2.1 Basic Definition, Classification and Terminologies

Linear DE

The general form of a linear differential equation can be expressed in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x)$$

where
$a_0(x),\ a_1(x),\ \cdots,\ a_n(x)$ are called coefficients and $a_n(x) \neq 0$
$f(x)$ is a function of x

Can you tell which one is the **independent variable** and **dependent variable** in the given linear differential equation?

Variable / Constant Coefficients

Variable coefficients

- Coefficients with variable such as x, t , z in it.
- Normally will be written as $a_n(x)$, $a_n(t)$, or $a_n(z)$

Constant coefficients

- Coefficients with constants such as 2, 5, $^2/_3$ in it.
- Will be written as a_n or a_0

Homogenous / Non-Homogenous Equation

Homogenous Equation
- When $f(x) = 0$

Non-Homogenous Equation
- When $f(x) \neq 0$

Example 2.1.1:

1. $y'' + 7y' - 10y = 0$

Linear / Non-linear
Constant Coefficients / Variable Coefficients
Homogenous / Non-homogenous Equation

2. $yy'' + 3y' + 5y^2 = 5x^2$

Linear / Non-linear
Constant Coefficients / Variable Coefficients
Homogenous / Non-homogenous Equation

3. $x^2y'' + xy' + y = \cos x$

Linear / Non-linear
Constant Coefficients / Variable Coefficients
Homogenous / Non-homogenous Equation

Linearly Dependent / Linearly Independent

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

Linearly independent

- $c_1 = c_2 = \cdots = c_n = 0$

Linearly dependent

- At least one of $c_1, c_2, \cdots, c_n \neq 0$

Example 2.1.2:

Determine whether or not the following functions are linearly independent.

a) $y_1 = \sin x$ and $y_2 = 4\sin x$

b) $y_1 = x$ and $y_2 = x^2$

Solution:

a) Given that $y_1 = \sin x$ and $y_2 = 4\sin x$. Equation $C_1 \sin x + 4C_2 \sin x = 0$ can be rewritten as

$$\sin x = -\frac{4C_2 \sin x}{C_1},$$

gives $C_1 = 4,\ C_2 = -1.$
Since $C_1,\ C_2 \neq 0$, this equation is linearly dependent

b) Given that $y_1 = x$ and $y_2 = x^2$. Equation $C_1x + C_2x^2 = 0$ has a solution only if $C_1 = 0,\ C_2 = 0.$
Since $C_1,\ C_2 = 0$, this equation is linearly independent

Linear Combination of Solutions

If y_1 and y_2 are linearly independent, and each of them are the solutions for the DE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

Then the general solution for the equation is

$$y = Ay_1 + By_2$$

Linear Superposition Principle

If $y_1, y_2, y_3, \cdots, y_n$ are linearly independent, and each of them are the solutions for the DE

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0\,y = 0$$

Then the general solution for the equation is

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

2.2 Solution of Homogenous Equation

How to find the solution of this equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0. \qquad \text{(i)}$$

1) Assume the solution in the form of $y=e^{mt}$. Thus, we have

$$y=e^{mt};\quad \frac{dy}{dt}=me^{mt};\quad \frac{d^2y}{dt^2}=m^2e^{mt} \qquad \text{(ii)}$$

2) The solution $y=e^{mt}$ must satisfy (i). Insert (ii) -> (i) gives the characteristic (auxiliary) equation:

$$am^2+bm+c=0 \qquad \text{(iii)}$$

3) Find the roots

$$m_1,m_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

For $b^2-4ac>0 \Rightarrow$ roots are real and distinct ($m_1 \neq m_2$)
For $b^2-4ac=0 \Rightarrow$ roots are real and equal ($m_1 = m_2$)
For $b^2-4ac<0 \Rightarrow$ roots are complex conjugate numbers

4) The solution of (i) depends on the type of roots.

Case 1: Roots are real and distinct, $m_1 \neq m_2$,

General solution: $y = Ae^{m_1 x} + Be^{m_2 x}$.

Example 2.2.1:

Solve $y'' + 5y' + 6y = 0.$

Solution:

1) Use the characteristic equation:

$$m^2 + 5m + 6 = 0.$$

2) Find the roots : $(m+3)(m+2) = 0 \Rightarrow m_1 = -3; \ m_2 = -2.$

3) The general solution is

$$y = Ae^{-3x} + Be^{-2x}$$

Example 2.2.2:

Solve $y'' - y' - 11y = 0.$

Solution:

1) Use the characteristic equation:

$$m^2 - m - 11 = 0.$$

2) Find the roots : $m = \frac{1}{2} \pm \frac{\sqrt{45}}{2}$

$$\Rightarrow m_1 = \frac{1+\sqrt{45}}{2}, \qquad m_2 = \frac{1-\sqrt{45}}{2}.$$

3) The general solution is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}.$$

Case 2: Roots are real and equal, $m_1 = m_2$
General solution: $y = (A + Bx)e^{m_1 x}$

Example 2.2.3:

Solve $y'' + 6y' + 9y = 0$

Solution:

1) Use the characteristic equation:
$$m^2 + 6m + 9 = 0$$
2) Find the roots : $(m+3)^2 = 0 \Rightarrow \quad m = -3, -3$
3) The general solution is
$$y(x) = (A + Bx)e^{-3x}$$

Case 3: Roots are complex numbers; $m = \alpha \pm i\beta$
General solution: $y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Example 2.2.4:

Solve $y'' - 10y' + 26y = 0.$

Solution:

1) Use the characteristic equation:
$$m^2 - 10m + 26 = 0.$$
2) Find the roots : $m = \dfrac{10 \pm \sqrt{100 - 4(26)}}{2}$
$$= \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm \sqrt{4i^2}}{2} = 5 \pm i$$
$$\Rightarrow \alpha = 5 \quad ; \quad \beta = 1$$
3) The general solution is
$$y(x) = e^{5x}(-A\cos x + B\sin x)$$

