1 Notes on Implementation

The Pricing Engine was implemented with the following specifications:

Language(s):	C/C++, Bash Shell Scripts
Compiler:	GNU GCC Compiler
Operating System:	Linux (Ubuntu 11.10)
Text Editor:	ViM Editor
Version Control:	GitHub
Linked Libraries:	fftw3

The program consists of header files in `include/` with source code implementations in `src/`, which define the interface and implementation of the Transform and Price Model Objects. The Pricing Engine is coded in `main.cpp`. The engine takes input files in a proprietary `.input` format and outputs convergence tables in the `results/` directory:

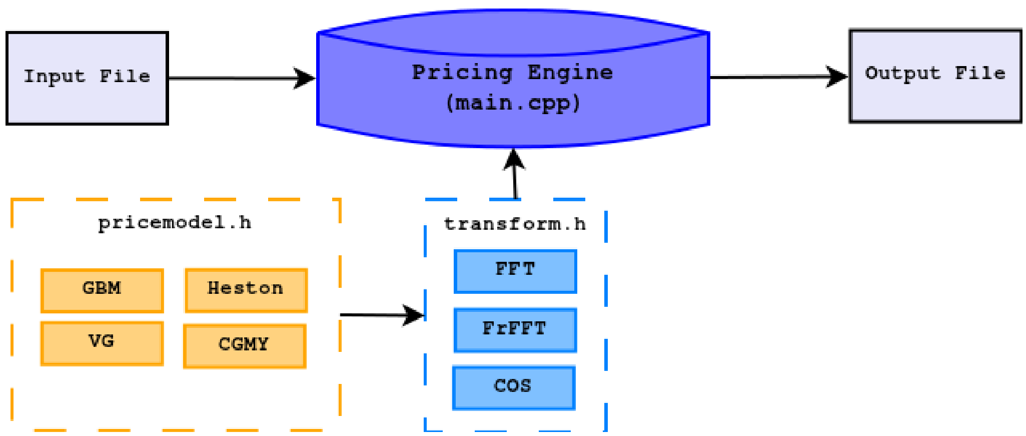

Figure 1: Schematic for Transform Method Pricing Engine

For more information, including code documentation and compilation instructions, please view the README in the `transform_model` directory.

2 Assumptions

In this case study, we attempt to price a put option contract with the terms specified in figure 2.

Contract:	Vanilla Put
Spot Price (S)**:**	1300
Strike Prices (K)**:**	$1, 1.05, 1.1, ..., 1.3$ ($\times 1000$)
Risk Free Rate (r)**:**	.25 %
Continuous Dividend (q)**:**	1.25%
Time To Expiry (T) **:**	.25 years

Figure 2: Option Contract Terms and Assumptions

3 Geometric Brownian Motion (GBM)

In this section we look at the results of 3 transform methods for the GBM pricing model with $\sigma = .35$. The results were compared to the analytical values computed using the Black-Scholes formula (see figure 3).

Strike	**Premium**
1000	6.02335
1050	11.27656
1100	19.42639
1150	31.16228
1200	47.01909
1250	67.31053
1300	92.10513

Figure 3: Black Scholes Theoretical Values.

3.1 Fast Fourier Transform

The FFT method has 3 parameters, α (damping factor), η (spacing factor) and N, a size parameter. In this section, I ran the routine once for each strike. I first set $\eta = .25$ and ran scenarios with N ranging from 4 to 4096, as well as α ranging from -0.5 to -25. For low levels of α in the neighborhood of -1, the model converged to a value other than the theoretical Black-Scholes value. For the strike price of 1000, the results were most pronounced.

When $\alpha = -1.25$, as the premium converged toward 7.80312, which is approx. 31% off the theoretical price (figure 4). As $K \to S$, this error seems to moderate. For the case where $S = K = 1300$, the calculated premium for $\alpha = -1.25$ converges to a value approx. 2.6% off the theoretical value (figure 4).

Setting $\eta = .1$ seemed to get rid of the above issue. For all strikes, convergence occurred

Strike	1000	1050	1100	1150	1200	1250	1300
N=4	28.0842	16.1107	9.8791	6.3917	4.3154	3.0125	2.1571
N=8	12.8554	7.8960	5.0989	3.4256	2.3727	1.6808	1.2092
N=16	2.2830	1.8547	1.4189	1.0596	0.7810	0.5694	0.4093
N=32	0.0184	0.1433	0.1757	0.1649	0.1373	0.1059	0.0767
N=64	0.3065	0.1706	0.1043	0.0690	0.0484	0.0355	0.0269
N=128	0.3104	0.1741	0.1059	0.0690	0.0477	0.0347	0.0264
N=4096	0.3104	0.1741	0.1059	0.0690	0.0477	0.0347	0.0264

Figure 4: FFT Relative Error for $\alpha = -1.25, \eta = .25$.

for $N = 128$ and $\alpha = -2$ (figure 5).

