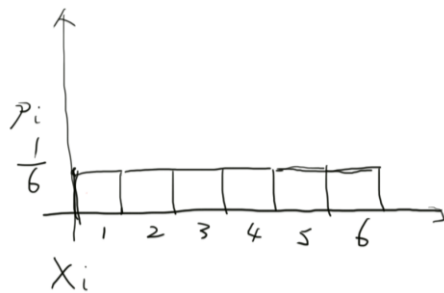


Monte Carlo: Sampling

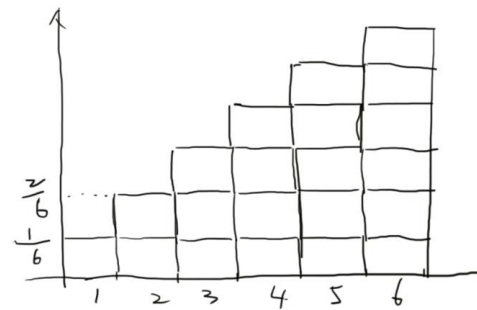
Review of Probability

- Discrete

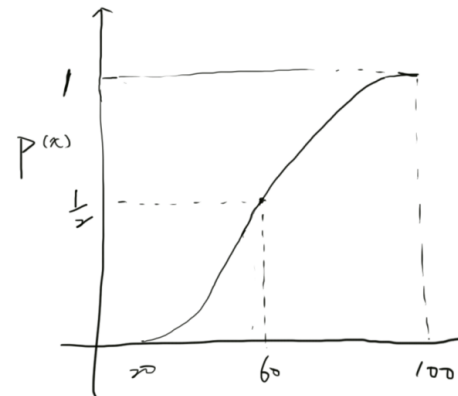
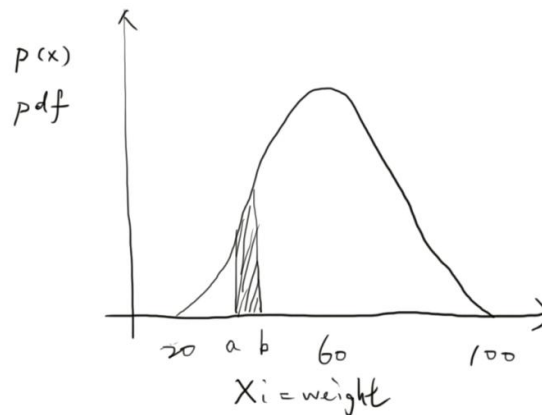
pdf



cdf



- Continuous



Review of Monte Carlo Integration

- Function to integrate:

$$\int_a^b f(x)dx$$

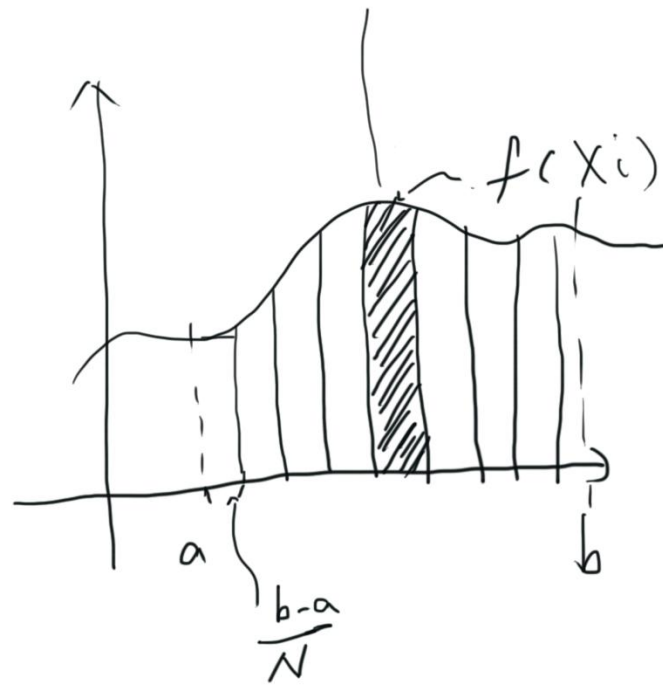
- Monte Carlo estimator

$$F_N = \frac{b-a}{N} \sum_{i=1}^N f(X_i) \text{ is the estimator of } \int_a^b f(x)dx$$

