Definition 3.1 (Laplace Transform)

The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\left(f(t)\right) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Remember !!!

Example 3.1.1:

Using definition of Laplace Transform, find $F(s)$ if $f(t) = a$, a is constant.

Solution:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \ f(t) \ dt \\ &= \int_0^\infty e^{-st} \ a \ dt \\ &= \left. \frac{ae^{-st}}{-s} \right|_0^\infty \\ &= -\frac{1}{s}\left[ae^{-st}\right]_0^\infty \\ &= -\frac{a}{s}\left[e^{-s\,\infty} - e^{-s\,0}\right] \\ &= -\frac{a}{s}[0 - 1] \quad \therefore \quad \mathcal{L}(a) = \frac{a}{s}, \quad s > 0 \end{aligned}$$

$$\mathcal{L}(1) = \frac{1}{s} \text{ and } \mathcal{L}(-3) = -\frac{3}{s}, \quad s > 0$$

Example 3.1.2:

Find $F(s)$ if $f(t)=e^{at}$, $\boldsymbol{a}$ is constant

Solution:

$$F(s)=\int_0^\infty e^{-st}\ e^{at}\ dt$$

$$=\int_0^\infty e^{-(s-a)t}\ dt$$

$$=\left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$=-\frac{1}{(s-a)}\left[0-1\right],\ s-a>0=\frac{1}{s-a}\ ,\ s>a \quad \therefore\ \mathcal{L}\left(e^{at}\right)=\frac{1}{s-a}$$

$$\mathcal{L}\left(e^{-2t}\right)=\frac{1}{s+2}\quad ,\quad s>-2$$

Example 3.1.3:

Find $\mathcal{L}\left(\sin\ at\right)$

Solution:

$$\mathcal{L}\left(\sin\ at\right)=\int_0^\infty e^{-st}\ \sin\ at\quad dt$$

$$=\left[\frac{e^{-st}}{s^2+a^2}\left(-s\ \sin\ at-a\ \cos\ at\right)\right]_0^\infty$$

$$=\left[0-\frac{e^0}{s^2+a^2}\left(-s\ \sin\ 0-a\ \cos\ 0\right)\right]=\frac{a}{s^2+a^2}\quad ,\quad s>0$$

$$\mathcal{L}\left(\sin\ 3t\right)=\frac{3}{s^2+9}$$

Example 3.1.4:

Find $\mathcal{L}\left(f\left(t\right)\right)$ if

$$f\left(t\right)=\begin{cases} 2 & , \quad 0<t<5 \\ 0 & , \quad 5<t<10 \\ e^{4t} & , \quad 10<t \end{cases}$$

Solution:

$$F\left(s\right)=\int_0^5 e^{-st}\ 2dt+\int_5^{10} e^{-st}\ 0dt+\int_{10}^{\infty} e^{-st}\ e^{4t}dt$$

$$=2\int_0^5 e^{-st}dt+\int_{10}^{\infty} e^{(4-s)t}\ dt$$

$$=2\left[\left.\frac{e^{-st}}{-s}\right|_0^5\right]+\left[\left.\frac{e^{(4-s)t}}{\left(4-s\right)}\right|_{10}^{\infty}\right]$$

$$=2\left[\frac{e^{-5s}}{-s}-\frac{e^{-0}}{-s}\right]+\left[0-\frac{e^{10(4-s)}}{\left(4-s\right)}\right]$$

$$=\frac{2}{s}-\frac{2e^{-5s}}{s}+\frac{e^{-10(s-4)}}{\left(s-4\right)},s>4$$

Elementary Laplace Transform

$\mathcal{L}(f(t)) = F(s)$		
$f(t)$	$F(s)$	Condition on s
a	$\dfrac{a}{s}$	$s>0$
$t^n,\ n=0,1,2,\ldots$	$\dfrac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\dfrac{1}{s-a}$	$s>a$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\dfrac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\dfrac{s}{s^2-a^2}$	$s>\lvert a\rvert$

Note:

This table will enable us to obtain the transforms of many other functions.

Definition 3.2 (Linearity Property of Laplace Transform)

Let *f*, f_1, f_2 be a functions whose Laplace Transforms exist for s > α and *c* be *a* constant. Then for s > α,

$$\mathcal{L}\left(f_1 \pm f_2\right) = \mathcal{L}\left(f_1\right) \pm \mathcal{L}\left(f_2\right)$$

$$\mathcal{L}\left(cf\right) = c\,\mathcal{L}\left(f\right)$$

Example 3.2.1:

Find $\mathcal{L}\left(1 + 5e^{4t} - 6\sin 2t\right)$

Solution:

$$\begin{aligned}\mathcal{L}\left(1 + 5e^{4t} - 6\sin 2t\right) &= \mathcal{L}(1) + \mathcal{L}\left(5e^{4t}\right) - \mathcal{L}\left(6\sin 2t\right) \\ &= \frac{1}{s} + 5\,\mathcal{L}\left(e^{4t}\right) - 6\,\mathcal{L}\left(\sin 2t\right) \\ &= \frac{1}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}\end{aligned}$$

Exercises 3.2.1:

1. Show $\mathcal{L}\left(\sinh at\right) = \dfrac{a}{s^2 - a^2}$ using the linearity property.
2. Find $\mathcal{L}\left(\sin^2 2t\right)$.
3. Find $\mathcal{L}\left(e^{3t}\cosh t\right)$.

