Classification by type

- **Ordinary Differential Equations (ODE)**
    - ❖ Contains one or more dependent variables
    - ❖ with respect to one independent variable

$$\frac{dy}{dx} \quad => \quad y'$$

$\boldsymbol{y}$ is the *dependent variable* while $\boldsymbol{x}$ is the *independent variable*

$\boldsymbol{u}$ is the *dependent variable* while $\boldsymbol{t}$ is the *independent variable*

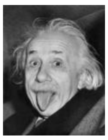
$$\frac{d^2u}{dt^2} \quad => \quad u''$$

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u = e^t \quad => \quad u'' + u' + u = e^t$$

Dependent Variable: u

Independent Variable: t

- **Partial Differential Equations (PDE)**
    - ❖ involve one or more dependent variables
    - ❖ and two or more independent variables



Can you determine which one is the DEPENDENT VARIABLE and which one is the INDEPENDENT VARIABLES from the following equations ???

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = 0 \quad => \quad w_x + w_t = 0$$

Dependent Variable: w

Independent Variable: x, t

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^{xy} \quad => \quad u_{xx} + u_{xy} + u_y = e^{xy}$$

Dependent Variable: u

Independent Variable: x, y

Classification by order / degree

- **Order of Differential Equation**
  - ❖ Determined by the highest derivative

- **Degree of Differential Equation**
  - ❖ Exponent of the highest derivative

Examples:

a) $\left(\frac{dy}{dx}\right)^2 + y = \sin x$	Order : 1	Degree: 2
b) $\frac{\partial^2 u}{\partial t^2} + u^2 = \cos t$	Order : 2	Degree: 1
c) $\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 + w = e^x$	Order : 2	Degree: 1
d) $\left(\frac{\partial^3 z}{\partial y^3}\right)^4 + \left(\frac{\partial z}{\partial y}\right)^2 + z = 0$	Order : 3	Degree: 4

Classification as linear / nonlinear

- **Linear Differential Equations**
  - ❖ Dependent variables and their derivative are of degree 1
  - ❖ Each coefficient depends only on the independent variable
  - ❖ A DE is linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Examples:

1) $\frac{dy}{dx} + y = \sin x$

2) $\frac{d^2y}{dx^2} + y = \sin^2 x$

3) $\frac{d^5y}{dx^5} + (\sin^2 x)y = \tan^2 x$

- **Nonlinear Differential Equations**
    - Dependent variables and their derivatives are not of degree 1

Examples:

1)	$\frac{dy}{dx} + y^2 = \sin x$	Order : 1 Degree: 1
2)	$\left(\frac{dy}{dx}\right)^2 + y = \sin x$	Order : 1 Degree: 2
3)	$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + \frac{y}{x^2+1} = e^x$	Order : 3 Degree: 2

Initial & Boundary Value Problems

Initial conditions : will be given on specified given point

Boundary conditions : will be given on some points

Examples :

1)	$y(0) = 1\ ;\ y'(0) = 2$	Initial condition
2)	$y(1) = 5\ ;\ y(2) = 2$	Boundary condition

Initial Value Problems (IVP)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y + \sin x$$

Initial Conditions:

$$y(0) = 0\ ;\quad y'(0) = 1$$

Boundary Value Problems (BVP)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y + \sin x$$

Boundary Conditions:

$$y(0) = 1\ ;\quad y(1) = 2$$

Solution of a Differential Equation

- **General Solutions**
  - ❖ Solution with arbitrary constant depending on the order of the equation
- **Particular Solutions**
  - ❖ Solution that satisfies given boundary or initial conditions

Examples:

$$y = A\cos x + B\sin x \qquad (1)$$

Show that the above equation is a solution of the following DE

$$y'' + y = 0 \qquad (2)$$

Solutions:

$$y' = -A\sin x + B\cos x \qquad (3)$$

$$y'' = -A\cos x - B\sin x \qquad (4)$$

Insert (1) and (4) into (2)

$$= -A\cos x - B\sin x + A\cos x + B\sin x$$

$$= 0$$

Proven that y is the solution for the given DE.

EXERCISE:

Show that $y = A\cos(\ln x) + B\sin(\ln x)$ is the solution of the following DE $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Forming a Differential Equation

Example 1:

Find the differential equation for $y = x - \frac{A}{x}$

Solution:

$$y = x - \frac{A}{x} \tag{1}$$

$$\frac{dy}{dx} = 1 + \frac{A}{x^2} \tag{2}$$

Try to eliminate A by,

a) Divide (1) with x :

$$\frac{y}{x} = 1 - \frac{A}{x^2} \tag{3}$$

b) (2)+(3) :

$$\frac{dy}{dx} + \frac{y}{x} = 2 \tag{4}$$

Example 2:

Form a suitable DE using $y = A\cos x + B\sin x$

Solutions:

$$y' = -A\sin x + B\cos x$$

$$y'' = -A\cos x - B\sin x$$

$$= -(A\cos x + B\sin x)$$

$$\therefore \quad y'' = -y \quad => \quad \frac{d^2y}{dx^2} + y = 0$$

Exercise:

Form a suitable Differential Equation using $y = Ax^2 + Bx^5$

Hints:

1. **Since there are two constants in the general solution, y has to be differentiated twice.**
2. **Try to eliminate constant A and B.**

1.2 First Order Ordinary Differential Equations (ODE)

Types of first order ODE

- Separable equation
- Homogenous equation
- Exact equation
- Linear equation
- Bernoulli Equation

1.2.1 Separable Equation

How to identify?

