Labor Markets with Endogenous Job Referral Networks: Theory and Empirical Evidence

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Abstract

This paper studies the emergence of job referral networks as an endogenous response to local labor market conditions. I build a frictional matching model in which workers directly exchange information about job opportunities. The model determines general equilibrium in the density of the job referral network workers use to share job information, jointly with unemployment, vacancies, and earnings. I introduce the concept of network balance as an equilibrium condition linking individual search decisions to aggregate network structure. In the model, referral networks are most dense at moderate levels of labor market tightness – that is, in labor markets where job information is hard, but not impossible, to come by through formal search. Using data on individual referral use from the Cornell National Social Survey, I provide evidence in support of the model's counterintuitive prediction that workers are less likely to find jobs through referral in markets where referrals are more widely used.

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1 Introduction

Economists have long recognized the substantial role of job referrals as a form of search (Rees 1966; Ioannides and Loury 2004). The spread of communications technology seems to have strengthened, rather than weakened, local social networks (Mok et al. 2010), reinforcing their role as intermediaries of labor market information. A growing set of recent studies demonstrate empirically that job referrals and local social interactions matter, both for finding a job, and in terms of what happens on the job(Loury 2006; Bayer et al. 2008; Hellerstein et al. 2008; Goel and Lang 2009; Beaman and Magruder 2010; Schmutte 2010; Dustmann et al. 2011). Incorporating job referrals and social interactions thus seems likely to be a fruitful approach to improving the explanatory power of labor market models.

This paper develops and tests the predictions of a model in which local labor market conditions and the density of job referral networks are jointly determined in dynamic equilibrium. The model combines two basic insights. The first is that a referral network, as an informal labor market intermediary, emerges endogenously in response to labor market conditions. Workers use referrals more when jobs are hard to find through other means. The underlying ebb and flow of job information through referral networks is therefore an emergent outcome of the choices of individual workers deciding how to participate in job search. The second insight is that the endogenous evolution of this intermediation service may exacerbate variation across time and space in aggregate labor market outcomes. Together, the model in this paper explains how social institutions interact with labor market conditions, and how variability in labor market conditions can be driven by social institutions.

Very few papers attempt to endogenize referral networks in any setting, and this is the first to do so in a dynamic labor market model. I closely follow Calvó-Armengol and Zenou (2005), altering their model to allow the density of the underlying social network to be determined endogenously with unemployment, the vacancy rate, and wages. Workers respond to labor market conditions by varying the intensity with which they search for employment by seeking referrals. I introduce "Network Balance" as a condition determining the equilibrium density of the job referral network. The Network Balance condition requires that the amount of job information flowing to workers seeking referral be equal to the amount of job information flowing from workers giving referral. By adding this condition, the density of the referral network is a well-defined equilibrium outcome of individual decisions about referral-seeking effort, and is jointly determined with the unemployment rate, vacancy rate, and wages. The referral network greases the wheels of the labor market, but in a manner that depends on, and hence can magnify, labor market conditions.

When job search through referrals is costly, congestion leads workers to search through the referral network less intensively. This key behavioral prediction has the testable implication that the probability of being hired through referral, conditional on being hired at all, is decreasing in the density of the local referral network. To test this prediction, I use individual-level data from the Cornell National Social Survey (CNSS) that include information on whether workers found their most recent job through referrals. The data also record the Census block group in which each individual resides. I combine these data with aggregate characteristics of the local labor market, including unemployment and job flow rates. The key variable for the test is a measure from the Current Population Survey (CPS) of the intensity with which people utilize referrals in their job search. Under the model, this CPS variable provides a measure of the density of the referral network. The model predicts that the probability that a worker in the CNSS is hired through referral is decreasing in the CPS measure of referral density.

In the preferred specification, as reported in Section 4.3, a one standard deviation increase in the density of the local referral network is associated with a five percentage point decrease in the probability of being hired through referral. The result is robust to the inclusion of controls for individual and labor market conditions that might be correlated with referral network density and individual referral use. Without a clear source of random variation in residential location, the main empirical result should be interpreted with some caution. However, since sorting on unobservables would predict a positive correlation between referral network density and hiring through referral, the robustness of the negative correlation is supportive of the theoretical model, and more broadly of interest to those working on the role of referrals in individual labor market outcomes.

In Section 2, I describe the contribution of this paper. I go on to develop the model in Section 3, focusing on the marginal contribution of the Network Balance condition to close the general equilibrium in labor market conditions and referral network density. Section 4 further develops the predictions of the model regarding the link between individual outcomes and local labor market conditions. I then describe the data and present the results of the empirical test. In Section 5, I report conclusions and discuss possible avenues for future work.

2 Related Literature

The present paper joins a broad theoretical literature that focuses on referral networks as conduits for information about employment opportunities. Arrow and Borzekowski (2004) and Calvo-Armengol and Jackson (2004) use partial equilibrium models to show that the structure of social networks can generate persistent differences in employment and earnings across groups of workers. Calvó-Armengol and Zenou (2005) and Fontaine (2008), push the same intuition into a general equilibrium framework in which it is possible to ask whether the partial equilibrium outcomes hold up and also how exogenous referral network structures affect the aggregate labor market. Galenianos (2011) considers a similar model, but allows for heterogeneous agents and the possibility that referrals assist with screening.

The challenge of endogenizing job referral networks has been addressed in a very nice, closely related, paper. Galeotti and Merlino (2010) build a static labor market model with endogenous job contact networks. Their paper provides a clear microfoundation of the network formation process as a strategic game played in advance of labor market matching. My approach to making referral networks endogenous is simpler. I introduce the concept of "network balance" as an equilibrium condition to endogenize referral network structure. This condition provides an explicit link between individual choices and the aggregate outcomes that are the focus of the model. Furthermore, as I demonstrate in the text, and exploit empirically, this formulation links the structure of the referral network to a measurable moment in aggregate data from the Current Population Survey (CPS). I trade off the clear, explicit, microfoundations of Galeotti and Merlino (2010) against the ability to complete a dynamic general equilibrium model in which referral network density is endogenously determined along with unemployment, vacancies, and wages. One key theoretical result, that referral network density is non-monotonic with respect to labor market tightness, is common across both papers. That these two complementary approaches arrive at similar conclusions reflects the strength of the underlying idea. The two approaches provide different options for future research that extends the applications of endogenous referral networks.

This paper also contributes to a group of microeconometric studies of the link between local labor market conditions and the productivity of referrals. Galeotti and Merlino (2010) also use data from the UK to find that productivity of referral is non-linear in the regional separation rate. I add to this literature by documenting that referral productivity is also decreasing in the intensity of referral use. My model also predicts a nonlinearity in the formal offer arrival rate, which I cannot directly measure. My estimates of the effect of the hiring rate, which is strongly positively correlated with the separation rate, are positive, but imprecisely estimated.

Wahba and Zenou (2005), using Egyptian data, find support for the prediction in Calvó-Armengol and Zenou (2005), that the probability of being hired through referral should be 'hump-shaped' in referral network density. In their model, referral networks are exogenous. Unlike their paper, I use reported referral use to measure referral network density rather than population density. In the empirical work, I draw on Wahba and Zenou (2005) by including a quadratic term in population density as an additional control for the cost of search through referral. Further progress in this area relies on finding sources of data that are both geographically detailed and include information on referral use.