Initial Value Problem

Example 2.2.5:
Solve the initial value problem: $y'' + 6y' + 16y = 0$; $y(0) = 1, y'(0) = 0.$

These are the initial conditions

Solution:

1)Use the characteristic equation:

$$m^2 + 6m + 16 = 0.$$

2) Find the roots : $m = \dfrac{-6 \pm \sqrt{36 - 4(16)}}{2} = \dfrac{-6 \pm \sqrt{-28}}{2}$

$$= -3 \pm \frac{\sqrt{28i^2}}{2} = -3 \pm \sqrt{7}i.$$

3)The general solution is

$$y(t) = e^{-3t}\left(A\cos\sqrt{7}t + B\sin\sqrt{7}t\right)$$

4) Using IC: i) $y(0) = 1$; $A = 1$;
ii) $y'(0) = 0$

$$y' = e^{-3t}\left(-A\cdot\sqrt{7}\sin\sqrt{7}t + \sqrt{7}B\cos\sqrt{7}t\right) + \left(A\cos\sqrt{7}t + B\sin\sqrt{7}t\right)(-3e^{-3t})$$

$0 = \sqrt{7}B - A\cdot 3$ $\quad\sqrt{7}B = +3$ $\quad\Rightarrow B = \dfrac{3}{\sqrt{7}}.$

5) The solution is

$$y(t) = e^{-3t}\left(\cos\sqrt{7}t + \frac{3}{\sqrt{7}}\sin\sqrt{7}t\right)$$

Exercise 2.2

1. $2y'' - 7y' + 3y = 0$ (Ans: $y(x) = c_1 e^{x/2} + c_2 e^{3x}$)
2. $y'' + 2y' + y = 0, y(0) = 5, y'(0) = -3$ (Ans: $y(x) = 5e^{-x} + 2xe^{-x}$)
3. $y'' - 4y' + 5y = 0$ (Ans: $y(x) = e^{2x}(A\cos x + B\sin x)$)

2.3 Solution of Non-Homogeneous Equation

How to find the solution of this equation

The solution of

$$ay'' + by' + cy = f(x)$$

is given by

$$y(x) = y_h(x) + y_p(x)$$

where

$y_h(x)$ is the solution of the homogeneous equation and $y_p(x)$ is called a particular integral.

Two methods to find $y_p(x)$:

1) **Method of Undetermined Coefficient**
2) **Method of Variation of Parameters**

2.3.1 Method of Undetermined Coefficient

The form of y_p depends on the form of $f(x)$.

Case 1: Polynomial of degree n ;

$$P_n = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$$

Case 2: Form of Ce^{mx}; where C and m are constants

Case 3: Sine/cosine function

Case 4: Sums or products of the above functions (Case 1,2,3)

Case 1: Polynomial $f(x)=A_n x^n + A_{n-1}x^{n-1}+\dots+A_1 x + A_0$
with n as highest degree.
Try $y_p(x)=x^r\left(C_n x^n + C_{n-1}x^{n-1}+\dots+C_1 x + C_0\right)$

Notes: Choose the smallest r, (r = 0,1,2). Chosen to ensure that no term in $y_p(x)$ is in the corresponding $y_h(x)$.

Example 2.3.1:
Solve $y''+y'-2y=2x^2-4x$ (i)

Solution:
Step 1 : Find $y_h(x)$:

$$m^2+m-2=0$$
$$(m+1)(m+2)=0 \Rightarrow m=-2,1$$
$$\therefore \quad y_h(x)=Ae^{-2x}+Be^{x}$$

Step 2: Find $y_p(x)$: take r = 0; try

$$y_p(x)=C_2x^2+C_1x+C_0 \qquad \text{(ii)}$$

Step 3: Compare y_p and y_h. Any similar term? YES/NO?

Step 4: If **YES** - do some modification on y_p by taking different values of r. Then repeat **Step 3**. If **NO** – Proceed to find y_p' & y_p''

$$\left.\begin{aligned} y_p'(x)&=2C_2x+C_1 \\ y_p''(x)&=2C_2 \end{aligned}\right\} \quad \text{(iii)}$$

Step 5: Substitute (ii,iii) into (i) and equate coefficient power of x.

$$2C_2+2C_2x+C_1-2\left(C_2x^2+C_1x+C_0\right)=2x^2-4x$$
$$(2C_2+C_1-2C_0)+x(2C_2-2C_1)-2C_2x^2=2x^2-4x$$
$$\Rightarrow 2C_2+C_1-2C_0=0; \quad (2C_2-2C_1)=-4; \quad -2C_2=2$$

Therefore,

$$C_2 = -1 \qquad C_1 = 1 \qquad C_0 = -\frac{1}{2}$$

$$\Rightarrow y_p(x) = -x^2 + x - \frac{1}{2}$$

Final Answer:

$$\therefore \quad y(x) = Ae^{-2x} + Be^{x} - x^2 + x - \frac{1}{2}$$

What is the $y_p(x)$ for this $f(x)$?

Example 2.3.2:

Solve $y'' + 2y' = x$

Solution:

Step 1 : $y_h = A + Be^{-2x}$

Step 2 : Try

$$\left.\begin{aligned} y_p &= C_1 x + C_0 \\ y_p' &= C_1 \\ y_p'' &= 0 \end{aligned}\right\}$$

Look! This term is already appears in y_h. We have to modify this y_p. Choose new y_p!!

Step 3 : Choose new y_p with $r = 1$; Try

$$y_p(x) = x\left[C_1 x + C_0\right]$$

Compare to y_h again and now, this y_p is correct!!

$$\Rightarrow C_1 = \frac{1}{4}; C_0 = -\frac{1}{4}$$

$$\therefore y_p = \frac{1}{4}(x^2 - x)$$

Final Answer:

$$\therefore \quad y(x) = A + Be^{-2x} + \frac{1}{4}(x^2 - x)$$

Case 2: For $f(x)$ is the form Ce^{mx}.

