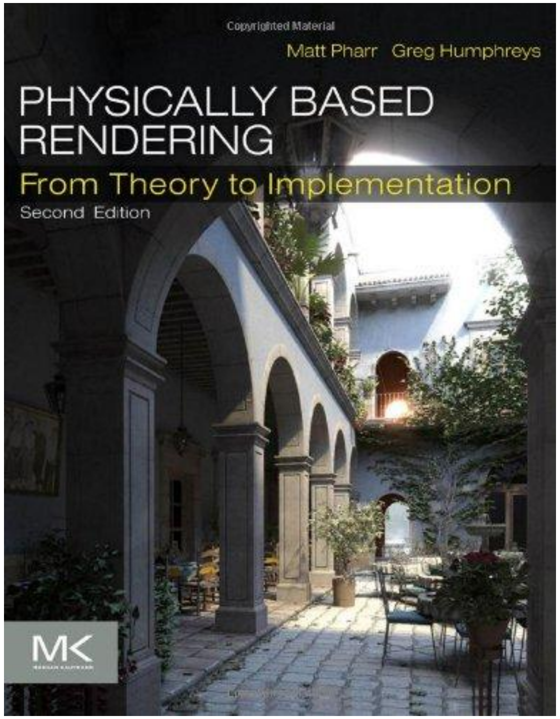
Path Tracing:
Monte Carlo integration

Physically based rendering

- Oscar winner



- https://www.youtube.com/watch?v=7d9juPsv1QU

Randomize algorithm

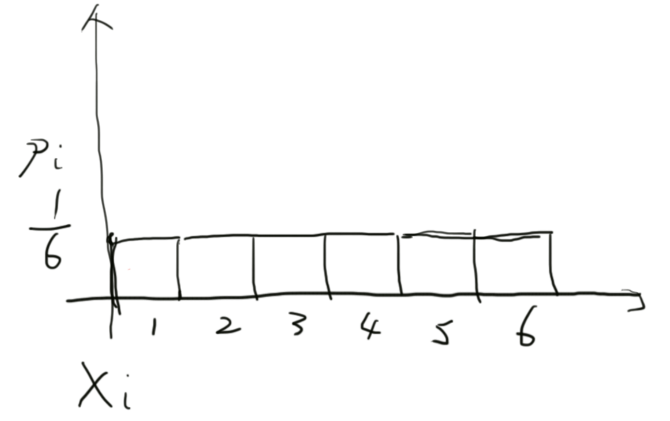
- Monte Carlo
  - Monte Carlo integration
- Las Vegas
  - Quick sort

Probability Review

- Random Variable:
  - X
  - A value chosen by some random process
- Discrete
  - Row a Dice and see it's outcome
- Continuous
  - A random person's weight

Dice

$$X_i = \{1,2,3,4,5,6\}$$



$$P_i = \frac{1}{6}$$

$$\sum_{i=1}^{6} P_i = 1$$

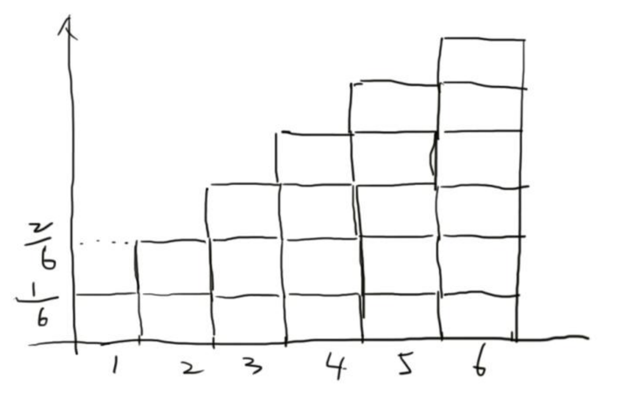
The *cumulative disitribution function* (CDF)