3 Model

The model is an extension of Calvó-Armengol and Zenou (2005) that allows for endogenous density of the job referral network. Interest centers on the steady-state equilibrium of a matching model augmented with endogenous job referral networks. Making the structure of the referral network endogenous requires an additional equilibrium condition, for which I introduce the concept of network balance. Network balance requires that the net intensity of requests for information within the network is balanced with the flow of information provided by the network. The equilibrium referral network responds to the extent of information flowing through the labor market, which is then determined by the optimizing behavior of firms maximizing the present value of vacancies. The model also predicts endogenous spillovers across the choice of referral-seeking intensity on the part of workers, and, in particular, that job-finding through referral is negatively related to the intensity of referral use in one's local labor market. I provide novel evidence in support of these implications in Section 4.

3.1 Setup of the Model

Workers begin each period either employed or unemployed, and choose a contact intensity that determines the amount of effort put into seeking referrals. Firms then decide whether or not to open vacancies. Having created vacancies, employers issue job offers according to a Poisson matching process. Workers who are unemployed and receive an offer retain it. An employed worker who receives a new job offer will transfer it to an unemployed worker requesting information from her. If there is more than one unemployed person requesting information, she chooses someone at random. At the end of this transfer stage, all unemployed workers with job offers become employed, and all job offers that have not been transferred by employed workers expire.

Time is discrete and workers and firms are infinitely lived. There is a continuum of measure μ of identical workers, a fraction of whom begin the period employed. Workers discount the future at rate ρ, maximize wealth, and receive utility b when unemployed. There is a continuum of measure 1 of identical firms. The unemployment rate at the beginning of any period t is u_{t-1}. Employers open a total of V_t vacancies. $\lambda_t = V_t/\mu$ is the formal recruiting intensity on the part of employers. Thus, an individual worker hears about at least one vacancy through the formal market with probability equal to $1 - e^{\lambda_t}$. Each filled vacancy produces y_0 units of output when initially occupied, and $y_1 \geq y_0$ in each subsequent period that the position is filled.

Workers also search for jobs through referral by asking other workers for job information. Each worker, i, chooses a contact intensity, γ_i, that determines how many workers he will query for a job offer. The number of workers contacted follows a Poisson distribution with mean γ_i. Both forms of search are undirected. Firms can not discriminate between employed and unemployed workers in sending job offers, and workers can not discriminate between unemployed and employed workers when searching for referrals[1]. Workers also do not know how many other workers are asking any given contact for a referral. They must forecast the expected number of competitors for information.

For a given worker, j, who is contacted by i for a referral, let X_j be the number of requests for referrals. X_j is Poisson with mean $E(X_j) = s$. All contacts are homogeneous, so we assume $E(X_j) = E(X_k) = s$. Note that the referral network at t is completely characterized by the single parameter s. Call s the 'density' of the referral network. In simple random graphs, this is the characteristic feature of the network in the sense that the average degree of nodes in a simple random graph is a sufficient statistic for the graph structure.

Once all job offers have been transferred to their final destinations, unemployed workers become employed if they receive at least one offer and begin to produce. In the first period of employment, they earn the 'training' wage, w_0 and earn w_1 in subsequent peri-

[1]The latter assumption may seem counterintuitive, since workers should know who is employed and therefore be able to direct their search for referrals. This assumption is not essential for the results, but makes the problem simpler to set up.

ods.

3.2 The Within-Period Flow of Job Information

I next focus on how job information moves through the labor market within each period. Workers are small relative to the market, and so do not take account of how their actions might affect aggregate outcomes. Taking the unemployment rate, vacancy rate, and referral network density as given, workers choose contact intensity to maximize the present value of lifetime utility. When the economy is in steady-state, there is no inter-temporal dimension to this choice. For the rest of this section, I suppress time subscripts.

The decision to use referrals is determined by how productive they are in generating offers. When a worker requests a referral, what is the probability that he receives one? It is the probability of the joint event that his contact is employed, is holding a job offer, and chooses to give him the offer from among all the other people asking for a referral. Assume the worker contacted for a referral is worker j. Since the process of contacting workers for referral is undirected, the probability that worker j is employed is simply $1-u$. Since the formal job search process is also undirected, given the Poisson assumption, the probability that j holds a job offer is $1 - e^{-\lambda}$.

The probability that worker j chooses to give worker i the offer from among all people asking her for a referral is $\mathrm{E}[\frac{1}{k+1}]$ where k is the number of competing referral requests.[2] k is also Poisson with parameter s, and $\mathrm{E}[\frac{1}{k+1}] = \frac{(1-e^{-s})}{s}$.(see Appendix A). To highlight the economic intuition, observe that the probability limit of receiving an offer, as the expected number of competitors, s, goes to 0, is 1. Likewise, as s goes to infinity, the probability of getting an offer through referral goes to zero. Also, it is easy to show that this probability is always decreasing in s, since $s > 0$ always.

The probability that a worker who was asked for a referral actually provides one is:

$$\pi(u, \lambda, s) = \frac{(1-u)(1-e^{-\lambda})(1-e^{-s})}{s} \tag{1}$$

This expression has a number of intuitively appealing properties. If $u = 1$, so that all workers are unemployed, the probability of getting an offer through referral is 0. Likewise, if there are no jobs coming from employers, $\lambda = 0$ and $\pi(u, 0, s) = 0$. An immediate implication is that the number of productive requests for job information is distributed

[2]Technically, an employed worker can receive more than one job offer, as can an unemployed worker. If we assume that contacts distribute all unused offers, then the productivity of referrals becomes more complicated. These complications do not add insight or change the qualitative results.

Poisson with parameter $\gamma_i \pi(u, v, s)$. Let $P(u, \lambda, s, \gamma_i)$ be the probability of getting at least one offer through referral. Then

$$P(u, \lambda, s, \gamma_i) = 1 - \exp[-\gamma_i \pi(u, \lambda, s)]. \tag{2}$$

Using the properties of Poisson random variables again, the number of job offers a worker receives from all sources is Poisson distributed with rate parameter $\lambda + \gamma_i \pi(u, \lambda, s)$. So, the probability that an unemployed worker gets hired is

$$h(u, \lambda, s, \gamma_i) = 1 - \exp[-(\lambda + \gamma_i \pi(u, \lambda, s))]. \tag{3}$$

3.3 The Effort Put into Seeking a Referral

As in Holzer (1988), the intensity of referral search is chosen to maximize individual utility, given the expected productivity of their use. Referral productivity depends on local labor market conditions and the extent of competition for referrals. It may also depend on individual characteristics that influence the cost of using referrals and the benefits of finding work. Let K_i denote the value to worker i of obtaining employment, and let c_i denote the constant marginal cost of intensifying the search for a referral.[3] The worker's choice problem is:

$$\max_{\gamma_i \geq 0} (h(u, \lambda, s, \gamma_i)) K_i - c_i \gamma_i. \tag{4}$$

The contact intensity, γ_i, is the solution to the Kuhn-Tucker condition:

$$\begin{aligned} \gamma_i \left[\frac{\partial h}{\partial \gamma_i} K_i - c_i \right] &= 0 \qquad (5) \\ \gamma_i &\geq 0. \qquad (6) \end{aligned}$$

[3]Here, I allow costs and benefits of search to vary across workers. In the next section, I assume that all workers are homogeneous for the purpose of deriving the steady-state equilibrium of the matching model. The constant marginal cost assumption is made for convenience of exposition, but is not required for the qualitative results.

