leq 2 \tag{1.19}$$

$$k\frac{2^{2m}\cos^{2m}(\omega h/2)}{h^{2m}} \leq 2 \tag{1.20}$$

$$k\frac{2^{2m}}{h^{2m}} \leq 2 \tag{1.21}$$

$$k \cdot 2^{2m-1} \leq h^{2m} \tag{1.22}$$

2. Problem #2

Consider the following problem

$$\begin{aligned} u_t(x,t) =& \partial_x^3 u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where f is 2π periodic.

Consider the following method for this problem

$$\begin{aligned} v_j^{n+1} =& v_j^n + kD_-D_+D_-v_j^n, \\ v_j^0 =& f_j. \end{aligned}$$

a) Calculate $\hat{Q}$.

First, let us find $D_-D_+D_-$

$$D_-D_+D_+ = \frac{E^0 - E^{-1}}{h} \cdot \frac{E^1 - E^0}{h} \cdot \frac{E^0 - E^{-1}}{h} \tag{2.1}$$

$$= \frac{(E^{-1} - 2E^0 + E)(E^0 - E^{-1})}{h^3} \tag{2.2}$$

$$= \frac{E^{-1} - 2E^0 + E - E^{-2} + 2E^{-1} - E^0}{h^3} \tag{2.3}$$

$$= \frac{E - 3E^0 + 3E^{-1} - E^{-2}}{h^3} \tag{2.4}$$

Now solve with $v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega)$

$$v_j^{n+1} = v_j^n + kD_-D_+D_-v_j^n \tag{2.5}$$

$$\frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^{n+1}(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega) + kD_-D_+D_- \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega) \tag{2.6}$$

$$e^{i\omega x_j} \hat{v}^{n+1}(\omega) = e^{i\omega x_j} \hat{v}^n(\omega) + kD_-D_+D_- e^{i\omega x_j} \hat{v}^n(\omega) \tag{2.7}$$

$$= e^{i\omega x_j} \hat{v}^n(\omega) + k\frac{E - 3E^0 + 3E^{-1} - E^{-2}}{h^3} e^{i\omega x_j} \hat{v}^n(\omega) \tag{2.8}$$

$$= e^{i\omega x_j} \hat{v}^n(\omega) + k\frac{e^{i\omega(x_j+h)} - 3e^{i\omega x_j} + 3e^{i\omega(x_j-h)} - e^{i\omega(x_j-2h)}}{h^3} \hat{v}^n(\omega) \tag{2.9}$$

$$\hat{v}^{n+1}(\omega) = \hat{v}^n(\omega) + k\frac{e^{i\omega h} - 3 + 3e^{-i\omega h} - e^{-2hi\omega}}{h^3} \hat{v}^n(\omega) \tag{2.10}$$

$$\hat{v}^{n+1}(\omega) = \left(1 + k\frac{e^{i\omega h} - 3 + 3e^{-i\omega h} - e^{-2hi\omega}}{h^3}\right) \hat{v}^n(\omega) \tag{2.11}$$

Thus

$$\hat{Q} = 1 + k\frac{e^{i\omega h} - 3 + 3e^{-i\omega h} - e^{-2hi\omega}}{h^3} \tag{2.12}$$

b) It there a conditon on k, h so that $|\hat{Q}| \leq 1$?

$$|\hat{Q}| = \left|1 + k\frac{e^{i\omega h} - 3 + 3e^{-i\omega h} - e^{-2hi\omega}}{h^3}\right| \tag{2.13}$$

$$= \left|1 + k\frac{-3 + 4\cos(\omega h) - i2\sin(\omega h) - e^{-2hi\omega}}{h^3}\right| \tag{2.14}$$

$$= \left|1 + k\frac{-3 + 4\cos(\omega h) - i2\sin(\omega h) - \cos(2\omega h) + i\sin(2\omega h)}{h^3}\right| \tag{2.15}$$