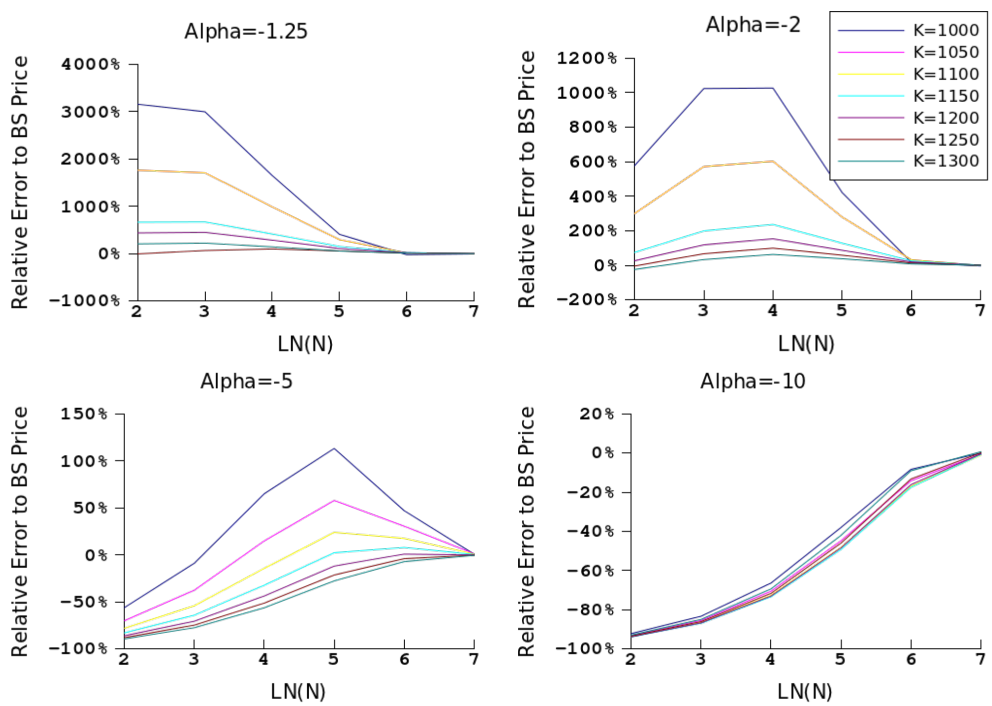

Figure 5: Convergence for FFT with GBM pricing model and various α.

3.2 Fractional Fourier Transform (FrFFT)

In the FrFFT method, λ is allowed to vary independently of η. Varying λ did not seem to impact the convergence for a particular strike price (it is a parameter for log-strike spacing).

I set $\lambda = .001$ for these tests. From testing, it became clear that very fast convergence could be achieved for high levels of η. However, as in the case for FFT, the convergence could be to the *wrong* value. I could mitigate this, provided that I set α high as well. For instance, when I set $\eta = 2$, α had to be in the range of -5 to -10 for convergence to the appropriate value (figure 6).

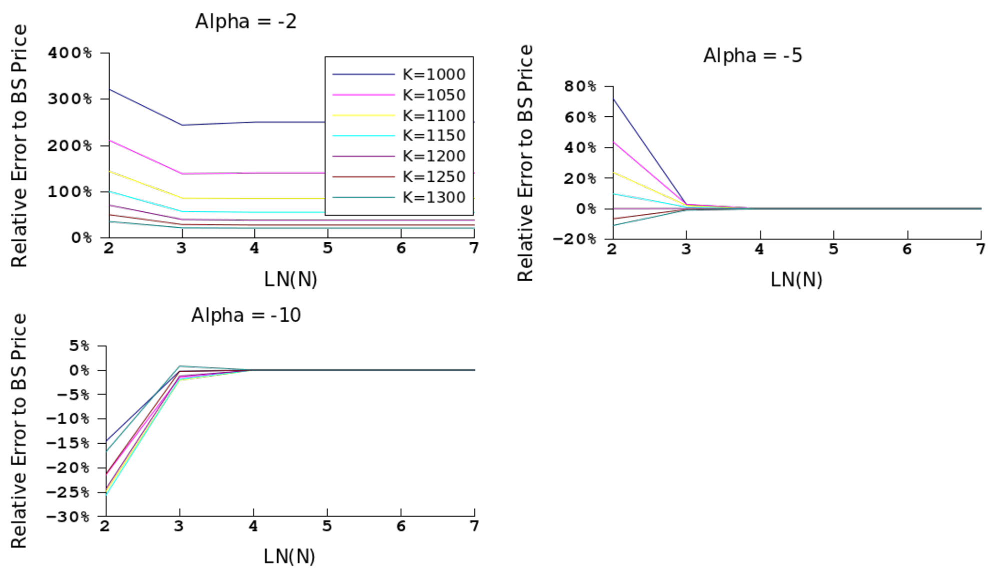

Figure 6: Convergence for FrFFT with GBM pricing model ($\eta = 2$).

Patterns of convergence mimicked those of the FFT method. In particular, for high levels of α (in norm), the values converged from below, whereas the values converged from above for low α levels. Additionally, convergence was much faster, typically requiring only $\ln(N) = 4$ to 6.

3.3 Cosine Transform Method (COS)

In the COS method, an interval $[a, b]$ is chosen to approximate the integral from $-\infty$ to ∞ in the fourier-cosine L^2 projection, and N is chosen as to truncate the fourier series expansion. I chose to look at $B_0(k)$ for $k = 10, 5, 2, 1$, where $B_p(r)$ is the ball about p of radius r. I additionally considered the bounds suggested by Fang and Oosterlee (2004) of

$$[a, b] = B_{c_1}\left(L\sqrt{c_2 + \sqrt{c_4}}\right).$$

For GBM, I found that a bound of $[-1, 1]$ performed better than the bound suggested above. Convergence occured quickly in N, sometimes for values as low as $\ln(N) = 3$ for $K = 1300$.

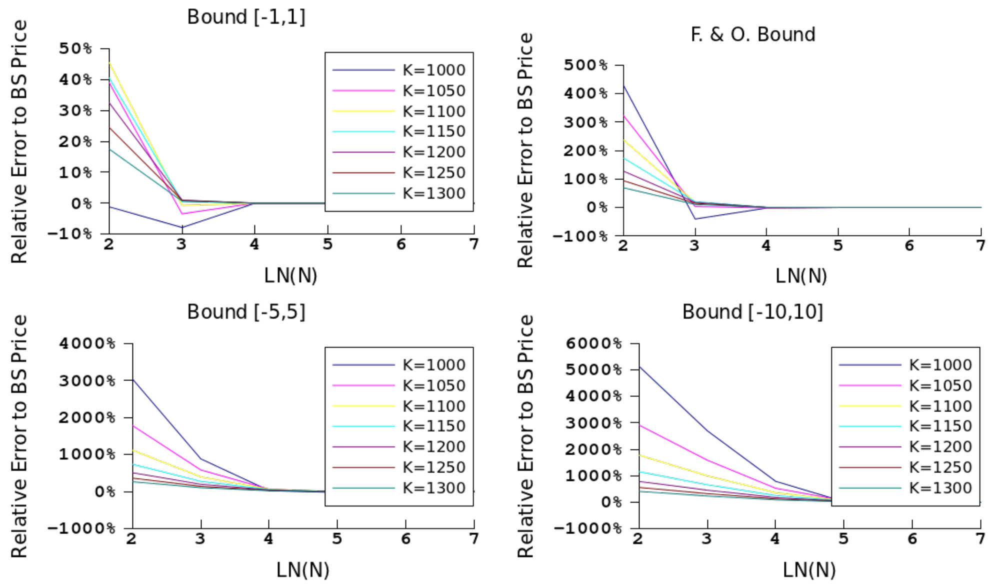

Figure 7: Convergence for COS Method with GBM pricing model.