At an interior solution, the marginal increase in the probability of being hired associated with increasing search effort is equated with the cost-benefit ratio:

$$\begin{aligned}\frac{\partial h}{\partial \gamma_i} &= \frac{c_i}{K_i} &(7)\\ \pi(u,\lambda,s)\left[1-h(u,\lambda,s,\gamma_i)\right] &= \frac{c_i}{K_i} &(8)\\ \pi(u,\lambda,s)\left[\exp(-(\lambda+\gamma_i\pi(u,\lambda,s)))\right] &= \frac{c_i}{K_i}. &(9)\end{aligned}$$

The second equation shows that the marginal impact of increasing contact intensity on the probability of being hired is increasing in referral productivity, π, and decreasing in the hiring probability. It is equal to the probability that a worker gets a referral in the case that he would not have otherwise been hired. The marginal benefit of increasing γ_i is the expected increase in probability of getting an offer through referral conditional on not receiving an offer through the formal market.

The third equation can be solved for γ_i at an interior solution:

$$\gamma_i = \frac{\ln \pi(u,\lambda,s) - \lambda - \ln \frac{c_i}{K_i}}{\pi(u,\lambda,s)}. \tag{10}$$

Increases in the productivity of referrals through π have opposing effects on contact intensity. At the intensive margin, an increase in π means that any person contacted for referral is more likely to be provide one. At the extensive margin, for any given number of referral contacts, a larger fraction of them are expected to be productive. These two effects create opposing incentives for the choice of contact intensity.

We can get some immediate predictions on referral use by noting that corner solutions are possible. $\gamma_i > 0$ requires

$$\pi(u,\lambda,s)e^{-\lambda} > \frac{c_i}{Ki}. \tag{11}$$

This result implies that there will be selection to referral use on the basis of $\frac{c_i}{K_i}$ for workers in the same labor market. Alternatively, among workers with the same relative cost of using referrals, those in labor markets with higher unemployment and more competition for referrals (s) will be less likely to use referrals. The relationship between γ_i and the arrival rate of offers, λ, is non-monotonic, increasing when λ is low, suggesting that at low levels increases in λ will push more workers to use referrals because a higher arrival rate makes them more productive. As the arrival rate increases, workers are increasingly likely to find employment through formal means, making the cost of referrals less attractive.[4]

3.4 The Equilibrium Network Density

I now describe how the overall network density is related to the individual contact intensities. Workers respond to expected competition for job information from each worker they contact. Thinking in terms of a referral network, where requests for job information are directed edges, the number of people requesting information from worker j is the 'in-degree' of j. A form of balance must obtain in such a network. All of the 'incoming' requests for information received by workers are also 'outgoing' requests for information sent by workers. This provides a balancing condition: the expected number of incoming requests for job information must equal the expected number of outgoing requests. It follows in this setting that

Claim 1

$$s = \bar{\gamma} u,$$

where $\bar{\gamma}$ is the common choice of γ among unemployed workers.

(Proof in Appendix)

Since $\bar{\gamma}$ is a function of s determined as the solution to the worker's single-period choice problem, we have

$$s = \bar{\gamma}(u, v, s)u. \tag{12}$$

We define the equilibrium network density s^* as the fixed point in s, if one exists, to the above expression.

Claim 2 *s^* exists and is unique.*

(Proof in Appendix)

3.5 Labor Market Equilibrium

We now embed the equilibrium network density into the dynamic optimization problem for firms and workers to obtain steady state equilibrium values $(u^*, v^*, w_1^*, \gamma^*)$.

[4] In evaluating these predictions in the data, there is a slight complication, since workers who choose positive, but small, values of γ may end up making zero contacts by chance. This means that there will be some one-sided noise around the selection margin. A stochastic frontier approach may be appropriate for dealing with the problem. This paper avoids the issue by assuming contacts are positive.

The Worker's problem

$I_{E,t}$ is the present value of the utility stream associated with being employed

$$I_{E,t} = w_1 + \frac{1}{1+r}\left[(1-\delta)\, I_{E,t} + \delta I_{U,t}\right]. \tag{13}$$

$I_{U,t}$ is the present value of the utility stream associated with being unemployed:

$$I_{U,t} = \max_{\gamma_t}\left\{b - c(\gamma_t) + h(u_{t-1}, v_t, \gamma_t, s_t)\left[w_0 + \frac{1}{1+r}\left((1-\delta)\, I_{E,t} + \delta I_{U,t}\right)\right]\right\}. \tag{14}$$

I already showed in Section 3.3 how the worker chooses γ_t for given $(u_{t-1}, v_t, w_{1,t})$ along with the constant K, which we now see includes the present discounted value of future earnings streams associated with being employed. The choice of $\gamma_{i,t}$ only affects this through the hiring probability h.

I proceed by substituting into Equation (14) the within-period solution γ_t that satisfies the Kuhn-Tucker condition and network equilibrium, $s_t = \gamma_t u_{t-1}$, so that

$$I_{U,t} = b - c(\gamma_t) + h(u_{t-1}, v_t, \gamma_t)\left[w_0 + \frac{1}{1+r}\left((1-\delta)\, I_{E,t} + \delta I_{U,t}\right)\right]. \tag{15}$$

I follow the literature in assuming that the outside option, $b = 0$, and competitive pressure dictates $w_0 = b = 0$. It follows that the steady state increment to utility associated with finding a job is

$$I_E - I_U = \frac{1+r}{r + \delta + (1-\delta)h(u, v, \gamma)}\left(w_1 - c(\gamma)\right). \tag{16}$$

The Employer's Problem

Firms open vacancies according to a free entry condition: the expected profit stream associated with the marginal vacancy is 0. Vacancies cost κ per period to keep open. A given vacancy is filled with probability $f(u_{t-1}, v_t, s_t)$.

The number of new job matches in the economy in period t is

$$u_{t-1}\mu h(u_{t-1}, v_t, \gamma_t), \tag{17}$$

so the rate at which vacancies are matched to unemployed workers must be

$$m(u_{t-1}, v_t, \gamma_t) = u_{t-1} h(u_{t-1}, v_t, \gamma_t). \tag{18}$$

Hence, the probability that a particular vacancy is filled is equal to

$$f(u_{t-1}, v_t, s_t) = \frac{m(u_{t-1}, v_t, \gamma_t)}{v_t}. \tag{19}$$

This implies

$$\frac{f(u_{t-1}, v_t, s_t)}{h(u_{t-1}, v_t, \gamma_t)} = \frac{u_{t-1}}{v_t}. \tag{20}$$

Here, as in Calvó-Armengol and Zenou (2005), because the matching function is not homogeneous of degree 1, the v/u ratio is not a sufficient statistic to characterize tightness.