$P(\mathrm{x}) = \Pr\{X \le x\}$

= outcome of the dice is less or equal to x

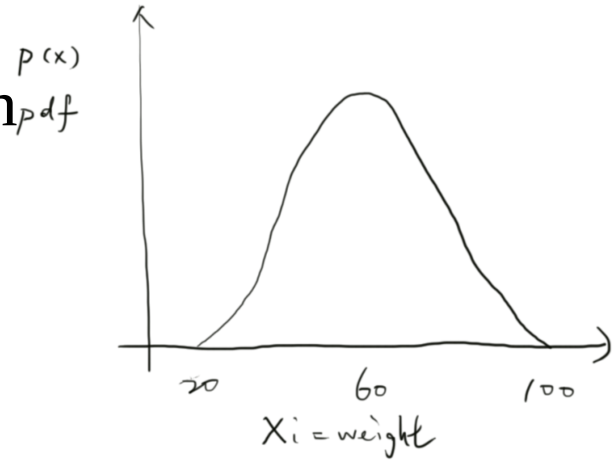
example

$$P(2) = \frac{1}{3}$$



Continuous random variables

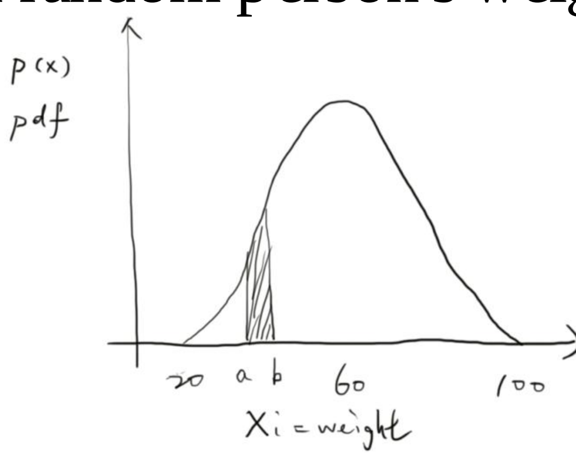
X_i = weight of a randomly chosen person



$p(x)$ = probability density function

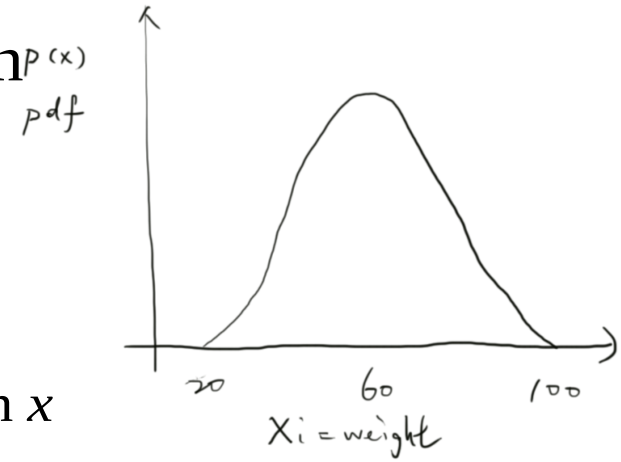
$$P(x \in [a,b]) = \int_a^b p(x)dx$$

= probability of a random person's weight is between a, and b



Continuous random variables

$X_i =$ weight of a randomly chosen person

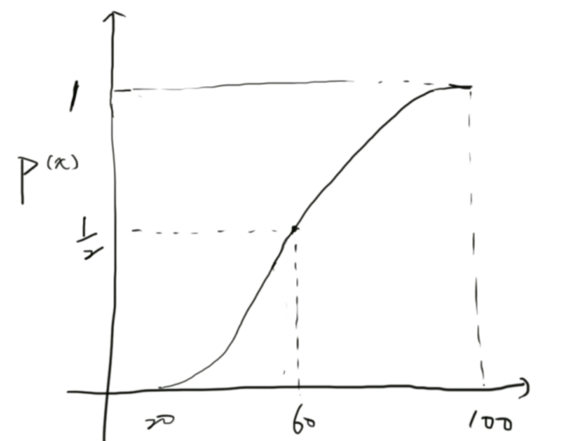


The *cumulative disitribution function* (CDF)