From the data, it appears that a larger bound leads to smoother, yet slower, convergence. For options that are increasingly out of the money, smoothness may be desired and therefore bounds should be set wider than normal. For most options fairly close to at-the-money (within approx. 30%) it appears that the bounds proposed by Fang and Oosterlee (2004) are appropriate for GBM.

4 Heston Stochastic Volatility Model (Heston)

In this section we look at the results of the 3 transform methods for the Heston pricing model with parameters $(\kappa, \theta, \sigma\rho, v_0) = (1, .25, .3, -.8, .1)$. Since the analytic price cannot be computed directly, we will set the value for the FFT method with $N = 4096$ as the denominator for relative error calculations.

4.1 FFT Method Results

Again we set $\eta = .1$ and ran tests for a range of α and N. The Heston model showed similar characteristics to the GBM model as far as convergence is concerned, converging for $N = 2^8$. Interestingly, the error for $\alpha = -1.25$ is almost non-existent ($O(2^{-5})$).

Strike	Premium
1000	8.14134
1050	13.5305
1100	21.4082
1150	32.3885
1200	47.0368
1250	65.8052
1300	88.9759

Figure 8: FFT Heston Put Premiums ($\alpha = -2, \eta = .1, N = 4096$).

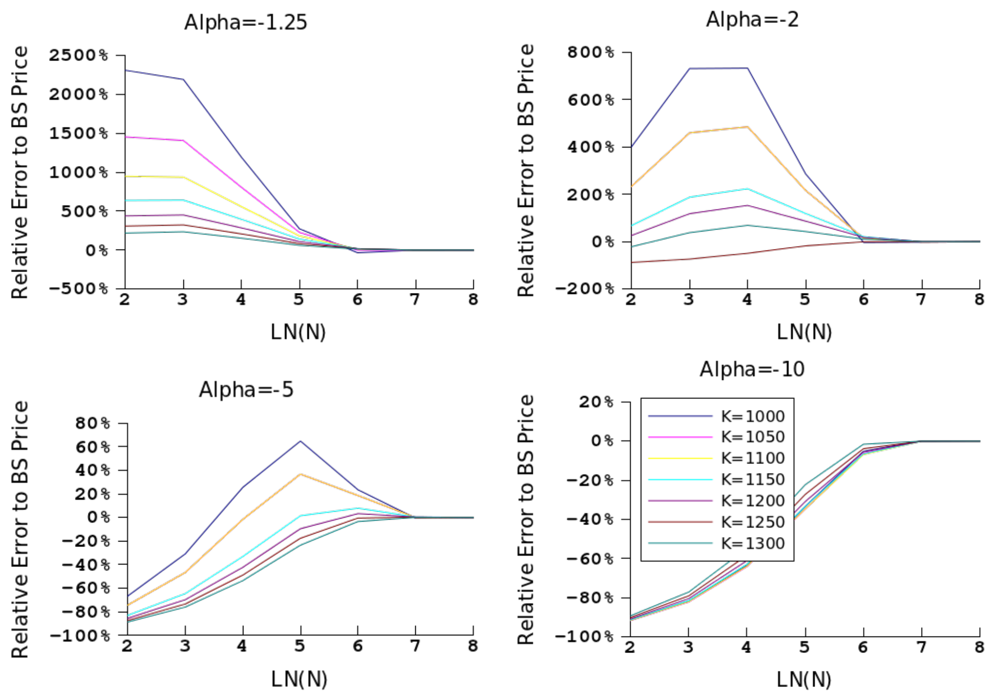

Figure 9: Convergence for FFT Method with Heston pricing model.

4.2 FrFFT Method Results

I set $\lambda = .001$ and $\eta = 1.5$ to compare to GBM results. In this case the model does not converge to the correct price for $\alpha = -1.25$ or $\alpha = -2$. This is similar to what was seen in the GBM model. Where convergence occurs, it is generally faster than the FFT method.