Seek $y_p = x^r\left(De^{mx}\right)$.

Notes: Choose the smallest r, (r = 0,1,2). Chosen to ensure that no term in $y_p(x)$ is in the corresponding $y_h(x)$.

Example 2.3.3:

Solve $y'' + y' - 2y = 5e^{3x}$. (i)

What is the $y_p(x)$ for this $f(x)$?

Solution:

Step 1: Find $y_h(x)$.

$$m^2 + m - 2 = 0$$

$$(m+2)(m-1) = 0 \quad \Rightarrow m = -2, 1,$$

$$\therefore \quad y_h(x) = Ae^{-2x} + Be^{x} \qquad \text{(ii)}$$

Step 2: Find $y_p(x)$. Try $r = 0$;

$$\left.\begin{aligned} y_p &= De^{3x} \\ y_p' &= 3De^{3x} \\ y_p'' &= 9De^{3x} \end{aligned}\right\}\text{(iii)}$$

Is there any term in y_p appears in y_h?? NO!! Then, continue..

Step 3: Substitute (iii) into (i)

$$9De^{3x} + 3De^{3x} - 2De^{3x} = 5e^{3x}$$

Equate coefficients:

$$(9D + 3D - 2D)e^{3x} = 5e^{3x}$$

$$10D = 5 \quad \Rightarrow D = \frac{1}{2}$$

$$\Rightarrow y_p = \frac{1}{2}e^{3x}$$

Final Answer:

$$\therefore \quad y(x) = y_h + y_p = Ae^{-2x} + Be^{x} + \frac{1}{2}e^{3x}$$

Example 2.3.4:

Solve $y'' + 2y' = 3e^{-2x}$.

What is the $y_p(x)$ for this $f(x)$?

Solution: Identify that $f(x)$ in form Ce^{-2x}.

Step 1: Find y_h :

$$m^2 + 2m = 0$$
$$m(m+2) = 0$$
$$m = 0, -2$$
$$\Rightarrow y_h(x) = A + Be^{-2x}$$

Look! This term is already appears in y_h. We have to modify this y_p. Choose new y_p!!

Step 2: Find y_p Try $r = 0$;

$$y_p = De^{-2x}$$

Step 3: Change y_p ; Try $r = 1$;

$$y_p = Dxe^{-2x}$$
$$y_p' = De^{-2x} - 2Dxe^{-2x}$$
$$y_p'' = -4De^{-2x} + 4Dxe^{-2}$$
$$\Rightarrow D = -\frac{3}{2}$$

Compare to y_h again and now, this y_p is correct!!

Final Answer:

$$\therefore \quad y(x) = A + Be^{-2x} - \frac{3}{2}xe^{-2x}$$

Case 3: $f(x) = A_1 \sin \beta x$ **or** $A_1 \cos \beta x$

Try $y_p = x^r \left(C \sin \beta x + D \cos \beta x \right)$

Notes: Choose the smallest r, (r = 0,1,2). Chosen to ensure that no term in $y_p(x)$ is in the corresponding $y_h(x)$.

Example 2.3.5:

Solve $y'' + y' - 2y = \sin x.$ (i)

What is the $y_p(x)$ for this $f(x)$?

Solution:

Identify that $f(x)$ in form $\sin x$.

Step 1: Find y_h :

$$\Rightarrow y_h(x) = Ae^{-2x} + Be^{x}$$

Step 2: Find y_p, try $r = 0$;

$$\left. \begin{aligned} y_p &= C \sin x + D \cos x \\ y_p' &= C \cos x - D \sin x \\ y_p'' &= -C \sin x - D \cos x \end{aligned} \right\} \quad \text{(ii)}$$

Step 3: Substitute (ii) into (i)

$$-C \sin x - D \cos x + C \cos x - D \sin x - 2(C \sin x + D \cos x) = \sin x$$

Equate coefficients:

$$\Rightarrow C = -\frac{3}{10}, D = -\frac{1}{10} \cos x$$

$$\therefore y_p = -\frac{3}{10} \sin x - \frac{1}{10} \cos x$$

Final Answer:

$$y(x) = Ae^{-2x} + Be^{x} - \frac{3}{10} \sin x - \frac{1}{10} \cos x$$

Case 4: Sums or products of the **Case 1,2 and 3**

1) Sums of the Case 1,2 and 3 : we have

$$f(x) = f_1(x) \pm f_2(x) \pm f_3(x) \pm ..... \pm f_n(x)$$

then

$$y_p(x) = y_{p1}(x) \pm y_{p2}(x) \pm y_{p3}(x) \pm ..... \pm y_{pn}(x).$$

The solution become:

$$y(x) = y_h(x) \pm y_{p1}(x) \pm y_{p2}(x) \pm y_{p3}(x) \pm ..... \pm y_{pn}(x)$$

2) : Products of the Case 1,2 and 3; $y_p(x)$ is given by the table

$f(x)$	$y_p(x)$
$P_n(x)e^{mx}$	$x^r\left(C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0\right)e^{mx}$
$P_n(x)\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^r\left(C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0\right)\cos\beta x$ $x^r\left(C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0\right)\sin\beta x$
$Ce^{mx}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^r e^{mx}\left(C\sin\beta x + D\cos\beta x\right)$
$P_n(x)e^{mx}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^r\left(C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0\right)e^{mx}\cos\beta x$ $x^r\left(C_n x^n + C_{n-1}x^{n-1} + ... + C_1 x + C_0\right)e^{mx}\sin\beta x$

Notes: Choose the smallest r, (r = 0,1,2). Chosen to ensure that no term in $y_p(x)$ is in the corresponding $y_h(x)$.