The present value of the profit stream associated with a filled vacancy is

$$I_{F,t} = y_1 - w_1 + \frac{1}{1+r}\left[(1-\delta)\, I_{F,t} + \delta I_{V,t}\right]. \tag{21}$$

The present value of the profit stream associated with an unfilled vacancy is

$$I_{V,t} = -\kappa + \frac{1}{1+r}\left\{(1 - f(u_{t-1}, v_t, s_t))\, I_{V,t} + f(u_{t-1}, v_t, s_t)\left[(1-\delta)\, I_{F,t} + \delta I_{V,t}\right]\right\}. \tag{22}$$

Free entry implies $I_{V,t} = 0$. Invoking the steady state assumption and rearranging terms yields

$$I_F = \left(\frac{1+r}{1-\delta}\right)\frac{\kappa}{f(u, v, s)} \tag{23}$$

and the "labor demand curve"

$$\frac{y_1 - w_1}{r + \delta} = \left(\frac{1}{1-\delta}\right)\frac{\kappa}{f(u, v, s)}. \tag{24}$$

Wages

Workers and firms bargain over the surplus associated with the match. The wage, w_1 acts to transfer the surplus wealth from the firm to the worker. The bargaining solution is determined as the solution to the cooperative Nash problem

$$w_1 = \arg\max\,(I_E - I_U)^{\beta}\,(I_F - I_v)^{(1-\beta)}. \tag{25}$$

The new parameter, $0 \le \beta \le 1$ is the worker's bargaining power. Solving this problem yields:

$$w_1 = \beta\left(y_1 + \kappa\frac{v}{u}\right) + (1-\beta)c(\gamma). \tag{26}$$

3.6 Steady State Labor Market Equilibrium

The steady state equilibrium is defined as a tuple $(u^*, v^*, w_1^*, \gamma^*)$ of unemployment rate, vacancy rate, wage, and referral network density that jointly satisfy the four equilibrium conditions:

Beveridge Curve

$$(1-\delta)m(u,v,s) = \delta(1-u) \tag{27}$$

Labor Demand

$$\frac{y_1 - w_1}{r+\delta} = \left(\frac{1}{1-\delta}\right)\frac{\kappa}{f(u,v,s)} \tag{28}$$

Wage Setting

$$w_1 = \beta\left(y_1 + \kappa\frac{v}{u}\right) + (1-\beta)c(\gamma) \tag{29}$$

Network Balance

$$\frac{(1-u)v(1-v)}{\gamma^* u + 1}\exp\left[-[\gamma^*\frac{(1-u)v}{\gamma^* u+1}]\right]K \leq c'(\gamma^*) \tag{30}$$

The evolution of unemployment over time is given by the difference between the flow into unemployment and the flow out of unemployment

$$u_t - u_{t-1} = \delta(1-u_{t-1}) - (1-\delta)u_{t-1}h(u_{t-1}, v_t, \gamma_t) \tag{31}$$

In steady state,

$$(1-\delta)m(u,v,s) = \delta(1-u). \tag{32}$$

The nature of the equilibrium relationships is illustrated in Figure 2. The figure is plotted in the (u, v) domain to highlight the similarity with a conventional matching model. The Beveridge Curve and Job Creation locus retain their conventional shape. In addition to u and v, we add the equilibrium referral network density parameter, γ.

Note that the Network Balance locus is 'hump-shaped' in (u, v). Any unemployment rate can support two different vacancy rates at a particular level of referral use. The intuition behind this result is related to the reason the probability of finding a job through referral is non-monotonic in the offer arrival rate. When the vacancy rate is low, increases in the vacancy rate tend to drive workers to seek referrals, because they are more productive. To offset this effect, unemployment must rise. Eventually, the vacancy rate becomes high enough that further increases drive workers away from referrals, because they are unnecessary. At that point, the unemployment rate must decrease to draw workers back to referral use.[5]

Network density, γ is increasing within the region bounded by the Network Balance curve. An inward shift of the Job Creation curve reduces vacancies, increases unemployment, and increases referral density. More plainly, a demand-driven recession increases referral network density in the simulated economy documented in Figure 2. By contrast, an outward shift of the Beveridge Curve should increase vacancies, increase unemployment, and reduce referral density. Thus a recession associated with structural changes that affect the matching technology will reduce referral use. Of course, these relationships are specific to the chosen example. The model also predicts that reservation wages, and hence bargained wages, will change over the business cycle as well. Wages are increasing in the referral network density, since the cost of search through referral also increases. This model therefore also provides an alternate, potentially complementary explanation for the Shimer puzzle, since wage expectations can change counter-cyclically, which means productivity shocks will be amplified relative to a case with a constant outside option. [6]

4 Does Congestion Matter? An Empirical Analysis

The model motivates an exploration of the relationship between local labor market conditions and the productivity of job referral. In the steady-state equilibrium, the model makes strong comparative static predictions based on the assumption that there are endogenous spillovers in referral use. It specifically predicts that the probability that a worker is hired through referral is negatively impacted by the density of the referral network. In this section, I use novel data from the Cornell National Social Survey, together with data on local labor market conditions, to test this prediction. I use data on the aggregate referral-seeking intensity of the unemployed collected in the Current Population Survey (CPS), and show how this data moment can be used to measure the density of the referral network.

[5]The non-monotonicity in the Network Balance equation is consistent with a similar finding by Galeotti and Merlino (2010). Using a static model and a different mechanism for making referrals endogenous, they show that referral use is non-monotonic in the economy's separation rate. The same kind of congestion-driven relationship also appears in a different form in Calvó-Armengol and Zenou (2005) and Wahba and Zenou (2005).

[6]Referrals are generally thought to be a relatively inexpensive form of search, so this mechanism is unlikely to explain a substantial portion of excess volatility in unemployment. Even less is known about the relative costs of search through different channels than is known about the presence of referrals. One study, Caliendo et al. (2010), finds that reservation wages are increasing in the size of the individuals referral network, albeit in a different context.

The model makes stark predictions on individual behavior, and in particular, on the relationship between aggregate variables, including aggregate referral use, and the probability that any individual is hired through referral. The essential counterintuitive prediction of the paper is that the probability of being hired through referral should be decreasing in referral network density. That is, when more people are searching through referral, the probability of being hired through referral falls.

4.1 Individual-Level Predictions

Substituting the solution value for the steady-state equilibrium referral network density, γ_i^*, from Equation 10, into the equation for the probability of obtaining a referral, $P(u, \lambda, s, \gamma_i^*)$, Equation 2, yields

$$P(u, \lambda, s, \gamma_i^*) = 1 - \frac{c_i e^{\lambda}}{K_i \pi(u, \lambda, s)}. \tag{33}$$

Recall that u and λ are steady state unemployment and recruiting intensity. The variable s is the worker's choice of referral-seeking intensity, c_i is the marginal cost of search through referral, and K_i is the value of employment. It follows that

$$R(u, \lambda, \bar{\gamma}, K_i, c_i; q) = (q + (1-q)e^{-\lambda})P(u, \lambda, s, \gamma_i^*) \tag{34}$$

is the probability of being hired through referral, where q is the probability that a worker prefers a referral when one is available.

