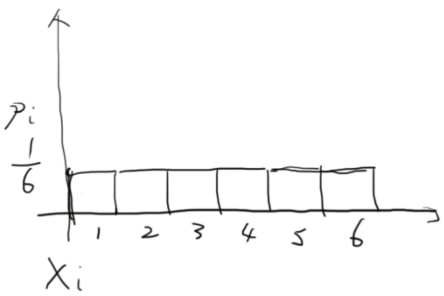
Monte Carlo: Sampling

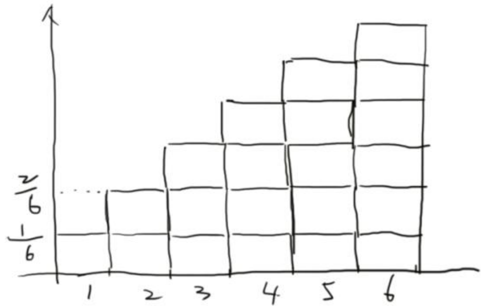
Review of Probability

- Discrete

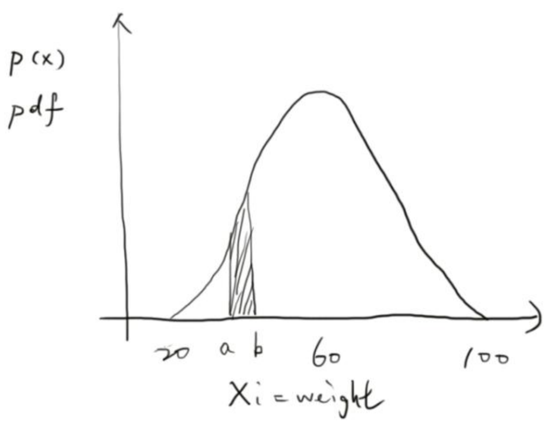
pdf

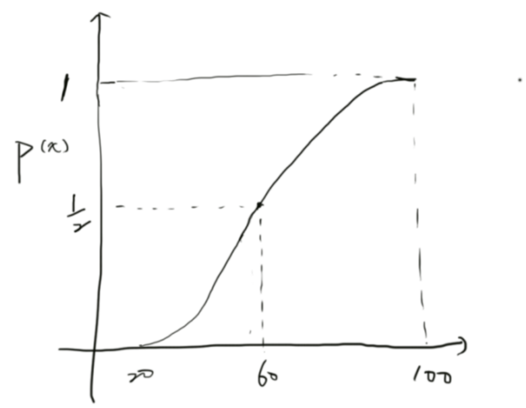


cdf



- Continuous





Review of Monte Carlo Integration

- Function to integrate:

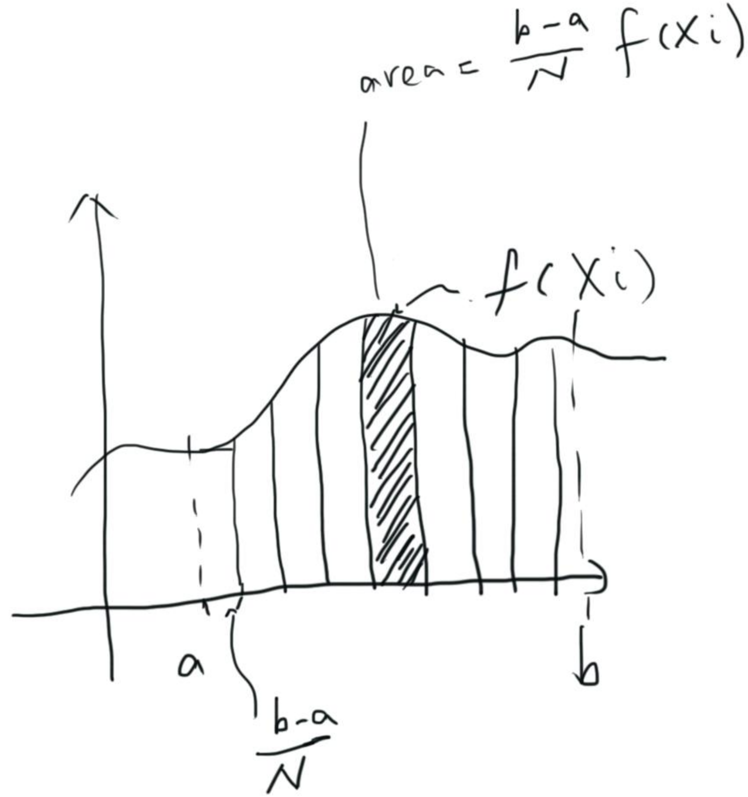
$$\int_a^b f(x)dx$$

- Monte Carlo estimator

$$F_N = \frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\ is\ \text{the estimator of}\ \int_a^b f(x)dx$$