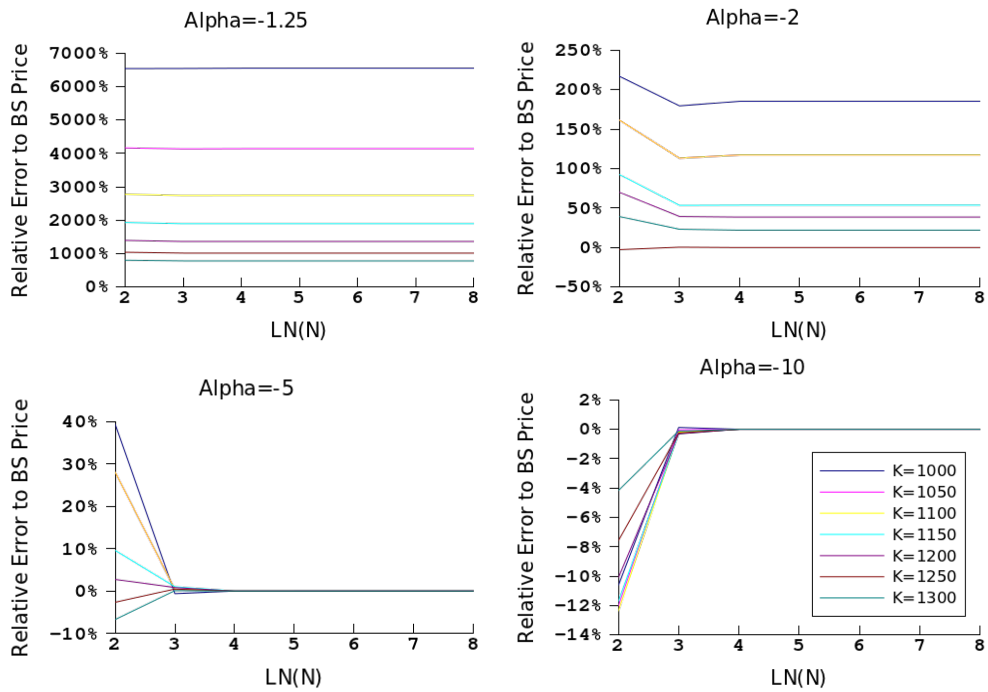

Figure 10: Convergence for FrFFT Method with Heston pricing model.

4.3 COS Method Results

Here we use the same bounds as set in the GBM case. As was seen in the GBM case, all bounds converge and the convergence is fast for bounds near the F&O bound $[-2.05, 2.03]$.

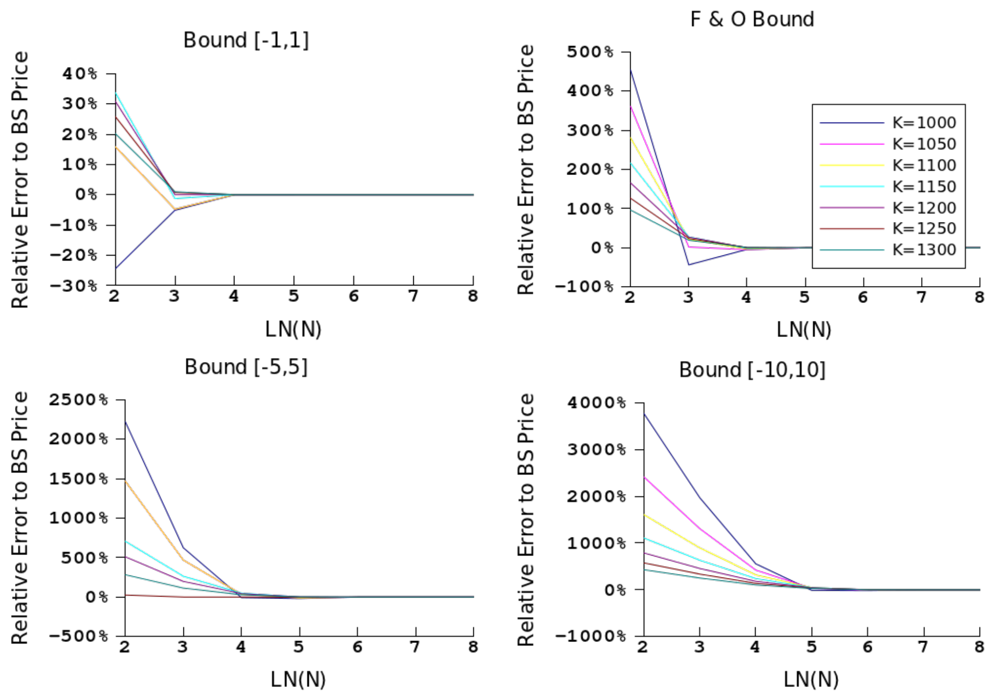

Figure 11: Convergence for COS Method with Heston pricing model.

5 Variance Gamma Model (VG)

In this section we look at the results of the 3 transform methods for a Variance Gamma pricing model with parameters $(\sigma, \theta, \nu) = (.25, -.25, .35)$. Again, we bootstrap the targeted price using the FFT method with $N = 4096$ as the denominator for relative error calculations.

Strike	Premium
1000	7.62612
1050	11.2079
1100	16.2229
1150	23.1707
1200	32.7174
1250	45.7647
1300	63.578

Figure 12: FFT VG Put Premiums ($\alpha = -2, \eta = .1, N = 4096$).

5.1 FFT Method Results

We set $\eta = .1$ and ran tests for a range of α and N. The VG model interestingly showed pathalogical behaviour different than that of the GBM or Heston models in that it converged correctly for small values of α, but failed to converge to the correct price for large α.

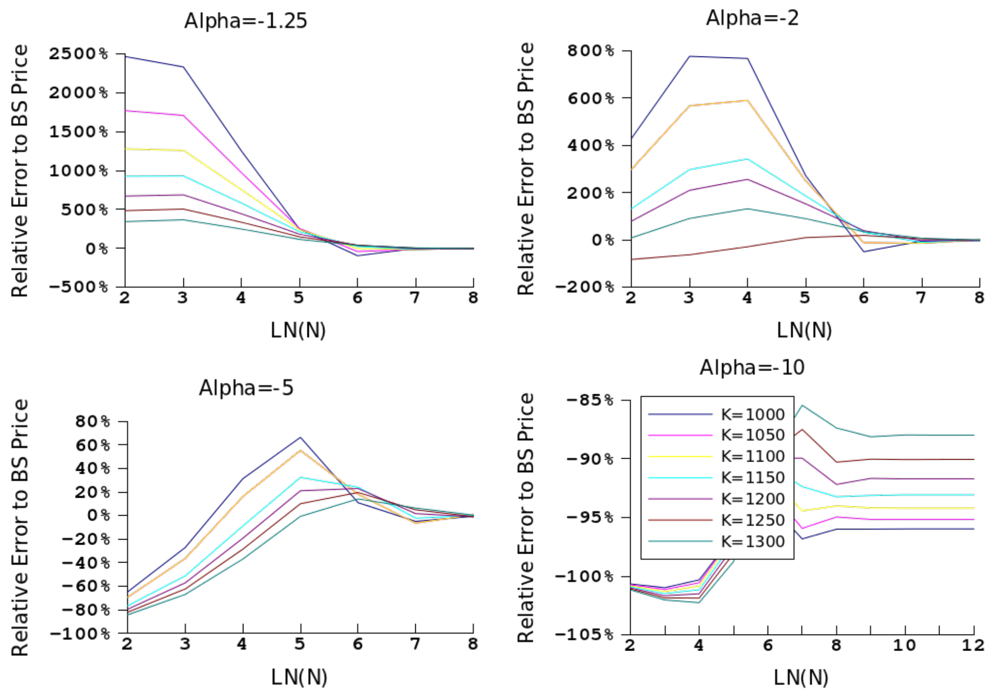

Figure 13: Convergence for FFT Method with VG pricing model.