Definition 3.3 (First Shifting Property)

If $F(s)=\mathcal{L}\left(f(t)\right)$,

then $\mathcal{L}\left(e^{at} f(t)\right)=F(s-a)$

Example 3.3.1:

Find $\mathcal{L}\left(e^{2t}\ t\right)$.

Solution

$$a=2 \quad , \quad f(t)=t \qquad \therefore \quad \mathcal{L}\left(e^{2t}\ t\right)=F(s-2)$$

where $$F(s)=\mathcal{L}\left(f(t)\right)=\mathcal{L}(t)=\frac{1}{s^2}$$

$$\therefore\ \mathcal{L}\left(e^{2t}\ t\right)=F(s-2)=\frac{1}{(s-2)^2}$$

Exercises 3.3.1:

Find $\mathcal{L}\left(e^{4t}\ \cos t\right)$.

Definition 3.4 (Derivatives of the Laplace Transform)

If $\mathcal{L}\left(f\left(t\right)\right)=F\left(s\right),$ then

$$\mathcal{L}\left(t^{n}\ f\left(t\right)\right)=(-1)^{n}\frac{d^{n}F}{ds^{n}},\quad n=1,2,3,\ldots$$

Example 3.4.1:

Find $\mathcal{L}\left(t\sin 6t\right)$.

Solution:

$$n=1\quad ,\quad f\left(t\right)=\sin 6t\quad ,\quad F\left(s\right)=\frac{6}{s^{2}+36}$$

$$\begin{aligned}\mathcal{L}\left(t\sin 6t\right)&=(-1)^{1}\frac{dF}{ds}\\&=-\frac{\left(s^{2}+36\right)(0)-6\left(2s\right)}{\left(s^{2}+36\right)^{2}}\\&=-\frac{-12s}{\left(s^{2}+36\right)^{2}}=\frac{12s}{\left(s^{2}+36\right)^{2}}\end{aligned}$$

Exercises 3.4.1:

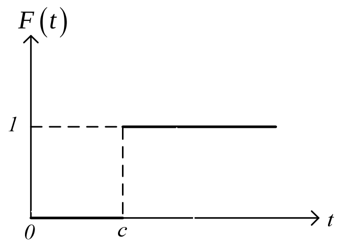
1. Find $\mathcal{L}\left(t\ e^{t}\right)$.
2. Find $\mathcal{L}\left(e^{-t}\ t\sin 2t\right)$.

Definition 3.5 (Heaviside Unit Step Function)

The function $H(t-a)$ is defined by

$$f(t) = H(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

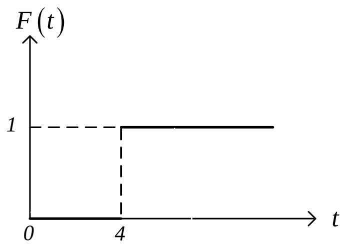
Graph of $f(t) = H(t-a)$



where $f(t) = 0, \quad 0 \le t < a$ and $f(t) = 1, \quad t \ge a$

Example 3.5.1:

$$\text{Draw } f(t) = H(t-4) = \begin{cases} 0 \ , & t < 4 \\ 1 \ , & t \ge 4 \end{cases}$$

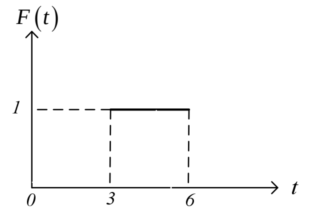


Exercises 3.5.1:

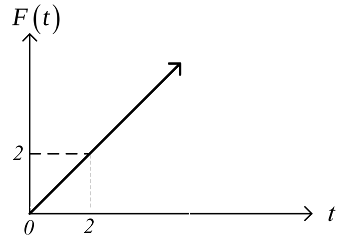
1. Draw $f_1(t) = e^{-t}$
2. Draw $f_2(t) = e^{-t}H(t-2)$
3. Draw $f_3(t) = e^{-(t-2)}H(t-2)$