The model produces the following comparative static predictions, which are derived in the Appendix:

1. $\frac{\partial R}{\partial \bar{\gamma}} < 0$. The probability of being hired through referral is strictly decreasing in referral network density.

2. $\frac{\partial R}{\partial u} < 0$. The probability of being hired through referral is strictly decreasing in the unemployment rate.

3. There exists a λ^* such that $\frac{\partial R}{\partial \lambda} > 0$ when $\lambda < \lambda^*$ and $\frac{\partial R}{\partial \lambda} < 0$ when $\lambda \geq \lambda^*$.

$\frac{\partial R}{\partial \bar{\gamma}} < 0$ occurs because the increase in referral-seeking on the part of unemployed workers unambiguously makes referrals less productive, and makes workers less likely to seek them. Similarly, $\frac{\partial R}{\partial u} < 0$ because higher rates of unemployment also increase the competition for referrals and make them less productive. Increases in the offer arrival rate, λ, have two offsetting effects. The direct effect of a higher offer rate is to increase the probability that a worker gets an offer directly, and so does not need referral. The indirect effect is that referrals, when used, are more likely to be productive. The indirect effect dominates the direct effect when the offer rate is very low. The critical value, λ^*, is the formal offer rate at which the direct effect begins to dominate. The extent to which workers prefer offers obtained through referral to offers obtained directly, measured by the variable q, affects the critical value. Specifically, if $q = 1$, then $\lambda^* > 0$. Increases in q decrease $\lambda^* > 0$, and there are parameterizations of the model with $q \in (0, 1)$ where $\lambda^* < 0$.

4.2 Data

Data to test the model's predictions combine measurements of job referral productivity collected in the Cornell National Social Survey (CNSS) with data on local labor market conditions from a variety of sources. The objective is, first, to assess whether local labor market conditions are important in determining referral use and productivity, and, second, to determine whether the structure of the job information network has an impact on the use and productivity of referrals for individual workers. The nature of the local job information network is captured by using information on the job search behavior of people in one's local labor market.

Cornell National Social Survey

The Cornell National Social Survey was a random digit dial survey collected by the Survey Research Institute at Cornell University in November and December 2008. The sample was meant to be representative of the total U.S. population. The survey asked respondents to report whether they obtained their most recent job through the referral of a friend or relative who already worked for the same employer. In addition, the survey collected a range of demographic, work history, and opinion questions.[7] Equally important for studying the relationship between job referrals and local labor market conditions is that that data have geographic detail that extends to the level of the Census block group. I use the geo-codes on the CNSS to merge data on local labor market conditions confronting each individual.

[7]In addition to a battery of standard demographic questions, the CNSS allowed various researchers from Cornell University to submit questions that would be administered as part of the survey. The crucial question used in this paper on referral behavior was sponsored by Brian Rubineau. I also use some of the work history questions sponsored by Yael Levitte and Matthew Freedman. I am indebted to all three for their generosity in letting me use these data for this analysis

To measure local labor market conditions at the time of referral, I need to know when the referral occurred and where the worker was searching for work at that time. The job referral question in the CNSS applies to a worker's most recent job. To determine when they were searching for that job, I use information from another question on completed tenure, which is available only for currently employed workers. The tenure question is only sufficiently finely-grained for workers who obtained their most recent job within the last two years. For those workers, I can observe whether they have moved since starting their job. Imposing these restrictions leaves a sample of 168 workers. Complete data on all local labor market characteristics are available for 142 of these. The small sample size limits the power of my statistical tests. Given the data, my finding that the effect of referral network density decreases the productivity of referrals is strong and robust, and, as I will argue, difficult to explain on the basis of alternative mechanisms.

Data on Local Labor Market Conditions

The local labor market variables include measures of unemployment, job accession and separation rates, and the intensity of the use of personal contacts to search for work. County-level unemployment rate data come from the Local Area Unemployment Statistics (LAUS) of the BLS. The job accession and separation data are from the Quarterly Workforce Indicators (QWI) developed by the Longitudinal Employer and Household Dynamics (LEHD) program of the U.S. Census Bureau and are also included as county level aggregates. Finally, data on referral use as a form of search are collected in the Basic Current Population Survey.

Referral Network Density

The other key variable in the model is the density of the referral network, which measures the individual's expectation of the amount of competition for referrals. I use data on the use of referrals in job search from the Basic Monthly Current Population Survey (CPS). As part of the standard battery of questions on job search, the CPS asks unemployed individuals to list various forms of search, including whether they have contacted friends and neighbors to find work. I aggregate these data to construct annual state-level measures of the fraction of unemployed workers who contacted at least one person as part of their job search. Let p_k^{ref} be the fraction of unemployed workers in state k who contacted at least one friend as measured in the CPS. From this, I can back out $\bar{\gamma}$ as

$$\bar{\gamma_k} = -\log(1 - p_k^{ref}), \tag{35}$$

which follows from the fact that the probability of contacting at least one friend, given a referral search intensity of $\bar{\gamma_k}$, is $1 - e^{-\bar{\gamma}_k}$.[8]

4.3 Results: The Probability of Being Hired By Referral

Table 1 presents summary statistics for the full CNSS and the final sample. The sample is, as expected, younger, but otherwise demographically similar to, the full population. About 30 percent of workers report being referred for their current job, which is consistent with the fraction reported in the survey in Ioannides and Loury (2004). The table also reports the sample mean of the local labor market variables across CNSS workers. The average worker confronts a market in which 22 percent of unemployed workers report using personal contacts to find work.

Table 2 presents results of estimating probit models that predict whether CNSS workers obtained their current job through referral. Of primary interest is the effect of other workers' decisions to use referral in job search on the probability of being hired through referral. The parameter on γ measures this effect. The data consistently show a strong negative correlation between referral network density and the probability of being hired through referral. In all specifications, the marginal effect of referral network density is around -1.1. At face value, this means that a one standard deviation increase in referral network density is associated with a five percentage point decrease in the probability of being hired through referral. Since the baseline probability of being hired through referral is about 30 percent, this is a large estimate.

The models in column (1) and column (2) compare the explanatory power of individual characteristics and local labor market conditions. Local labor market conditions are at least as good at predicting variation in the productivity of referral as individual demographic and human capital characteristics.

Of course there is concern that these models are plagued by self-selection and omitted variables. The measure of referral network density could simply be absorbing omitted factors that make the probability of being hired through referral less likely. If so, then controlling for factors that directly affect the probability of being hired through referral

[8] The empirical results are qualitatively the same whether I use p_k^{ref} directly or the transformation to $\bar{\gamma}$. I report my results using $\bar{\gamma}$ to maintain consistency with the model.

should attenuate the omitted variable bias.

Columns (3), (4) and (5) attempt to control for factors that influence the costs and benefits of search through referral. Population density was suggested by Wahba and Zenou (2005) as a proxy for network density. I use tract-level population density as a proxy for the cost of contact. Workers in higher density tracts will find it easier to search through referral, all else the same. The coefficient on density is positive, although imprecisely estimated, and the quadratic term is negative, as in Wahba and Zenou (2005). In column (5), I reintroduce individual-level unobservables correlated with referral use and productivity. This attenuates the estimate of γ, but only modestly.