$$(\text{when N} \to \infty, F_N = \int_a^b f(x)dx)$$

$$(E[F_N] = \int_a^b f(x)dx))$$

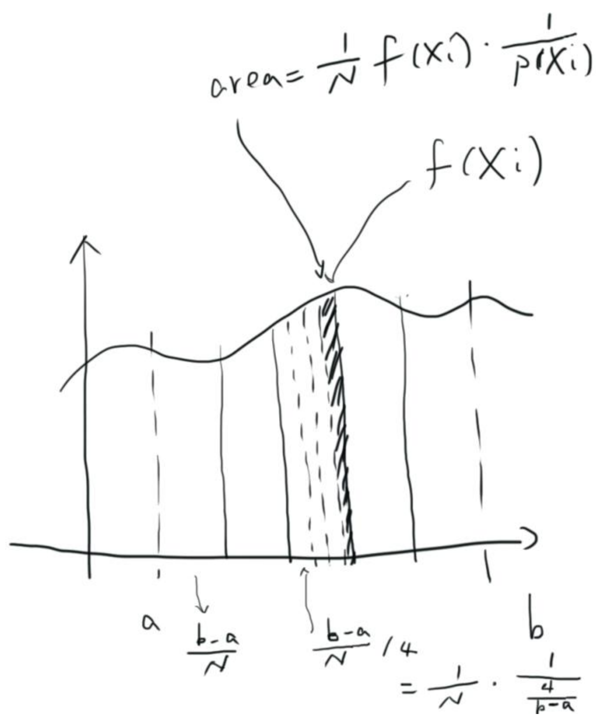


area = $\frac{b-a}{N} f(x_i)$
$f(x_i)$
a
b
$\frac{b-a}{N}$

Review of Monte Carlo Integration

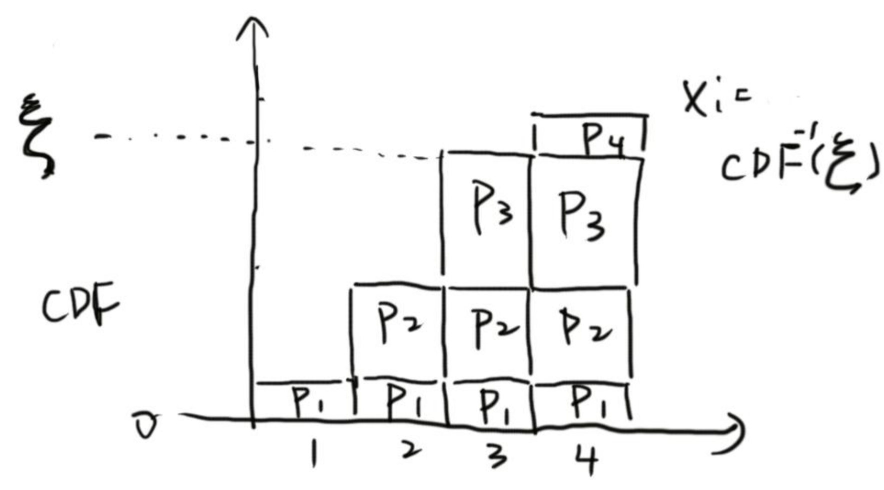
- More general form

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} \textit{ is } \text{the estimator of } \int_a^b f(x)dx$$



Review of Inversion method

1. Compute the CDF $P(x) = \int_0^x p(x')dx'$
2. Compute the inverse $P^{-1}(x)$
3. Obtain a uniformly distributed random number ξ
4. Compute $X_i = P^{-1}(\xi)$



