Uncertainty Quantification (ACM41000)
Exercises – Set 2

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1. Let $P(t)$ be the average concentration of a pollutant in a particular domain at time t. The pollutant naturally degrades over time at a rate k but the domain is subject to a pollution source, so that pollutant enters the domain at a constant (positive) rate s. These effects are summarised in the following ordinary differential equation (ODE):

$$\frac{dP}{dt} = -k(P - P_0) + s, \qquad k, s \in \mathbb{R}^+, \tag{1}$$

   where P_0 is the background level of pollution. Let $P(0) = P_0$ be the initial pollution level. Using the integrating-factor technique, show that

$$P(t) = \frac{s}{k}\left(1 - \mathrm{e}^{-kt}\right) + P_0. \tag{2}$$

2. Show that

$$\lim_{t\to\infty} P(t) = P_0 + \frac{s}{k},$$

   independent of the initial level of pollution. Hence, make a rough sketch of the solution $P(t)$ coming from Part (a).

3. A powerplant emits nitrous oxides (NOx) at a rate s according to Equation (**??**). The factory is fined by the Environmental Protection Agency if the pollution level (even momentarily) exceeds twice the background level, i.e. a fine is imposed if $P(t) > 2P_0$. Show that the factory should emit at a rate

$$s < P_0 k$$

   to avoid the fine.

Question 2.

IV

(a) $$\frac{dP}{dt} = -kP + (S + kP_0)$$

i.e. $$\frac{dP}{dt} + kP = (S + kP_0)$$

$$\mu = e^{\int k\,dt} = e^{kt}$$

$$\frac{d}{dt}(\mu P) = \mu(S + kP_0)$$

$$\frac{d}{dt}(P e^{kt}) = \underbrace{(S + kP_0)}_{\text{const.}} e^{kt}$$

$$\Rightarrow e^{kt} P(t) = C + \left(\frac{S + kP_0}{k}\right) e^{kt}$$

$$P(t) = Ce^{-kt} + \left(\frac{S}{k} + P_0\right), \quad C = \text{const.}$$

$$P(0) = C + \frac{S}{k} + \cancel{P_0} = \cancel{P_0} \text{ ... given}$$

$$\Rightarrow C = -\frac{S}{k}.$$

$$P(t) = \frac{S}{k} + Ce^{-kt} + P_0$$

$$= \frac{S}{k} - \frac{S}{k} e^{-kt} + P_0$$

$$= \frac{S}{k}(1 - e^{-kt}) + P_0$$

$$\boxed{P(t) = \frac{S}{k}(1 - e^{-kt}) + P_0}$$

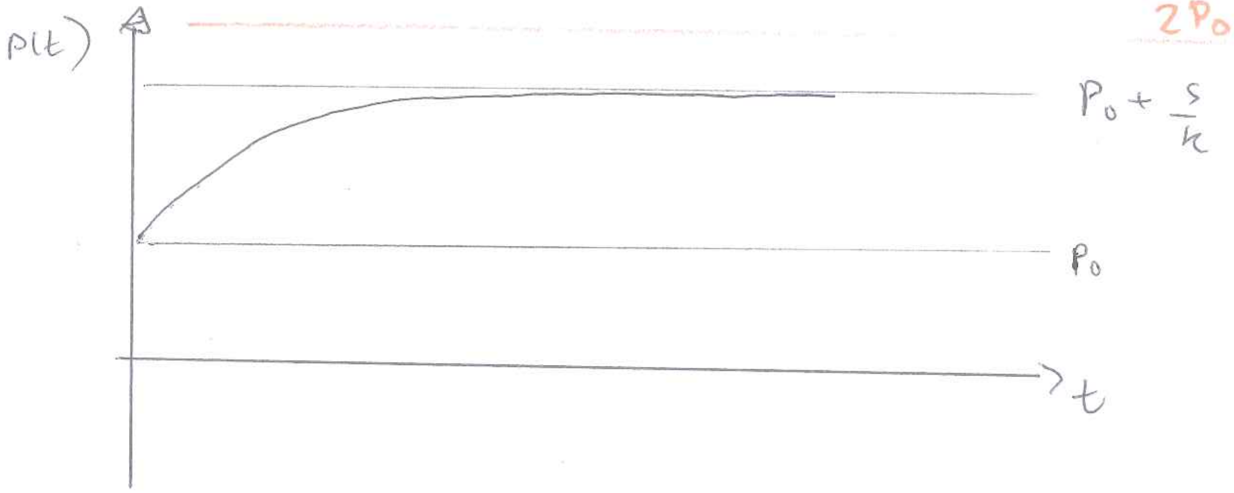
(b) $$\lim_{t\to\infty} P(t) = \frac{s}{k} - \lim_{t\to\infty}\left(\frac{s}{k}e^{-kt}\right) + P_0$$

$$= \frac{s}{k} + P_0 .$$

$P(0) = P_0$.

$\lim_{t\to\infty} P(t) = P_0 + \frac{s}{k}$.

By inspection, $P(t)$ is an increasing f^n.



(c) By inspection of the graph in Part (b), we need

$P_0 + \frac{s}{k} < 2P_0 \implies \boxed{\frac{s}{k} < P_0}$.

Hence, $s < P_0 k$. ∎