STAT40810 — Stochastic Models

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Week 9

Poisson Process

Big O, Little o

- A common notation in mathematics, physics, computer science and statistics is the Big O, little o notation.
- When we write $O(h)$, as $h \to 0$, we mean that something is approximately proportional to h when h tends to zero.
- When we write $o(h)$, as $h \to 0$, we mean that something is smaller than h as h tends to zero.
- More formally, ...

$$f(h) = O(h), \text{ if } \lim_{h \to 0} \frac{f(h)}{h} = C \neq 0$$

$$f(h) = o(h), \text{ if } \lim_{h \to 0} \frac{f(h)}{h} = 0$$

Counting processes

Counting processes are increasing processes $\{X_t, t \in \mathbb{R}_+\}$ with $X_t \in \mathbb{N}$.
They are an important class of processes in continuous time.

Notation

The idea is to count something over time (eg, the arrivals of customers): X_t is the number of arrivals between time 0 and time t, so naturally we set $X_0 = 0$.
Let $N_{(s,t]}$ be the number of arrivals during the interval $(s, t]$:

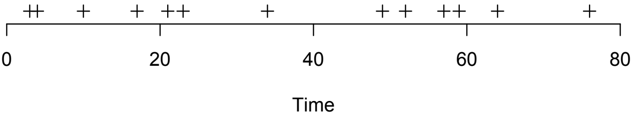
$$N_{(s,t]} = X_t - X_s.$$

Note that $X_t = N_{(0,t]}$.

Example: Ireland vs New Zealand

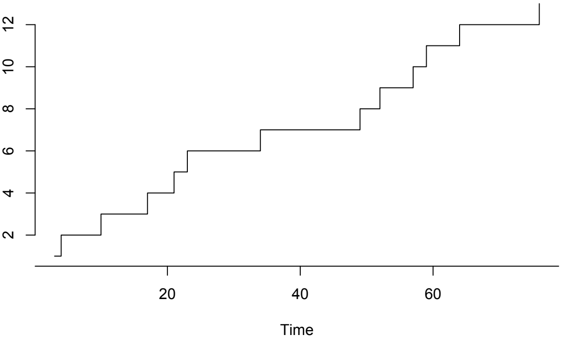
Let's consider the times where there were scores in the weekend game between Ireland and New Zealand.

The scores were at these times...



Example: Ireland vs New Zealand

The counting process looks like this...



Definition of a Poisson process

Poisson process

A Poisson process with *intensity* $\lambda > 0$ is a counting process $\{X_t\}$ with

1. independent increments;
2. $\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$ when $h \to 0$;
3. $\mathbb{P}(X_{t+h} - X_t > 1) = o(h)$ when $h \to 0$.

Some remarks about (1)

By "independent increments" we mean that, for any $t_1 < t_1 \leq t_3 < t_4$,

$$X_{t_2} - X_{t_1} \text{ indep. of } X_{t_4} - X_{t_3}.$$

In other words:

$$N_{(t_1,t_2]} \text{ is indep. of } \quad N_{(t_3,t_4]}.$$

This is sometimes referred as the *loss of memory* property.

Remarks on (2) and (3)

Recall (2) and (3):

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h),$$

$$\mathbb{P}(X_{t+h} - X_t > 1) = o(h).$$

We can write (3) in a different way:

$$\frac{\mathbb{P}(X_{t+h} - X_t > 1)}{h} \to 0, \text{ as } h \to 0.$$

Examples of Poisson Property

- In the rugby game, two scores will never happen *exactly* at the same time.
- In a shop/bank two customers will never arrive *exactly* at the same time, and the probability that two customers arrive at a very small period of time is negligible.
- Remark that (2) and (3) imply that:

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h).$$

About Poisson

Siméon Denis Poisson (1781-1840), the French mathematician and physicist, who received the Copley medal from the Royal Society of London in 1832.

