Verification: Rotation Matrix Forms a Group

The rotation matrix is given by:

$$Rotation(a) = \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix}. \tag{1}$$

The set of all such matrices corresponds to rotations by any angle $a \in \mathrm{R}$. We verify the group properties under matrix multiplication:

1. Closure

If $Rotation(a)$ and $Rotation(b)$ are two rotation matrices, their product is:

$$\begin{aligned} Rotation(a) \cdot Rotation(b) &= \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix} \cdot \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix} \\ &= \begin{bmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{bmatrix}. \end{aligned}$$

This is another rotation matrix $Rotation(a+b)$. Therefore, the set is closed under multiplication.

2. Associativity

Matrix multiplication is associative. Hence, for all $Rotation(a), Rotation(b), Rotation(c)$,

$$Rotation(a) \cdot (Rotation(b) \cdot Rotation(c)) = (Rotation(a) \cdot Rotation(b)) \cdot Rotation(c). \tag{2}$$

3. Identity Element

The identity element is the rotation matrix corresponding to $a = 0$:

$$Rotation(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Multiplying $Rotation(0)$ with any $Rotation(a)$ leaves $Rotation(a)$ unchanged:

$$Rotation(a) \cdot Rotation(0) = Rotation(0) \cdot Rotation(a) = Rotation(a). \tag{4}$$

4. Inverse Element

The inverse of $Rotation(a)$ is $Rotation(-a)$, the rotation by the negative angle:

$$Rotation(-a) = \begin{bmatrix} \cos(-a) & -\sin(-a) \\ \sin(-a) & \cos(-a) \end{bmatrix} = \begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix}. \tag{5}$$

Multiplying $Rotation(a) \cdot Rotation(-a)$ or $Rotation(-a) \cdot Rotation(a)$ results in the identity matrix.

Conclusion

The set of all 2×2 rotation matrices forms a **group** under matrix multiplication. This group is called the **special orthogonal group in two dimensions**, denoted as $SO(2)$. It represents all possible rotations in R^2 and satisfies all group properties.