APMA 2550, Homework 2: Due Thursday, Oct. 20. at 4pm

In the following problems we will consider the heat equation

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t) \quad && -\infty < x < \infty, t > 0 \\ u(x,0) =& f(x) && -\infty < x < \infty \end{aligned}$$

where f is 2π periodic.

Problem 1 Let u be the solution to the heat equation above. What function does $u(x,t)$ as t tends to infinity? Prove it.

Problem 2 Prove that $\|\frac{\partial^k u(\cdot,t)}{\partial^k x}\|$ is bounded for any k and any $t > 0$.

Problem 3 Consider the Crank-Nicolson method for the heat equation.

$$v_j^{n+1} = v_j^n + \frac{k}{2}(D_+D_-v_j^n + D_+D_-v_j^{n+1}),$$

or can be written as

$$(I - \frac{k}{2}D_+D_-)v_j^{n+1} = (I + \frac{k}{2}D_+D_-)v_j^n.$$

a) Prove that the method is unconditionally stable.

b) Bound the truncation error. Is it better than the Euler method?

Problem 4 Write a computer code for the Backward Euler method, Crank-Nicolson method, the forward Euler method and the Dufort-Frankel method . Choose initial condition to be $f(x) = x - \pi$. Your are going to run your four codes by fixiing $k/h \approx 1/3$, then $k/h^{3/2} \approx 1/3$ and finally $k/h^2 \approx 1/3$. First take $h = 1/500$ and run the Crank-Nicolson method with $k/h^2 \approx 1/3$. Call this approximation the "true solution" (although you know it is not exact). Now compare for all the four methods run them with $h = 1/100$ and the different combinations of k/h^s up to time $T = 2$. Now compare the error at time $T = 2$ of these approximations to the "true solution".

Problem 5 In this problem you will study a nonlinear convection diffusion equation. Consider the problem viscid Burger's equation.

$$\begin{aligned} u_t(x,t) + (\frac{1}{2}u^2)_x =& \eta u_{xx}(x,t) \quad && -\infty < x < \infty, t > 0, \\ u(x,0) =& f(x) && -\infty < x < \infty, \end{aligned}$$

where f is 2π periodic. Note that $(\frac{1}{2}u^2)_x = uu_x$.

We will consider several methods for this problem.

First, we consider a fully implicit method:

$$(I - \eta k D_+D_-)v_j^{n+1} = v_j^n - \frac{\lambda}{2}v_j^n(v_{j+1}^{n+1} - v_{j-1}^{n+1})$$

where $\lambda = \frac{k}{h}$. Note that in this case you will have to invert different matrices in each step.

The second method is a semi-implicit method:

$$(I - \eta k D_+D_-)v_j^{n+1} = v_j^n - \frac{\lambda}{2}v_j^n(v_{j+1}^n - v_{j-1}^n)$$

Note that in this problem you will have the same matrix to invert in each step (so should invert only once) but with different right-hand sides.

Test both methods with $\lambda \approx 1/3$ $h = 100$, $\eta = 1$ and to $T = 50$. With initial condition $f(x) = x - \pi$. Which method was faster? Now repeat the problem when $\eta = .005$. Did any of the methods do good in this case?

Problem 6 Consider the second-order wave equation

$$\begin{aligned} \partial_t^2 u(x,t) =& c^2 \partial_x^2 u(x,t) \qquad -\infty < x < \infty, t > 0, \\ u(x,0) =& f(x) \qquad -\infty < x < \infty, \\ \partial_t u(x,0) =& g(x) \qquad -\infty < x < \infty, \end{aligned}$$

where f and g are 2π periodic.

In this problem you will compare the performance of two methods. The first one is the leap frog method given in class. The second one is the Newmark method. The idea behind the Newmark method is to define a new variable $q = \partial_t u$. Then, the equivalent system is

$$\begin{aligned} \partial_t q(x,t) =& c^2 \partial_x^2 u(x,t) \qquad -\infty < x < \infty, t > 0, \\ q(x,t) =& \partial_t u(x,t) \qquad -\infty < x < \infty, t > 0, \\ u(x,0) =& f(x) \qquad -\infty < x < \infty, \\ q(x,0) =& g(x) \qquad -\infty < x < \infty. \end{aligned}$$

The numerical method reads

$$\begin{aligned} w_j^{n+1} =& w_j^n + \frac{kc^2}{2} D_+ D_- (u_j^{n+1} + u_j^n) \\ \frac{w_j^{n+1} + w_j^n}{2} =& \frac{u_j^{n+1} - u_j^n}{k} \\ u_j^0 =& f_j \\ w_j^0 =& g_j \end{aligned}$$

Impement the two methods with $c = 1$ and $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Run until time $T = 1$ and choose $\frac{k}{h} \approx \frac{1}{2}$ and $\frac{k}{h} \approx 2$ while setting $h = 1/100$. Plot solutions at time $T = 1$. Does one method seem to do better than the other?