Fundamental theorem

A reminder: the Poisson distribution

$$\mathbb{P}\{X = k\} = \frac{\gamma^k}{k!} e^{-\gamma}.$$

Then:

- $\mathbb{E}(X) = \mathbb{V}\mathrm{ar}(X) = \gamma$.

Theorem

Let $\{X_t\}$ be a Poisson process with intensity λ and $N_{(s,t]} = X_t - X_s$ for $s < t$.
Then:

$$N_{(s,t]} \sim \mathcal{P}(\lambda(t - s)).$$

An immediate consequence

The arrival rate

The arrival rate between times s and t is:

$$\frac{X_t - X_s}{t - s}.$$

Consequence of the theorem

For a Poisson process, the expected arrival rate is constant, equal to λ:

$$\mathbb{E}\left[\frac{X_t - X_s}{t - s}\right] = \lambda.$$

Arrival times

Definition

Let S_1, S_2, ... be the arrival times:

$$S_k = \min\{t \geq 0 : \quad X_t = k\}$$

(and by convention $S_0 = 0$).

Note that:

$$X_t = \sum_{i=1}^{\infty} \mathbf{1}_{S_i \leq t}.$$

Distribution of the arrival times

Theorem

For any $k \geq 1$, $S_k \sim \Gamma(k, \lambda)$ where we recall that the $\Gamma(k, \lambda)$ distribution has density:

$$f(x) = \frac{\lambda^k}{\Gamma(k)} e^{-\lambda x} x^{k-1}.$$

Gap between arrival times

Definition

For any $k \geq 1$, $T_k = S_k - S_{k-1}$.

Theorem

The T_k are iid with distribution $\mathcal{E}(\lambda)$, where we recall that the exponential distribution $\mathcal{E}(\lambda)$ has density

$$f(x) = \lambda e^{-\lambda x}.$$

In particular, $\mathbb{E}(T_k) = 1/\lambda$ and $\mathrm{Var}(T_k) = 1/\lambda^2$.

Marked Poisson process

Defintion - marked Poisson process

A marked Poisson process is defined as:

- a Poisson process $\{X_t, t \geq 0\}$ with arrival times S_1, S_2, ...
- a collection of iid random variables M_1, M_2, ... indep. of $\{X_t\}$.

For example:

- S_k is the time of the k-th score in a rugby game.
- M_k is the points awarded for the k-th score.

Or:

- S_k is the arrival time of the k-th customer.
- M_k is the money spent by the k-th customer.

Or:

- S_k is the time of occurence of the k-th earthquake;
- M_k is its magnitude.

...

Thinned Poisson process

Definition - thinned Poisson process

Let $\{X_t\}$ be a marked Poisson process with arrival times S_1, S_2, ... and marks M_1, M_2, ... $\sim \mathcal{B}e(p)$.
We define the thinned Poisson process $\{Y_t\}$ by

$$Y_t = \sum_{i=1}^{\infty} \mathbf{1}_{S_i \leq t, M_i = 1}.$$

In other words, $\{Y_t\}$ is obtained from $\{X_t\}$ by erasing the arrivals S_i with $M_i = 0$.

Theorem

The process $\{Y_t\}$ is actually a Poisson process with parameter $p\lambda$.

Superposition of Poisson process

Theorem

Let $\{X_t\}$ and $\{Y_t\}$ be two Poisson processes independent of each other, with intensity given by λ and μ, respectively. Let us put $Z_t = X_t + Y_t$. Then $\{Z_t\}$ is a Poisson process with intensity $\lambda + \mu$.

Poisson Process: Estimation

- Estimating the intensity of a Poisson process is straightforward.
- Suppose we observe a process for the time interval $(0, T]$.
- We know that the number of events has a Poisson(λT) distribution.
- Thus,

$$\hat{\lambda} = \frac{\{\#\text{Events observed}\}}{T}.$$

- For the rugby game we get $\hat{\lambda} = 0.1625$.
- That is, we would expect to see 0.1625 scores per minute.
- The time between scores should be exponentially distributed with rate 0.1625 (mean=6 minutes and 9 seconds)