5.2 FrFFT Method Results

I set $\lambda = .001$ and $\eta = 1.5$ for comparability. However, the high η lead to terrible convergence results. So I set $\eta = .25$ and the results were much better. In this case the VG model does not converge to the correct price for $\alpha = -10$, which we would expect given our results from the FFT trial. Where convergence occurs, the added speed of convergence is small compared to the efficiency gains for other price models (i.e., GBM and Heston).

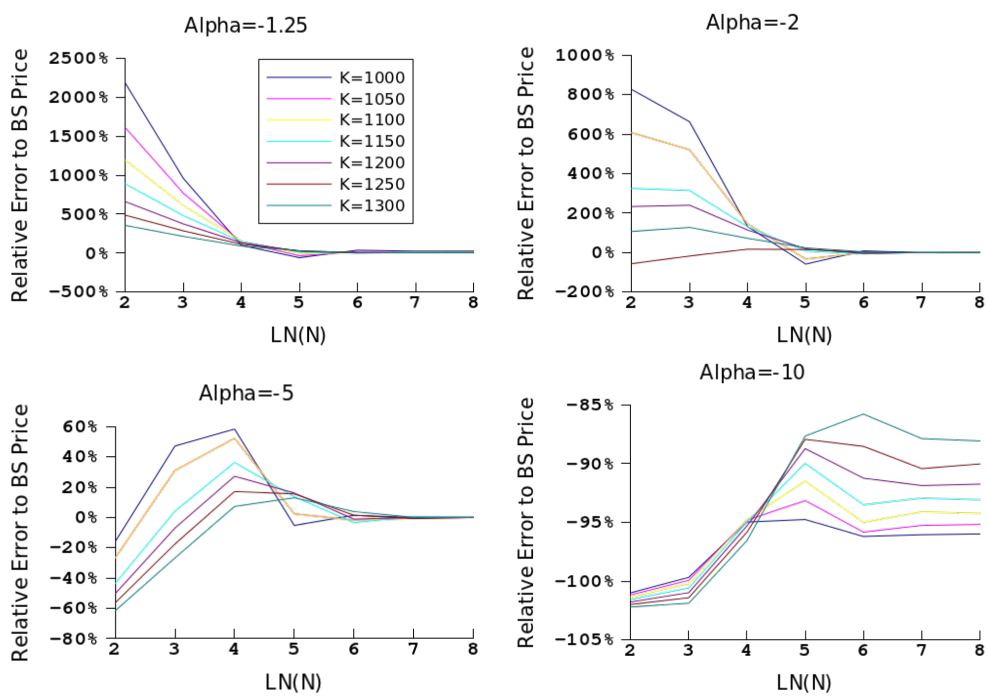

Figure 14: Convergence for FrFFT Method with VG pricing model.

5.3 COS Method Results

We again use the same bounds as set in the original GBM case for consistency. Here the F&O bound, $[-2.768, 2.643]$ looks best. In all cases convergence occurs and it is faster than FFT or FrFFT methods.

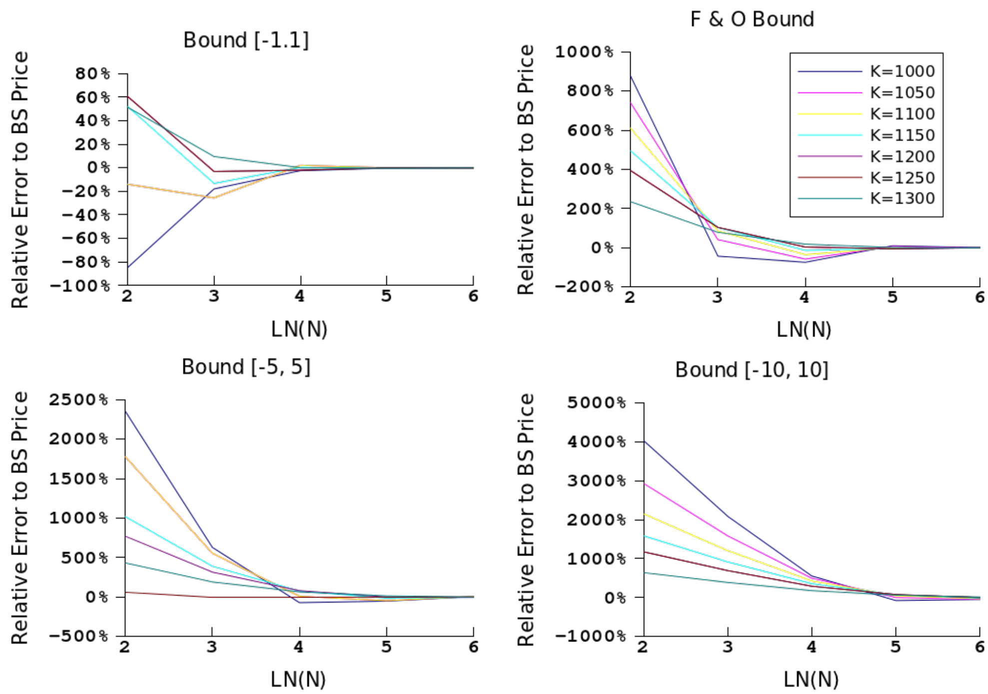

Figure 15: Convergence for COS Method with VG pricing model.