$$(\text{when } N \rightarrow \infty, F_N = \int_a^b f(x)dx)$$

$$(E[F_N] = \int_a^b f(x)dx)$$

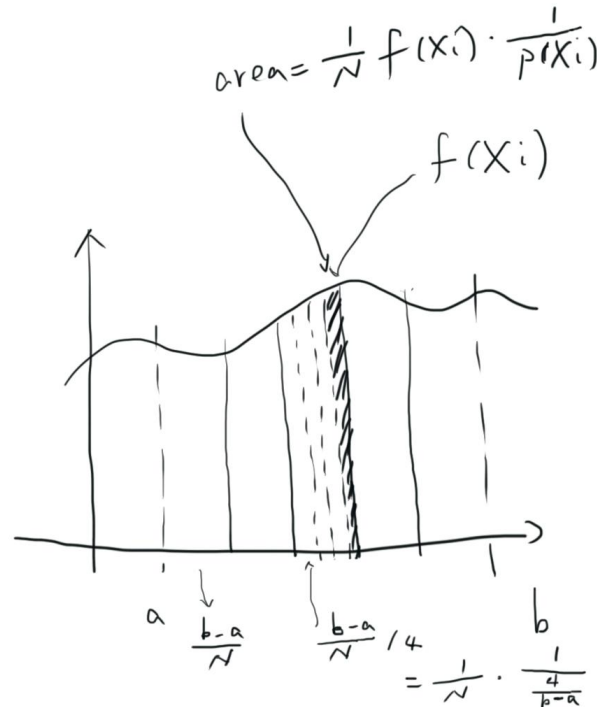
$$\text{area} = \frac{b-a}{N} f(x_i)$$



Review of Monte Carlo Integration

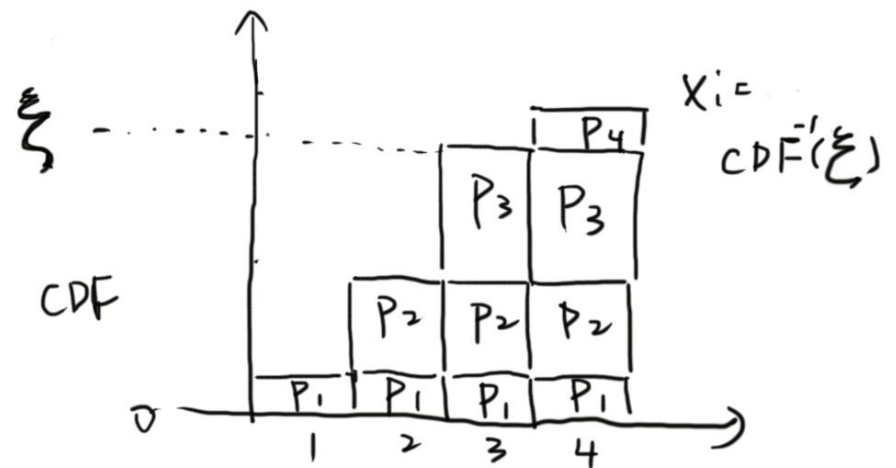
- More general form

$$F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)} \text{ is the estimator of } \int_a^b f(x) dx$$



Review of Inversion method

1. Compute the CDF $P(x) = \int_0^x p(x')dx'$
2. Compute the inverse $P^{-1}(x)$
3. Obtain a uniformly distributed random number ξ
4. Compute $X_i = P^{-1}(\xi)$



Example

- Blinn NDF
- Piecewise-constant 1D function

Sampling Blinn NDF

$$D(\omega) = \frac{(n+2)}{2\pi} (\cos \theta_h)^n$$

- Sample $\cos(n \cdot h)$

$p(\omega) = p(\theta, \phi)$ only depend on θ , $p(\phi)$ can be sampled by $2\pi\xi_2$

$$p(\theta) = c \cdot \frac{(n+2)}{2\pi} (\cos \theta_h)^n$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} c \cdot \frac{(n+2)}{2\pi} (\cos \theta_h)^n \sin \theta_h d\theta_h = 1$$

$$= \frac{c(n+2)}{2\pi} \int_0^{\frac{\pi}{2}} (\cos \theta_h)^n \sin \theta_h d\theta_h$$

$$= \frac{c(n+2)}{2\pi} \int_0^1 u^n du$$

$$= \frac{c(n+2)}{2\pi} \frac{u^{n+1}}{n+1} \Big|_0^1 = \frac{c(n+2)}{2\pi(n+1)} = 1 \Rightarrow c = \frac{2\pi(n+1)}{(n+2)} \Rightarrow p(\cos \theta_h) = (n+1)(\cos \theta_h)^n$$