Example

- Blinn NDF
- Piecewise-constant 1D function

Sampling Blinn NDF

$$D(\omega) = \frac{(n+2)}{2\pi}(\cos\theta_h)^n$$

- Sample cos(n・h)

$$p(\omega) = p(\theta,\phi) \text{only depend on } \theta,\ p(\phi) \text{ can be sampled by } 2\pi\xi_2$$

$$p(\theta) = c \cdot \frac{(n+2)}{2\pi}(\cos\theta_h)^n$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} c \cdot \frac{(n+2)}{2\pi}(\cos\theta_h)^n \sin\theta_h d\theta_h = 1$$

$$= \frac{c(n+2)}{2\pi}\int_0^{\frac{\pi}{2}} (\cos\theta_h)^n \sin\theta_h d\theta_h$$

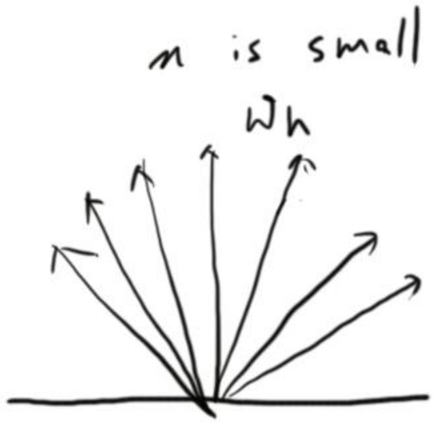
$$= \frac{c(n+2)}{2\pi}\int_0^1 u^n du$$

$$= \frac{c(n+2)}{2\pi}\left.\frac{u^{n+1}}{n+1}\right|_0^1 = \frac{c(n+2)}{2\pi(n+1)} = 1 \Rightarrow c = \frac{2\pi(n+1)}{(n+2)} \Rightarrow p(\cos\theta_h) = (n+1)(\cos\theta_h)^n$$

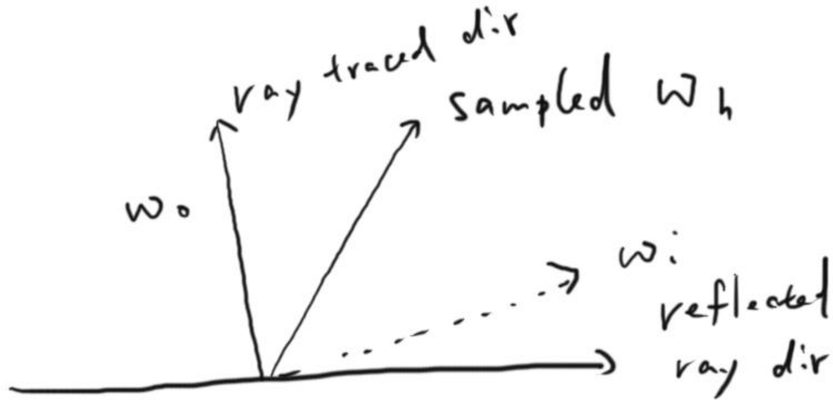
$$\Rightarrow \cos\theta_h = \sqrt[n+1]{\xi_1}$$

Blinn NDF

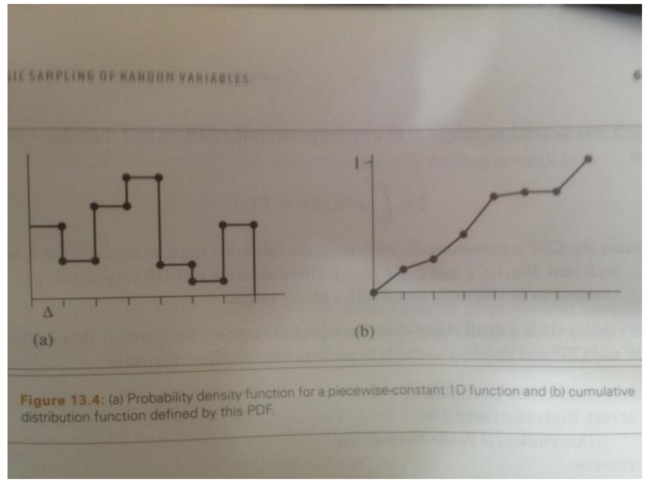
- Sample reflection direction



$$\cos\theta_h = \sqrt[n+1]{\xi_1}$$



Piecewise constant 1D function



$$f(x) = \begin{cases} v_0 \quad x_0 \leq x < x_1 \\ v_1 \; x_0 \leq x < x_1 \\ \vdots \end{cases}$$