$P(x) = \Pr\{X \leq x\}$

= A randomly chosen person's weight is less than x

$= \int_{-\infty}^{x} p(x)dx$

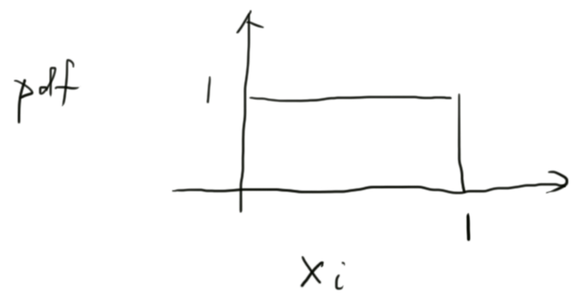


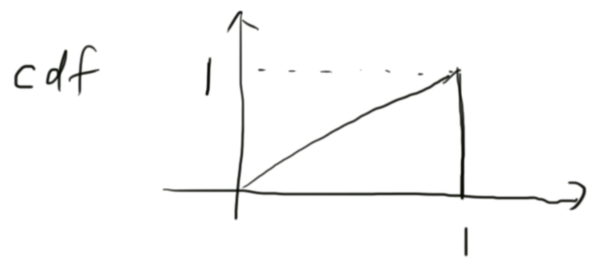
Uniform random variable

canonical uniform random variable ξ

(pronounced as Xi)

$$p(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$





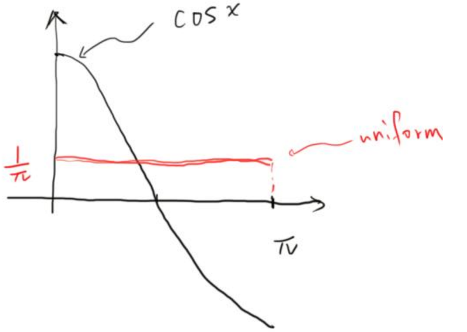
Expected values

- Expected value of a function f
  - Average value of the function over some distribution p

$$E_p[f(x)] = \int_D f(x)p(x)dx$$

  - If p is uniform, we drop the subscript p from Ep
- Example
  - Expected value of cos(x), x between 0 and Pi

$$E[\cos(x)] = \int_0^{\pi} \frac{\cos(x)}{\pi} dx = \frac{1}{\pi}(-\sin\pi + \sin 0) = 0$$



Properties of E

$$E[af(x)] = aE[f(x)]$$

$$E[\sum_i f(X_i)] = \sum_i E[f(X_i)]$$

The Monte Carlo estimator

- Suppose we want to evaluate the integral
  $\int_a^b f(x)dx$
- Given uniform random variables $X_i \in [a,b]$
- We claim

$$F_N = \frac{b-a}{N}\sum_{i=1}^{N} f(X_i) \ is \ \text{the estimator of} \int_a^b f(x)dx$$

$$(\text{when N} \to \infty, F_N = \int_a^b f(x)dx)$$

$$(E[F_N] = \int_a^b f(x)dx)$$

Proof

$$E[F_N] = E[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)]$$

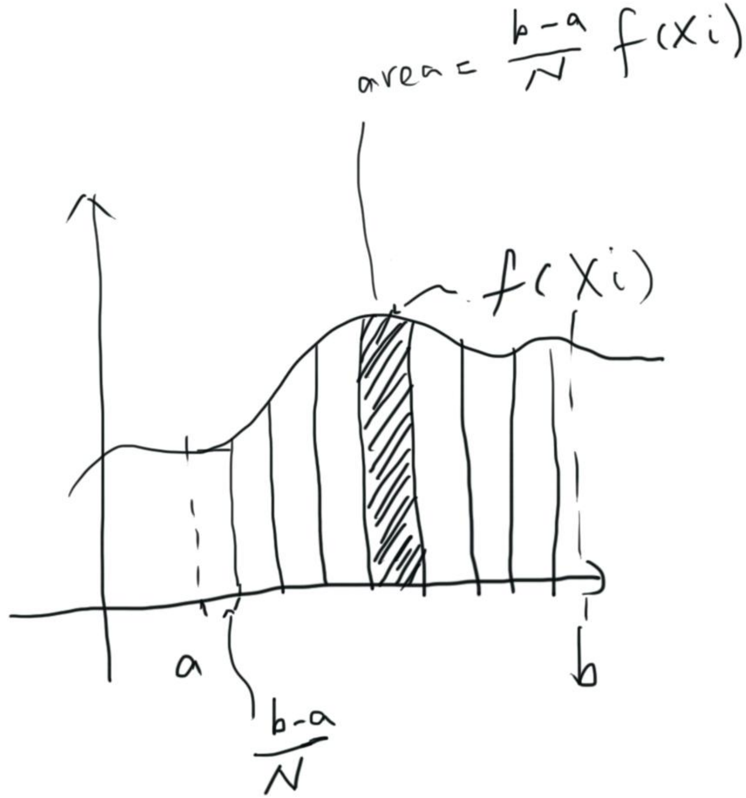
$$= \frac{b-a}{N}\sum_{i=1}^{N} E[f(X_i)]$$

