Modelling the orientation dynamics of “chaotic mixing“/ turbulence.

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Msc Data and Computational Science

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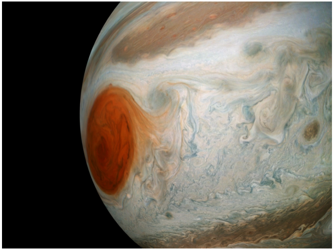
Introduction : Why study turbulence



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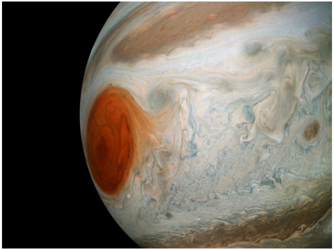

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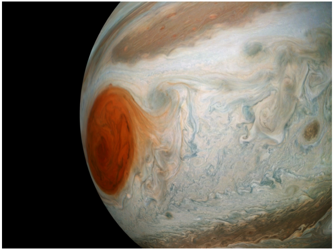

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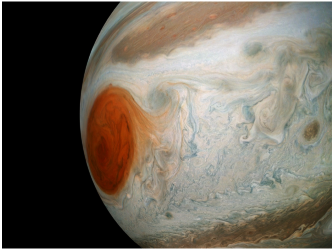

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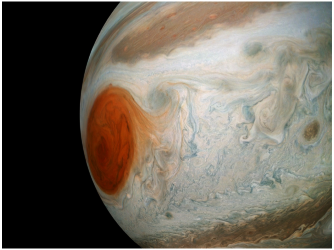
"Turbulence is the most important unsolved problem of classical physics."

- Richard P. Feynman
In The Feynman Lectures on Physics (1964).



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- Compare the PDF of the orientation angle from the vorticity simulation vs and FP model.
- Use uncertainty quantification methods to fit angle and growth rate PDF parameters to the simulated data.

Advection-Diffusion equation

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The model for the orientation dynamics will be dervied from the Advection-Diffusion equation

Vorticity Equation

$$\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = (-1)^p\nu_p\nabla^{2p}\omega + \nabla\times\mathbf{F} - \nabla\times\mathbf{D}$$

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- ω is the vorticity, describes the rotation of a fluid particle some point
- $\mathbf{u}(x, t)$ is the velocity
- ν is the viscosity
- $\mathbf{F}$ is a forcing term
- $\mathbf{D}$ is a damping term

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The growth rate of the tracer gradient is defined as

$$\boxed{\Lambda = -2\mu \sin\zeta}$$

- μ is the strain rate
- ζ orientation angle

Stochastic Model Equation Model

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$$\gamma \frac{dX}{dt} = w + (-cos(X) + k\sqrt{\delta})Y + Z$$

$$\frac{dY}{dt} = -\frac{Y}{\tau} + \frac{\sqrt{D_Y}}{\tau}\xi_Y$$

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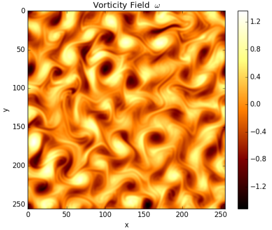
Corresponding Fokker-Planck equation

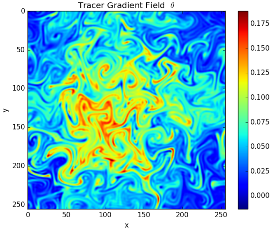
$$\frac{\partial P}{\partial t} = \mathcal{L}_{OU}P - \frac{\partial}{\partial t}(VP)$$

where

$$\mathcal{L}_{OU}P = \frac{1}{\tau_Y}\frac{\partial}{\partial t}(y\circ) + \frac{1}{\tau_Y^2}\frac{\partial^2}{\partial y^2} + \frac{1}{\tau_Z}\frac{\partial}{\partial Z}(z\circ) + \frac{\rho}{\tau_Z^2}\frac{\partial^2}{\partial z^2} + \frac{2c\sqrt{\rho}}{\tau_Y\tau_Z}\frac{\partial^2}{\partial y \partial z}$$
$$V = 2(w + y\cos x + z)$$