Piecewise constant 1D function

- Compute PDF

$$\text{the integral } c = \int_0^1 f(x)dx = \sum_{i=0}^{N-1} \Delta v_i = \sum_{i=0}^{N-1} \frac{v_i}{N}$$

$$\text{the PDF } p(x) = \frac{f(x)}{c}$$

- Compute CDF

$$P(x_0) = 0$$

$$P(x_1) = \int_{x_0}^{x_1} p(x)dx = \frac{v_0}{Nc} = P(x_0) + \frac{v_0}{Nc}$$

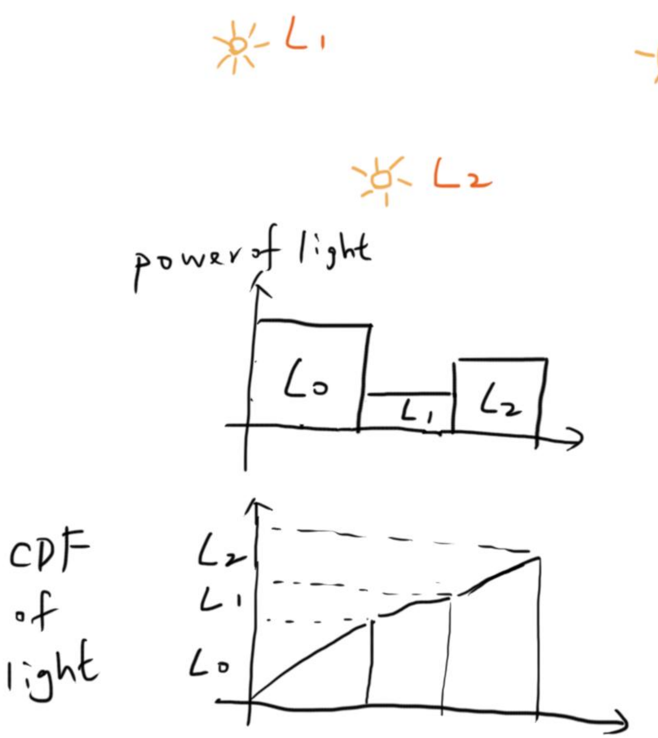
$$P(x_2) = \int_{x_1}^{x_2} p(x)dx = \int_{x_0}^{x_1} p(x)dx + \int_{x_1}^{x_2} p(x)dx = P(x_1) + \frac{v_1}{Nc}$$

$$P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{Nc}$$

- Sampling by binary search the interval, then interpolate the pdf

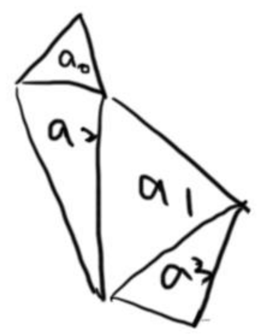
Piecewise constant 1D function

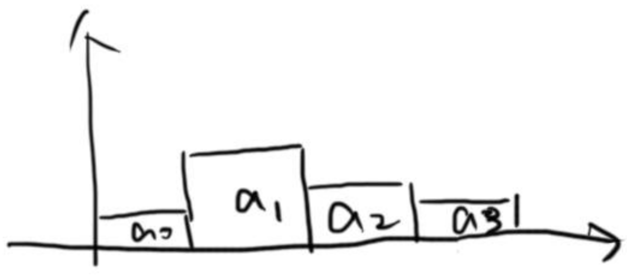
- Example: Sample multiple lights



Piecewise constant 1D function

- Example: Sampling mesh surface





2D Sampling

- Hemisphere
- Disk
- Triangle
- Mesh

2D sampling

- How to sample a 2D joint density function?
  - ex. Hemisphere direction $p(\theta,\phi)$
- Simpler case, if the density function is separable $p(x,y) = p_x(x)p_y(y)$
- Then random variable (X, Y) can be found by independently sampling X from p_x and Y from p_y
  - Example: Blinn NDF