6 CGMY Model

In this section we look at the results of the 3 transform methods for a two CGMY models, CGMY1 and CGMY2. The parameters for CGMY1 are $(C, G, M, Y) = (1.2, 8.5, 6.5, .05)$. For CGMY2, the same parameters are used except that $Y = .95$. Again, we bootstrap the targeted price using the FFT method with $N = 4096$ as the denominator for relative error calculations. In the case where $Y = .05$, prices for out of the money put options became negligible at 1150.

6.1 FFT Method Results

We set $\eta = .1$ and ran tests for a range of α and N.

6.1.1 CGMY2

For CGMY2 we see the usual pattern of results, though we note that convergence occurs to the same price for all levels of α chosen.

Strike	CGMY1 Premium	CGMY2 Premium
1000	NM	22.2624
1050	NM	34.1759
1100	NM	49.4374
1150	NM	68.1189
1200	4.21541	90.1662
1250	25.4646	115.429
1300	56.4439	143.693

Figure 16: FFT CGMY Put Premiums ($\alpha = -2, \eta = .1, N = 4096$).

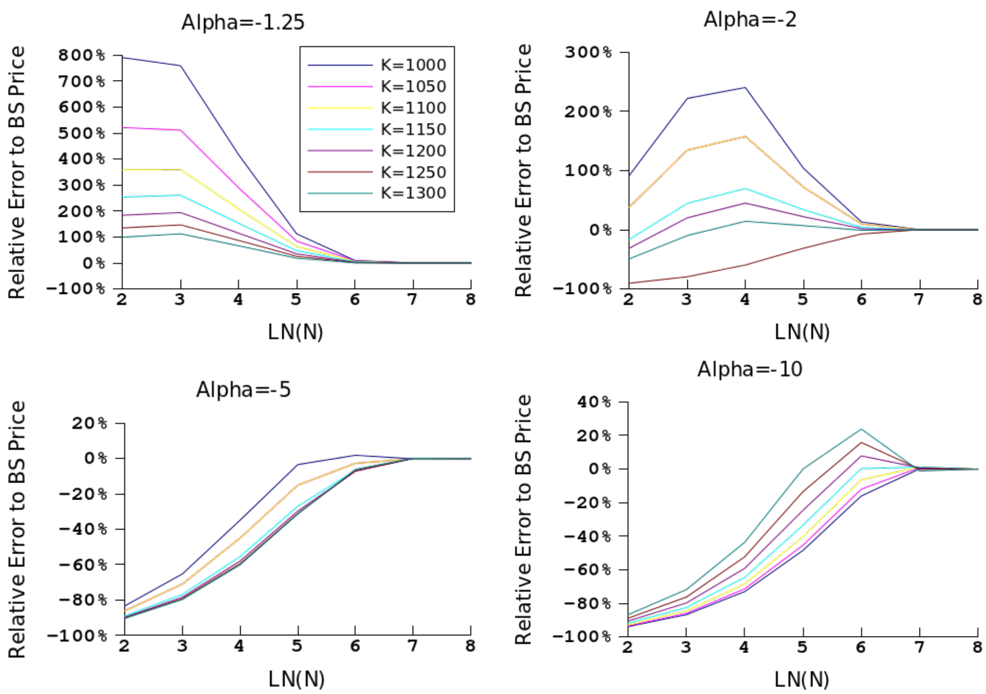

Figure 17: Convergence for FFT Method with CGMY(1.2, 6.5, 8.5, .95).

6.1.2 CGMY1

For CGMY1, it took considerably higher N to converge, and as α increased, a wave pattern in the convergence emerged, indicating that for very high α (in norm), convergence could be an issue. I tested this, and it is true for $\alpha \approx -100$. However, convergence was still fine for $\alpha = -50$, so this is not a real concern.

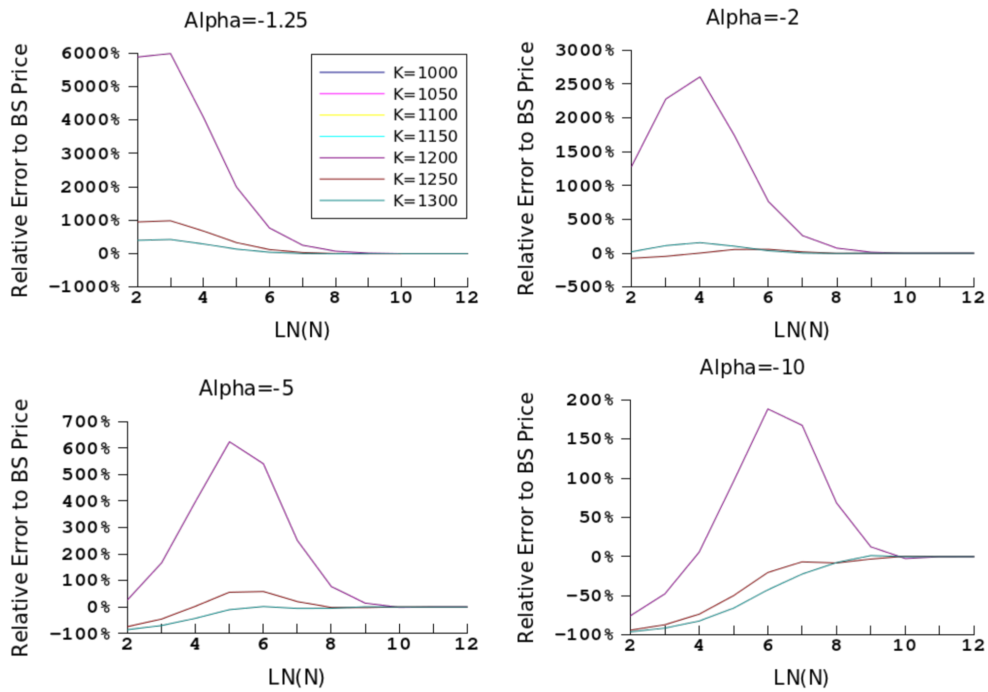

Figure 18: Convergence for FFT Method with CGMY$(1.2, 6.5, 8.5, .05)$.