Suppose $f(x,y) = \frac{dy}{dx} = u(x)v(y)$

Hence this become a **SEPARABLE EQUATION** if it can be written as

$$\frac{dy}{v(y)} = u(x)dx$$

Method of Solution : integrate both sides of equation

$$\int \frac{1}{v(y)} dy = \int u(x)dx$$

Example 1:

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \quad , \qquad y(0) = 1$$

Solution:

i) Separate the functions

$$\left(\frac{1+2y^2}{y}\right) dy = (\cos x)\, dx$$

ii) Integrate both sides

$$\int \frac{1+2y^2}{y} dy = \int \cos x \;\, dx$$

Use your Calculus knowledge to solve this problem !

Answer: $\ln y + y^2 = \sin x + C$

iii) Use the initial condition given, $y(0) = 1$

$$\ln 1 + 1^2 = \sin 0 + C$$

$$\therefore \;\; C = 1$$

iv) Final answer

$$\ln y + y^2 = \sin x + 1$$

Note: Some DE may not appear separable initially but through appropriate substitutions, the DE can be separable.

Example 2:

Show that the DE $\frac{dy}{dx} = (x+y)^2$ can be reduced to a separable equation by using substitution $z = x + y$. Then obtain the solution for the original DE.

Solutions:

i) Differentiate both sides of the substitution wrt x

$$z = x + y \qquad \rightarrow (1)$$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx} \qquad => \quad \frac{dy}{dx} = \frac{dz}{dx} - 1 \qquad \rightarrow (2)$$

ii) Insert (2) and (1) into the DE

$$\frac{dz}{dx} - 1 = z^2$$

$$\frac{dz}{dx} = z^2 + 1 \qquad \rightarrow (3)$$

iii) Write (3) into separable form

$$\frac{1}{z^2+1} dz = dx$$

iv) Integrate the separable equation

$$\int \frac{1}{z^2+1} dz = \int dx$$

Final answer : $y = \tan(x + C) - x$

Exercise

1) Solve the following equations

a. $x\frac{dy}{dx} = \cot y$

b. $\frac{dy}{dx} = \frac{y}{x(x+1)}$

c. $\frac{dy}{dx} + (1 + y^2) = 0, \quad y(0) = 0$

d. $\sqrt{xy}\frac{dy}{dx} = \sqrt{4 - x}$

2) Using substitution $z = xy$, convert

$$x\frac{dy}{dx} + y = 2x\sqrt{1 - x^2y^2}$$

to a separable equation. Hence solve the original equation.

1.2.2 Homogenous Equation

How to identify?

Suppose $f(x, y) = \frac{dy}{dx}$, $f(x, y)$ is homogenous if

$$f(\lambda x, \lambda y) = f(x, y)$$

for every real value of λ

Method of Solution :

i) Determine whether the equation homogenous or not

ii) Use substitution $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the original DE

iii) Separate the variable x and v

iv) Integrate both sides

v) Use initial condition (if given) to find the constant value

Separable equation method

Example 1:

Determine whether the DE is homogenous or not

a) $$\frac{dy}{dx} = \frac{x^2 + y^2}{(x-y)(x+y)}$$

b) $$x\frac{dy}{dx} - y = x\sqrt{x^2 + y^2}$$

Solutions:

a) $$f(x,y) = \frac{dy}{dx} = \frac{x^2 + y^2}{(x-y)(x+y)}$$

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 - (\lambda y)^2}$$

$$= \frac{\lambda^2(x^2 + y^2)}{\lambda^2(x^2 - y^2)} = f(x,y)$$

$\therefore$ this differential equation is homogenous

b) $$f(x,y) = \frac{dy}{dx} = \frac{y}{x} + \sqrt{x^2 + y^2}$$

$$f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} + \sqrt{(\lambda x)^2 + (\lambda y)^2}$$

$$= \frac{y}{x} + \lambda\sqrt{x^2 + y^2} \neq f(x,y)$$

$\therefore$ this differential equation is non-homogenous

Example 2:

Solve the homogenous equation

$$(y^2 + xy)\,dx - x^2\,dy = 0$$

Solutions:

i) Rearrange the DE

$$x^2\,dy = (y^2 + xy)\,dx$$

$$\frac{dy}{dx} = \frac{(y^2 + xy)}{x^2} \qquad \rightarrow (1)$$

ii) Test for homogeneity

$$f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \frac{\lambda^2(y^2 + xy)}{\lambda^2(x^2)} = f(x, y)$$

∴ this differential equation is homogenous

iii) Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into (1)

$$v + x\frac{dv}{dx} = \frac{\left((vx)^2 + x(vx)\right)}{x^2} = \frac{v^2x^2 + vx^2}{x^2} = v^2 + v$$

$$x\frac{dv}{dx} = v^2 + v - v = v^2$$

iv) Solve the problem using the separable equation method

$$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

Final answer : $x = Ae^{-x/y}$

Note:

Non-homogenous can be reduced to a homogenous equation by using the right substitution.

Example 3:

Find the solution for this non-homogenous equation

$$\frac{dy}{dx} = \frac{y-2}{x+y-5} \qquad (1)$$

by using the following substitutions

$$x = X + 3, \qquad y = Y + 2 \qquad (2),(3)$$

Solutions:

i) **Differentiate (2) and (3)**

$$dx = dX\ , \quad dy = dY$$

and substitute them into (1),

$$\frac{dY}{dX} = \frac{Y}{X+Y}$$

ii) **Test for homogeneity,** $f(\lambda x, \lambda y) = f(x, y)$

iii) **Use the substitutions** $\boldsymbol{Y = VX}$ **and** $\boldsymbol{\frac{dY}{dX} = V + X\frac{dV}{dX}}$

$$V + X\frac{dV}{dX} = \frac{Y}{X+Y} = \frac{VX}{X+VX} = \frac{V}{1+V}$$

$$X\frac{dV}{dX} = \frac{V}{1+V} - V = -\frac{V^2}{1+V}$$

REMEMBER!