Let $c = \frac{h^3}{k}$

$$c|\hat{Q}| = |c - 3 + 4\cos(\omega h) - i2\sin(\omega h) - \cos(2\omega h) + i\sin(2\omega h)| \tag{2.16}$$

$$c^2|\hat{Q}|^2 = |c - 3 + 4\cos(\omega h) - \cos(2\omega h) + i(\sin(2\omega h) - 2\sin(\omega h))|^2 \tag{2.17}$$

$$= (c - 3 + 4\cos(\omega h) - \cos(2\omega h))^2 + (\sin(2\omega h) - 2\sin(\omega h))^2 \tag{2.18}$$

$$= \left(c - 2 + 4\cos(\omega h) - 2\cos^2(\omega h)\right)^2 + (\sin(2\omega h) - 2\sin(\omega h))^2 \tag{2.19}$$

$$\begin{aligned} &= (c-2)^2 + 16\cos^2(\omega h) + 4\cos^4(\omega h) \\ &\quad + 8(c-2)\cos(\omega h) - 16\cos^3(\omega h) - 4(c-2)\cos^2(\omega h) \\ &\quad + \sin^2(2\omega h) - 4\sin(2\omega h)\sin(\omega h) + 4\sin^2(\omega h) \end{aligned} \tag{2.20}$$

$$\begin{aligned} &= (c-2)^2 + 16\cos^2(\omega h) + 4\cos^4(\omega h) \\ &\quad + 8(c-2)\cos(\omega h) - 16\cos^3(\omega h) - 4(c-2)\cos^2(\omega h) \\ &\quad + \sin^2(2\omega h) - 8\sin^2(\omega h)\cos(\omega h) + 4\sin^2(\omega h) \end{aligned} \tag{2.21}$$

$$\begin{aligned} &= (c-2)^2 + 16\cos^2(\omega h) + 4\cos^4(\omega h) \\ &\quad + 8(c-2)\cos(\omega h) - 16\cos^3(\omega h) - 4(c-2)\cos^2(\omega h) \\ &\quad + 1 - \cos^2(2\omega h) - 8\cos(\omega h) + 8\cos^3(\omega h) + 4 - 4\cos^2(\omega h) \end{aligned} \tag{2.22}$$

$$\begin{aligned} &= (c-2)^2 + 16\cos^2(\omega h) + 4\cos^4(\omega h) \\ &\quad + 8(c-2)\cos(\omega h) - 16\cos^3(\omega h) - 4(c-2)\cos^2(\omega h) \\ &\quad + 1 - (2\cos^2(\omega h) - 1)^2 - 8\cos(\omega h) + 8\cos^3(\omega h) + 4 - 4\cos^2(\omega h) \end{aligned} \tag{2.23}$$

$$\begin{aligned} &= (c-2)^2 + 16\cos^2(\omega h) + \cancel{4\cos^4(\omega h)} \\ &\quad + 8(c-2)\cos(\omega h) - 16\cos^3(\omega h) - 4(c-2)\cos^2(\omega h) \\ &\quad + 1 - \cancel{4\cos^4(\omega h)} + \cancel{4\cos^2(\omega h)} - 1 - 8\cos(\omega h) + 8\cos^3(\omega h) + 4 - \cancel{4\cos^2(\omega h)} \end{aligned} \tag{2.24}$$

$$\begin{aligned} &= (-16 + 8)\cos^3(\omega h) + (16 - 4(c-2))\cos^2(\omega h) \\ &\quad + (8(c-2) - 8)\cos(\omega h) + ((c-2)^2 + 1 - 1 + 4) \end{aligned} \tag{2.25}$$

$$c^2|\hat{Q}| = -8\cos^3(\omega h) + (24 - 4c)\cos^2(\omega h) + (8c - 24)\cos(\omega h) + (c^2 - 4c + 8) \tag{2.26}$$