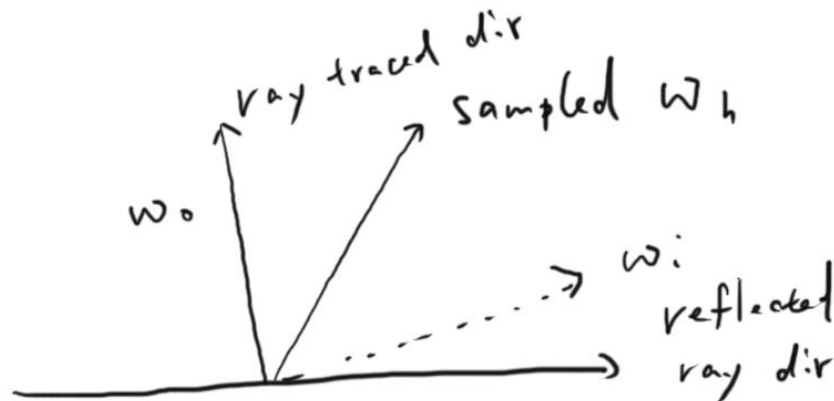
$$\Rightarrow \cos \theta_h = \sqrt[n+1]{\xi_1}$$

Blinn NDF

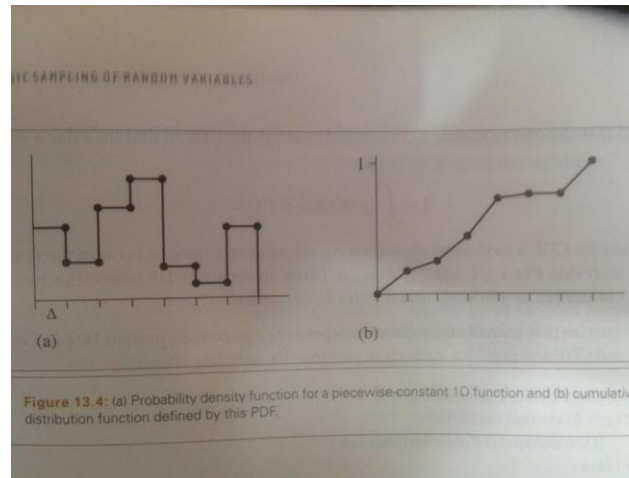
- Sample reflection direction



$$\cos \theta_h = \sqrt[n+1]{\xi_1}$$



Piecewise constant 1D function



$$f(x) = \begin{cases} v_0 & x_0 \leq x < x_1 \\ v_1 & x_0 \leq x < x_1 \\ \vdots \end{cases}$$

Piecewise constant 1D function

- Compute PDF

$$\text{the integral } c = \int_0^1 f(x) dx = \sum_{i=0}^{N-1} \Delta v_i = \sum_{i=0}^{N-1} \frac{v_i}{N}$$