Now we use $\boldsymbol{X, Y}$ instead of $\boldsymbol{x, y}$

iv) **Use the separable equation method to solve the problem**

$$-\int \frac{1+V}{V^2}\,dV = \int \frac{1}{X}\,dX$$

Ans: $\boldsymbol{y = 2 + Ae^{(x-3)/(y-2)}}$

LASTLY, do not forget to replace $\boldsymbol{X, Y}$ with $\boldsymbol{x, y}$

1.2.3 Exact Equation

How to identify?

Suppose $f(x, y) = -\frac{M(x,y)}{N(x,y)}$,

Therefore the first order DE is given by

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$$=> \underbrace{M(x, y)}_{\frac{\partial u}{\partial x}} \, dx + \underbrace{N(x, y)}_{\frac{\partial u}{\partial y}} \, dy = 0$$

Condition for an exact equation.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Method of Solution (Method 1):

i) Write the DE in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

And test for the exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ii) If the DE is exact, then

$$M = \frac{\partial u}{\partial x} \quad , \quad N = \frac{\partial u}{\partial y} \qquad (1), (2)$$

To find $u(x, y)$, integrate (1) wrt x to get

$$u(x, y) = \int M(x, y)\, dx + \phi(y) \qquad (3)$$

iii) To determine $\phi(y)$, differentiate (3) wrt y to get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + \phi'(y) = N$$

iv) Integrate $\phi'(y)$ to get $\phi(y)$

v) Replace $\phi(y)$ into (3). If there is any initial conditions given, substitute the condition into the solution.

vi) Write down the solution in the form

$$u(x,y) = A, \text{ where A is a constant}$$

Method of Solution (Method 2):

i) Write the DE in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

And test for the exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ii) If the DE is exact, then

$$M = \frac{\partial u}{\partial x} \quad , \quad N = \frac{\partial u}{\partial y} \qquad (1), (2)$$

iii) To find $u(x,y)$ from M, integrate (1) wrt x to get

$$u(x,y) = \int M(x,y)\,dx + \phi_1(y) \qquad (3)$$

iv) To find $u(x,y)$ from N, integrate (2) wrt y to get

$$u(x,y) = \int N(x,y)\,dy + \phi_2(x) \qquad (4)$$

v) Compare (3) and (4) to get value for $\phi_1(y)$ and $\phi_2(x)$.

vii) Replace $\phi_1(y)$ into (3) **OR** $\phi_2(x)$ into (4).

viii) If there are any initial conditions given, substitute the conditions into the solution.

ix) Write down the solution in the form

$$u(x,y) = A, \text{ where A is a constant}$$

Example 1:

Solve $(2xy+3)\,dx+(x^2-1)\,dy=0$

Solution (Method 1):

i) Check the exactness

$$\frac{\partial M}{\partial y}=2x, \qquad \frac{\partial N}{\partial x}=2x$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, this equation is exact.

ii) Find $u(x,y)$

$$\frac{\partial u}{\partial x}=2xy+3=M(x,y) \tag{1}$$

$$\frac{\partial u}{\partial y}=x^2-1=N(x,y) \tag{2}$$

To find $u(x,y)$, integrate either (1) or (2), let's say we take (1)

$$\int \partial u=\int 2xy+3\;\partial x$$

$$u(x,y)=x^2y+3x+\;\phi(y) \tag{3}$$

iii) Now we differentiate (3) wrt y to compare with $\frac{\partial u}{\partial y}=N$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^2y+3x+\;\phi(y)\right)=x^2+\phi'(y) \tag{4}$$

Now, let's compare (4) with (2)

$$x^2+\phi'(y)=x^2-1$$

$$\phi'(y)=-1$$

iv) Find $\phi(y)$

$$\int \phi'(y)=-\int 1\;\;dy \quad\rightarrow\quad \phi(y)=-y+B$$

v) Now that we found $\phi(y)$, our new $u(x,y)$ should looks like this

$$u(x,y)=x^2y+3x-y+B$$

vi) Write the solution in the form $u(x, y) = A$

$$u(x, y) = x^2y + 3x - y + B = A$$

$$x^2y + 3x - y = A - B$$

$$x^2y + 3x - y = C, \text{ where } C = A - B \text{ is a constant}$$

Exercise :

Try to solve Example 1 by using Method 2

Answer:

i) Check the exactness

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, this equation is exact.

ii) Find $u(x, y)$

$$\frac{\partial u}{\partial x} = 2xy + 3 = M(x, y) \qquad (1)$$

$$\frac{\partial u}{\partial y} = x^2 - 1 = N(x, y) \qquad (2)$$

To find $u(x, y)$, integrate both (1) and (2),

$$\int \partial u = \int 2xy + 3 \ \partial x$$

$$u(x, y) = x^2y + 3x + \phi_1(y) \qquad (3)$$

$$\int \partial u = \int x^2 - 1 \ \partial y$$

$$u(x, y) = x^2y - y + \phi_2(x) \qquad (4)$$

iii) Compare $u(x, y)$ to determine the value of $\phi_1(y)$ and $\phi_2(x)$

$$x^2y + 3x + \phi_1(y) = x^2y - y + \phi_2(x)$$

Hence,

$$\phi_1(y) = -y \text{ and } \phi_2(x) = 3x$$

iv) Replace $\phi_1(y)$ into (3) OR $\phi_2(x)$ into (4)

$$u(x, y) = x^2y - y + 3x$$

v) Write the solution in the form $u(x, y) = A$

$$x^2y - y + 3x = A$$

Note:

Some non-exact equation can be turned into exact equation by multiplying it with an integrating factor.