Effect of the Unit Step Function

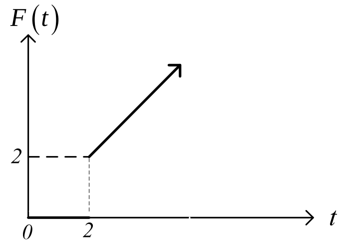
1) $$f(t) = H(t-3) - H(t-6)$$



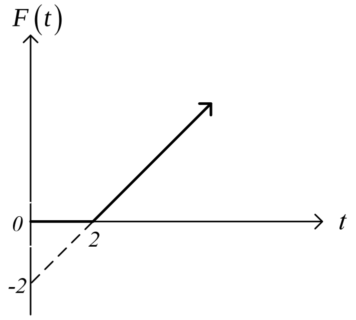
2) $$f(t) = t = \begin{cases} 0 & , \quad t < 0 \\ t & , \quad t \geq 0 \end{cases}$$



3) $$f(t) = t \cdot H(t-2) = \begin{cases} 0 & , \quad t < 2 \\ t & , \quad t \geq 2 \end{cases}$$



4) $$f(t) = (t-2)H(t-2) = \begin{cases} 0 & , \quad t < 2 \\ t-2 & , \quad t \geq 2 \end{cases}$$



Definition: Step Function

A step function is a piecewise continuous function of the form

$$g(t) = \begin{cases} g_1, & 0 \leq t < a_1 \\ g_2, & a_1 \leq t < a_2 \\ \quad \vdots & \\ g_{n-1}, & a_{n-2} \leq t < a_{n-1} \\ g_{n,} & t \geq a_{n-1} \end{cases}$$

The step functions can be expressed into the unit step functions forms.

Given a step function

$$g(t) = \begin{cases} g_1, & 0 \leq t < a \\ g_2, & t \geq a \end{cases}$$

The Heaviside unit step function is

$$g(t) = g_1 + \begin{cases} 0, & 0 \leq t < a \\ [g_2 - g_1], & t \geq a \end{cases}$$

$$g(t) = g_1 + [g_2 - g_1] \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

$$g(t) = g_1 + [g_2 - g_1]H(t-a)$$

<u>Example 3.5.2:</u>

Express the following functions in terms of unit step functions.

$$f(t) = \begin{cases} t-4, & 0 \leq t < 2 \\ t, & 2 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

Solution:

$$f(t) = f_1 + [f_2 - f_1]H(t-a_1) + [f_3 - f_2]H(t-a_2)$$

$$f(t) = (t-4) + [t-(t-4)]H(t-2) + [0-t]H(t-4)$$

$$f(t) = (t-4) - 4H(t-2) - tH(t-4)$$

Example 3.5.3:

Express the following unit step functions into the step functions form.

$$f(t) = (t+1) + [1-t]H(t-2) + [t^2]H(t-4)$$

Solution:

$$f(t) = \underbrace{(t+1)}_{f_1} + \underbrace{[1-t]}_{f_2-f_1} H(t-2) + \underbrace{[t^2]}_{f_3-f_2} H(t-4)$$

$f_1 = t+1$

$$f_2 - f_1 = 1 - t \;\Rightarrow\; f_2 = (1-t) + f_1$$
$$f_2 = (1-t) + (t+1)$$
$$f_2 = 2$$

$$f_3 - f_2 = t^2 \;\Rightarrow\; f_3 = t^2 + f_2$$
$$f_3 = t^2 + 2$$

$$\therefore \qquad f(t) = \begin{cases} t+1, & 0 \le t < 2 \\ 2, & 2 \le t < 4 \\ t^2 + 2, & t \ge 4 \end{cases}$$

Laplace Transform of *H(t – a)*

Theorem: Laplace Transform of Unit Step Functions

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}, \qquad a > 0$$

Inverse Laplace Transform of Unit Step Functions

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = H(t-a)$$

Example 3.5.4:

Find $\mathfrak{L}\left\{f(t)\right\}$ if $f(t)=H(t-2)-H(t-4)$.

Solution:

a) Using the Laplace Transform Table

$$\begin{aligned}\mathfrak{L}\left\{f(t)\right\}&=\mathfrak{L}\left\{H(t-2)-H(t-4)\right\}\\&=\mathfrak{L}\left\{H(t-2)\right\}-\mathfrak{L}\left\{H(t-4)\right\}\\&=\frac{e^{-2s}}{s}-\frac{e^{-4s}}{s}\end{aligned}$$

b) Using the Laplace Transform definition

- Change the unit step function into the step function form

$$\mathcal{L}\{\mathrm{H(t-2)-H(t-4)}\}=\underbrace{0}_{f_1}+\underbrace{[1]}_{f_2-f_1}\ H(t-2)+\underbrace{[-1]}_{f_3-f_2}H(t-4)$$

$$f_1=0;\quad f_2=1+f_1=1;\quad f_3=-1+f_2=0.$$

$$\text{Hence } f(t)=\begin{cases}0, & 0\le t<2\\ 1, & 2\le t<4\\ 0, & t\ge 4\end{cases}$$