6.2 FrFFT Method Results

Again, for consistency we set $\eta = 1.5$ and $\lambda = .001$ and ran tests for a range of α and N. We see the same pattern we have seen for the other pricing models in CGMY2. In CGMY1, we needed a high α in order to force convergence to correct values.

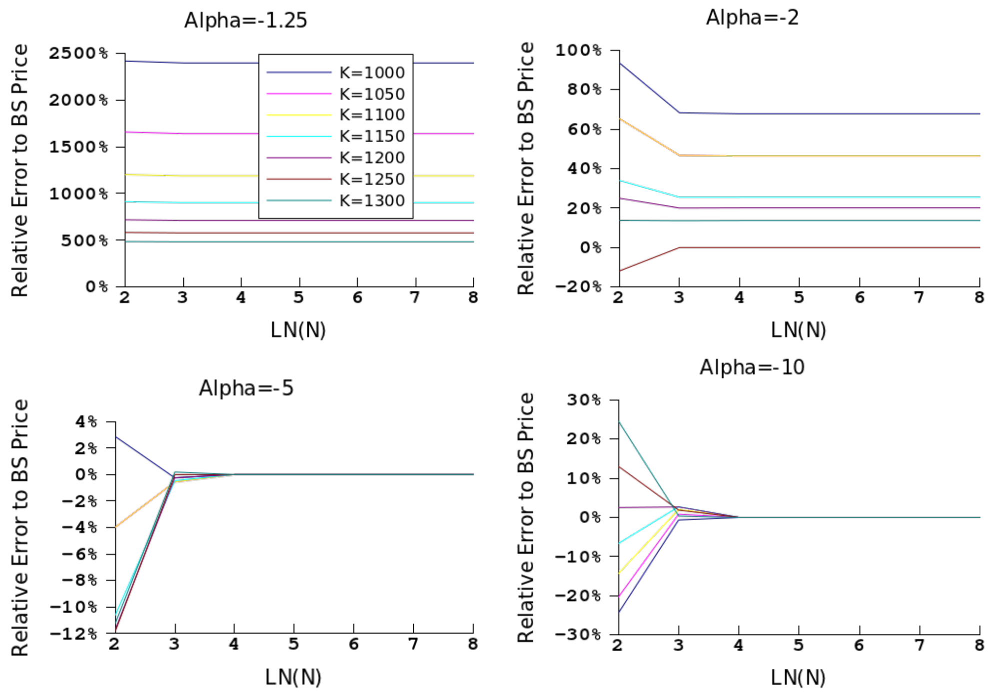

Figure 19: Convergence for FrFFT Method with CGMY$(1.2, 6.5, 8.5, .95)$.

6.3 COS Method Results

With the same bounds as we used, earlier we ran the COS Method for the two CGMY models. The F&O optimal bound for CGMY1 was $[-3.137, 2.963]$ and for CGMY2 was $[-14.6, -6.4]$. The Cumulant-derived F&O bound produced some nasty results (convergence to values on the order of 1000). I omitted the relative error graph for it because it was not meaningful.

7 Concluding Remarks

From the data, we see that the FFT method performs well for moderate α in the range of -2 to -5 and $\eta = .25$.

The FrFFT method can produce more efficient results if η is raised in most models, however this causes terrible convergence results in the Variance Gamma Model. For the Variance Gamma Model running FrFFT with $\eta = .25$ seemed to work well.

Convergence was in general the fastest for the COS model and the bounds suggested by

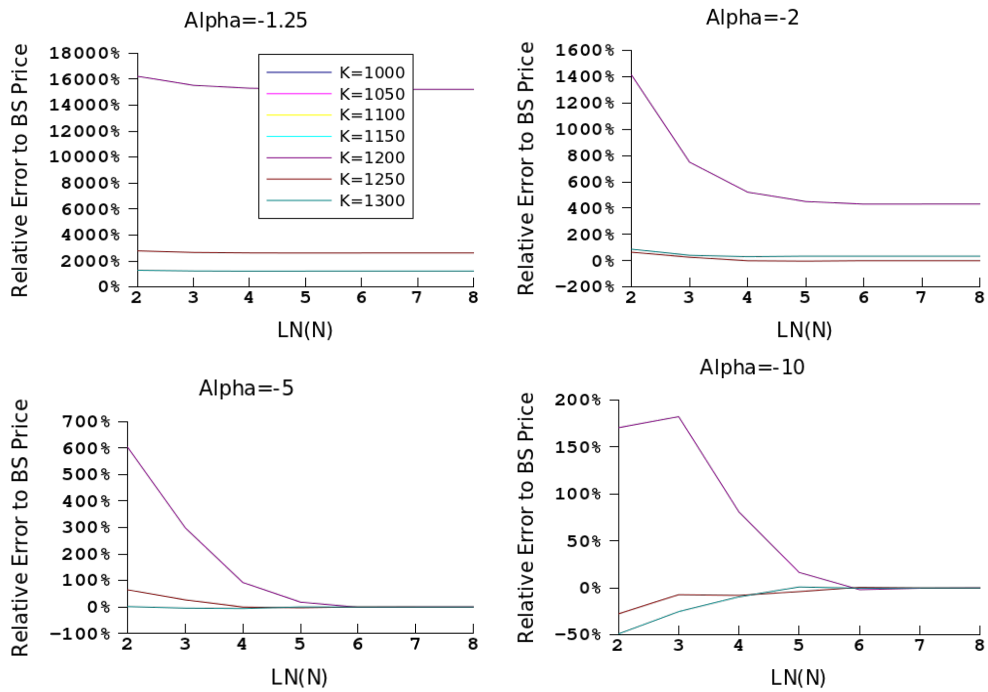

Figure 20: Convergence for FrFFT Method with CGMY$(1.2, 6.5, 8.5, .05)$.