Example 2:

$$(xy + y^2 + y)\, dx + (x + 2y)\, dy = 0$$

Show that the following equation is not exact. By using integrating factor, $\mu(x, y) = e^x$, solve the equation.

Solution:

i) Show that it is not exact

$$\frac{\partial M}{\partial y} = x + 2y + 1, \qquad \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this equation is not exact.

ii) Multiply $\mu(x,y)$ into the DE to make the equation exact

$$\underbrace{(xy+y^2+y)(e^x)}_{M}\ dx+\underbrace{(x+2y)(e^x)}_{N}\ dy=0$$

iii) Check the exactness again

$$\frac{\partial M}{\partial y}=xe^x+2e^xy+e^x$$

$$\frac{\partial N}{\partial x}=(1)e^x+(x+2y)e^x$$

$$\frac{\partial N}{\partial x}=xe^x+2e^xy+e^x$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, this equation is exact.

iv) Find $u(x,y)$

$$\frac{\partial u}{\partial x}=(xy+y^2+y)(e^x)=M(x,y) \qquad (1)$$

$$\frac{\partial u}{\partial y}=(x+2y)(e^x)=N(x,y) \qquad (2)$$

To find $u(x,y)$, integrate either (1) or (2), let's say we take (2)

$$\int \partial u=\int (x+2y)(e^x)\ \partial y$$

$$u(x,y)=(xy+y^2)(e^x)+\ \phi(x) \qquad (3)$$

v) Find $\phi(x)$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left((xy+y^2)(e^x)+\ \phi(x)\right) \qquad (4)$$

$$=(xy+y^2+y)(e^x)+\phi'(x)$$

Now, let's compare (4) with (1)

$$(xy+y^2+y)(e^x)+\phi'(x)=(xy+y^2+y)(e^x)$$

$$\phi'(x)=0$$

$$\phi(x)=B$$

vi) Write $\boldsymbol{u(x,y) = A}$

$$u(x,y) = (xy + y^2)(e^x) + B = A$$

$u(x,y) = (xy + y^2)(e^x) = C$, where $C = A - B$

Exercises :

1. **Try solving Example 2 by using method 2.**
2. **Determine whether the following equation is exact. If it is, then solve it.**
   a. $(2x + y)dx + (x - 2y)\,dy = 0$
   b. $(\cos x\ \cos y + 2x)\,dx - (\sin x\ \sin y + 2y)\,dy = 0$
   c. $\cos\theta\,dr - \left(r\sin\theta - e^{\theta}\right)d\theta = 0$
3. **Given the differential equation**

$$(x - 2y)\ dx + (y - 2x)\ dy = 0$$

   i. **Show that the differential equation is exact. Hence, solve the differential equation by the method of exact equation.**
   ii. **Show that the differential equation is homogeneous. Hence, solve the differential equation by the method of homogeneous equation. Check the answer with 3i.**

1.2.4 Linear First Order Differential Equation

How to identify?

The general form of the first order linear DE is given by

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

When the above equation is divided by $a(x)$,

$$\frac{\mathbf{a(x)}}{\mathbf{a(x)}}\frac{dy}{dx} + \frac{\mathbf{b(x)}}{\mathbf{a(x)}}y = \frac{\mathbf{c(x)}}{\mathbf{a(x)}}$$

$$(\mathbf{1})\frac{dy}{dx} + \mathbf{p(x)}\,y = \mathbf{q(x)} \qquad (1)$$

NOTE:

Must be + here!!

Where $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{c(x)}{a(x)}$

Method of Solution :

i) Determine the value of $p(x)$ dan $q(x)$ such the the coefficient of $\frac{dy}{dx}$ is 1.

ii) Calculate the integrating factor, $\mu(x)$

$$\mu(x) = e^{\int p(x)dx}$$

iii) Write the equation in the form of

$$\frac{d}{dx}[\mu(x)y] = \mu(x)q(x)$$

$$\mu(x)y = \int \mu(x)\,q(x)\;\;dx$$

iv) The general solution is given by

$$y = \frac{1}{\mu(x)}\int \mu(x)\,q(x)\;\;dx$$

Example 1:

Solve this first order DE

$$\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = \frac{e^x}{x}$$

Solution:

i) Determine $p(x)$ and $q(x)$

$$p(x) = \frac{1+x}{x}, \qquad q(x) = \frac{e^x}{x}$$

ii) Find integrating factor, $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int \frac{1+x}{x} dx} \\ &= e^{\int \left(\frac{1}{x}+1\right)\, dx} \\ &= e^{(\ln x + x)} \\ &= e^{(\ln x)} \cdot e^{(x)} = xe^x\end{aligned}$$

iii) Write down the equation

$$\frac{d}{dx}(xe^x y) = (xe^x)\left(\frac{e^x}{x}\right)$$

$$xe^x y = \int e^{2x}\, dx$$

$$y = \frac{1}{xe^x}\left[\frac{e^{2x}}{2} + C\right]$$

iv) Final answer

$$y = \frac{e^x}{2x} + \frac{C}{xe^x}$$

Note:

Non-linear DE can be converted into linear DE by using the right substitution.

Example 2:

Using $z = y^2$, convert the following non-linear DE into linear DE.

$$x^2 y \frac{dy}{dx} - xy^2 = 1; \qquad y(1) = 1$$

Solve the linear equation.