Using the Laplace transform definition

$$\mathcal{L}\{f(t)\}=\int_0^2 e^{-st}\cdot 0\,dt+\int_2^4 e^{-st}\cdot 1\,dt+\int_4^\infty e^{-st}\cdot 0\,dt$$

$$=\left[-\frac{e^{-st}}{s}\right]_2^4=\left[-\frac{e^{-4s}}{s}-\left(-\frac{e^{-2s}}{s}\right)\right]=\frac{e^{-2s}}{s}-\frac{e^{-4s}}{s}$$

Laplace Transform of *H(t – a). F(t – a)*

Theorem: Second-shift Property

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$$

Inverse Laplace Transform with Second-shift Property

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$$

Example 3.5.5:

Find $\mathfrak{L}\left\{(t-4)^2 H(t-4)\right\}$.

Solution

$$c = 4,\ f(t) = t^2,\ F(s) = \frac{2!}{s^3}$$

$$\mathfrak{L}\left\{(t-4)^2 H(t-4)\right\} = F(s)e^{-4s} = \frac{2!}{s^3}e^{-4s}$$

$$\therefore\quad \mathfrak{L}\left\{(t-4)^2 H(t-4)\right\} = \frac{2e^{-4s}}{s^3}$$

Exercises 3.5.2:

1. Find $\mathfrak{L}\left\{\sin(t-3)H(t-3)\right\}$.
2. Find $\mathfrak{L}\left\{e^{(t-5)}H(t-5)\right\}$.
3. Find $\mathcal{L}\left\{\cos\ 2\left(t+{}^{\pi}/_{2}\right)H\left(t-{}^{\pi}/_{2}\right)\right\}$

Example 3.5.6

Find the function whose transform is $\dfrac{e^{-4s}}{s^2}$.

Solution

The numerator corresponds to e^{-as} where $a = 4$ and therefore indicate $H(t-4)$.

Then $\dfrac{1}{s^2} = F(s) = \mathfrak{L}\{t\} \quad \therefore\ f(t) = t \qquad \therefore\ \mathfrak{L}^{-1}\left\{\dfrac{e^{-4s}}{s^2}\right\} = (t-4)H(t-4)$

Note:

Remember that in writing the final result $f(t)$ is replaced by $f(t-c)$.

Exercises 3.5.3:

1. A function $f(t)$ is defined by $f(t) = \begin{cases} 4 & , \quad 0 < t < 2 \\ 2t-3 & , \quad t > 2 \end{cases}$

   Sketch the graph of the function, expressing the function $f(t)$ in unit step form and determine its Laplace transforms.

2. Write the following $f(t)$ in terms of unit step functions and determine the Laplace transforms.

$$f(t) = \begin{cases} e^{-t} & , \quad 0 < t < 2 \\ 2t-1 & , \quad t \ge 2 \end{cases}$$

Answer: $\mathfrak{L}\{f(t)\} = \dfrac{1}{s+1} + e^{-2s}\left(\dfrac{3}{s} + \dfrac{2}{s^2}\right) - \dfrac{e^{-2(s+1)}}{s+1}$

3. A function $f(t)$ is defined by

$$f(t)=\begin{cases}6 & , \quad 0<t<1 \\ 8-2t & , \quad 1<t<3 \\ 4 & , \quad t>3\end{cases}$$

Sketch the graph and find the Laplace transform of the function.

Answer: $\mathfrak{L}\{f(t)\}=\frac{6}{s}-\frac{2e^{-2s}}{s^2}+\frac{2e^{-3s}}{s^2}+\frac{2e^{-3s}}{s}$

4. Given

$$f(t)=\begin{cases}0 & , \quad 0<t<2 \\ t & , \quad 2<t<5 \\ e^{2t} & , \quad t>5\end{cases}$$

Find the Laplace transform of the function.

Answer: $\mathfrak{L}\{f(t)\}=\frac{e^{-2s}}{s^2}+\frac{e^{10}\cdot e^{-5s}}{s-2}-\frac{e^{-5s}}{s^2}+\frac{2e^{-2s}}{s}-\frac{5e^{-5s}}{s}$

5. Determine the function $f(t)$ for which,

$$\mathfrak{L}\{f(t)\}=\frac{3}{s}-\frac{4e^{-s}}{s^2}+\frac{5e^{-2s}}{s^2}.$$

Find its inverse transform and sketch the graph of $f(t)$.