$$= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)p(x)dx$$

$$= \frac{1}{N}\sum_{i=1}^{N} \int_a^b f(x)dx$$

$$= \int_a^b f(x)dx$$

Geometric interpretation



More general form

- Given a probability density function $p(x)$

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} \textit{ is } \text{the estimator of } \int_a^b f(x)dx$$

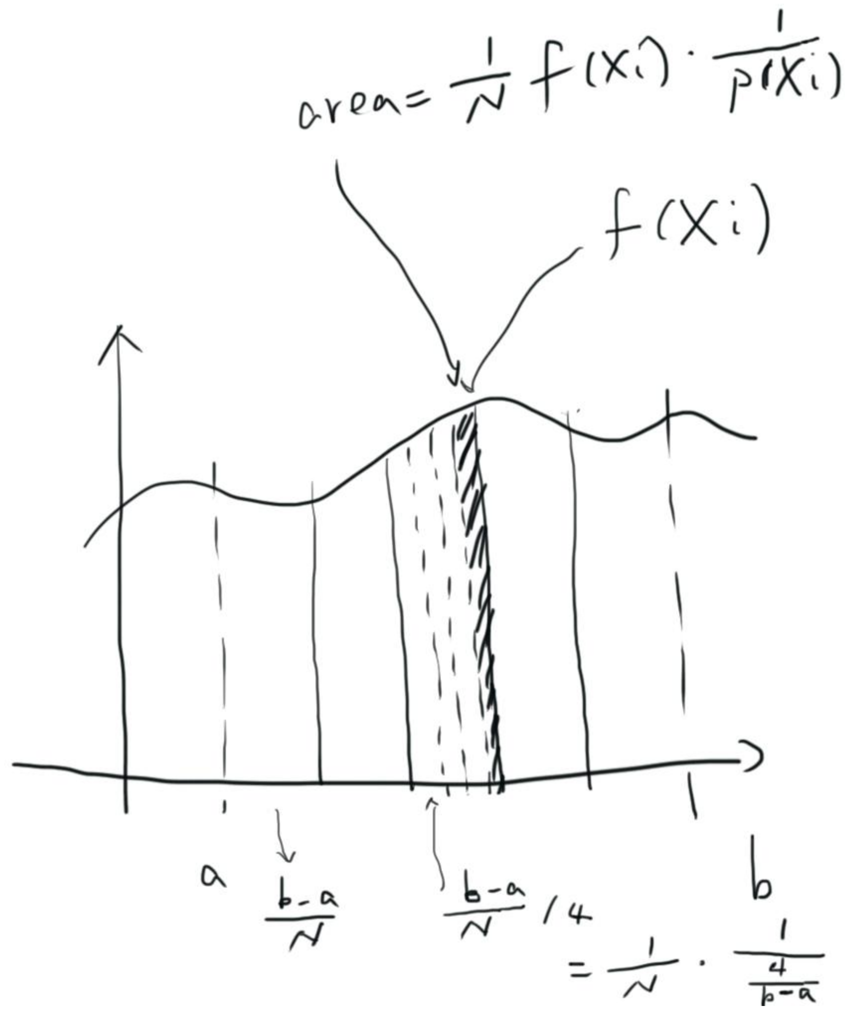
$$proof$$

$$E[F_N] = E\left[\frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_a^b \frac{f(x)}{p(x)}p(x)dx$$