Solutions:

i) Differentiate $z = y^2$ to get $\frac{dy}{dx}$ and replace into the non-linear equation.

$$\frac{dz}{dx} = 2y\frac{dy}{dx} \qquad => \qquad \frac{1}{2}\frac{dz}{dx} = y\frac{dy}{dx}$$

$$x^2\left(\frac{1}{2}\frac{dz}{dx}\right) - x(z) = 1$$

ii) Change the equation into the general form of linear equation & determine $p(x)$ and $q(x)$

$$\frac{dz}{dx} - \left(\frac{2}{x^2}\right)(xz) = \left(\frac{2}{x^2}\right)1$$

$$\frac{dz}{dx} \underbrace{-\left(\frac{2}{x}\right)}_{p} z = \underbrace{\frac{2}{x^2}}_{q}$$

iii) Find the integrating factor, $\mu(x) = e^{\int p(x)dx}$

$$\mu(x) = e^{-\int\left(\frac{2}{x}\right)dx}$$
$$= e^{-2\ln x} = e^{\ln x^{-2}}$$
$$= x^{-2} = \frac{1}{x^2}$$

iv) Find y

$$\frac{d}{dx}\left(\frac{1}{x^2}z\right) = \left(\frac{1}{x^2}\right)\left(\frac{2}{x^2}\right) = \frac{2}{x^4}$$

$$\frac{z}{x^2} = \int \frac{2}{x^4}\ dx$$

$$z = x^2\left[-\frac{2}{3x^3} + C\right]$$

Since $z = y^2$,

$$y^2 = -\frac{2}{3x} + Cx^2$$

v) Use the initial condition given, $\boldsymbol{y(1) = 1}$.

$$1^2 = -\frac{2}{3(1)} + C(1^2)$$

$$C = \frac{5}{3}$$

$$\therefore \qquad y^2 = -\frac{2}{3x} + \frac{5}{3}x^2$$

1.2.5 Equations of the form $\frac{dy}{dx} = G(ax + by)$

How to identify?

When the DE is in the form

$$\frac{dy}{dx} = G(ax + by) \quad (1)$$

use substitution

$$z = ax + by \quad (2)$$

to turn the DE into a separable equation

Method of Solution :

i) Differentiate (2) wrt x (to get $\frac{dy}{dx}$)

$$\frac{dz}{dx} = a + b\frac{dy}{dx} \quad (3)$$

ii) Replace (3) into (1)

iii) Solve using the separable equation solution

Example 1:

Solve $\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$

Solution:

i) Write the equation as a function of $x - y$

$$\frac{dy}{dx} = -(x - y) - 1 + (x - y + 2)^{-1} \qquad (1)$$

ii) Let $z = x - y$ and differentiate it to get $\frac{dy}{dx}$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx} \qquad (2)$$

iii) Replace (2) into (1)

$$1 - \frac{dz}{dx} = -(z) - 1 + (z + 2)^{-1}$$

$$\frac{dz}{dx} = z + 2 - \frac{1}{(z + 2)}$$

$$\frac{dz}{dx} = \frac{(z + 2)^2 - 1}{(z + 2)}$$

iv) Solve using the separable equation solution

$$\int \frac{(z + 2)}{(z + 2)^2 - 1}\, dz = \int dx$$

Substitution Method

$$u = (z + 2)^2 - 1$$

$$du = 2(z + 2)\, dz \quad \Rightarrow \quad (z + 2)\, dz = \frac{du}{2}$$

$$\frac{1}{2}\int \frac{1}{u}\, du = \int dx$$

$$\frac{1}{2}\ln u = x + C$$

$$\sqrt{u} = Ae^{x} \quad => \quad u = Ae^{2x}$$

Since $u = (z+2)^2 - 1$,

$$(z+2)^2 - 1 = Ae^{2x}$$

$$z = \sqrt{Ae^{2x}+1} - 2$$

Since $z = x - y$,

$$x - y = \sqrt{Ae^{2x}+1} - 2$$

$$\therefore \quad y = (x+2) - \sqrt{Ae^{2x}+1}$$

1.2.6 Bernoulli Equation

How to identify?

The general form of the Bernoulli equation is given by

$$\frac{dy}{dx} + b(x)\ y = c(x)\ y^n \qquad (1)$$

where $n \neq 0,\ n \neq 1$

To reduce the equation to a linear equation, use substitution

$$z = y^{1-n} \qquad (2)$$

Method of Solution :

iv) Divide (1) with y^n

$$y^{-n}\frac{dy}{dx} + b(x)\, y^{1-n} = c(x) \qquad (3)$$

v) Differentiate (2) wrt x (to get $\frac{dy}{dx}$)

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{1}{(1-n)}\frac{dz}{dx} = y^{-n}\frac{dy}{dx} \qquad (4)$$

vi) Replace (4) into (3)

$$\frac{1}{(1-n)}\frac{dz}{dx} + b(x)\, z = c(x)$$

$$\frac{dz}{dx} + \underbrace{(1-n)b(x)}_{p(x)}\ z = \underbrace{(1-n)c(x)}_{q(x)}$$

vii) Solve using the linear equation solution

- Find integrating factor, $\mu(x) = e^{\int p(x)dx}$
- Solve $\frac{d}{dx}(\mu(x)\, z) = \mu(x)\, q(x)$

Example 1:

Solve

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4 \qquad (1)$$

Solutions:

i) **Determine $n = 4$**

ii) **Divide (1) with y^4**

$$\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3}y^{-3} = e^x \qquad (2)$$

iii) **Using substitution, $z = y^{-3}$**

$$\frac{dz}{dx} = -3\,y^{-4}\frac{dy}{dx}$$

$$\frac{1}{y^4}\;\frac{dy}{dx} = -\frac{1}{3}\;\frac{dz}{dx} \qquad (3)$$

iv) **Replace (3) into (2) and write into linear equation form**

$$-\frac{1}{3}\;\frac{dz}{dx} + \frac{1}{3}z = e^x$$

$$\frac{dz}{dx} - z = -3\,e^x$$

$$p(x) = -1, \qquad q(x) = -3\,e^x$$

v) **Find the integrating factor**

$$\begin{aligned}\mu(x) &= e^{-\int 1\,dx}\\ &= e^{-x}\end{aligned}$$

vi) **Solve the problem**

$$\frac{d}{dx}(e^{-x}\,z) = e^{-x}(-3\,e^x)$$

$$\begin{aligned}z &= -\frac{1}{e^{-x}}\int 3\;dx\\ &= -e^x[3x + C]\end{aligned}$$

vii) Since $z = y^{-3}$

$$y^{-3} = Ce^{x} - 3xe^{x}$$

Exercise: **Answer:**

1. $\frac{dy}{dx} - \frac{1}{x}y = xy^2$ $\quad\quad$ $\frac{1}{y} = -\frac{x^2}{3} + \frac{C}{x}$

2. $\frac{dy}{dx} + \frac{y}{x} = y^2$ $\quad\quad$ $\frac{1}{y} = x(C - \ln x)$

3. $x\frac{dy}{dx} + y = xy^3$ $\quad\quad$ $y^2 = \frac{1}{2x + Cx^2}$

4. $\frac{dy}{dx} + \frac{2}{x}y = -x^2y^2 \cos x$ $\quad\quad$ $\frac{1}{y} = x^2(\sin x + C)$

5. $2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x}y^3$ $\quad\quad$ $\frac{1}{y^2} = \frac{(4x+5)^3}{12\cos x} + \frac{C}{\cos x}$

6. Given the differential equation,

$$\left(2x^4 y\right) dy + (4x^3y^2 - x^3)dx = 0.$$

Show that the equation is exact. Hence solve it.

7. The equation in Question 6 can be rewritten as a Bernoulli equation,

$$2x\frac{dy}{dx} + 4y = \frac{1}{y}.$$

By using the substitution $z = y^2$, solve this equation. Check the answer with Question 6.

1.3 Applications of the First Order ODE

The Newton's Law of Cooling

The Newton's Law of Cooling is given by the following equation

$$\frac{dT}{dt} = -k(T - T_s)$$

Do you know what type of DE is this?

Where

k is a constant of proportionality

T_s is the constant temperature of the surrounding medium

General Solution

Q1: Find the solution for $T(t)$

It is a separable equation. Therefore

$$\int \frac{dT}{T - T_s} = \int -k \ dt$$

$$\ln|T - T_s| = -kt + C$$

$$T - T_s = e^{-kt+C}$$

$$T = T_s + Ae^{-kt} \quad where\ A = e^C$$

Q2: Find A when $t = 0,\ T = T_0$

$$T_0 = T_s + A(1) \quad => \quad A = T_0 - T_s$$

$$T = T_s + (T_0 - T_s)e^{-kt}$$

Q3: Find k. Given that $T_s = 70°,\ T_0 = 100°,\ T(6) = 80°$

$$80 = 70 + (100 - 70)e^{-6k}$$

$$e^{-6k} = \frac{10}{30}$$

$$\ln e^{-6k} = \ln\frac{1}{3}$$

$$k = -\frac{1}{6}\ln\frac{1}{3} = 1.098612$$

$$T = T_s + (T_0 - T_s)e^{-1.0986t} = 70 + 30e^{-1.0986t}$$

Example 1:

According to Newton's Law of Cooling, the rate of change of the temperature T satisfies

$$\frac{dT}{dt} = -k(T - T_s)$$

Where T_s is the ambient temperature, k is a constant and t is time in minutes. When object is placed in room with temperature 10°C, it was found that the temperature of the object drops from 90°C to 30°C in 30 minutes. Then determine the temperature of an object after 20 minutes.

Solution:

i) **Determine all the information given.**

Room temperature = $T_s = 10°\text{C}$

When $t = 0,\ T_0 = 90°\text{C}$

When $t = 30,\ T_{30} = 30°\text{C}$

Question: Temperature after 20 minutes, $t = 20,\ T = ?$

ii) **Find the solution for $T(t)$**

$$T = T_s + Ae^{-kt}$$

iii) **Use the conditions given to find A and k**

When $t = 0,\ T_0 = 90°\text{C}\ ,\ T_s = 10°\text{C}$

$$90 = 10 + A \quad => \quad A = 80$$

$$T = 10 + 80e^{-kt}$$

When $t = 30,\ T_{30} = 30°\text{C}$

$$30 = 10 + 80e^{-30k}$$

$$e^{-30k} = \frac{20}{80}$$

$$k = -\frac{1}{30}\ln\frac{1}{4} = 0.04621$$
$$T = 10 + 80e^{-0.04621t}$$

iv) $\boldsymbol{t = 20,\ T = ?}$

$$T_{20} = 10 + 80e^{-0.04621(20)} = 41.75°\text{C}$$

Exercise:

1. **A pitcher of buttermilk initially at 25^{0}C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the buttermilk has dropped to 15°C after 20 minutes. When will it be at °C ?**

2. **Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70^{0}F. At 12 noon, the temperature of the body is 80^{0}F and at 1pm it is 75^{0}F. Assume that the temperature of the body at the time of death was 98.6^{0}F and that it has cooled in accord with Newton's Law. What was the time of death?**

Natural Growth and Decay

The differential equation

$$\frac{dx}{dt} = kx$$

Do you know what type of DE is this?