To understand the significance of this finding, note that selection on unobservables would predict that workers in states with high levels of referral use will also use referrals a lot. The persistent negative correlation between referral network density and referral productivity is counterintuitive, lending support to the model proposed in this paper. Of course, other models are consistent with this finding. For instance, if employers are less likely to hire through referral when the number of people seeking referrals is high, we would see the same negative correlation.

Note also that the results do not uphold the other comparative static predictions. In the model, the unemployment rate is predicted to decrease the probability of being hired through referral. In the data, however, there is essentially no relationship between county-level unemployment and being hired through referral. We do not have a good proxy for the vacancy rate nor for the formal offer arrival rate. The county-level hiring rate is a potential proxy for the overall rate of recruiting intensity. This has a positive but insignificant point estimate. Since the rates of hiring and separation tend to move together, this finding is somewhat at odds with Galeotti and Merlino (2010) who find a negative relationship between the separation rate and hiring through referral.

5 Conclusion

I have presented a frictional matching model with an endogenous job referral network. The key structural feature of the referral network determined in the model is its steady-state density. Workers adjust their referral-seeking behavior in response to local labor market conditions, including an endogenous spillover mediated through the referral network density. In equilibrium, referral network density increases when markets are slack, consistent with the intuition that referral networks as a social institution emerge endogenously to facilitate exchange.

As a check of the modeling assumptions, I have also provided new information on the link between referral productivity and aggregate referral use. The data provide consistent evidence that referral productivity is negatively correlated with aggregate referral use. This stylized fact is consistent with the model, and should be of interest to those studying the microeconomic determinants of referral use and productivity.

Empirical work in this area is compromised by the lack of data that combine good geographic detail with information on referral use. Nevertheless, it should be possible to use the existing theoretical framework with aggregate data on referral use from the CPS to evaluate the model across variation in employment conditions and referral use across U.S. States. The implications of this model for explaining local labor market variability and volatility, and the effects of social networks on the efficacy of labor market policy are topics for future research.

References

Arrow, K. J. and Borzekowski, R. (2004). Limited network connections and the distribution of wages, *Finance and economics discussion series*, Washington: Board of Governors of the Federal Reserve System.

Bayer, P., Ross, S. L. and Topa, G. (2008). Place of work and place of residence: Informal hiring networks and labor market outcomes, *Journal of Political Economy* **116**: 1150–1194.

Beaman, L. and Magruder, J. (2010). Who Gets the Job Referral? Evidence from a Social Networks Experiment.

Caliendo, M., Schmidl, R. and Uhlendorff, A. (2010). Social networks, job search methods and reservation wages: Evidence for Germany, *IZA discussion papers*, Institute for the Study of Labor (IZA).

Calvo-Armengol, A. and Jackson, M. O. (2004). The effects of social networks on employment and inequality, *American Economic Review* **94**(3): 426–454.

Calvó-Armengol, A. and Zenou, Y. (2005). Job matching, social network and word-of-mouth communication, *Journal of Urban Economics* **57**: 500–522.

Dustmann, C., Glitz, A. and Schnberg, U. (2011). Referral-based job search networks, *IZA discussion papers*, Institute for the Study of Labor (IZA).

Fontaine, F. (2008). Why are similar workers paid differently? The role of social networks, *Journal of Economic Dynamics and Control* **32**(12): 3960–3977.

Galenianos, M. (2011). Hiring through referrals, *IZA discussion papers*, Economics Department, Penn State University.

Galeotti, A. and Merlino, L. (2010). Endogenous job contact networks, *ISER working papers*, Institute for Social and Economic Research.

Goel, D. and Lang, K. (2009). Social ties and the job search of recent immigrants, *NBER Working Papers*, National Bureau of Economic Research, Inc.

Hellerstein, J. K., McInerney, M. and Neumark, D. (2008). Measuring the importance of labor market networks, *Technical report*, National Bureau of Economic Research.

Holzer, H. J. (1988). Search method use by unemployed youth, *Journal of Labor Economics* **6**(1): 1–20.

Ioannides, Y. M. and Loury, L. D. (2004). Job information networks, neighborhood effects, and inequality, *Journal of Economic Literature* **42**: 1056–1093.

Loury, L. D. (2006). Some contacts are more equal than others: Informal networks, job tenure, and wages, *Journal of Labor Economics* **24**: 299–318.

Mok, D., Wellman, B. and Carrasco, J. (2010). Does distance still matter in the age of the internet?, *Urban Studies* **47**(13): 2747–2783.

Rees, A. (1966). Information networks in labor markets, *The American Economic Review* **56**: 559–566.

Schmutte, I. (2010). Job referral networks and the determination of earnings in local labor markets, *Working Papers 10-40*, Center for Economic Studies, U.S. Census Bureau.

Wahba, J. and Zenou, Y. (2005). Density, social networks and job search methods: Theory and application to Egypt, *Journal of Development Economics* **78**: 443–473.

Figures and Tables

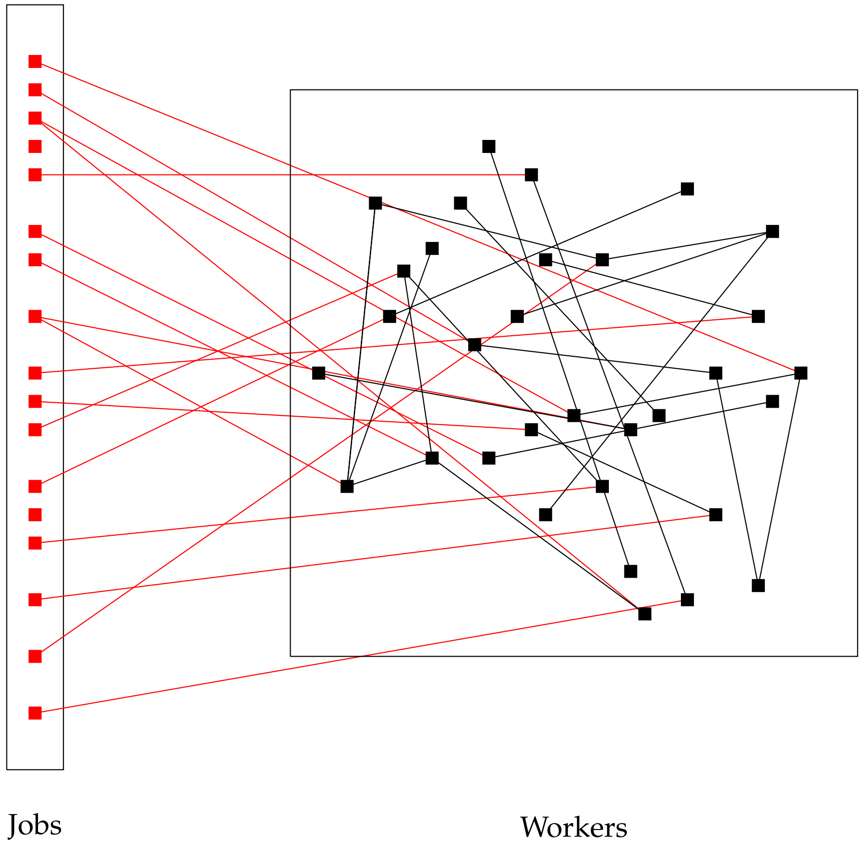

Figure 1: The two essential networks of the model. The nodes on the left, labeled in red, represent employers who are searching for workers. The nodes on the right in black represent workers. The black edges represent contact amongst workers seeking job related information.