$$\text{the PDF } p(x) = \frac{f(x)}{c}$$

- Compute CDF

$$P(x_0) = 0$$

$$P(x_1) = \int_{x_0}^{x_1} p(x) dx = \frac{v_0}{Nc} = P(x_0) + \frac{v_0}{Nc}$$

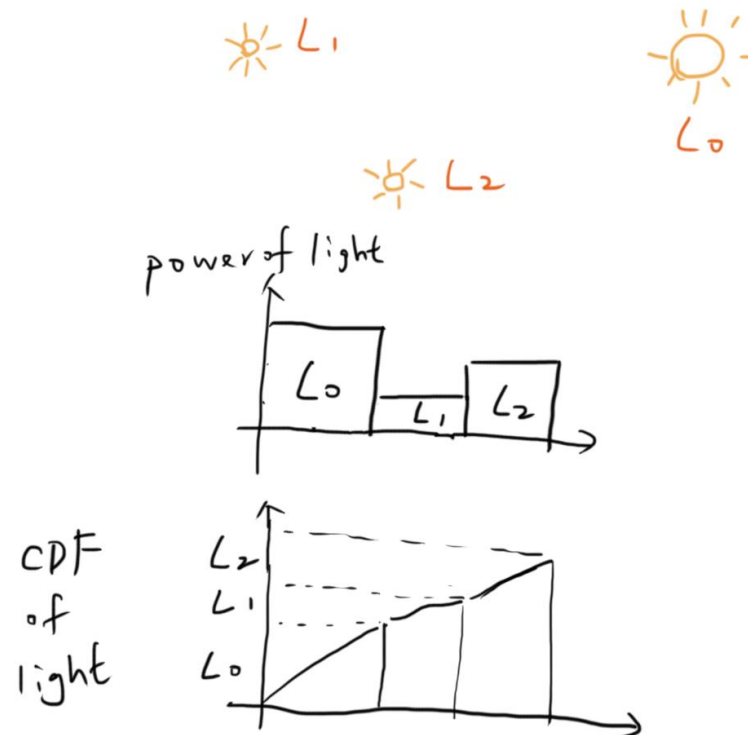
$$P(x_2) = \int_{x_1}^{x_2} p(x) dx = \int_{x_0}^{x_1} p(x) dx + \int_{x_1}^{x_2} p(x) dx = P(x_1) + \frac{v_1}{Nc}$$

$$P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{Nc}$$

- Sampling by binary search the interval, then interpolate the pdf

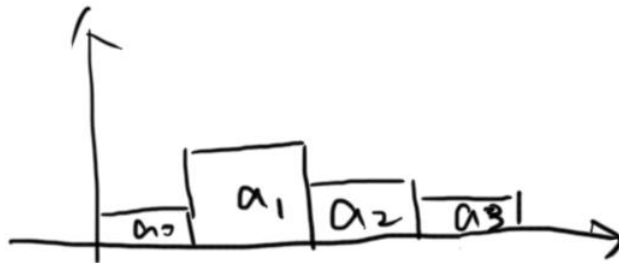
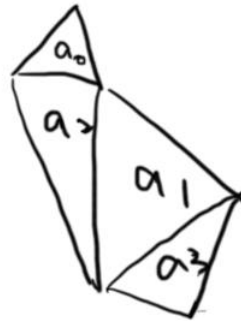
Piecewise constant 1D function

- Example: Sample multiple lights



Piecewise constant 1D function

- Example: Sampling mesh surface



2D Sampling

- Hemisphere
- Disk
- Triangle
- Mesh

2D sampling

- How to sample a 2D joint density function?
 - ex. Hemisphere direction $p(\theta, \phi)$
- Simpler case, if the density function is separable $p(x, y) = p_x(x)p_y(y)$
- Then random variable (X, Y) can be found by independently sampling X from p_x and Y from p_y
 - Example: Blinn NDF

2D sampling

- General case
- First calculate *marginal density* function

$$p(x) = \int p(x, y) dy$$

– Average density for a particular x over all possible y

- Then we can calculate the conditional density function $p(y | x) = \frac{p(x, y)}{p(x)}$