Where

k is a constant

x is the size of population / number of dollars / amount of radioactive

The problems:

1. Population Growth
2. Compound Interest
3. Radioactive Decay
4. Drug Elimination

Example 1:

A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What populations can its city planners expect in the year 2000?

Solution:

1) **Extract the information**

$t = 0, \quad P_0 = 25000$

$t = 10, \quad P_{10} = 30000$

$t = 40, \quad P_{40} = ?$

2) **Solve the DE**

$$\frac{dP}{dt} = kP$$

Separate the equation and integrate

$$\int \frac{dP}{P} = \int k\, dt \qquad => \qquad P = Ae^{kt}$$

3) Use the initial & boundary conditions

$\underline{t = 0, \quad P_0 = 25000}$

$A = 25000 \qquad => \therefore \quad P = 25000\, e^{kt}$

$\underline{t = 10, \quad P_{10} = 30000}$

$30000 = 25000\, e^{10k}$

$$10k = \ln \left| \frac{30000}{25000} \right| \qquad => \quad k = 0.01823$$

$\therefore \; P = 25000\, e^{0.01823t}$

$\underline{t = 40, \quad P_{40} = ?}$

$P_{40} = 25000\, e^{(0.01823)(40)} = 51840$

In the year 2000, the population size is expected to be 51840

Exercise:

1) (Compounded Interest) Upon the birth of their first child, a couple deposited RM5000 in an account that pays 8% interest compounded continuously. The interest payments are allowed to accumulate. How much will the account contain on the child's eighteenth birthday? **(ANS: RM21103.48)**

2) (Drug elimination) Suppose that sodium pentobarbitol is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45mg of sodium pentobarbitol per kg of the dog's body weight. Suppose also that sodium pentobarbitol is eliminated exponentially from the dog's bloodstream, with a half-life of 5 hours. What single dose should be administered in order to anesthetize a 50-kg dog for 1 hour? **(ANS: 2585 mg)**

3) (Half-life Radioactive Decay) A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years, it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining. **(ANS: 24180 years)**

Electric Circuits - RC

Given that the DE for an RL-circuit is

$$\boldsymbol{L\frac{dI}{dt} + RI = E(t)}$$

Do you know what type of DE is this?

Where

$E(t)$ is the voltage source

L is the inductance

R is the resistance

CASE 1 : $E(t) = E_0$ (constant)

$$L\frac{dI}{dt} + RI = E_0 \qquad (1)$$

i) Write in the linear equation form

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}$$

$$p(t) = \frac{R}{L}, \qquad q(t) = \frac{E_0}{L}$$

ii) Find the integrating factor, $\mu(t)$

$$\mu(t) = e^{\int\left(\frac{R}{L}\right)dt}$$

$$= e^{Rt/L}$$

iii) Multiply the DE with the integrating factor

$$\frac{d}{dt}\left(e^{Rt/L}\, I\right) = \left(e^{Rt/L}\right)\left(\frac{E_0}{L}\right)$$

iv) Integrate the equation to find I

$$\left(e^{Rt/L}\, I\right) = \int \left(e^{Rt/L}\right)\left(\frac{E_0}{L}\right)\, dt$$

$$I = \frac{1}{e^{Rt/L}}\left[\left(e^{Rt/L}\right)\left(\frac{E_0}{L}\right)\left(\frac{L}{R}\right) + C\right] = \frac{E_0}{R} + Ce^{-Rt/L}$$

CASE 2 : $E(t) = E_0 \sin wt$ or $E(t) = E_0 \cos wt$

Consider $E(t) = E_0 \sin wt$, the DE can be written as

$$L\frac{dI}{dt} + RI = E_0 \sin wt$$

i) Write into the linear equation form and determine $p(t)$ and $q(t)$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}\ \sin wt$$

$$p(t) = \frac{R}{L}\ ,\quad q(t) = \frac{E_0}{L}\sin wt$$

ii) Find integrating factor, $\mu(t)$

$$\mu(t) = e^{\int\left(\frac{R}{L}\right)dt}$$
$$= e^{Rt/L}$$

iii) Multiply the DE with the integrating factor

$$\frac{d}{dt}\left(e^{Rt/L}\, I\right) = \left(e^{Rt/L}\right)\frac{E_0}{L}\ \sin wt$$

iv) Integrate the equation to find I

$$I = \left(\frac{1}{e^{Rt/L}}\right)\left(\frac{E_0}{L}\right)\int \left(e^{Rt/L}\right)\ \sin wt\ dt \qquad (1)$$

Tabular Method

Differentiate	Sign	Integrate
$e^{Rt/L}$	+	$\sin wt$
$\left(\frac{R}{L}\right)e^{Rt/L}$	-	$-\left(\frac{1}{w}\right)\cos wt$
$\left(\frac{R}{L}\right)^2 e^{Rt/L}$	+	$-\left(\frac{1}{w}\right)^2 \sin wt$

$$I = \left(\frac{1}{e^{Rt/L}}\right)\left(\frac{E_0}{L}\right)\left[\left(\left(e^{Rt/L}\right)\left(-\frac{1}{\mathrm{w}}\right)\cos wt\right) - \left(-\frac{\mathrm{R}}{\mathrm{w}^2\mathrm{L}}\right)\left(e^{Rt/L}\right)\sin wt + \left(\left(\frac{R}{L}\right)^2\left(-\left(\frac{1}{\mathrm{w}}\right)^2\right)\int\left(e^{Rt/L}\right)\sin wt\ dt\right)\right]$$

$$I = \left(-\frac{E_0}{\mathrm{w}L}\cos wt\right) + \left(\frac{E_0\mathrm{R}}{\mathrm{wL}^2}\right)\sin wt - \left[\left(\frac{\mathrm{R}}{\mathrm{wL}}\right)^2\left(\frac{E_0}{Le^{Rt/L}}\right)\int\left(e^{Rt/L}\right)\sin wt\right] \tag{2}$$