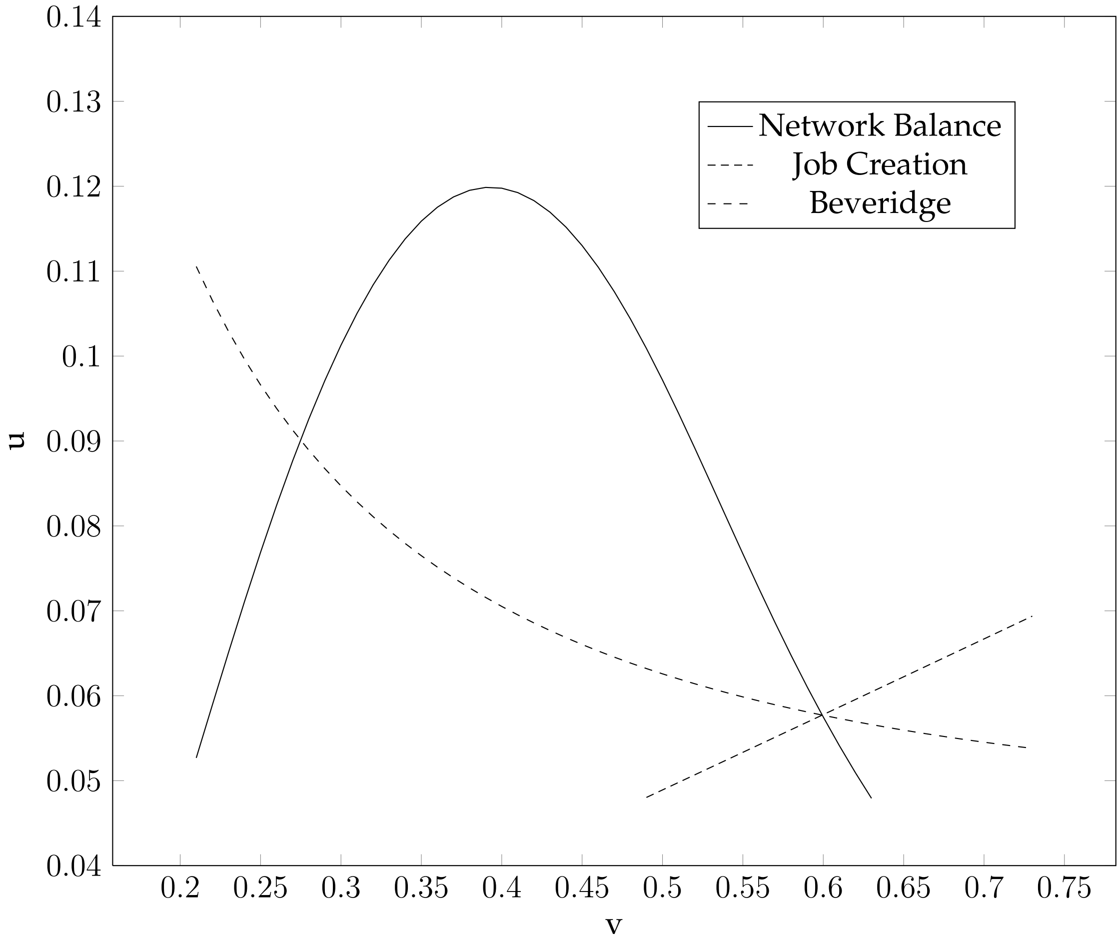

Figure 2: Simulated Equilibrium Relationships. $\gamma = 2.1$ in this equilibrium. The simulated equilibrium uses the parameter values $\delta = 0.05, r = 0.05, c = 1, \kappa = 1, \beta = 0.5,$ and $y1 = 15$

Table 1: Summary Statistics

	Full Sample			Analysis Sample		
Variable	N	Mean	Std Dev	N	Mean	Std Dev
Referred to current job	899	0.280	0.449	168	0.298	0.459
Native-born	1000	0.907	0.291	168	0.869	0.338
Age in 2000	977	42.426	15.417	166	34.861	13.295
Less than High School	977	0.061	0.240	168	0.042	0.200
High School Graduate	977	0.204	0.403	168	0.196	0.398
Some College	977	0.292	0.455	168	0.292	0.456
College	977	0.247	0.431	168	0.250	0.434
Post Graduate	977	0.197	0.398	168	0.220	0.416
White	967	0.807	0.395	162	0.772	0.421
Black	967	0.115	0.319	162	0.123	0.330
In Labor Force	999	0.791	0.407	168	1	0
Employed	790	0.727	0.446	168	1	0
Male	1000	0.457	0.498	168	0.488	0.501
Changed Jobs in last 2 yrs.	570	0.295	0.456			
Accession Rate				152	0.226	0.053
Separation Rate				152	0.225	0.051
Tract Pop. Density per KM				164	1.420	2.441
County Unemployment Rate				168	4.597	1.216
Asked Friends				168	0.220	0.052

Summary Statistics for data used to analyze individual-level probability of being hired through referral. The panel on the left describes the full sample from the Cornell National Social Survey. The panel on the right describes the sample used in the final analysis. The final sample is restricted to employed individuals who reported starting their most recent job within two years of their interview. All variables are from the CNSS except those describing local economic conditions, which are linked from other sources described in the text. 'Asked Friends' is the fraction of unemployed workers in the state who reported in the Current Population Survey asking friends and neighbors as part of their job search.

Table 2: Results

	(1)	(2)	(3)	(4)	(5)
State Referral Network Density		-3.29*	-3.85**	-4.3***	-3.77*
		(1.754)	(1.818)	(1.818)	(1.987)
County Unemployment		0.10	0.09	0.09	0.08
		(0.090)	(0.091)	(0.092)	(0.098)
County Accession Rate		1.62	2.08	2.29	2.40
		(1.571)	(1.624)	(1.651)	(1.73)
Tract Pop. Density			0.06	0.19	0.19
			(0.044)	(0.116)	(0.141)
Tract Pop. Density^2				-0.01	-0.01
				(0.008)	(0.010)
White	-0.075				0.05
	(0.281)				(0.310)
Male	-0.071				-0.10
	(0.224)				(0.233)
High School	-0.196				-0.18
	(0.623)				(0.658)
Some College	-0.184				-0.24
	(0.612)				(0.645)
College	-0.283				-0.30
	(0.624)				(0.660)
Post Graduate	0.068				-0.10
	(0.623)				(0.685)
Age in 2000	-0.013				-0.01
	(0.009)				(0.009)
Married	0.083				0.14
	(0.232)				(0.241)
Native	0.032				0.065
	(0.376)				(0.424)
N	142	142	142	142	142
Log-likelihood	-89.233	-88.225	-87.419	-86.659	-85.803

Probit model estimates for the CNSS sample of recent job changers. The dependent variable in each model is whether the worker was referred to their current job. The table reports coefficient estimates with standard errors in parentheses. One, two, and three stars indicate that the point estimate is significantly different from zero at the 10, 5, and 2.5 percent level of confidence.