Thus $|\hat{Q}| \leq 1$ if

$$-\frac{8}{c^2}\cos^3(\omega h)+\left(\frac{24}{c^2}-\frac{4}{c}\right)\cos^2(\omega h)+\left(\frac{8}{c}-\frac{24}{c^2}\right)\cos(\omega h)+1-\frac{4}{c}+\frac{8}{c^2}\leq 1 \tag{2.27}$$

$$-\frac{8}{c^2}\cos^3(\omega h)+\left(\frac{24}{c^2}-\frac{4}{c}\right)\cos^2(\omega h)+\left(\frac{8}{c}-\frac{24}{c^2}\right)\cos(\omega h)+\left(\frac{8}{c^2}-\frac{4}{c}\right)\leq 0 \tag{2.28}$$

$$-8\cos^3(\omega h)+(24-4c)\cos^2(\omega h)+(8c-24)\cos(\omega h)+(8-4c)\leq 0 \tag{2.29}$$

$$2\cos^3(\omega h)+(c-6)\cos^2(\omega h)+(6-2c)\cos(\omega h)+(c-2)\geq 0 \tag{2.30}$$

$$(\cos(\omega h)-1)(2\cos^2(\omega h)+(c-4)\cos(\omega h)+(2-c))\geq 0 \tag{2.31}$$

$$(\cos(\omega h)-1)^2(2\cos(\omega h)+(c-2))\geq 0 \tag{2.32}$$

$$2\cos(\omega h)+(c-2)\geq 0 \tag{2.33}$$

$$c\geq 2-2\cos(\omega h) \tag{2.34}$$

In order for this to be true of all ωh, then $c \geq 4$. Thus

$$c=\frac{h^3}{k}\geq 4 \implies |\hat{Q}|\leq 1 \tag{2.35}$$

in other words:

$$h^3\geq 4k \implies |\hat{Q}|\leq 1 \tag{2.36}$$

3. Problem #3

Consider the problem

$$\begin{aligned} u_t(x,t) =& \partial_x u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where f is 2π periodic. Consider the method of lines:

$$v_j'(t) = \frac{1}{h}(\frac{-1}{2}v_{j+2}(t) + 2v_{j+1}(t) + av_j(t)).$$

Therefore, the corresponding $Q = \frac{1}{h}(\frac{-1}{2}E^2 + 2E + aE^0)$. Can you find a so that

$$|(Q - \partial_x)(e^{i\omega x_j})| \leq O(\omega^3 h^2)?$$

$$(Q - \partial_x)(e^{i\omega x_j}) = Qe^{i\omega x_j} - \partial_x e^{i\omega x_j} \tag{3.1}$$

$$= \frac{-0.5E^2 + 2E + aE^0}{h}e^{i\omega x_j} - \partial_x e^{i\omega x_j} \tag{3.2}$$

$$= \frac{-0.5e^{i\omega(x_j+2h)} + 2e^{i\omega(x_j+h)} + ae^{i\omega x_j}}{h} - i\omega e^{i\omega x_j} \tag{3.3}$$

$$\tag{3.4}$$

We will use

$$e^{i\omega(x_j+h)} = \sum_{k=0}^{\infty} \frac{(i\omega)^k e^{i\omega x_j}}{k!} h^k \tag{3.5}$$

$$e^{i\omega(x_j+2h)} = \sum_{k=0}^{\infty} \frac{(i\omega)^k e^{i\omega x_j}}{k!} 2^k h^k \tag{3.6}$$

Thus

$$(3.7)\quad (Q-\partial_x)(e^{i\omega x_j}) = \frac{-0.5e^{i\omega(x_j+2h)} + 2e^{i\omega(x_j+h)} + ae^{i\omega x_j}}{h} - i\omega e^{i\omega x_j}$$

$$(3.8)\quad = \frac{-0.5\sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}2^k h^k + 2\sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}h^k + ae^{i\omega x_j}}{h} - i\omega e^{i\omega x_j}$$

$$(3.9)\quad = -0.5\sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}2^k h^{k-1} + 2\sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}h^{k-1} + \frac{a}{h}e^{i\omega x_j} - i\omega e^{i\omega x_j}$$

$$(3.10)\quad = -\sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}2^{k-1} h^{k-1} + \sum_{k=0}^{\infty}\frac{(i\omega)^k e^{i\omega x_j}}{k!}2h^{k-1} + \frac{a}{h}e^{i\omega x_j} - i\omega e^{i\omega x_j}$$