From (1)

$$\int\left(e^{Rt/L}\right)\ sin\ wt\ \ dt = \left(e^{Rt/L}\right)\left(\frac{L}{E_0}\right)I \tag{3}$$

Replace (3) into (2)

$$I = \left(-\frac{E_0}{wL}\cos wt\right) + \left(\frac{E_0R}{wL^2}\right)\sin wt - \left(\frac{R}{wL}\right)^2 I$$

$$\left[1 + \left(\frac{\mathrm{R}}{\mathrm{wL}}\right)^2\right]I = \left(-\frac{E_0}{\mathrm{w}L}\cos wt\right) + \left(\frac{E_0R}{wL^2}\right)\sin wt$$

$$I = \frac{1}{\left[1 + \left(\frac{\mathrm{R}}{\mathrm{wL}}\right)^2\right]}\left[\left(-\frac{E_0}{\mathrm{w}L}\cos wt\right) + \left(\frac{E_0R}{wL^2}\right)\sin wt\right]$$

Exercise:

1) A 30-volt electromotive force is applied to an LR series circuit in which the inductance is 0.1 henry and the resistance is 15 ohms. Find the curve $\boldsymbol{i(t)}$ if $\boldsymbol{i(0) = 0}$. Determine the current as $\boldsymbol{t} \to \infty$.

2) An electromotive force

$$\boldsymbol{E(t)}\begin{cases}\mathbf{120}, & \mathbf{0 \le t \le 20} \\ \mathbf{0}, & \mathbf{t \ge 20}\end{cases}$$

is applied to an LR series circuit in which the inductance is 20 henries and the resistance is 2 ohms. Find the current $\boldsymbol{i(t)}$ if $\boldsymbol{i(0) = 0}$.

Vertical Motion – Newton's Second Law of Motion

The Newton's Second Law of Motion is given by

$$m\frac{dV}{dt} = F$$

Where
F is the external force
m is the mass of the body
v is the velocity of the body with the same direction with F
t is the time

Example 1:

A particle moves vertically under the force of gravity against air resistance kv^2, where k is a constant. The velocity v at any time t is given by the differential equation

$$\frac{dv}{dt} = g - kv^2.$$

If the particle starts off from rest, show that

$$v = \frac{\lambda\left(e^{2\lambda kt} - 1\right)}{\left(e^{2\lambda kt} + 1\right)}$$

Such that $\lambda = \sqrt{\frac{g}{k}}$. Then find the velocity as the time approaches infinity.

Solution:

i) Extract the information from the question

Initial Condition $t = 0, v = 0$

ii) Separate the DE

$$\frac{1}{k\left(\frac{g}{k} - v^2\right)}\,dv = \,dt$$

$$\frac{1}{\left(\sqrt{\frac{g}{k}}\right)^2 - v^2}\,dv = \,k\,dt$$

Let $\lambda = \sqrt{\frac{g}{k}}$,

$$\frac{1}{\lambda^2 - v^2}\,dv = k\,dt$$

iii) Integrate the above equation

$$\int \frac{1}{\lambda^2 - v^2}\,dv = \int k\,dt$$

$$\frac{1}{\lambda^2 - v^2} = \frac{1}{(\lambda + v)(\lambda - v)}$$

Using Partial Fraction

$$\frac{1}{(\lambda + v)(\lambda - v)} = \frac{1}{2\lambda(\lambda + v)} + \frac{1}{2\lambda\,(\lambda - v)}$$

$$\frac{1}{2\lambda}\int \frac{1}{(\lambda + v)} + \frac{1}{(\lambda - v)}\,dv = kt + C$$

$$\frac{1}{2\lambda}[\,\ln|\lambda + v| - \ln|\lambda - v|] = kt + C$$

$$\ln\frac{|\lambda + v|}{|\lambda - v|} = 2\lambda kt + 2\lambda C$$

iv) Use the initial condition, $t = 0, v = 0$

$$\ln|1| = 2\lambda C \quad => \quad C = 0$$

$$\ln\frac{|\lambda + v|}{|\lambda - v|} = 2\lambda kt$$

v) Rearrange the equation

$$\frac{\lambda + v}{\lambda - v} = e^{2\lambda kt}$$

$$\lambda + v = e^{2\lambda kt}(\lambda - v)$$

$$e^{2\lambda kt}v + v = \lambda e^{2\lambda kt} - \lambda$$

$$v\left(e^{2\lambda kt} + 1\right) = \lambda\left(e^{2\lambda kt} - 1\right)$$

$$v = \frac{\lambda\left(e^{2\lambda kt} - 1\right)}{(e^{2\lambda kt} + 1)}$$

vi) Find the velocity as the time approaches infinity.

$$v = \frac{\lambda\left(1 - \frac{1}{e^{2\lambda kt}}\right)}{\left(1 + \frac{1}{e^{2\lambda kt}}\right)}$$

When $t \to \infty, \frac{1}{e^{2\lambda kt}} \to 0$ => $v \approx \lambda = \sqrt{\frac{g}{k}}$