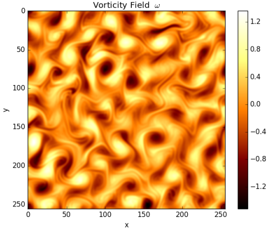
Numerical Simulation : Vorticity Equation

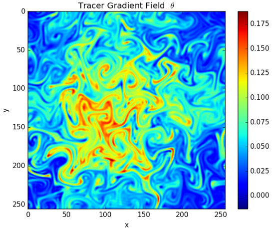




Numerical Simulation : Vorticity Equation

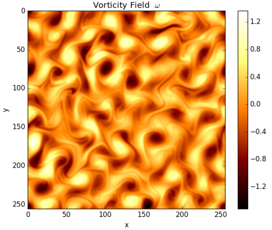
- Periodic Boundary Conditions

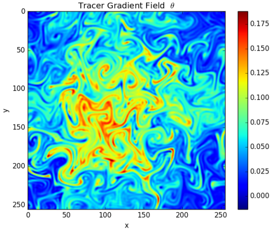




Numerical Simulation : Vorticity Equation

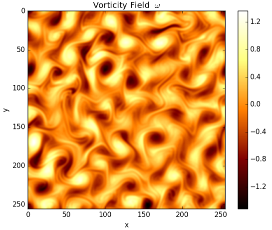
- Periodic Boundary Conditions
- Discretise the Vorticity equation

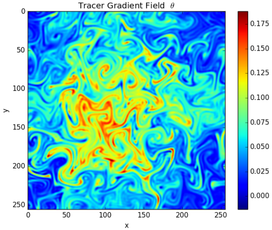




Numerical Simulation : Vorticity Equation

- Periodic Boundary Conditions
- Discretise the Vorticity equation
- Convert discretised vorticity equation to fourier space





Numerical Simulation : Vorticity Equation

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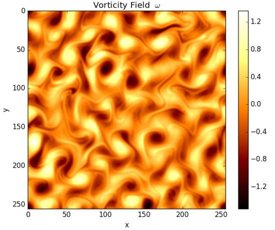
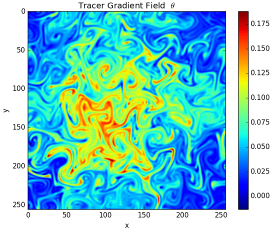
- Save snapshots of ω at regular intervals





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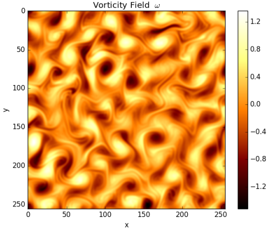
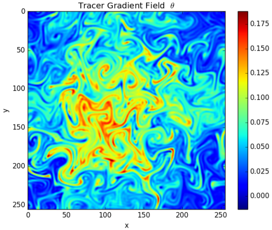
- Extract emperical PDF of angle X statistically stable dataset





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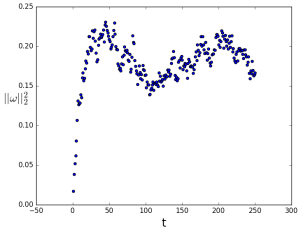
- Compute the PDF of the growth rate Λ using the joint PDF of XY

Analysis to date

- Verify simulation of vorticity has reached a steady state



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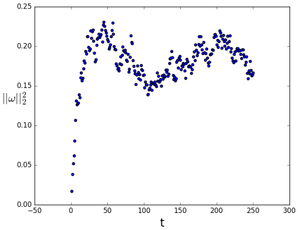

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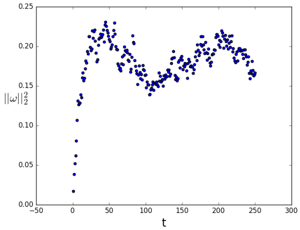

Outstanding Work

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- Apply uncertainty quantification methods to the SDE/Fokker-Planck equation to fit the model parameters

Conclusions

The End