F& O seem to be suitable, although they did not work for the CGMY model with small Y parameters.

In summary, I propose heuristically selected parameter choices (figure 11) and ranges for $\ln(N)$ to get convergence to an error of $< .01\%$ (figure 12). Preference for the upper end of the range is given for options $> 15\%$ out of the money and also for CGMY models with small Y paremeters.

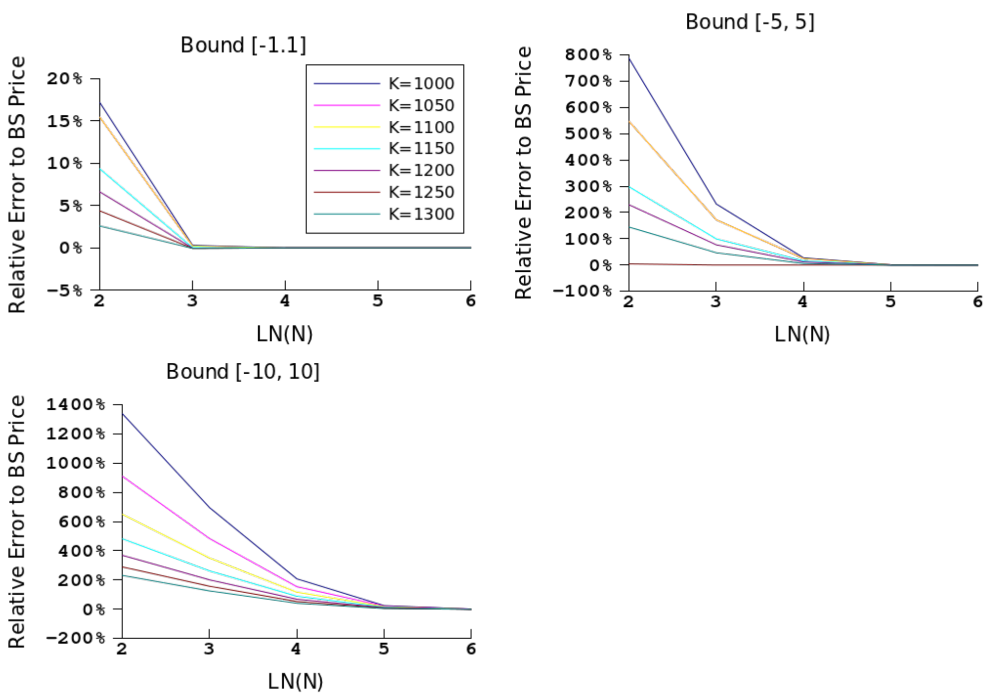

Figure 21: Convergence for COS Method with CGMY$(1.2, 6.5, 8.5, .95)$.

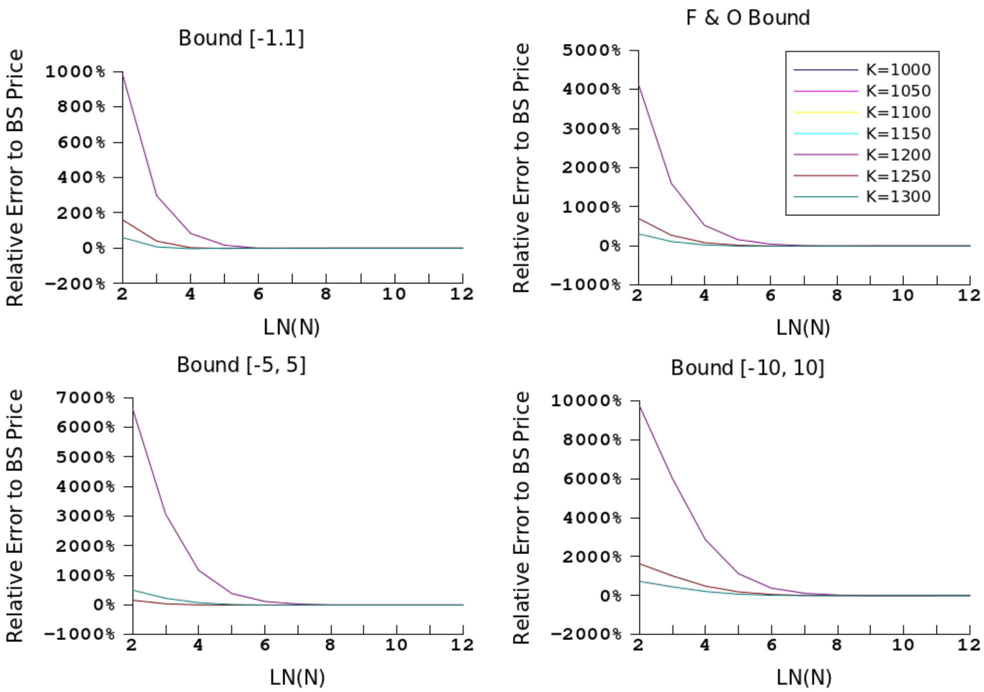

Figure 22: Convergence for COS Method with CGMY$(1.2, 6.5, 8.5, .05)$.

	GBM	**Heston**	**VG**	**CGMY**
FFT	$\alpha = -2, \eta = .1$			
FrFFT	$\alpha = -5, \eta = 1.5$		$\alpha = -2, \eta = .25$	$\alpha = -5, \eta = 1.5$
COS	F &O bound			$[-1, 1]$

Figure 23: FFT CGMY Put Premiums ($\alpha = -2, \eta = .1, N = 4096$).

	GBM	**Heston**	**VG**	**CGMY**
FFT	$7-8$	$7-8$	$8-10$	$6-12$
FrFFT	$4-5$	$4-5$	$6-8$	$4-8$
FrFFT	$4-5$	$3-5$	$4-6$	$6-8$

Figure 24: $\ln(N)$ convergence range.