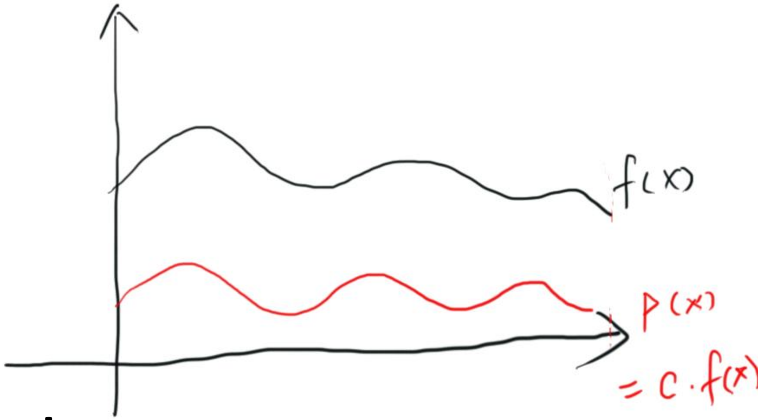
$$= \frac{1}{N}\sum_{i=1}^{N}\int_a^b \frac{f(x)}{p(x)}dx = \int_a^b f(x)dx$$

Geometric interpretation



Why we need this form

- Suppose we have a pdf $p(x) = c \cdot f(x)$



$$\Rightarrow \int p(x)dx = \int c \cdot f(x)dx = 1$$

$$\Rightarrow c = \frac{1}{\int f(x)dx}$$

- Then the Monte Carlo estimator

$$\frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{c} = \int f(x)dx,$$

Perfect estimate!!

Extending to multiple dimensions

- Extending to multiple dimensions is straightforward
- Consider the three-dimensional integral

$$\int_{x0}^{x1}\int_{y0}^{y1}\int_{z0}^{z1} f(x,y,z)\,dxdydz$$

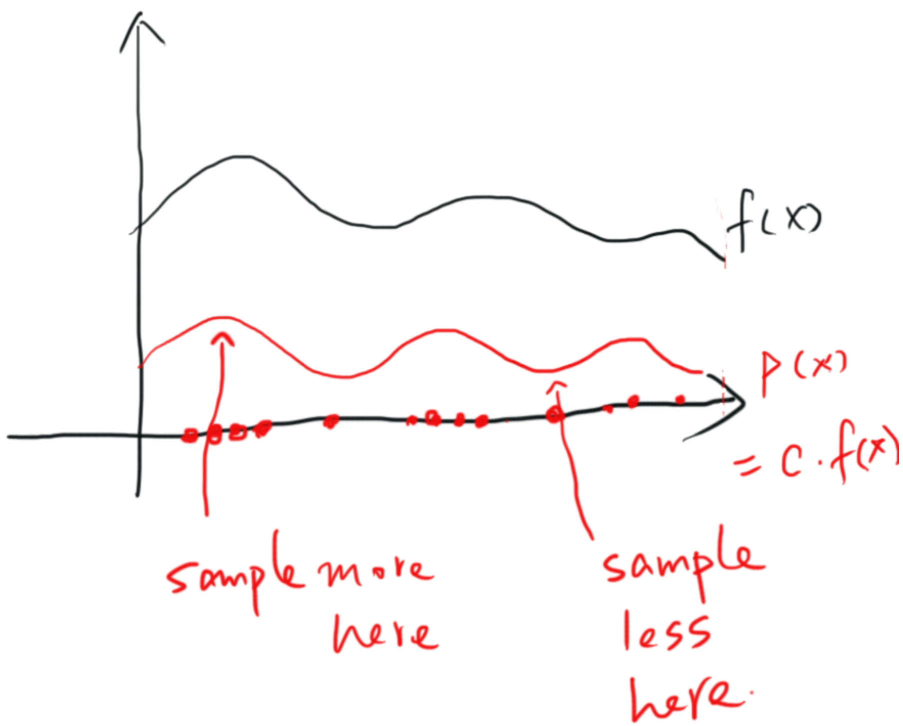
- If we use uniform sampling $p(x,y,z) = \frac{1}{(x_1 - x_0)}\frac{1}{(y_1 - y_0)}\frac{1}{(z_1 - z_0)}$

then the estimator is $\frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{\mathrm{N}}\sum_i f(X_i)$

