STAT40810 — Stochastic Models

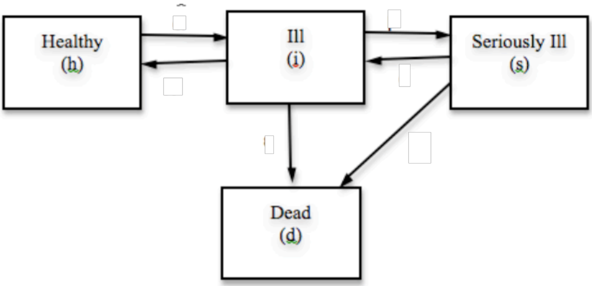
Brendan Murphy

Week 10

Multistate Markov Models

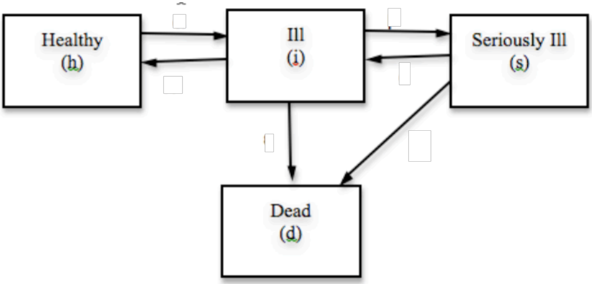
Multistate Markov Model

- Consider the following process being proposed to model a health/illness scenario.



Multistate Markov Model

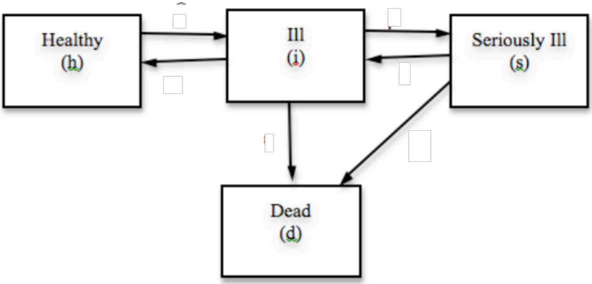
- Consider the following process being proposed to model a health/illness scenario.



- The process has four states:
  - healthy (h)
  - ill (i)
  - seriously ill (s)
  - dead (d)

Multistate Markov Model

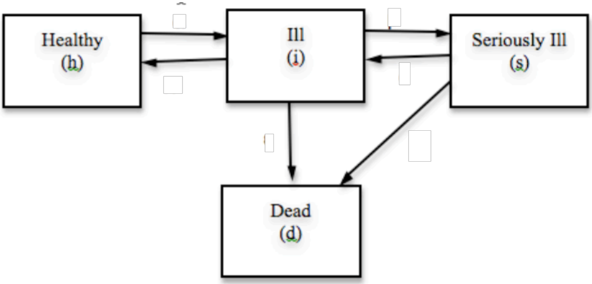
- Consider the following process being proposed to model a health/illness scenario.



- The process has four states:
  - healthy (h)
  - ill (i)
  - seriously ill (s)
  - dead (d)
- And people can move between states in *continuous time*.

Multistate Markov Model

- Consider the following process being proposed to model a health/illness scenario.



- The process has four states:
  - healthy (h)
  - ill (i)
  - seriously ill (s)
  - dead (d)
- And people can move between states in *continuous time*.
- The flow between states is controlled by unknown parameters.

Assumptions

- Consider the process at time t and what happens before time $t + dt$.
- We make the following assumptions:
  1. The probabilities of being in any state at any subsequent time depends only on the time involved and on the state currently occupied.

Assumptions

- Consider the process at time t and what happens before time $t + dt$.
- We make the following assumptions:
  1. The probabilities of being in any state at any subsequent time depends only on the time involved and on the state currently occupied.
  2. We assume a constant transition intensity λ_{gh}.

Assumptions

- Consider the process at time t and what happens before time $t + dt$.
- We make the following assumptions:
  1. The probabilities of being in any state at any subsequent time depends only on the time involved and on the state currently occupied.
  2. We assume a constant transition intensity λ_{gh}.
  3. The probability of transitioning from state g to state h between time t and $t + dt$ is $\lambda_{gh} dt + o(dt)$ as $dt \to 0$.

Assumptions

- Consider the process at time t and what happens before time $t + dt$.
- We make the following assumptions:
  1. The probabilities of being in any state at any subsequent time depends only on the time involved and on the state currently occupied.
  2. We assume a constant transition intensity λ_{gh}.
  3. The probability of transitioning from state g to state h between time t and $t + dt$ is $\lambda_{gh} dt + o(dt)$ as $dt \to 0$.
  4. The probability of transitioning between two or more states is $o(dt)$ as $dt \to 0$.