Answer: $f(t)=\begin{cases}3 & , \quad 0<t<1 \\ 7-4t & , \quad 1<t<2 \\ t-3 & , \quad t>2\end{cases}$

Definition 3.6 (Dirac Delta Function)

The Dirac Delta function $\delta(t-a)$ is defined by

$$\delta(t-a) = \begin{cases} \infty, & if \quad t = a \\ 0, & otherwise \end{cases}$$

And

$$\int_0^\infty \delta(t-a)\, dt = 1$$

$$\int_0^\infty f(t)\delta(t-a)\, dt = f(a)$$

THEOREM: Laplace Transform for the Dirac Delta Function

For $a > 0$, $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

and the inverse Laplace Transform is

$$\mathcal{L}^{-1}\{e^{-as}\} = \delta(t-a)$$

For $a > 0$, $\mathcal{L}\{f(t)\delta(t-a)\} = f(a)e^{-as}$

Example 3.6.1:

$$\mathfrak{L}\{6\cdot\delta(t-a)\} = 6e^{-as}$$

Example 3.6.2:

Given $f(t) = \begin{cases} t, & 0 \le t < 3 \\ 5, & t \ge 3 \end{cases}$

$$\mathfrak{L}\{f(t)\cdot\delta(t-3)\} = f(3)e^{-3s} = 5e^{-3s}$$

$$\mathfrak{L}\{f(t)\cdot\delta(t-2)\} = f(2)e^{-2s} = 2e^{-2s}$$

Definition 3.7 (Periodic Function)

In many technological problems, we are dealing with forms of mechanical vibrations or electrical oscillations and the necessity to express such periodic functions in Laplace transforms soon arises. Let $f(t)$ be a periodic function of period T i.e. $f(t) = f(t+T)$, $T \neq 0$.

Example 3.7.1:

Show that $f(t) = \sin 2\pi t$ is a periodic function.

Solution:

$$
\begin{aligned}
f(t) &= \sin 2\pi t \\
f(t+T) &= \sin 2\pi (t+T) \\
f(t) &= f(t+T) \\
\sin 2\pi t &= \sin 2\pi (t+T) \\
&= \sin 2\pi t \cos 2\pi T + \cos 2\pi t \sin 2\pi T
\end{aligned}
$$

$$\cos 2\pi T = 1, \quad \sin 2\pi T = 0 \Rightarrow T = 1, 2, 3, \ldots$$

Exercises 3.7.1:

1. Show that $f(t) = t^2$ is not a periodic function.
2. Sketch the following periodic function.

$$f(t) = \begin{cases} 1 \quad , & 0 < t < 1 \\ -1 \quad , & 1 < t < 2 \end{cases}$$

$$f(t) = f(t+2)$$

Laplace Transform of Periodic Function

If $f(t)$ is a periodic function of period *T*,

then $\mathfrak{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \quad ; s > 0$.

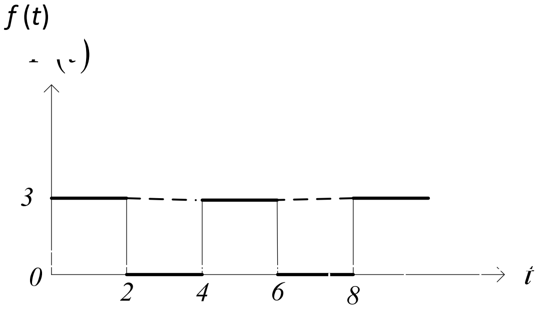
Example 3.7.2:

Sketch the following periodic function and find its Laplace transform.

$$f(t) = \begin{cases} 3 & , \quad 0 < t < 2 \\ 0 & , \quad 2 < t < 4 \end{cases}$$

$$f(t) = f(t+4)$$

Solution:



The expression for

$$\begin{aligned} \mathfrak{L}\{f(t)\} &= \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\,dt \\ &= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} \cdot 3\,dt + \int_2^4 e^{-st} \cdot 0\,dt \right] \\ &= \frac{3}{s\left(1+e^{-2s}\right)} \end{aligned}$$

Exercises 3.7.2:

Sketch the following periodic function and find its Laplace transform.

$$f(t) = \begin{cases} t & , \quad 0 < t < 1 \\ 1 & , \quad 1 < t < 2 \end{cases}$$

$$f(t) = f(t+2)$$

Definition 3.8 (Inverse Laplace Transform)

Recall that,

Notation for Laplace Transform; $\mathcal{L}\left(f(t)\right)=F(s)$

So, the inverse form; $\mathcal{L}^{-1}\left(F(s)\right)=f(t)$

Linearity property of Inverse Laplace Transform

If $\mathcal{L}^{-1}\left(F(s)\right)=f(t)$ and $\mathcal{L}^{-1}\left(G(s)\right)=g(t)$

with α and β is a constants, then

$$\begin{aligned}\mathcal{L}^{-1}\left(\alpha F(s)\pm\beta G(s)\right)&=\alpha\mathcal{L}^{-1}\left(F(s)\right)\pm\beta\mathcal{L}^{-1}\left(G(s)\right)\\&=\alpha f(t)\pm\beta g(t)\end{aligned}$$

First shifting property for Inverse Laplace Transform

If $\mathcal{L}^{-1}\left(F(s)\right)=f(t)$ with a as constant, then

$$\mathcal{L}^{-1}\left(F(s-a)\right)=e^{at}\,f(t)$$

or we can write as $\mathcal{L}^{-1}\left(F(s-a)\right)=e^{at}\,\mathcal{L}^{-1}\left(F(s)\right)$.

