Modelling the orientation angles and growth rates of tracer gradient using a stochastic differential equation.

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Msc Data and Computational Science

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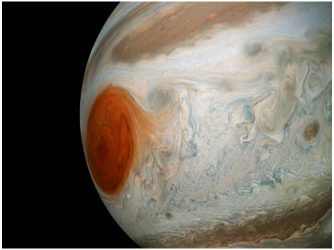
Introduction : Why study turbulence



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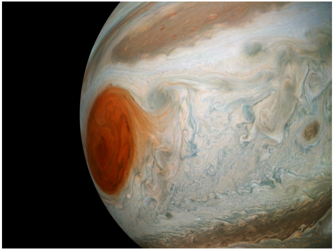

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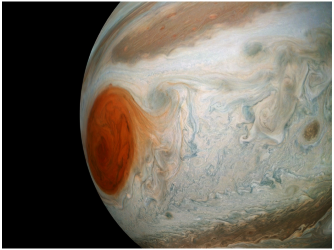

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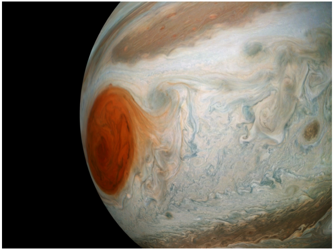

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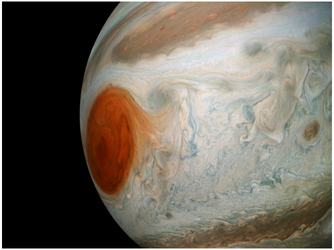
"Turbulence is the most important unsolved problem of classical physics."

- Richard P. Feynman
In The Feynman Lectures on Physics (1964).



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- Compare the PDF of the orientation angles from the vorticity simulation emoperical data and the SDE model.
- Use uncertainty quantification methods to fit angle and growth rate PDF parameters to the simulated data.

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$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = (-1)^p \nu_p \nabla^{2p} \omega + \nabla \times \mathbf{F} - \nabla \times \mathbf{D}$$

Orientation Dynamics :: WIP

ToDo

Stochastic Model :: WIP

Stochastic Differential Equation Model

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Stochastic Differential Equation Model

$$\gamma \frac{dX}{dt} = w + (-cos(X) + k\sqrt{\delta})Y + Z$$
$$\frac{dY}{dt} = -\frac{Y}{\tau} + \frac{\sqrt{D_Y}}{\tau}\xi_Y$$
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$$

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$$\frac{\partial P}{\partial t} = \mathcal{L}_{OU}P - \frac{\partial}{\partial t}(VP)$$

where

$$\mathcal{L}_{OU}P = \frac{1}{\tau_Y}\frac{\partial}{\partial t}(y\bigcirc) + \frac{1}{\tau_Y^2}\frac{\partial^2}{\partial y^2} + \frac{1}{\tau_Z}\frac{\partial}{\partial Z}(Z\bigcirc) + \frac{\rho}{\tau_Z^2}\frac{\partial^2}{\partial z^2} + \frac{2c\sqrt{\rho}}{\tau_Y\tau_Z}\frac{\partial^2}{\partial y\partial z}$$

Numerical Simulation : Vorticity Equation

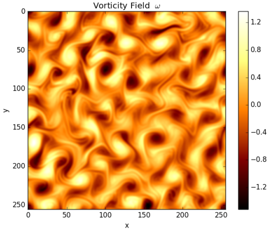
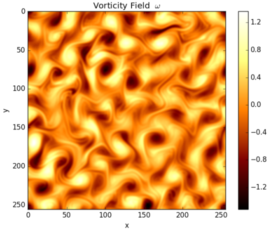


Figure: ω at T = 250

Numerical Simulation : Vorticity Equation

- Periodic Boundary Conditions



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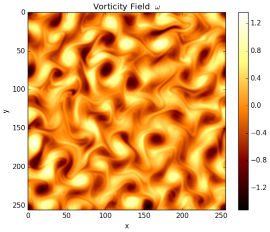

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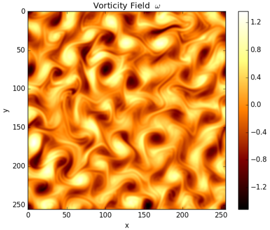

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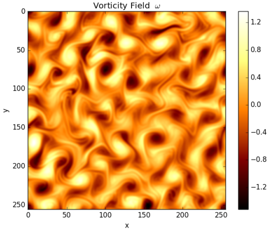

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Numerical Simulation : Vorticity Equation

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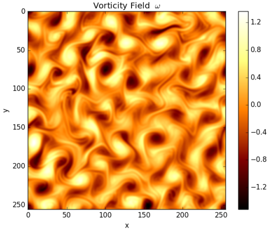
- Save snapshots of ω at regular intervals
- Extract emperical PDF of angle X from statistically stable



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Numerical Simulation : Fokker Planck

- Periodic Boundary Conditions
- Solve for PDF of the Fokker-Plank
- Extract marginal probability of X for the PDF of the angle X
- Compute the PDF of the growth rate using the PDF of X

Analysis

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Conclusions