Data

- Suppose for a cohort of people we observed the following data:

Data

- Suppose for a cohort of people we observed the following data:

Transitions	Count
h to i	n_{hi}
i to h	n_{ih}
i to s	n_{is}
s to i	n_{si}
i to d	n_{id}
s to d	n_{sd}

Data

- Suppose for a cohort of people we observed the following data:

Transitions	Count
h to i	n_{hi}
i to h	n_{ih}
i to s	n_{is}
s to i	n_{si}
i to d	n_{id}
s to d	n_{sd}

- And...

State	Time
h	t_h
i	t_i
s	t_s
d	t_d

Data

- Suppose for a cohort of people we observed the following data:

Transitions	Count
h to i	n_{hi}
i to h	n_{ih}
i to s	n_{is}
s to i	n_{si}
i to d	n_{id}
s to d	n_{sd}

- And...

State	Time
h	t_h
i	t_i
s	t_s
d	t_d

- We have enough information to form the likelihood function.

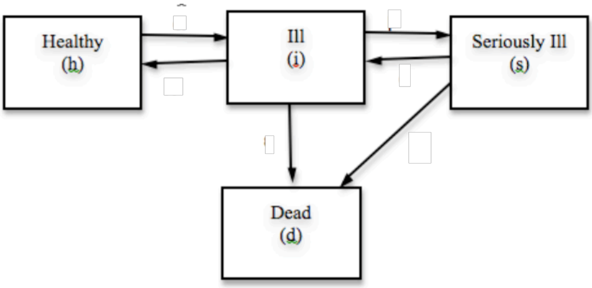
Example

- The following data were collected from 100 individuals who were monitored:
  - total waiting time in state h: 4,100 months
  - total waiting time in state i: 1,000 months
  - total waiting time in state s: 500 months
  - total number of transfers form state h to state i: 110
  - total number of transfers from state i to state h: 90
  - total number of transfers from state i to state s: 40
  - total number of transfers from state s to state i: 20
  - total number of transfers from state i to state d: 7
  - total number of transfers from state s to state d: 10

Likelihood

- The likelihood is of the form

$$
\begin{aligned}
L &= \exp(-\lambda_{hi} t_h) \exp(-(\lambda_{ih} + \lambda_{is} + \lambda_{id}) t_i) \exp(-(\lambda_{si} + \lambda_{sd}) t_s) \\
&\quad \lambda_{hi}^{n_{hi}} \lambda_{ih}^{n_{ih}} \lambda_{is}^{n_{is}} \lambda_{id}^{n_{id}} \lambda_{si}^{n_{si}} \lambda_{sd}^{n_{sd}}
\end{aligned}
$$



- Note that t_d doesn't enter the expression.

Parameter Estimates

- The model parameters can be easily found using maximum likelihood.
- It turns out (check) that

$$\hat{\lambda}_{gh} = \frac{n_{gh}}{t_g}$$

  for any pair of states g and h.

Parameter Estimates

- The model parameters can be easily found using maximum likelihood.
- It turns out (check) that

$$\hat{\lambda}_{gh} = \frac{n_{gh}}{t_g}$$

  for any pair of states g and h.
- Furthermore,

$$SE(\hat{\lambda}_{gh}) = \frac{\sqrt{n_{gh}}}{t_g}.$$

Estimates

- Hence, the MLEs are

$$\hat{\lambda}_{hi} = \frac{n_{hi}}{t_h} = \frac{110}{4100}$$

$$\hat{\lambda}_{ih} = \frac{n_{ih}}{t_i} = \frac{90}{1000}$$

$$\hat{\lambda}_{is} = \frac{n_{is}}{t_i} = \frac{40}{1000}$$

$$\hat{\lambda}_{id} = \frac{n_{id}}{t_i} = \frac{7}{1000}$$

$$\hat{\lambda}_{si} = \frac{n_{si}}{t_s} = \frac{20}{500}$$

$$\hat{\lambda}_{sd} = \frac{n_{sd}}{t_s} = \frac{10}{500}$$

Confidence Intervals

- Hence, approximate 95% confidence intervals for the intensities are

$$\lambda_{hi} \in (0.022, 0.032)$$

$$\lambda_{ih} \in (0.071, 0.109)$$

$$\lambda_{is} \in (0.028, 0.052)$$

$$\lambda_{id} \in (0.002, 0.012)$$

$$\lambda_{si} \in (0.022, 0.058)$$

$$\lambda_{sd} \in (0.008, 0.032)$$