STAT40810 — Stochastic Models

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Week 8

(Finite) Markov Chains

Andrey Markov

Andrey Andreyevich Markov (1856-1922), Russian mathematician, famous for Markov inequality, Markov processes, Markov chains, ... and his political opinions - among others, his support for the writer Leo Tolstoy led to his excommunication in 1912. His brother and his son were famous mathematicians too.

Reminder - Markov property

Definition - Markov property

Let $\{X_t, t \in \mathcal{T}\}$ be a discrete time process with $\mathcal{T} = \mathbb{N}$. It is said to have the Markov property if for any $n \in \mathcal{T}$,

$$p(X_{n+1}|X_n, X_{n-1}, X_{n-2}, \dots) = p(X_{n+1}|X_n).$$

In other words, to make predictions about the future of the process, it suffices to consider the present state, and not the past history.

In other other words, the state of the system is important but now how it arrived at that state.

Definition of a finite Markov chain

Definition - finite Markov chain

The process $\{X_t, t \in \mathbb{N}\}$ is said to be a finite Markov chain if:

1. it satisfies the Markov property,
2. the state space $\mathcal{X}$ is finite. We will usually take $\mathcal{X} = \{1, ..., M\}$.

Markov property for a finite Markov chain

The Markov property can be written:

$$\mathbb{P}\left(X_{n+1} = i_{n+1} | X_n = i_n, \ldots, X_1 = i_1, X_0 = i_0\right) \\ = \mathbb{P}\left(X_{n+1} = i_{n+1} | X_n = i_n\right).$$

Homogeneous finite Markov chain

Definition - homogeneous finite Markov chain

The Markov chain is said to be homogeneous if:

$$\mathbb{P}\left(X_{n+1} = j | X_n = i\right) = p(i,j) \text{ does not depend on } n.$$

We will only consider homogeneous finite Markov chains, which we abbreviate to "Markov chains".

Note that $p(i,j) \geq 0$ and

$$\sum_{j=1}^{M} p(i,j) = 1.$$

Basic properties

Definition

We put $\pi_n(i) = \mathbb{P}(X_n = i)$.

We have:

$$\begin{aligned}
\mathbb{P}&\left(X_n = i_n, \ldots, X_0 = i_0\right) \\
&= \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \\
&\qquad \times \mathbb{P}\left(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right) \\
&= p(i_{n-1}, i_n)\mathbb{P}\left(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right).
\end{aligned}$$

By recurrence:

$$\mathbb{P}\left(X_n = i_n, \ldots, X_0 = i_0\right) = \pi_0(i_0)p(i_0, i_1)\ldots p(i_{n-1}, i_n).$$

Transition matrix

Definition - Transition matrix

$$P = \begin{pmatrix} p(1,1) & \dots & p(1,M) \\ \vdots & \dots & \vdots \\ p(M,1) & \dots & p(M,M) \end{pmatrix}.$$

Remember that any transition matrix satisfies $\sum_{j=1}^{M} p(i,j) = 1$ and $p(i,j) \geq 0$. Such a matrix is called a *stochastic matrix*.

Notation

It is also usual to represent the probability distribution of X_n by a row vector

$$\pi_n = (\pi_n(1), \dots, \pi_n(M)).$$

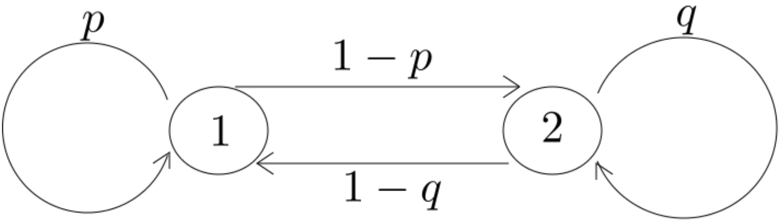
Example 1 - two state Markov chain

Example: the weather in Dublin is either "sunny" or "rainy". For the sake of brevity, we say that the weather is in state 1 when it is sunny, 2 when it is rainy. We assume that when it is sunny, the probability that is is still sunny the next day is p. When it is rainy, the probability that it is still rainy the next day is q.
This gives a Markov chain with transition matrix:

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}.$$

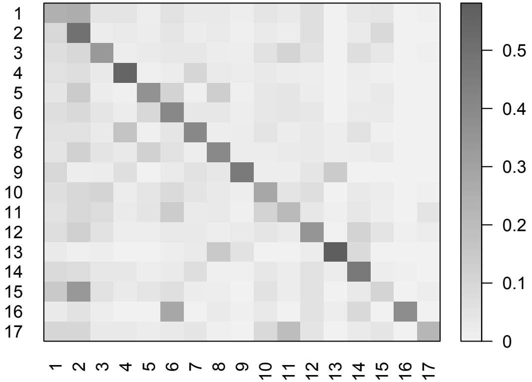
Example 1 - transition graph of the chain

An alternative way to represent the transition matrix P:



MSNBC Clickstream

- A Markov chain transition matrix was estimated from the MSNBC data.



- There are interesting patterns in the transitions.