2D sampling

- General case
- First calculate *marginal density* function

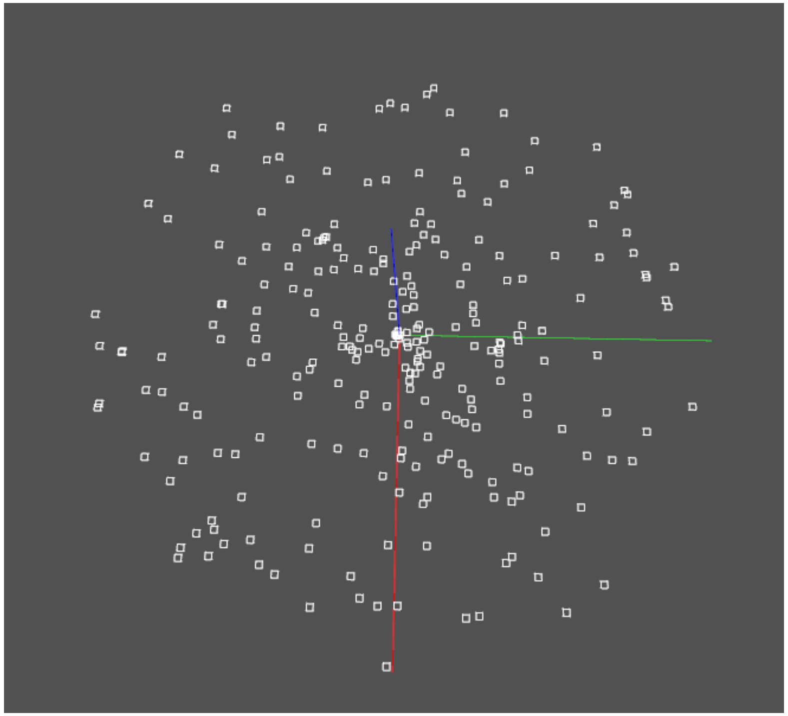
$$p(x) = \int p(x, y)dy$$

  – Average density for a particular x over all possible y
- Then we can calculate the conditional density function $p(y \mid x) = \dfrac{p(x, y)}{p(x)}$
  – Density function for y given some particular x has been chosen

Uniform sampling disk

- Obvious but Wrong approach

$$r = \xi_1$$
$$\theta = 2\pi\xi_2$$



Uniform sampling disk

- Uniform sampling respect to area, pdf must be a constant

$$p(x,y)=\frac{1}{\pi}$$

transform to polar coordinate (explained later)

$$\mathrm{p}(\mathrm{r},\theta)=\frac{\mathrm{r}}{\pi}$$

- Compute marginal and conditional densities

$$p(r)=\int_0^{2\pi}p(r,\theta)d\theta=2r$$

transform to polar coordinate (explained later)

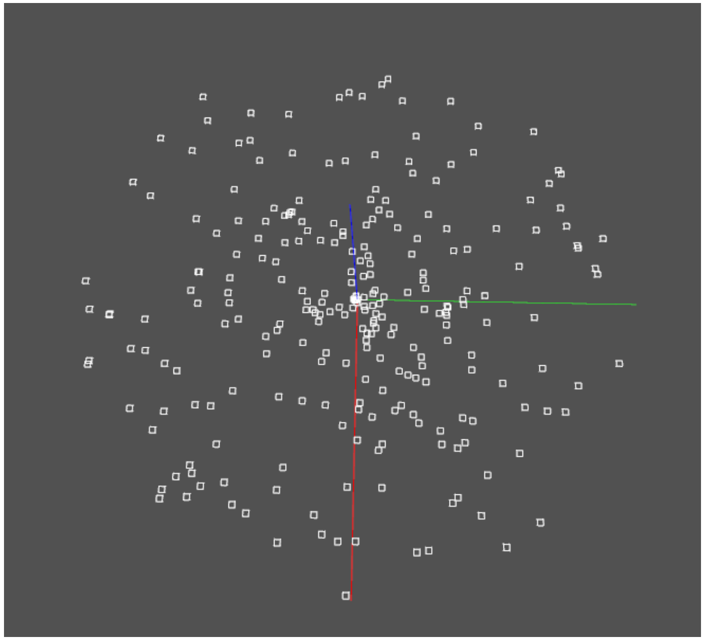
$$p(\theta\,|\,r)=\frac{p(r,\theta)}{p(r)}=\frac{1}{2\pi}$$