– Density function for y given some particular x has been chosen

Uniform sampling hemisphere

- Uniform means $p(\omega)$ is constant

$$p(\omega) = c$$

$$\int_{\mathbf{H}} c d\omega = 1 \Rightarrow c = \frac{1}{2\pi}$$

$$p(\omega) = \frac{1}{2\pi}$$

$$p(\theta, \phi) = \sin(\theta) p(\omega) = \frac{\sin(\theta)}{2\pi}$$

Uniform sampling hemisphere

- Sampling θ by θ 's marginal density function $p(\theta)$

$$p(\theta) = \int_0^{2\pi} \frac{\sin \theta}{2\pi} d\phi = \sin \theta$$

- Compute conditional density for ϕ

$$p(\phi | \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi}$$

Uniform sampling hemisphere

- Use the inverse method, integrate to get CDF

$$P(\theta) = \int_0^\theta \sin \theta' d\theta' = 1 - \cos \theta$$

$$P(\phi | \theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}$$

- Then inverse

$$\theta = \cos^{-1} \xi_1$$

$$\phi = 2\pi\xi_2$$

- Then transform to cartesian coordinate

$$x = \sin \theta \cos \phi = \cos(2\pi\xi_2) \sqrt{1 - \xi_1^2}$$

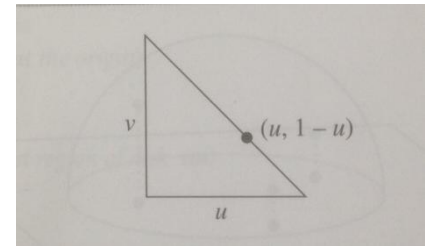
$$y = \sin \theta \sin \phi = \sin(2\pi\xi_2) \sqrt{1 - \xi_1^2}$$

$$z = \cos \theta = \xi_1$$

Uniform sampling triangle

- We will assume we are sampling an isosceles right triangle of area $\frac{1}{2}$
- Output is barycentric coordinate, so work for any triangle

$$p(u, v) = 2$$



Uniform sampling triangle

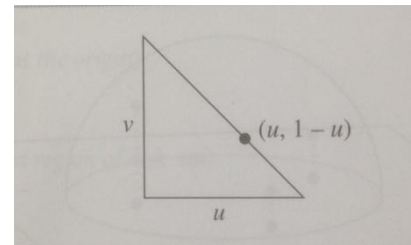
$$p(u, v) = 2$$

- First, find the marginal density

$$p(u) = \int_0^{1-u} p(u, v) dv = 2 \int_0^{1-u} dv = 2(1-u)$$

- And conditional density

$$p(v | u) = \frac{p(u, v)}{p(u)} = \frac{2}{2(1-u)} = \frac{1}{1-u}$$



Sampling triangle

- Inverse method as always

$$P(u) = \int_0^u p(u') du' = 2u - u^2$$

$$P(v) = \int_0^u p(v'|u) dv' = \frac{v}{1-u}$$

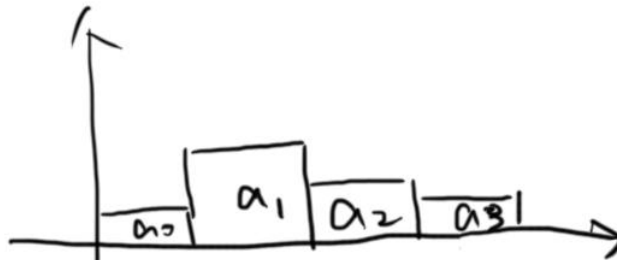
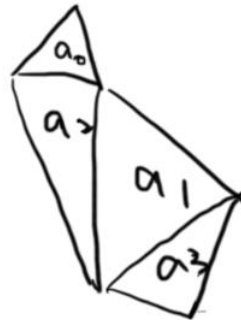
\Rightarrow