Example 2.3.6:

What is the $y_p(x)$ for this $f(x)$?

Solve $$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-8y=xe^{-x} \qquad \text{(i)}$$

Solution:

Step 1: Find $y_h(x)$

$$m^2+2m-8=0$$
$$(m-2)(m+4)=0 \quad \Rightarrow m=2,-4$$

$$\therefore \quad y_h(x)=Ae^{2x}+Be^{-4x} \qquad \text{(ii)}$$

Step 2: Find $y_p(x)$

Try: $y_p(x)=e^{-x}(C_1x+C_0)$ (iii)

Step 3: Solve using undetermined coefficient method.

$$y_p'(x)=-e^{-x}C_1x+e^{-x}C_1-C_0e^{-x}$$
$$y_p''(x)=e^{-x}C_1x-e^{-x}C_1-e^{-x}C_1+C_0e^{-x}$$

Substitute into (i):

$$e^{-x}C_1x-e^{-x}C_1-e^{-x}C_1+C_0e^{-x}$$
$$+2(-e^{-x}C_1x+e^{-x}C_1-C_0e^{-x})-8e^{-x}(C_1x+C_0)=xe^{-x}$$
$$\Rightarrow C_1=-\frac{1}{9}, C_0=0.$$

Final Answer :

$$y(x)=Ae^{2x}+Be^{-4x}-\frac{1}{9}xe^{-x}$$

Example 2.3.7:

What is the $y_p(x)$ for this $f(x)$?

Solve $y'' - y' - 2y = e^{-x} + x$ (i)

Solution:

Step1: Characteristic Equation:

$$y_h(x) = Ae^{2x} + Be^{-x} \qquad \text{(ii)}$$

Step 2: Write $y_p = y_{p1} + y_{p2}$

Step 3: Find y_{p1}

Try $y_{p1} = De^{-x}$

Look!! This is same with y_h.

Try new y_p

$$y_{p1} = xDe^{-x}$$

$$y_{p1}' = De^{-x} - xDe^{-x}$$

$$y_{p1}'' = -2De^{-x} + xDe^{-x}$$

Subst. this into (i) and solve it, gives

$$y_{p1} = \frac{1}{3}xe^{-x}$$

Step 4: Find y_{p2}, try

$$y_{p2} = C_1x + C_0 \quad \Rightarrow C_1 = -\frac{1}{2} \quad , \quad C_0 = \frac{1}{4}$$

Step 5: Final Answer

$$y(x) = y_h(x) + y_{p1}(x) + y_{p2}(x)$$

$$= Ae^{2x} + Be^{-x} + \frac{1}{3}xe^{-x} - \frac{1}{2}x + \frac{1}{4}$$

Exercise 2.3.1:

1. Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

a) $y''+2y'-y=x^{-1}e^{x}$ [Ans: NO]

b) $5y''-3y'+2y=x^{3}\cos 5x$ [Ans: Yes]

c) $y''+4y=3\cos(e^{x})$ [Ans: NO]

d) $y''+3y'-y=\cos^{2} x$ [Ans: Yes]

2. Find the form for y_p to

$$y''+2y'-3y=f(x)$$

where $f(x)$ equals

a) $7\cos 3x$ [Ans: $y_p = C\sin 3x + D\cos 3x$]

b) $2xe^{x}\sin x$ [Ans: $y_p = (C_1x+C_0)e^{x}\cos x+(B_1x+B_0)e^{x}\sin x$]

c) $x^{2}e^{x}$ [Ans: $y_p = x(C_2x^2+C_1x+C_0)e^{x}$]

d) $\sin x+\cos 2x$ [Ans: $y_p = A\sin x + B\cos x + C\sin 2x + D\cos 2x$]

e) $3xe^{x}$ [Ans: $y_p = x(C_1x+C_0)e^{x}$]

3. Solve $y'' + 2y' = 6.$ [Ans: $y = A + Be^{-2x} + 3x$]

4. Solve $y'' + y = e^{3x}.$ [Ans: $y(x) = A\cos x + B\sin x + \frac{1}{10}e^{3x}$]

5. Solve $y'' + y' - 2y = \sin^2 x.$ [Hint : use trigonometry identity]

6. Solve $y'' + 4y = 3\cos 2x.$ [Hint: $y_p = x\left[C\sin 2x + D\cos 2x\right]$, $y_h = A\sin 2x + B\cos 2x$]

2.3.2 Method of Variation of Parameters

Variation of parameters can be used to determine a particular solution. This method applies even when the coefficients of the d.e. are a function of *x*.
Obviously it can solve non-homogeneous second ode with constant coefficients.

$$ay'' + by' + cy = f(x)$$

Solve all types of *f(x)* such as $\tan x, \sec x, 1/x^2, \ln x, \sin(e^x)$ and etc.

Method of variation of parameters:

Step 1 : Identify a and $f(x)$

Step 2 : Consider homogeneous equation. Choose y_1 and y_2.

Step 3 : Find Wronskian, W for y_1 and y_2.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Step 4 : Find u and v, where

$$u(x) = -\int \frac{y_2 f(x)}{aW} dx + C \; ; \qquad v(x) = \int \frac{y_1 F(x)}{aW} dx + D$$

Step 5 : The general solution for particular solution of differential equation is given by

$$y = uy_1 + vy_2$$

Example 2.3.8:

Find the solution for $y'' + y = \sec x$ (i)

Solution :

Step 1 : Identify $a = 1$, $f(x) = \sec x$.

Both are arbitrary choices, where we can let $y_1(x) = \sin x$ and $y_2(x) = \cos x$ and we can let arbitrary constants A and B equal to 1.