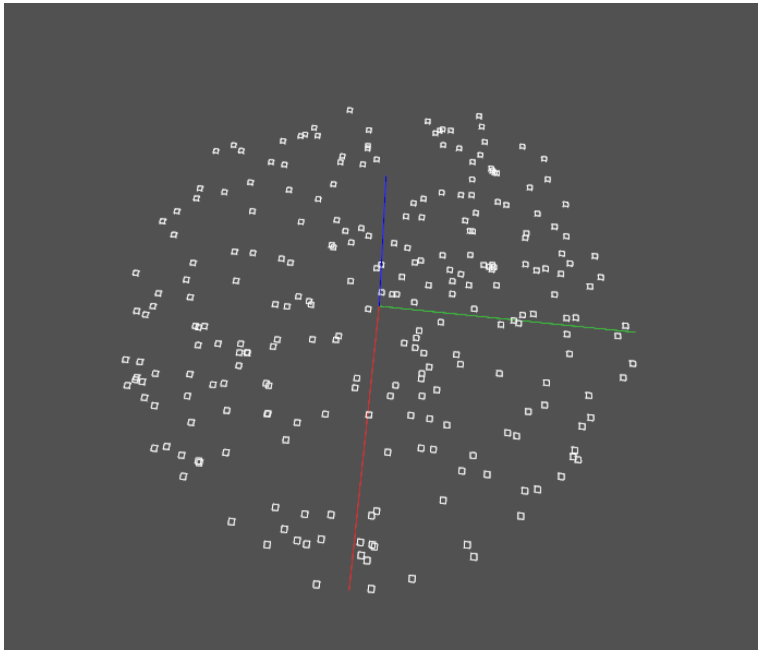
Uniform sampling disk

- Integrating and inverting to find $P(r), P^{-1}(r), P(\theta)$ and $P^{-1}(\theta)$

$$r = \sqrt{\xi}$$

$$\theta = 2\pi$$





Uniform sampling hemisphere

- Uniform means $p(\omega)$ is constant

$$p(\omega) = \mathrm{c}$$

$$\int_{\mathrm{H}} \mathrm{cd}\omega \ = 1 \Rightarrow c = \frac{1}{2\pi}$$

$$p(\omega) = \frac{1}{2\pi}$$

$$d\omega = \sin\theta d\theta d\phi$$

$$p(\theta,\phi) d\theta d\phi = p(\omega) d\omega$$

$$p(\theta,\phi) = \sin(\theta)\, p(\omega) = \frac{\sin(\theta)}{2\pi}$$

Uniform sampling hemisphere

- Sampling θ by $\theta's$ marginal density function $p(\theta)$

$$p(\theta) = \int_0^{2\pi} \frac{\sin\theta}{2\pi} d\phi = \sin\theta$$

- Compute conditional density for ϕ

$$p(\phi \mid \theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi}$$

Uniform sampling hemisphere

- Use the inverse method, integrate to get CDF

$$P(\theta) = \int_0^\theta \sin\theta' \, d\theta' = 1 - \cos\theta$$

$$P(\phi \mid \theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}$$

- Then inverse

$$\theta = \cos^{-1} \xi_1$$

$$\phi = 2\pi\xi_2$$

- Then transform to cartesian coordinate

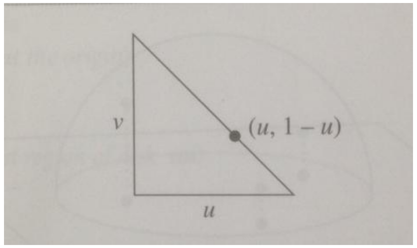
$$x = \sin\theta\cos\phi = \cos(2\pi\xi_2)\sqrt{1 - \xi_1^{\,2}}$$

$$y = \sin\theta\sin\phi = \sin(2\pi\xi_2)\sqrt{1 - \xi_1^{\,2}}$$

$$z = \cos\theta = \xi_1$$

Uniform sampling triangle

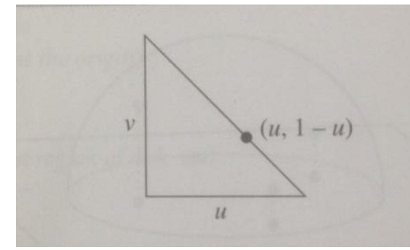
- We will assmue we are sampling an isosceles right triangle of area ½
- Output is barycentric coordinate, so work for any triangle