Example 3.8.1:

$$F(s)=\frac{4}{s^2+9}=\frac{4}{3}\left(\frac{3}{s^2+9}\right)\Rightarrow f(t)=\frac{4}{3}\sin 3t$$

Example 3.8.2:

$$F(s)=\frac{1}{s^5}=\frac{1}{4!}\left(\frac{4!}{s^5}\right)\Rightarrow f(t)=\frac{1}{24}t^4$$

Example 3.8.3:

$$F(s)=\frac{6}{(s-1)^4}$$

By shifting property, $G(s-a)=G(s-1),\, a=1$.

$$G(s)=\frac{6}{s^4}=\frac{3!}{s^4}$$

$$\Rightarrow g(t)=t^3\Rightarrow f(t)=e^t\; t^3$$

Exercises 3.8.1:

Determine

1. $\mathfrak{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\}$

2. $\mathfrak{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}$

Example 3.8.4:

$$\mathfrak{L}^{-1}\left\{\frac{s}{s^2+25}\right\} = \mathfrak{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} = \cos 5t$$

We can write down the corresponding function in *t,* provided we can recognize it from our table of transforms.

But, what about $\mathfrak{L}^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$?

Solution:

$$\frac{3s+1}{s^2-s-6} = \frac{3s+1}{(s+2)(s-3)} = \frac{1}{s+2} + \frac{2}{s-3}$$

$$\therefore L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\} = L^{-1}\left\{\frac{1}{s+2} + \frac{2}{s-3}\right\}$$

from table: $= e^{-2t} + 2e^{3t}$

$$\frac{3s+1}{s^2-s-6} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$A(s-3) + B(s+2) = 3s+1$$

$$s = 3 : 5B = 10 \Rightarrow B = 2$$

$$s = -2 : -5A = -5 \Rightarrow A = 1$$

$$\frac{3s+1}{s^2-s-6} = \frac{1}{s+2} + \frac{2}{s-3}$$

The two simpler functions of $\frac{1}{s+2}$ and $\frac{2}{s-3}$ are called the **partial fractions** of $\frac{3s+1}{s^2-s-6}$.

Therefore, u need to know partial fractions!!

Partial Fractions

There are few types of denominator that u should know:

1) A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where *A* is a constant to be determined.
2) A repeated factor $(s+a)^2$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}$.
3) Similarly $(s+a)^3$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$.
4) A quadratic factor $\left(s^2+ps+q\right)$ gives $\dfrac{Ps+Q}{s^2+ps+q}$.
5) Repeated quadratic factors $\left(s^2+ps+q\right)^2$ gives

$$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{\left(s^2+ps+q\right)^2}.$$

Example 3.8.5:

Determine $\mathfrak{L}^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$. **Answer**: $3e^{-3t}+2e^{4t}$

$$\frac{5s+1}{s^2-s-12}=\frac{A}{s+3}+\frac{B}{s-4}$$

$$A(s-4)+B(s+3)=5s+1$$

$$s=-3:-7A=-14\Rightarrow A=2;\quad s=4:7B=21\Rightarrow B=3$$

$$\frac{5s+1}{s^2-s-12}=\frac{2}{s+3}+\frac{3}{s-4}$$

$$\mathfrak{L}^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}=\mathfrak{L}^{-1}\left\{\frac{2}{s+3}+\frac{3}{s-4}\right\}=2e^{-3t}+3e^{4t}$$

Exercises 3.8.2:

Determine

1. $\mathfrak{L}^{-1}\left\{\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$. **Answer**: $3e^{-t}+\cos 2t-3\sin 2t$

2. $\mathfrak{L}^{-1}\{F(s)\}$ for $F(s)=\dfrac{s+1}{s^2+2s+10}$

3. $\mathfrak{L}^{-1}\{F(s)\}$ for $F(s)=\dfrac{2s-3}{s^2+s-2}$

4. $\mathfrak{L}^{-1}\left\{\dfrac{2s+3}{(s+4)^3}\right\}$ Answer: $\therefore\ f(t)=e^{-4t}\left(2t-\dfrac{5}{2}t^2\right)$

Convolution Theorem

$$\mathfrak{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)\,du$$

Example 3.8.6:

Find the inverse Laplace transform for $\dfrac{1}{s(s^2+4)}$ using Convolution theorem.