Step 2: Find root $y'' + y = 0$:

$$m^2 + 1 = 0 \Rightarrow m^2 = -1 = i^2 \Rightarrow m = \pm i.$$

$$\Rightarrow y_h(x) = A\cos x + B\sin x \qquad \text{(ii)}$$

From eqn (ii), we choose $y_1(x)$ and $y_2(x)$.

$$\text{Let} \quad y_1(x) = \cos x \quad , \; y_2(x) = \sin x \qquad \text{(iii)}$$

Step 3: Find **Wronskian, *W***

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos x \cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1 \qquad \text{(iv)}$$

Step 4: Find $u(x)$ and $v(x)$.

$$u(x) = -\int \frac{y_2 F(x)}{aW} dx \quad \Rightarrow u = -\int \frac{\sin x \sec x}{1} dx = -\int \tan x dx$$

$$\Rightarrow u(x) = -\int \frac{\sin x}{\cos x} dx = \ln|\cos x| + C.$$

$$v(x) = \int \frac{y_1 F(x)}{aW} dx \quad \Rightarrow v = \int \cos x . \sec x dx = \int 1 dx = x + D.$$

Step 5 : The general solution

$$y(x) = uy_1 + vy_2 = A\cos x + B\sin x + \ln|\cos x|\cos x + x\sin x$$

Notes: Constant a is the coefficients for y'' in actual equations. We can let the actual equations in easier form so that $a = 1$. **For example,** if we are given this function $3y'' + y = \sec x$, we can write the equation in this form $y'' + \frac{1}{3}y = \frac{\sec x}{3}$.

Example 2.3.9:

Find the solution for $y'' - 4y' + 4y = (x+1)e^{2x}$ by using variation of parameters.

Solution:

Step 1 : $a = 1$ and $f(x) = (x+1)e^{2x}$

Step 2 : Identify $y_1(x)$ and $y_2(x)$.

Characteristic eqn: $m^2 - 4m + 4 = 0 \Rightarrow m = 2,2$

$$\Rightarrow \quad y_h(x) = (A + Bx)e^{2x}.$$

Set $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$

Step 3 : Find the Wronskian, $W(x)$:

$$W(x) = y_1 y_2' - y_2 y_1'$$
$$= e^{2x}.\left[e^{2x} + 2xe^{2x}\right] - xe^{2x}.2e^{2x} = e^{4x}$$

Step 4 : Find $u(x)$ and $v(x)$:

$$u(x) = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{xe^{2x}(x+1)e^{2x}}{e^{4x}} dx = -\int x(x+1)dx = -\int (x^2 + x)dx$$

$$\Rightarrow u(x) = -\frac{x^3}{3} - \frac{x^2}{2} + C$$

$$v(x) = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^{2x}(x+1)e^{2x}}{e^{4x}} dx = \int (x+1)dx$$

$$\Rightarrow v(x) = \frac{x^2}{2} + x + D$$

Check your answer by using the method of undetermined coefficients.

Step 5 : General solution:

$$y = uy_1 + vy_2 = e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2}\right) + Ce^{2x} + Dxe^{2x}$$

Exercise 2.3.2

Solve the following differential equations

1. $y'' + 4y = \tan 2x$

2. $y'' + 2y' - 3y = 6$

3. $y'' + 4y = 2\sin x$

4. $y'' + 2y' + 2y = e^{-x}\cos x$

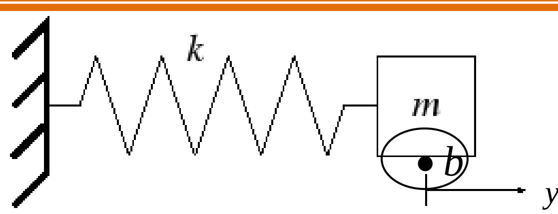
5. $2x'' - 2x' - 4x = 2e^{3t}$

6. $x'' - 4x' + 4x = te^{2t}$

7. $y'' + 3y' + 2y = \sin(e^{x})$

2.4 Second ODE : Applications

2.4.1 MECHANICAL SYSTEM



The governing equation is

$$my'' + by' + ky = f(t)$$

Do you know what type of equations are these?

where

m = inertia

b = damping

k=stiffness

$f(t)$ = force vibration.

	Undamped; $b = 0$	**Damped;** $b \neq 0$
Free vibration; $f(t) = 0$ **->Homogeneous**	$my'' + ky = 0$	$my'' + by' + ky = 0$
Force vibration; $f(t) \neq 0$ **->Non-Homogeneous**	$my'' + ky = f(t)$	$my'' + by' + ky = f(t)$

Case 1: UNDAMPED FREE VIBRATIONS (When $b = 0$; $f(t) = 0$)

The equation is given by:

$$m\frac{d^2y}{dt^2} + ky = 0 \qquad \text{(i)}$$

Divide by m;

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

ω is an angular frequency and period = $2\pi/\omega$

Rewrite,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}\,. \qquad \text{(ii)}$$

Step 1: Using characteristic equation:

$$r^2 + \omega^2 = 0 \qquad \Rightarrow \qquad r = \pm\omega i$$

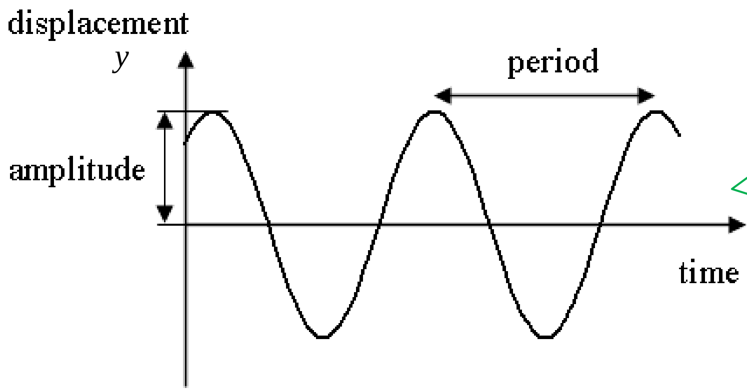
Step 2: General solution to (ii) is

$$y(t) = k_1 \cos\omega t + k_2 \sin\omega t$$

We also can express $y(t)$ in the more convenient form

$$y(t) = \sqrt{k_1^2 + k_2^2}\,\sin(\omega t + \phi)$$

$\sqrt{k_1^2 + k_2^2}$ is an amplitude



The motion of a mass in case 1 is called **simple harmonic motion**

Notes:

Given $k_1 \cos\omega t + k_2 \sin \omega t$, we can write this as

$$\sqrt{k_1{}^2 + k_2{}^2}\left(\frac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}}\cos\omega t + \frac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}}\sin\omega t\right)$$

i.e

$$k_1 \cos\omega t + k_2 \sin\omega t = \sqrt{k_1{}^2 + k_2{}^2}\left(\frac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}}\cos\omega t + \frac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}}\sin\omega t\right)$$

where

$$\sin\phi = \frac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}} \quad , \quad \cos\phi = \frac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\dfrac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}}}{\dfrac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}}} \qquad \Rightarrow \phi = \arctan\frac{k_1}{k_2}$$

$$\therefore \quad k_1 \cos\omega t + k_2 \sin\omega t$$

$$= \sqrt{k_1{}^2 + k_2{}^2}\left(\sin\phi\cos\omega t + \cos\phi\sin\omega t\right)$$

$$= \sqrt{k_1{}^2 + k_2{}^2}\ \sin\left(\omega t + \phi\right)$$

Notes: Alternative representation:

$$\cos\phi = \frac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}} \quad , \quad \sin\phi = \frac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}}$$

$$\therefore \quad k_1 \cos\omega t + k_2 \sin\omega t$$

$$= \sqrt{k_1{}^2 + k_2{}^2}\left(\cos\phi\cos\omega t + \sin\phi\sin\omega t\right)$$

$$= \sqrt{k_1{}^2 + k_2{}^2}\ \cos\left(\phi - \omega t\right)$$

Case 2: DAMPED FREE VIBRATIONS (When $b \neq 0$; $f(t) = 0$)

The equation is given by:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad \text{(i)}$$

Step 1: Characteristic equation for (i) is

$$mr^2 + br + k = 0$$

Its roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk}$$

Step 2: There are **3 cases** related to these roots:

Complex roots

1) $b^2 < 4mk$ Underdamped or Oscillatory Motion

The roots are $\alpha \pm i\beta$, where

$$\alpha := -\frac{b}{2m} \qquad \beta := \frac{1}{2m}\sqrt{4mk - b^2}$$

The general solution to (i) is

$$y(t) = e^{\alpha t}(k_1 \cos \beta t + k_2 \sin \beta t)$$

can express in

$$y(t) = \sqrt{k_1^2 + k_2^2}\, e^{\alpha t} \sin(\beta t + \phi)$$

$\sqrt{k_1^2 + k_2^2}\, e^{(-b/2m)t}$ is damping factor

2) $b^2 > 4mk$ Overdamped Motion

Two distinct roots

The roots are

$$r_1 = -\frac{b}{2m}, \quad r_2 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2 - 4mk}$$

The general solution to (i) is

$$y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t}$$

3) $b^2 = 4mk$ **Critical damped Motion**

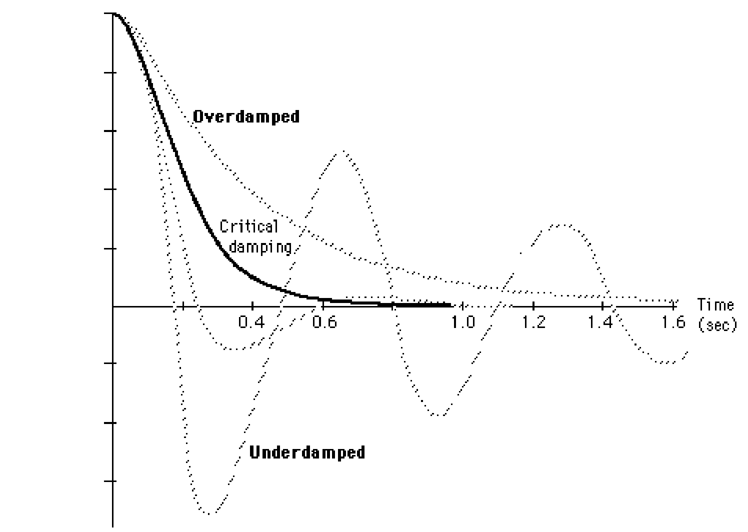
Two repeated roots

The roots are

$$r = r_1 = r_2 = -\frac{b}{2m}$$

The general solution to (i) is

$$y(t) = (k_1 + k_2 t)e^{rt}$$



Case 3: UNDAMPED FORCED VIBRATIONS (When $b = 0$; $f(t) \neq 0$)

Example 2.4.1
Solve the **initial boundary problem** for undamped mechanical system.

$$m\frac{d^2y}{dt^2} + ky = F_\circ \cos\gamma t \qquad \text{(i)}$$

$$y(0) = y'(0) = 0 \qquad \text{where } \gamma \neq \omega = \sqrt{k/m}$$

Solution:
Find homogeneous solution of (i).
Rewrite (i) as

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = \frac{F_\circ}{m}\cos\gamma t \qquad \text{(ii)}$$

Set $\omega^2 = k/m$, $\omega = \sqrt{k/m}$

Step 1: Using characteristic Equation:

$$m^2 + \omega^2 = 0 \quad \Rightarrow \quad m = \pm\omega i$$

$$y_h(t) = A\cos\omega t + B\sin\omega t \qquad \text{(iii)}$$

Step 2: Find particular solution of (ii)

Try $y_p(t) = C\cos\gamma t + D\sin\gamma t$ (iv) a

$$y_p' = -C\gamma\sin\gamma t + D\gamma^2\cos\gamma t \qquad \text{(iv) b}$$

$$y_p'' = -C\gamma^2\cos\gamma t - D\gamma^2\sin\gamma t \qquad \text{(iv) c}$$

Note: Since $\gamma \neq \omega$, system not at resonance.