$$p(u,v) = 2$$

Uniform sampling triangle

$$p(u,v) = 2$$



- First, find the marginal density

$$p(u) = \int_0^{1-u} p(u,v)\,dv = 2\int_0^{1-u} dv = 2(1-u)$$

- And conditional density

$$p(v \mid u) = \frac{p(u,v)}{p(u)} = \frac{2}{2(1-u)} = \frac{1}{1-u}$$

Sampling triangle

- Inverse method as always

$$P(u) = \int_0^u p(u')du' = 2u - u^2$$

$$P(v) = \int_0^u p(v'|u)dv' = \frac{v}{1-u}$$

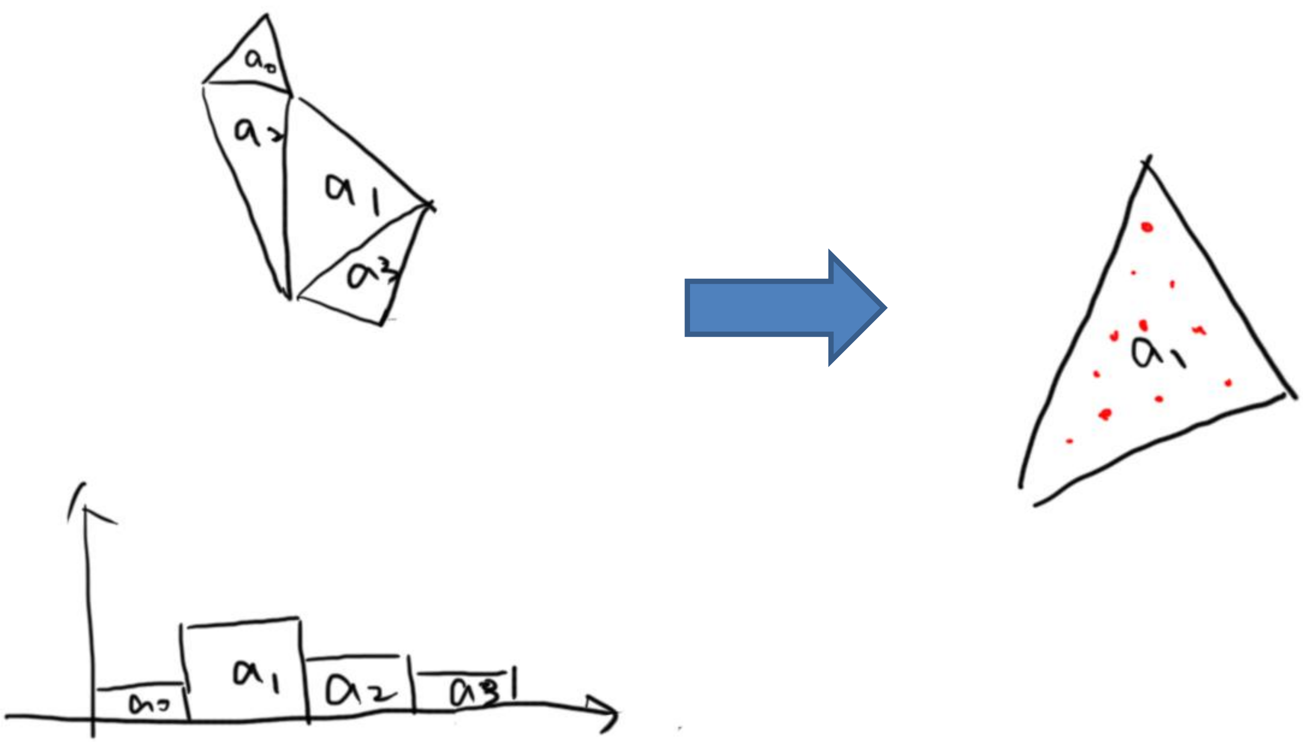
$$\Rightarrow$$

$$u = 1 - \sqrt{\xi_1}$$

$$v = \xi_2 \sqrt{\xi_1}$$

Uniform sampling mesh

- Uniform sample triangle index, then uniform sample the triangle

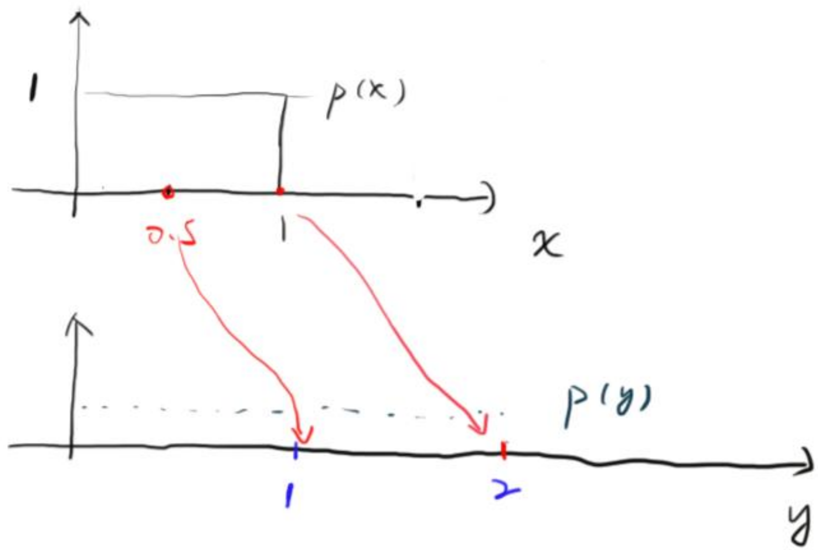


Transforming between distributions

- If we are given random variables X_i that are already drawn from some PDF $p(X_i)$
- Give another random variable $Y_i = y(X_i)$

- What is $p(Y_i)$?

Transforming between distributions

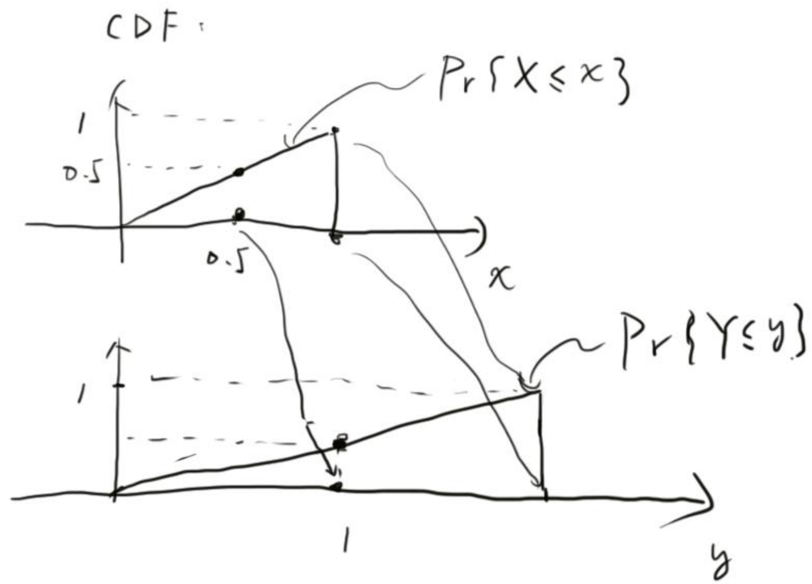


$y = 2x$

Q: $p(y)$?

$y(x)$ must be a one to one transformation

Transforming between distributions



$y = 2x$

$$\Pr\{Y \le y(x)\} = \Pr\{X \le x\}$$

$$P_y(y) = P_y(y) = P_y(y(x)) = P_x(x)$$

differentialting