$$u = 1 - \sqrt{\xi_1}$$

$$v = \xi_2 \sqrt{\xi_1}$$

Uniform sampling mesh

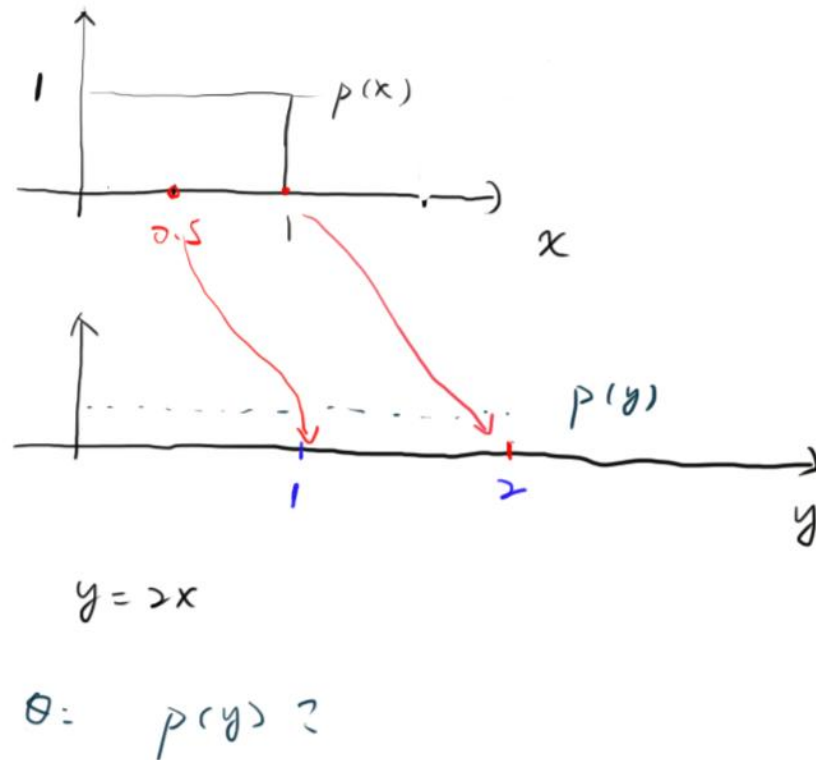
- Uniform sample triangle index, then uniform sample the triangle



Transforming between distributions

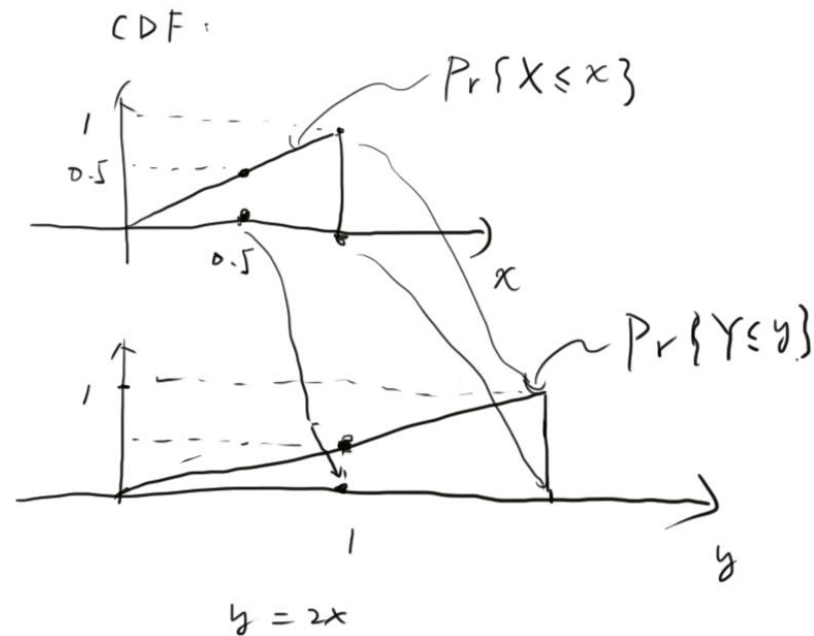
- If we are given random variables X_i that are already drawn from some PDF $p(X_i)$
- Give another random variable $Y_i = y(X_i)$
- What is $p(Y_i)$?

Transforming between distributions



$y(x)$ must be a one to one transformation

Transforming between distributions



$$\Pr\{Y \leq y(x)\} = \Pr\{X \leq x\}$$

$$P_y(y) = P_y(y) = P_y(y(x)) = P_x(x)$$

differentiating

$$p_y(y) \left| \frac{dy}{dx} \right| = p(x)$$

$$\Rightarrow p_y(y) = \left| \frac{dy}{dx} \right|^{-1} p(x)$$

Transformation in multiple dimensions

- If x is an n -dimensional random variable X with density function $p_x(x)$ and

$$Y = T(X) \text{ where } T \text{ is a bijection}$$

then the densities are related by

$$p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|J_T(x)|}$$

where $|J_T(x)|$ is the absolute value of the determinant of T 's Jacobian matrix, which is

$$\begin{pmatrix} \partial T_1 / \partial x_1 & \cdots & \partial T_1 / \partial x_n \\ \vdots & \ddots & \vdots \\ \partial T_n / \partial x_1 & \cdots & \partial T_n / \partial x_n \end{pmatrix}$$