Multiple dimensions

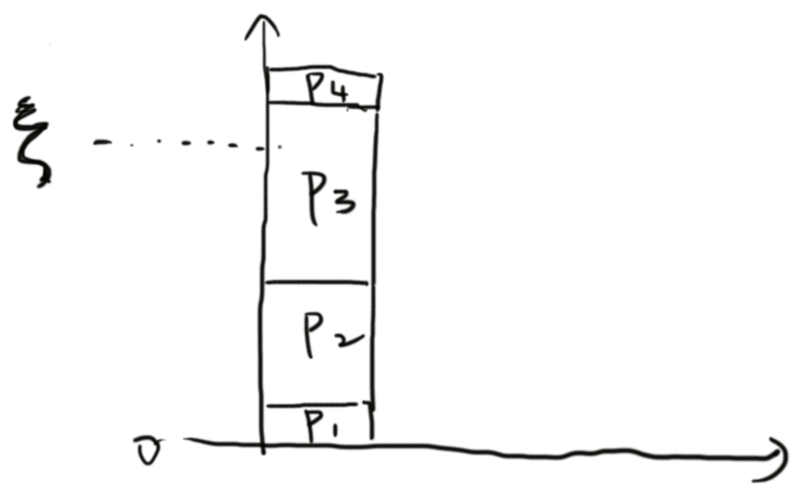
- The number of samples N is independent of the dimensionality of the integral
- Error in Monte Carlo estimator decreases at a rate of $O(\sqrt{N})$

Sampling of random variables



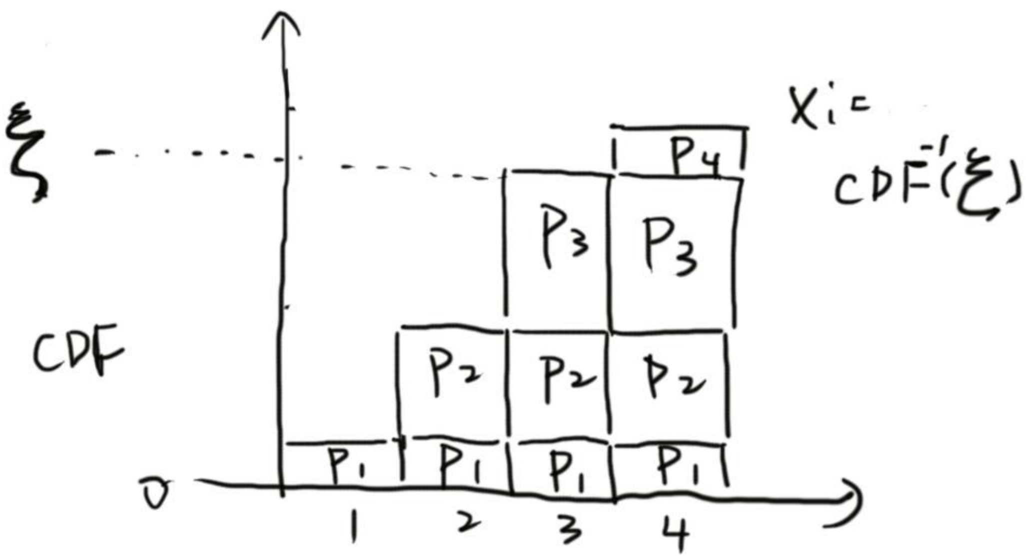
How to sample from a PDF

how to sample?



Inversion method

how to sample?



Inversion method

1. Compute the CDF $P(x) = \int_0^x p(x')dx'$
2. Compute the inverse $P^{-1}(x)$
3. Obtain a uniformly distributed random number ξ
4. Compute $X_i = P^{-1}(\xi)$