$$p_y(y)\left|\frac{dy}{dx}\right| = p(x)$$

$$\Rightarrow p_y(y) = \left|\frac{dy}{dx}\right|^{-1} p(x)$$

Transformation in multiple dimensions

- If x is an n-dimensional random variable X with density function $p_x(x)$ and

$$Y = T(X) \text{ where T is a bijection}$$

then the densities are related by

$$\mathrm{p_y(y)} = \mathrm{p_y}(T(x)) = \frac{p_x(x)}{\left|J_T(x)\right|}$$

where $\left|J_T(x)\right|$ is the absolute value of the determinant of T's Jacobian matrix,

which is

$$\begin{pmatrix} \partial T_1 / \partial x_1 & \cdots & \partial T_1 / \partial x_n \\ \vdots & \ddots & \vdots \\ \partial T_n / \partial x_1 & \cdots & \partial T_n / \partial x_n \end{pmatrix}$$

Example

- Polar coordinate

$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Suppose we draw samples from some density $\mathrm{p}(\mathrm{r},\theta)$

- What is the corresponding density $\mathrm{p}(\mathrm{x}, y)$?

- The Jacobian of the transform is

$$J_T = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \Rightarrow \text{determinant is } \mathrm{r}(\cos^2\theta + \sin^2\theta) = \mathrm{r}$$

$$\Rightarrow p(x, y) = p(r,\theta)/r$$