A Theory Appendix

Claim 3 (Poisson Lottery) *Suppose you compete for an even chance at a prize with an unknown number of competitors. If the number of your competitors is a random variable with Poisson distribution and parameter s, then the probability you get the prize is $\frac{1-e^{-s}}{s}$*

Proof. The probability of this event is the expected value of $\frac{1}{k+1}$ where k is the number of other competitors for the prize. Given the assumption that the number of competitors is Poisson, we have

$$E\left[\frac{1}{k+1}\right] = \sum_{k=0}^{\infty}(\frac{1}{k+1})\frac{e^{-s}s^k}{k!} \tag{36}$$

$$= \textstyle\sum_{k=0}^{\infty} \frac{e^{-s}s^k}{k+1!} \tag{37}$$

$$= \textstyle\frac{1}{s}\sum_{k=0}^{\infty} \frac{e^{-s}s^{k+1}}{k+1!} \tag{38}$$

$$= \textstyle\frac{1}{s}\sum_{k=1}^{\infty} \frac{e^{-s}s^k}{k!} \quad = \displaystyle\frac{1}{s}(1-e^{-s}) \tag{39}$$

■

Proof of Claim 1:

Proof. $s = E(X)$ where X is the number of requests for referrals directed to a randomly selected worker i.

Define X_j to as the number of information requests to i that come from worker j. Recall that γ_j is the parameter on the number of information requests made by j, which is a random variable distributed as Poisson. Due to homogeneity, all unemployed workers choose the same contact intensity. Thus X_j will be distributed as Poisson with parameter $\frac{\bar{\gamma}}{\mu}$ where μ is the total measure of workers.

Finally, note that $X = \sum X_j$ so

$s = E(X) = E(\sum X_j) = \sum E(X_j) = \sum \frac{\bar{\gamma}}{\mu} = u\gamma$.

Since $\gamma_j \neq 0$ just when j is unemployed, and $\gamma_i = \gamma_{i'} = \gamma$ when i and i' are both unemployed. ■

Proof of Claim 2:

Proof. Showing that s^*exists is equivalent to showing that there exists γ^* such that γ^* is the solution to the Kuhn-Tucker condition in equation 4 where we replace s with $\gamma^* u$.

$$\frac{(1-u)v(1-v)}{\gamma^* u+1}e^{-[\gamma^* \frac{(1-u)v}{\gamma^* u+1}]}K \leq c'(\gamma^*) \tag{40}$$

A non-negative solution fails to exist if both of the following conditions are satisfied:

1. $\gamma^* > 0 \Rightarrow \frac{(1-u)v(1-v)}{\gamma^* u+1}e^{-[\gamma^* \frac{(1-u)v}{\gamma^* u+1}]}K \neq c_w^{'}(\gamma)$

2. $\gamma^* = 0 \Rightarrow (1-u)v(1-v)K > c_w^{'}(0)$

The proof follows by demonstrating that if condition (2) holds, condition (1) cannot hold so that both cannot be true at the same time. Let $b(\gamma^*)$ denote the left-hand side expression in condition (1). We observe that $b(\gamma)$ is continuous and decreasing in γ^*. Furthermore, we have assumed that $c(\gamma)$ is weakly convex and continuously differentiable, so $c_w^{'}(\gamma) \geq c_w^{'}(0)$ and is weakly increasing for all $\gamma > 0$.

If condition (2) holds, then we have $b(0) > c'(0)$. But then by the above argument, $b()$ and $c^{'}()$ should cross somewhere on the positive line. I.e., there exists γ such that (1) is violated. ■

Comparative statics

Claim 4 $\frac{\partial R}{\partial \bar{\gamma}} < 0$

Proof. Differentiating R with respect to $\bar{\gamma}$ yields

$$\frac{\partial R}{\partial \bar{\gamma}} = \left[q + (1-q)e^{\lambda}\right]\left(1 - \frac{c_i e^{\lambda}}{K_i \pi(u, \lambda, \bar{\gamma}u)} \frac{\partial \pi(u, \lambda, \bar{\gamma}u)}{\partial \bar{\gamma}}\right). \tag{41}$$

We need $\frac{\partial \pi}{\partial \bar{\gamma}}$.

$$\frac{\partial \pi}{\partial \bar{\gamma}} = \frac{(1-u)(1-e^{\lambda})\left[(\bar{\gamma}u+1)e^{-\bar{\gamma}u} - 1\right]}{u\bar{\gamma}^2} \tag{42}$$

$\frac{\partial R}{\partial \bar{\gamma}}$ shares the sign of $\frac{\partial \pi}{\partial \bar{\gamma}}$. Suppose $\frac{\partial \pi}{\partial \bar{\gamma}} > 0$. Then $(\bar{\gamma}u+1)e^{-\bar{\gamma}u} - 1 > 0$ which requires $\bar{\gamma} < 0$, a contradiction. ■

Claim 5 $\frac{\partial R}{\partial u} < 0$

Proof. The proof is omitted, since it is nearly identical to the proof of the preceding claim.
■

Claim 6 *There exists* λ^* *such that* $\frac{\partial R}{\partial \lambda} > 0$ *when* $\lambda < \lambda^*$ *and* $\frac{\partial R}{\partial \lambda} < 0$ *otherwise.*

Proof.

$$\frac{\partial R}{\partial \lambda} = (q-1)e^{-\lambda} P(u, \lambda, \bar{\gamma}u, \gamma_i^*) + \left[q + (1-q)e^{-\lambda}\right] \frac{\partial P}{\partial \lambda} \tag{43}$$

$$\frac{\partial P}{\partial \lambda} = \frac{c_i}{K_i} \left(\frac{e\lambda(\frac{\partial \pi}{\partial \lambda} - \pi(u, \lambda, \bar{\gamma}u))}{\pi(u, \lambda, \bar{\gamma}u)^2} \right) \tag{44}$$

$$\frac{\partial \pi}{\partial \lambda} = e^{-\lambda}(1-u)\frac{1 - e^{-\bar{\gamma}u}}{\bar{\gamma}u}. \tag{45}$$

It follows that

$$\frac{\partial \pi}{\partial \lambda} - \pi(u, \lambda, \bar{\gamma}u = \left(\frac{e^{-\lambda}}{1 - e^{\lambda}} \right) \pi(u, \lambda, \bar{\gamma}u). \tag{46}$$

Let $\lambda^* = \ln 2$. It is straightforward to verify that $\frac{\partial P}{\partial \lambda} > 0$ for $\lambda < \lambda^*$ and $\frac{\partial P}{\partial \lambda} < 0$ for $\lambda > \lambda^*$.
■