Columbia University
IEOR 4721: Computational Methods in Derivatives Pricing

Case Study 1 (Due on Wednesday Feb 15, 2012)

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Problem 1:

(a) Geometric Brownian Motion (GBM): the characteristic function of the log of stock price in Black-Scholes framework is:

$$\begin{aligned}\mathbb{E}(e^{iu\ln S_t}) &= \mathbb{E}(e^{ius_t}) \\ &= \exp\left(i(s_0 + (r - q - \frac{\sigma^2}{2})t)u - \frac{1}{2}\sigma^2 u^2 t\right)\end{aligned}$$

Throughtout this case study assume $\sigma = 0.35$.

(b) In Heston stochastic volatility model, stock price follows the process:

$$\begin{aligned}dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)},\end{aligned}$$

where the two Brownian components $W_t^{(1)}$ and $W_t^{(2)}$ are correlated with rate ρ. The parameters κ, θ, and σ have certain physical meanings: κ is the mean reversion speed, θ is the long run variance, and σ is the volatility of the volatility. The characteritics function for the log of stock price process is given by

$$\begin{aligned}\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \frac{\exp\{iu\ln S_0 + i(r-q)tu + \frac{\kappa\theta t(\kappa - i\rho\sigma u)}{\sigma^2}\}}{(\cosh\frac{\gamma t}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma t}{2})^{\frac{2\kappa\theta}{\sigma^2}}} \exp\left\{-\frac{(u^2+iu)v_0}{\gamma\coth\frac{\gamma t}{2} + \kappa - i\rho\sigma u}\right\},\end{aligned}$$

where $\gamma = \sqrt{\sigma^2(u^2 + iu) + (\kappa - i\rho\sigma u)^2}$, and S_0 and v_0 are the initial values for the price process and the volatility process, respectively. Derivation of Heston characteristic function is given in the notes. Throughtout this case study assume volatility of volatility, $\sigma = 30\%$; $\kappa = 1.0$; $\theta = 0.25$; $\rho = -0.8$; $v_0 = 0.10$.

(c) Let $b(t; \theta, \sigma) \equiv \theta t + \sigma W(t)$ be a Brownian motion with constant drift rate θ and volatility σ, where $W(t)$ is a standard Brownian motion. Denote by $\gamma(t; \nu)$, the gamma process with independent gamma increments of mean h and variance νh over non-overlapping intervals of length h.

The three parameter *VG process* $X(t; \sigma, \theta, \nu)$ is defined by

$$X(t; \sigma, \theta, \nu) = b(\gamma(t; \nu), \theta, \sigma).$$

We see that the process $X(t)$ is a Brownian motion with drift evaluated at a gamma time change. The characteristic function for the time t level of the VG process is

$$\phi_{X(t)}(u) = \mathbb{E}(e^{iuX(t)}) = \left(\frac{1}{1 - iu\theta\nu + \sigma^2 u^2 \nu/2}\right)^{\frac{t}{\nu}}. \tag{0.1}$$

The VG dynamics of the stock price mirrors that of geometric Brownian motion for a stock paying a continuous dividend yield of q in an economy with a constant continuously compounded interest rate of r. The risk neutral drift rate for the stock price is $r - q$ and the forward stock price is modeled as the exponential of a VG process normalized by its expectation. Let $S(t)$ be the stock price at time t. The VG risk neutral process for the stock price is given by

$$S(t) = S(0)e^{(r-q)t + X(t) + \omega t}, \tag{0.2}$$

where the normalization factor $e^{\omega t}$ ensures that $\mathbb{E}_0[S(t)] = S(0)e^{(r-q)t}$. It follows from the characteristic function evaluated at $-i$ that

$$\omega = \frac{1}{\nu}\ln(1 - \sigma^2\nu/2 - \theta\nu).$$

Therefore the characteristic function for the log of stock price process is given by

$$\begin{aligned}\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp\left(iu(\ln S_0 + (r - q + \frac{1}{\nu}\ln(1 - \sigma^2\nu/2 - \theta\nu))t)\right)\left(\frac{1}{1 - iu\theta\nu + \sigma^2u^2\nu/2}\right)^{\frac{t}{\nu}}\end{aligned}$$

Throughtout this case study assume $\sigma = 25\%$, $\nu = 0.35$, $\theta = -0.25$.

(d) Characteristic function of the CGMY process with parameters C, G, M, and Y is

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{Ct\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)}$$

The CGMY risk neutral process for the stock price is given by

$$S(t) = S(0)e^{(r-q)t+X(t)+\omega t}, \quad (0.3)$$

where the normalization factor $e^{\omega t}$ ensures that $\mathbb{E}_0[S(t)] = S(0)e^{(r-q)t}$. It follows from the characteristic function evaluated at $-i$ that

$$\omega = -C\Gamma(-Y)\left\{(M-1)^Y - M^Y + (G+1)^Y - G^Y\right\}.$$

Therefore the characteristic function for the log of stock price process is given by

$$\begin{aligned}\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp\left(iu(\ln S_0 + (r-q+\omega)t)\right) e^{Ct\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)}\end{aligned}$$

Throughtout this case study assume $C = 1.2$, $G = 6.5$, $M = 8.5$ and $Y = 0.05\&0.95$

For the following parameters: spot price, $S_0 = \$1300$; maturity, $T = 0.25$ year; risk-free interest rate, $r = 0.25\%$, continuous dividend rate, $q = 1.25\%$ and strike range of $K = 1000, 1050, 1100, 1150, 1200, 1250, 1300$ price European put options for all four models via the following techniques:

(a) Fast Fourier transform (FFT): consider various values for damping factor, α, and for $N = 2^n$.

(b) Fractional fast Fourier transform (FrFFT): consider various values for damping factor, α, and for $N = 2^n$.

(c) Fourier-cosine (COS) method: consider different values for the interval $[a \quad b]$ and find the sensitivity of your results to the choice of $[a \quad b]$.

Compare and conclude.