STAT40810 — Stochastic Models

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Week 4

Spline Smoothing

Spline Smoothing: Motivation

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- Alternatively, we could try to estimate $f(x)$ without putting many restrictions on its functional form.
- Minimizing $\sum_{i=1}^{n}[y_i - f(x_i)]^2$ would give a well fitting model. However, without restrictions, the $f(x)$ that minimizes this criterion interpolates the data (unless the x_i values are non-unique).

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- Minimizing $\sum_{i=1}^{n}[y_i - f(x_i)]^2$ would give a well fitting model. However, without restrictions, the $f(x)$ that minimizes this criterion interpolates the data (unless the x_i values are non-unique).
- If we put a penalty for the "roughness of our estimate", then we may get a more sensible result.

Roughness Penalty

- One possibility is to penalize functions that are two "rough" or "wiggly".
- One criterion for measuring roughness is,

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- It is worth noting that if

$$f(x) = \beta_0 + \sum_{p=1}^{P} \beta_p x^p,$$

that is that $f(x)$ is a polynomial of order P.
Then,

$$R(f^{(Q)}) = 0 \text{ for } Q > P$$

Spline Smoothing

- Suppose we want to find the $f(x)$ that minimizes

$$\sum_{i=1}^{n}[y_i - f(x_i)]^2 + \lambda \int [f^{(j)}(x)]^2 dx.$$

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- The case of most interest is when $j = 2$ and this gives cubic splines.

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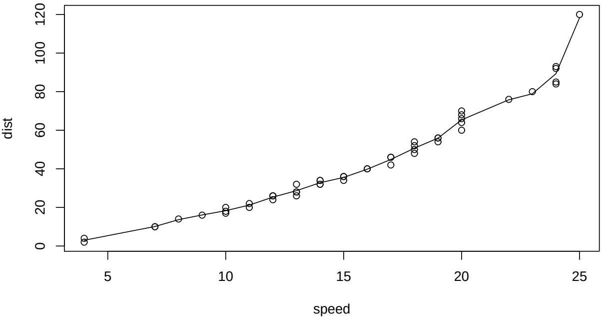
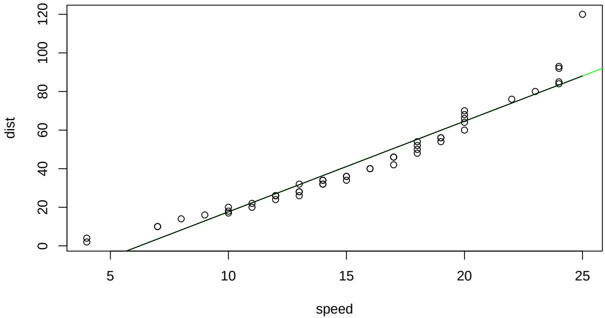
Example: Breaking Distance

- Suppose that we minimize the criterion with $\lambda = 0.2$, then we get a flexible fit.



Example: Breaking Distance

- Suppose that we minimize the criterion with $\lambda = 1.5$, then we get a curve that's almost a linear fit.



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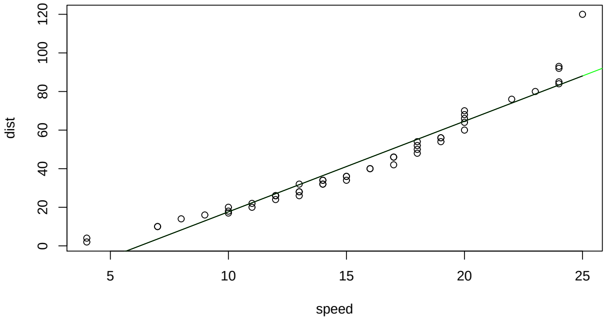

- The choice of λ drives the balance between fit and smoothness.