$$(3.11)\quad = \sum_{k=0}^{\infty}\left(\frac{(i\omega)^k e^{i\omega x_j}}{k!}2h^{k-1} - \frac{(i\omega)^k e^{i\omega x_j}}{k!}2^{k-1}h^{k-1}\right) + \frac{a}{h}e^{i\omega x_j} - i\omega e^{i\omega x_j}$$

$$(3.12)\quad = e^{i\omega x_j}\left(\frac{a}{h} - i\omega + \sum_{k=0}^{\infty}\frac{(i\omega)^k h^{k-1}}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.13)\quad = \frac{e^{i\omega x_j}}{h}\left(a - i\omega h + \sum_{k=0}^{\infty}\frac{(i\omega)^k h^k}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.14)\quad = \frac{e^{i\omega x_j}}{h}\left(a - i\omega h + \frac{3}{2} + \sum_{k=1}^{\infty}\frac{(i\omega)^k h^k}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.15)\quad = \frac{e^{i\omega x_j}}{h}\left(a - i\omega h + \frac{3}{2} + i\omega h + \sum_{k=2}^{\infty}\frac{(i\omega)^k h^k}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.16)\quad = \frac{e^{i\omega x_j}}{h}\left(a + \frac{3}{2} + \sum_{k=3}^{\infty}\frac{(i\omega)^k h^k}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.17)\quad = \frac{e^{i\omega x_j}}{h}\left(a + \frac{3}{2} + \frac{i\omega^3 h^3}{3} + \sum_{k=4}^{\infty}\frac{(i\omega)^k h^k}{k!}\left(2-2^{k-1}\right)\right)$$

$$(3.18)\quad (Q-\partial_x)(e^{i\omega x_j}) = e^{i\omega x_j}\left(\frac{a}{h} + \frac{3}{2h} + \frac{i\omega^3 h^2}{3} + \sum_{k=4}^{\infty}\frac{(i\omega)^k h^{k-1}}{k!}\left(2-2^{k-1}\right)\right)$$

Thus, if we let $a = -3/2$.

$$|(Q-\partial_x)(e^{i\omega x_j})| = \left|e^{i\omega x_j}\left(\frac{a}{h}+\frac{3}{2h}+\frac{i\omega^3h^2}{3}+\sum_{k=4}^{\infty}\frac{(i\omega)^kh^{k-1}}{k!}\left(2-2^{k-1}\right)\right)\right| \tag{3.19}$$

$$= \left|e^{i\omega x_j}\right|\left|\left(\frac{a}{h}+\frac{3}{2h}+\frac{i\omega^3h^2}{3}+\sum_{k=4}^{\infty}\frac{(i\omega)^kh^{k-1}}{k!}\left(2-2^{k-1}\right)\right)\right| \tag{3.20}$$

$$= \left|\left(\frac{a}{h}+\frac{3}{2h}+\frac{i\omega^3h^2}{3}+\sum_{k=4}^{\infty}\frac{(i\omega)^kh^{k-1}}{k!}\left(2-2^{k-1}\right)\right)\right| \tag{3.21}$$

$$= \left|\left(-\frac{3}{2h}+\frac{3}{2h}+\frac{i\omega^3h^2}{3}+\sum_{k=4}^{\infty}\frac{(i\omega)^kh^{k-1}}{k!}\left(2-2^{k-1}\right)\right)\right| \tag{3.22}$$

$$= \left|\left(\frac{i\omega^3h^2}{3}+\sum_{k=4}^{\infty}\frac{(i\omega)^kh^{k-1}}{k!}\left(2-2^{k-1}\right)\right)\right| \tag{3.23}$$

$$|(Q-\partial_x)(e^{i\omega x_j})| \le O(\omega^3h^2) \tag{3.24}$$

$$\tag{3.25}$$

Thus if $a=-3/2$ then $|(Q-\partial_x)(e^{i\omega x_j})| \le O(\omega^3h^2)$

Problem #4

Consider the problem

$$u_t(x,t) = -5\partial_x u(x,t)$$
$$u(x,0) = \sin(x)$$

You should be able to figure out the exact solution. Choose $N = 100$, and so $h = 2\pi/101$. Let $k = 1/M$.