To find const and D and C, substitute (iv) a,b,c into (ii). Equate coefficients of $\cos\gamma t$ and $\sin\gamma t$

$$m(-C\gamma^2\cos\gamma t - D\gamma^2\sin\gamma t) + k(C\cos\gamma t + D\sin\gamma t) = F_0\cos\gamma t$$

$$(-Cm\gamma^2 + Ck)\cos\gamma t + (Dk - Dm\gamma^2)\sin\gamma t = F_0\cos\gamma t$$

$$\Rightarrow D = 0 \quad , \quad C = \frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}$$

$$\therefore \quad y(x) = A\cos\omega t + B\sin\omega t + \frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}\cos\gamma t \qquad \text{(v)}$$

Find A & B using initial condition.

$$\Rightarrow \quad y(0) = 0$$

$$0 = A + \frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}$$

$$\therefore \quad A = -\frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}$$

$\Rightarrow \quad y'(0) = 0$, Find B?

B=0.

Final Answer:

$$\therefore \quad y(t) = -\frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}\cos\omega t + \frac{F_\circ}{m\left(\omega^2 - \gamma^2\right)}\cos\gamma t$$

Case 4: DAMPED FORCED VIBRATIONS (When b≠0, $f(t)\neq 0$)

The equation is given by:

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=F_\circ\cos\gamma t$$

The general solution is

$$y(t)=Ae^{-(b/2m)t}\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t+\phi\right)+\frac{F_\circ}{\sqrt{(k-m\gamma^2)+b^2\gamma^2}}\sin(\gamma t+\phi)$$

Exercise 2.4.1:

1. A 4 pound weight is attached to a spring whose spring constant is 16 lb/ft. what is the period of simple harmonic motion?

   [ans : period = $\frac{2\pi}{\omega}\to\omega=\sqrt{\frac{k}{m}}$ $\quad\therefore$ period $=\frac{\sqrt{2}}{8}\pi$]

2. A 24-pound weight, is attached to the end of a spring, stretches it 4 inches. Find the equation of motion if the weight is released from rest from a point 3 inches above the equilibrium position.

   [ans : $y(t)=-\frac{1}{4}\cos 4\sqrt{6}t$]

3. A 10 - pound weight attached to a spring stretches it 2 feet. The weight is attached to a dashpot damping device that offers a resistance numerically equal to $\beta(\beta>0)$ times the instantaneous velocity. Determine the values of the damping constant β so that the subsequent motion is a) overdamped; b)critically damped; c) underdamped.

   [ans : $a)\beta>\frac{5}{2};\quad b)\beta=\frac{5}{2};\quad c)0<\beta<\frac{5}{2}$]

4. A 16 - pound weight stretches a spring 8/3 feet. Initially the weight starts from rest 2 feet below the equilibrium position and the subsequent motion takes place in a medium that offers a damping force numerically equal to ½ the instantaneous velocity. Find the equation of motion if the weight is driven by an external force equal to $f(t) = 10\cos 3t$.

$$[\text{ans}: \; y(t) = e^{-t/2}\left(-\frac{4}{3}\cos\frac{\sqrt{47}}{2}t - \frac{64}{3\sqrt{47}}\sin\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}(\cos 3t + \sin 3t)]$$

5. The motion of a mass-spring system with damping governed by:

$$\frac{d^2y}{dt^2} + by' + 16y = 0; \quad y(0) = 1, \quad y'(0) = 0.$$

Find the equation of motion and sketch its graph for $b = 0, 6, 8$ and 10.

6. Determine the equation of motion for an undamped system at resonance governed by:

$$\frac{d^2y}{dt^2} + y = 5\cos t; \quad y(0) = 1, \quad y'(0) = 0.$$

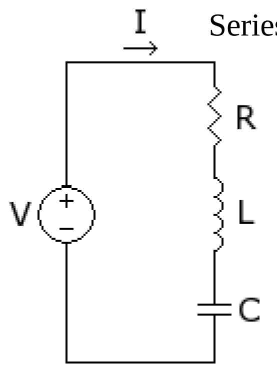
Sketch the solution.

7. The response of an overdamped system to constant force is governed by:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 6y = 18; \quad y(0) = 0, \quad y'(0) = 0.$$

Compute and sketch the displacement $y(t)$. What is the limiting value of $y(t)$ at $t \Rightarrow +\infty$?

2.4.2 RLC – Circuit



Series RLC Circuit notations:

V - the voltage of the power source (measured in volts V)
I - the current in the circuit (measured in amperes A)
R - the resistance of the resistor (measured in ohms = V/A);
L - the inductance of the inductor (measured in henrys = H = V·s/A)
C - the capacitance of the capacitor (measured in farads = F = C/V = A·s/V)
q - the charge across the capacitor (measured in coulombs C)

The equation for RLC circuit is given by:

$$L\frac{dI}{dt}+RI+\frac{q}{c}=E(t);$$

with the initial conditions : $q(0)=q_\circ \quad , \quad I(0)=I_\circ$

where $I=\dfrac{dq}{dt}$.

Example 2.4.3:

The series RLC circuit has a voltage source given by $E(t)=100\,\text{V}$, a resistor $R=20\Omega$, an inductor $L=10\,\text{H}$, and a capacitor $c=(6260)^{-1}$. If the initial current and the initial charge on the capacitor are both zero, determine the current in the circuit for t>0.