Example 3 - random walks

Example: random walk with reflecting boundaries

$$p(i, i+1) = p, \quad p(i, i-1) = 1 - p,$$

$$p(0, 1) = 1 \text{ and } p(M, M-1) = 1.$$

Example: random walk with absorbing boundaries

$$p(i, i+1) = p, \quad p(i, i-1) = 1 - p,$$

$$p(0, 0) = 1 \text{ and } p(M, M) = 1.$$

Looking n steps ahead

Definition

Let us define

$$p_n(i,j) = \mathbb{P}(X_n = j | X_0 = i).$$

Theorem

$$p_{m+n}(i,j) = \sum_{k=1}^{N} p_m(i,k) p_n(k,j).$$

Theorem

$$\Big(p_n(i,j)\Big)_{1 \le i,j \le N} = P^n.$$

Looking n steps ahead

Theorem

$$\pi_n = \pi_0 P^n.$$

Example: Dublin weather example with $p = 0.2$, $q = 0.7$,

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.28 & 0.72 \\ 0.27 & 0.73 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0.272 & 0.728 \\ 0.273 & 0.727 \end{pmatrix},$$

$$\cdots \quad P^n \simeq \begin{pmatrix} 0.272727\ldots & 0.727272\ldots \\ 0.272727\ldots & 0.727272\ldots \end{pmatrix}$$

Note that this implies that $\pi_n \simeq (0.2727\ldots, 0.7272\ldots)$ does not depend on π_0 when n is large!

Some questions and a first answer

1. Do we have, for some probability distribution $\pi = (\pi(1), \ldots, \pi(N))$,

$$\pi_n = \pi_0 P^n \to \pi,$$

   as $n \to \infty$?

2. If this is the case, does π depend on π_0?

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Theorem

If $\pi_n \to \pi$, as $n \to \infty$, then π satisfies $\pi P = \pi$.

Definition

Any π such that $\pi P = \pi$ is called a "stationary" or "equilibrium" or "invariant" distribution.

Reminder on eigenvalues and eigenvectors

For a matrix M, if there is $\lambda \in \mathbb{R}$ and a column vector $v \in \mathbb{R}^n$ such that

$$Mv = \lambda v$$

then λ is an "eigenvalue" and v is a "eigenvector".

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Theorem

Any transition matrix P has an invariant distribution.

Example: MSNBC

- The stationary distribution is:

```
1: frontpage (0.093)
2: news (0.197)
3: tech (0.059)
4: local (0.075)
5: opinion (0.048)
6: on-air (0.081)
7: misc (0.059)
8: weather (0.052)
9: msn-news (0.026)
10: health (0.053)
11: living (0.040)
12: business (0.070)
13: msn-sports (0.021)
14: sports (0.071)
15: summary (0.041)
16: bbs (0.005)
17: travel (0.010)
```

Example: social mobility

Data: 3500 pairs (father, adult son) from England and Wales early 1950's (a time of low unemployment). They have been classified according to 7 categories:

1. Professional and high administrative
2. Managerial and executive
3. Higher-grade supervisory and non-manual
4. Lower-grade supervisory and non-manual
5. Skilled manual and routine non-manual
6. Semi-skilled manual
7. Unskilled manual

The results are given in table in form of probabilities $p(i,j) = n_{ij} / \sum_k n_{ik}$, where n_{ij} is the number of pairs with father being from category i and son from j.

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Social mobility: the data

F\S	1	2	3	4	5	6	7
1	0.388	0.146	0.202	0.062	0.140	0.047	0.015
2	0.107	0.267	0.227	0.120	0.206	0.053	0.020
3	0.035	0.101	0.188	0.191	0.357	0.067	0.061
4	0.021	0.039	0.112	0.212	0.430	0.124	0.062
5	0.009	0.024	0.075	0.123	0.473	0.171	0.125
6	0.000	0.013	0.041	0.088	0.391	0.312	0.155
7	0.000	0.008	0.036	0.083	0.364	0.235	0.274

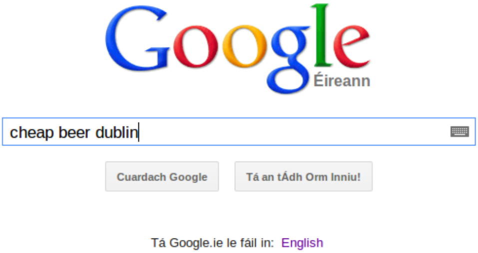
Social mobility - limiting and actual distributions

This table shows the calculated limiting distribution π for the corresponding Markov chain and the actual distributions for fathers and sons.

Class	π	Actual distribution: Father	Actual distribution: Son
1	0.023	0.037	0.029
2	0.042	0.043	0.046
3	0.088	0.098	0.094
4	0.127	0.148	0.131
5	0.409	0.432	0.409
6	0.182	0.131	0.170
7	0.129	0.111	0.121

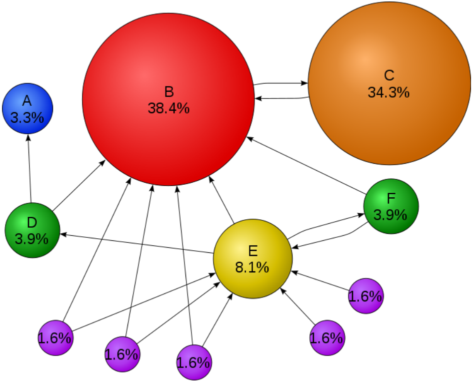
It is tempting (although probably incorrect) to conclude that English/Welsh society was stable in 1950's, for so many generations the limiting distribution was already being approached.
This might be not true if subsequent developments on the job market are taken into account.

Example 2


Google
Éireann
cheap beer dublin
Cuardach Google
Tá an tÁdh Orm Inniu!
Tá Google.ie le fáil in: English

Example 2: Google's PageRank algorithm

How does Google rank web pages?



Source: http://en.wikipedia.org/wiki/PageRank