Solution:

Let $F(s)=\dfrac{1}{s}$ and $G(s)=\dfrac{1}{s^2+4}$, then

$$f(t)=1,\ g(t)=\frac{1}{2}\sin 2t.$$

$$\Rightarrow \quad f(u)=1,\ g(t-u)=\frac{1}{2}\sin 2(t-u).$$

$$\begin{aligned}
\mathfrak{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} &= \int_0^t (1)\frac{1}{2}\sin 2(t-u)\,du \\
&= \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t \\
&= \frac{1}{4}\left[\cos 2(0)-\cos 2t\right] \\
&= \frac{1}{4}[1-\cos 2t]
\end{aligned}$$

Exercises 3.8.3:

Find $\mathfrak{L}^{-1}\left\{\dfrac{s^2}{\left(s^2+4\right)^2}\right\}$ using Convolution theorem.

Find $\mathfrak{L}^{-1}\left\{\dfrac{1}{\left(s^2+1\right)^2}\right\}$ using Convolution theorem.

Definition 3.9 (Solution of Differential equations by Laplace Transforms)

To solve a differential equation by Laplace transforms, we go through four distinct stages.

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

Transforms of Derivatives

If $\mathfrak{L}\{y(t)\} = Y(s)$, then

$$\mathfrak{L}\{y'(t)\} = sY(s) - y_0$$

$$\mathfrak{L}\{y''(t)\} = s^2Y(s) - sy_0 - y'_0$$

$$\mathfrak{L}\{y'''(t)\} = s^3Y(s) - s^2y_0 - sy'_0 - y''_0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mathfrak{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y_0 - s^{n-2}y'_0 - \cdots - y_0^{(n-1)}$$

Solution of 1st Order Differential equations

Example 3.9.1:

Solve the equation $\dfrac{dy}{dt}-2y=4$, given that at $t=0, y=1$.

Solution:

(a) Rewrite the equation in Laplace transforms using the last notation

$$\mathfrak{L}\left\{\frac{dy}{dt}-2y\right\}=\mathfrak{L}\{4\} \quad \Rightarrow \quad \mathfrak{L}\left\{\frac{dy}{dt}\right\}-2\mathfrak{L}\{y\}=\mathfrak{L}\{4\}$$

We have

$$\mathfrak{L}\left\{\frac{dy}{dt}\right\}=\mathfrak{L}\{y'(t)\}=sY(s)-y_0,$$
$$\mathfrak{L}\{y(t)\}=Y(s),$$
$$\mathfrak{L}\{4\}=\frac{4}{s}$$

.

Then the equation becomes

$$\Rightarrow \quad \left(sY(s)-y_0\right)-2Y(s)=\frac{4}{s}.$$

(b) Insert the initial condition that at $t=0, y=1$ i.e. $y_0=1$.

$$\Rightarrow \quad \left(sY(s)-1\right)-2Y(s)=\frac{4}{s}$$

(c) Now we rearrange this to give an expression for $Y(s)$

$$\Rightarrow \quad (s-2)Y(s)=\frac{4}{s}+1=\frac{4+s}{s} \quad \therefore \quad Y(s)=\frac{4+s}{s(s-2)}$$

(d) Finally we take inverse transforms to obtain *x*

$$\frac{s+4}{s(s-2)}=\frac{A}{s}+\frac{B}{s-2}$$

$$\therefore \quad s+4=A(s-2)+B(s)$$

i) Let $s=2 \quad \Rightarrow 6=2B \quad \therefore B=3$

ii) Let $s=0 \quad \Rightarrow 4=A(-2) \quad \therefore A=-2$

$$\therefore \quad Y(s)=\frac{s+4}{s(s-2)}=\frac{3}{s-2}-\frac{2}{s}$$

Therefore, taking inverse transforms

$$y(t)=\mathfrak{L}^{-1}\{Y(s)\}=\mathfrak{L}^{-1}\left\{\frac{s+4}{s(s-2)}\right\}$$

$$=\mathfrak{L}^{-1}\left\{\frac{3}{s-2}-\frac{2}{s}\right\}$$

$$\therefore \quad y(t)=3e^{2t}-2$$

<u>Example 3.9.2:</u>

Solve the equation $\frac{dx}{dt}-4x=2e^{2t}+e^{4t}$ given that at $t=0, x=0$.

<u>Solution:</u>

$$\mathfrak{L}\left\{\frac{dx}{dt}-4x\right\}=\mathfrak{L}\left\{2e^{2t}+e^{4t}\right\}$$

$$\Rightarrow \quad \left(sX(s)-x_0\right)-4X(s)=\frac{2}{s-2}+\frac{1}{s-4}.$$

$t=0, x=0$

$$\Rightarrow \quad \left(sX(s)-0\right)-4X(s)=\frac{2}{s-2}+\frac{1}{s-4}$$

$$\Rightarrow \quad \left(s-4\right)X(s)=\frac{2}{s-2}+\frac{1}{s-4}$$

$$\therefore \quad X(s)=\underbrace{\frac{2}{\left(s-2\right)\left(s-4\right)}}_{\text{can use partial fraction}}+\underbrace{\frac{1}{\left(s-4\right)^2}}_{\substack{\text{can transform}\\\text{directly from}\\\text{the table}}}$$