Solution:

The equation :

$$L\frac{dI}{dt}+RI+\frac{q}{c}=E(t) \qquad \text{(i)}$$

The initial conditions: $q(0)=0 \quad , \quad I(0)=0$

The information:

$$L=10 \quad , \quad R=20 \quad , \quad E(t)=100 \quad , \quad c=(6260)^{-1}$$

Method 1: Change the equation into homogeneous equation

Step 1: Differentiate eqn (i) with t, we have

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{c}\frac{dq}{dt}=\frac{d}{dt}(E(t))$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{c}I=\frac{d}{dt}(E(t)) \qquad \text{(ii)}$$

In our case $L=10$, $R=20$, $E(t)=100$, $c=(6260)^{-1}$

Eqn. (ii) becomes

$$10\frac{d^2I}{dt^2}+20\frac{dI}{dt}+6260\,I=0$$

Or

$$\frac{d^2I}{dt^2}+2\frac{dI}{dt}+626I=0 \qquad \text{(iii)}$$

This is a **homogenous equation**. Can you remember how to solve this?? **Use characteristic equation!!!**

Step 2: Find I(t),

$$m^2+2m+626=0$$

$$\Rightarrow \quad m=-1\pm 25\,i$$

$$\Rightarrow I(t)=e^{-t}(A\cos 25t+B\sin 25t) \qquad \text{(iv)}$$

This is the general solution of $I(t)$. Use the **initial conditions** given to find A and B.

Step 3: Need to find A & B. Substitute the Initial condition given into (iv):

$$I(0)=0 \quad \Rightarrow \quad 0=A$$

$$\therefore \quad I(t)=Be^{-t}\sin 25t \qquad \text{(v)}$$

What is B? How to find B?

From I.C. $q(0)=0$, we have to find $\dfrac{dI}{dt}=?$ at $t=0$.

From (i), at $t = 0$,

$$L\left.\frac{dI}{dt}\right|_{t=o}+RI(0)+\frac{q(0)}{c}=100$$

New initial condition $t = 0$, $I = 0$.

$$\Rightarrow \quad L\left.\frac{dI}{dt}\right|_{t=o}=100 \quad \Rightarrow \quad \boxed{\left.\frac{dI}{dt}\right|_{t=o}=\frac{100}{L}=10}$$

From (iv),

$$\frac{dI}{dt}=B\left(e^{-t}\sin 25t+e^{-t}25\cos 25t\right)$$

$$\Rightarrow \quad 100=B(25) \quad \Rightarrow \quad B=\frac{10}{25}=\frac{2}{5}$$

Final answer:

$$I(t)=\frac{2}{5}e^{-t}\sin 25t$$

Method 2 : Solve the non-homogeneous equation

Step 1:

We know $I=\dfrac{dq}{dt}\Rightarrow \dfrac{dI}{dt}=\dfrac{d^2q}{dt^2}$, substitute to equation (i):

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E(t) \qquad \text{(vi)}$$

In our case $L=10$, $R=20$, $E(t)=100$, $c=(6260)^{-1}$,

equation (vi) become

$$10\frac{d^2q}{dt^2}+20\frac{dq}{dt}+6260q=100$$

Or

$$\frac{d^2q}{dt^2}+10\frac{dq}{dt}+626q=10$$

This is a **nonhomogenous equation.** Can you remember how to solve this?? **You have to find** q_h and q_p **!!!**

Step 2:

Then find q_h and q_p

$$q_h=e^{-t}\left(A\cos 25t+B\sin 25t\right) \quad , \quad q_p=\frac{10}{626}$$

$$\Rightarrow q(t)=e^{-t}\left(A\cos 25t+B\sin 25t\right)+\frac{10}{626} \qquad \text{(vii)}$$

This is the general solution of $q(t)$. Use the **initial conditions** given to find A and B.

Step 3:

i)Given initial charge is 0

$$\Rightarrow \quad q=0 \quad \text{at} \quad t=0$$

$$\Rightarrow \quad 0=A+\frac{10}{626} \quad \therefore \quad A=-\frac{10}{626}$$

ii)Initial current is 0

$$\Rightarrow \quad I=\frac{dq}{dt}=0 \quad \text{at} \quad t=0$$

Differentiate (vii)

$$\frac{dq}{dt}=e^{-t}\left(-25A\cos 25t+25B\sin 25t\right)$$
$$+\left(A\cos 25t+B\sin 25t\right)\cdot -e^{-t}$$

$$\frac{dq}{dt}=e^{-t}\left[\left(-25A+B\right)\sin 25t+\left(-25A+B\right)\sin 25t\right] \quad \text{(iii)}$$

From I.C: $\dfrac{dq}{dt}=0 \quad \text{at} \quad t=0$

$$\Rightarrow \quad 0=25B-A$$

$$\therefore \quad B=\frac{A}{25}=-\frac{10}{625\times 25}=-\frac{2}{3130}=-\frac{1}{1505}$$

The final answer :

$$\therefore \quad I(t)=\frac{2}{5}e^{-t}\sin 25t$$

Exercise 2.4.2

1. An RLC series circuit has a voltage source given by $E(t)=20$ V , a resistor of 100 Ω, an inductor of 4 H and a capacitor of 0.01 F. If the initial current is zero and the initial charge on the capacitor is 4 C, determine the current in the circuit for $t>0$.

2. An RLC series circuit has a voltage source given by of $E(t)=10\cos 20t$ V , a resistor of 120 Ω, an inductor of 4 H and a capacitor of $(2200)^{-1}$ F. Find the steady state current (solution) for this circuit.

3. An RLC series circuit has a voltage source given by of $E(t)=30\sin 50t$ V , no resistor, an inductor of 2 H and a capacitor of 0.02 F. What is the current in this circuit for $t>0$ if at $t=0$, $I(0)=q(0)=0$?