(1) What is the minum integer value for M so that the Lax-Wendroff method is stable? Call this value $\bar{M}$

The Lax-Wendroff method is usually defined by

$$v_j^{n+1} = (I + kD_0)v_j^n + \sigma khD_+D_-v_j^n \qquad j \in \mathbb{Z}, n \geq 0 \tag{3.26}$$
$$v_j^0 = f_j \qquad j \in \mathbb{Z} \tag{3.27}$$

and for this version of the problem, we have

$$v_j^{n+1} = (I - 5kD_0)v_j^n + \sigma khD_+D_-v_j^n \qquad j \in \mathbb{Z}, n \geq 0 \tag{3.28}$$
$$v_j^0 = f_j \qquad j \in \mathbb{Z} \tag{3.29}$$

Thus, for this case,

$$v_j^{n+1} = (I - 5kD_0)v_j^n + \sigma khD_+D_-v_j^n \tag{3.30}$$
$$\hat{v}^{n+1} = (1 - i5\lambda\sin(\xi) - 4\sigma\lambda\sin^2(\xi/2))\hat{v}^n \tag{3.31}$$

and

$$\hat{Q} = 1 - i5\lambda\sin(\xi) - 4\sigma\lambda\sin^2(\xi/2) \tag{3.32}$$

thus

$$|\hat{Q}|^2 = |1 - i5\lambda\sin(\xi) - 4\sigma\lambda\sin^2(\xi/2)|^2 \tag{3.33}$$
$$= 25\lambda^2\sin^2(\xi) + 16\sigma^2\lambda^2\sin^4(\xi/2) - 8\sigma\lambda\sin^2(\xi/2) + 1 \tag{3.34}$$
$$= 100\lambda^2\sin^2(\xi/2)(1 - \sin^2(\xi/2)) + 16\sigma^2\lambda^2\sin^4(\xi/2) - 8\sigma\lambda\sin^2(\xi/2) + 1 \tag{3.35}$$
$$= 100\lambda^2\sin^2(\xi/2) - 100\lambda^2\sin^4(\xi/2) + 16\sigma^2\lambda^2\sin^4(\xi/2) - 8\sigma\lambda\sin^2(\xi/2) + 1 \tag{3.36}$$
$$= (16\sigma^2 - 100)\lambda^2\sin^4(\xi/2) + (100\lambda^2 - 8\sigma\lambda)\sin^2(\xi/2) + 1 \tag{3.37}$$

Case 1 (Lax-Wendroff): $16\sigma^2 - 100 \leq 0 \implies \sigma \leq \frac{5}{2}$

$$|\hat{Q}| \leq (100\lambda^2 - 8\sigma\lambda)\sin^2(\xi/2) + 1 \leq 1 \tag{3.38}$$

$$(100\lambda^2 - 8\sigma\lambda)\sin^2(\xi/2) \leq 0 \tag{3.39}$$

$$100\lambda^2 - 8\sigma\lambda \leq 0 \tag{3.40}$$

$$100\lambda^2 \leq 8\sigma\lambda \tag{3.41}$$

$$\lambda \leq 2\sigma/25 \tag{3.42}$$

(3.43)

Thus, we will let $\sigma = \frac{25\lambda}{2} = \frac{25k}{2h}$

Since $\sigma \leq \frac{5}{2}$, and $h = 2\pi/101$ then $\frac{25k}{2\cdot\frac{2\pi}{101}} \leq \frac{5}{2} \implies k \leq \frac{2\pi}{5\cdot 101}$.

Therefore

$$k \leq \frac{2\pi}{5\cdot 101} \implies \bar{M} = \left\lceil \frac{505}{2\pi} \right\rceil \tag{3.44}$$

(2) Run the Lax-Wendroff method with $k = 1/\bar{M}$ then calculate $\|u(\cdot, 1) - v^{\bar{M}}\|_h$.