Partial Fraction
$\frac{2}{\left(s-2\right)\left(s-4\right)}=\frac{A}{s-2}+\frac{B}{s-4}=\frac{-1}{s-2}+\frac{1}{s-4}$

$$x(t)=\mathfrak{L}^{-1}\left\{\frac{2}{\left(s-2\right)\left(s-4\right)}+\frac{1}{\left(s-4\right)^2}\right\}=\mathfrak{L}^{-1}\left\{\frac{-1}{s-2}+\frac{1}{s-4}+\frac{1}{\left(s-4\right)^2}\right\}$$

$$\therefore \quad x(t)=-e^{2t}+e^{4t}+te^{4t}$$

$$=e^{4t}\left(t+1\right)-e^{2t}$$

<u>Exercises 3.9.1:</u>

Solve the equation $\frac{dx}{dt}+2x=10e^{3t}$ given that at $t=0, x=6$.

Solution of 2nd Order Differential equations

The method is, in effect, the same as before, going through the same four distinct stages.

Example 3.9.3:

Solve the equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2e^{3t}$ given that $y(0) = 5$ and $y'(0) = 7$.

Solution:

(a) We rewrite the equation in terms of its transforms, remembering that

$$\mathfrak{L}\{y\} = Y(s)$$

$$\mathfrak{L}\{y'(t)\} = sY(s) - y_0$$

$$\mathfrak{L}\{y''(t)\} = s^2Y(s) - sy_0 - y_0'$$

The equation becomes

$$\Rightarrow \quad \underbrace{\left(s^2Y(s) - sy_0 - y_0'\right)}_{\mathfrak{L}\{y''(t)\}} - 3\underbrace{\left(sY(s) - y_0\right)}_{\mathfrak{L}\{y'(t)\}} + 2\underbrace{Y(s)}_{\mathfrak{L}\{y(t)\}} = \frac{2}{s-3}$$

(b) Insert the initial conditions. In this case $y_0 = 5$ and $y_0' = 7$.

$$\Rightarrow \quad s^2Y(s) - 5s - 7 - 3\left(sY(s) - 5\right) + 2Y(s) = \frac{2}{s-3}$$

(c) Rearrange to obtain $Y(s)$

$$\left(s^2-3s+2\right)Y(s)-5s-7+15=\frac{2}{s-3}$$
$$(s-1)(s-2)Y(s)=\frac{2}{s-3}+5s+7-15=\frac{2}{s-3}+5s-8$$
$$\mathrm{Y(s)}=\frac{2}{(s-3)(s-1)(s-2)}+\frac{5s-8}{(s-1)(s-2)}$$

(d) Now for partial fractions

$$\frac{2}{(s-3)(s-1)(s-2)}+\frac{5s-8}{(s-1)(s-2)}$$
$$=\frac{1}{(s-3)}+\frac{1}{(s-1)}+\frac{-2}{(s-2)}+\frac{3}{(s-1)}+\frac{2}{(s-2)}=\frac{4}{(s-3)}+\frac{1}{(s-1)}$$

Therefore, taking inverse transforms

$$y(t)=\mathfrak{L}^{-1}\left\{\frac{2}{(s-3)(s-1)(s-2)}+\frac{5s-8}{(s-1)(s-2)}\right\}=\mathfrak{L}^{-1}\left\{\frac{4}{(s-3)}+\frac{1}{(s-1)}\right\}$$

$$\therefore \quad y(t)=4e^{3t}+e^{t}$$

<u>Exercises 3.9.2</u>:

1. Solve the equation $\frac{d^2x}{dt^2}-4x=24\cos 2t$ given that at $x(0)=3$ and $x'(0)=4$.
2. Solve the boundary value problem equation

$$y'-3y=\delta(t-4)\cosh t,\quad y(5)=0.$$

REFERENCES:

1. Nagle, Saff and Snider, ***Fundamentals Of Differential Equations 5th Edition***, Addison Wesley Longman; 2000.

2. Alan Jeffrey, ***Advanced Engineering Mathematics***, Academic Press; 2002.

3. Abd. Wahid Md. Raji & Mohd Nor Mohamad, ***Persamaan Pembeza Biasa***; Jabatan Matematik; UTM;Skudai 2002.

4. Glynn James et al., ***Advanced Modern Engineering Mathematics***, Addison Wesley; 1993.

5. Boyce and di Prima, ***Elementary Differential Equations and Boundary Value Problems***, 4th edition, John Wiley and Sons; 1986.

6. K. A. Stroud, ***Advanced Engineering Mathematics***; MacMillan Ltd.; London; 1996.

7. Normah Maan, Halijah Osman et al., ***Differential Equations Module***, Jabatan Matematik; UTM; Skudai 2007.