DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITI TEKNOLOGI MALAYSIA

SSCE 1793 DIFFERENTIAL EQUATIONS **TUTORIAL 1**

1. Classify each of the following equations as an ordinary differential equation (ODE) or a partial differential equation(PDE), give the order, and indicate the independent and dependent variables. If the equation is an ODE, indicate whether the equation is linear or nonlinear.

(a) $3\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 9x = 2\cos 3t$ (mechanical vibration, electrical circuit, seismology)

(b) $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$ (competition between 2 species)

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation, potential theory,electricity)

(d) $\dfrac{dp}{dt} = kp(P-p)$ where P and k are constants (logistic curve, epidemiology,economics)

(e) $\dfrac{dx}{dt} = (4-x)(1-x)$ (chemical reaction rates)

(f) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$ (aerodynamics, stress analysis)

(g) $8\dfrac{d^4y}{dx^4} = x(1-x)$ (deflection of beams)

2. Determine whether the given equation is separable, linear, neither or both.

a. $\dfrac{dy}{dx} = \sin x + y.$

b. $\dfrac{dy}{dx} = \dfrac{ye^{x+y}}{x^2+2}.$

c. $x\dfrac{dx}{dt} + t^2x = \sin t.$

d. $3t = e^t\dfrac{dy}{dt} + y\ln t.$

e. $(t^2+1)\dfrac{dy}{dt} = yt - y.$

3. Solve the following separable ODEs.

a. $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x^2}$

b. $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$

c. $\dfrac{dx}{dt} + x^2 = x$

d. $\dfrac{dy}{dx} = 3x^2(1+y^2)$

e. $y^{-1}dy + ye^{\cos x}\sin x dx = 0$

f. $(x+xy^2)dx + e^{x^2}ydy = 0$

4. Solve the following Initial Value Problems.

a. $y' = x^3(1-y), \quad y(0) = -3$

b. $\dfrac{dy}{dx} = (1+y^2)\tan x, \quad y(0) = \sqrt{3}$

c. $\dfrac{dy}{dx} = 2\sqrt{y+1}(\cos x), \quad y(\pi) = 0$

d. $\dfrac{dy}{dx} = 2x\cos^2 y, \quad y(0) = \dfrac{\pi}{4}$

5. Obtain the general solution to the following ODE.

a. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

b. $(t+y+1)dt - dy = 0$

c. $(x^2+1)\dfrac{dy}{dx} + xy = x$

d. $(x^2+1)\dfrac{dy}{dx} = x^2 + 2x - 1 - 4xy$

6. Solve the following Initial Value Problem.

a. $\dfrac{dy}{dx} + 4y = e^{-x}, \quad y(0) = \dfrac{4}{3}$

b. $t^3\dfrac{dx}{dt} + 3t^2 x = t, \quad x(2) = 0$

c. $\dfrac{dy}{dx} + \dfrac{3y}{x} + 2 = 3x, \quad y(1) = 1$

d. $\sin x\dfrac{dy}{dx} + y\cos x = x\sin x, \quad y(\dfrac{\pi}{2}) = 2$

7. Classify the equation as separable, linear, exact or none of these. Notice that some equations may have more than one classifications.

a. $(x^2y + x^4\cos x) - x^3dy = 0.$ **b.** $(x^{\frac{10}{3}} - 2y)dx + xdy = 0.$

c. $\sqrt{-2y - y^2}dx + (3 + 2x - x^2)dy = 0.$ **d.** $y^2dx + (2xy + \cos y)dy = 0.$

e. $\theta dr + (3r - \theta - 1)d\theta = 0$

8. Classify the equation as separable, linear, exact or none of these. Notice that some equations may have more than one classifications.

(a) $(2xy + 3)dx - (x^2 - 1)dy = 0.$

(b) $(\cos x\cos y + 2x)dx - (\sin x\sin y + 2y)dy = 0.$

(c) $\frac{t}{y}dy + (1 + \ln y)dt = 0.$

(d) $e^t(y - t)dt + (1 + e^t)dy = 0.$

(e) $(2x + \frac{y}{1 + x^2y^2})dx + (\frac{x}{1 + x^2y^2} - 2y)dy = 0$

9. Solve the initial value problem:

(a) $(ye^{xy} - \frac{1}{y})dx + (xe^{xy} + \frac{x}{y^2})dy = 0, \;\; y(1) = 1.$

(b) $(y^2\sin x)dx + (\frac{1}{x} - \frac{y}{x})dy = 0, \quad y(\pi) = 1.$

10. For each of the following equations, find the most general function $M(x, y)$ or $N(x, y)$ respectively so that the equation is exact.

(a) $M(x, y)dx + (\sec^2 y - \frac{x}{y})dy = 0.$

(b) $(y\cos(xy) + e^x)dx + N(x, y)dy = 0.$

11. Consider the equation

$$(y^2 + 2xy)dx - x^2dy = 0$$

(a) Show that this equation is not exact.

(b) Show that multiplying both sides of the equation by y^{-2} yields anew equation that is exact.

(c) Use the solution of the resulting exact equation to solve the original equation.

(d) Were any solutions lost in the process?

12. Use the method discussed under "Homogeneous Equations" to solve:

(a) $(3x^2 - y^2)dx + (xy - x^3y^{-1})dy = 0.$

(b) $(x^2 + y^2)dx + 2xydy = 0..$

(c) $\frac{dy}{d\theta} = \frac{\theta\sec(\frac{y}{\theta}) + y}{\theta}.$

(d) $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}.$

13. Use the substitution $z = ax + by$ (a, b are suitable constants) to solve:

a. $\frac{dy}{dx} = \sqrt{x + y} - 1.$ **b.** $\frac{dy}{dx} = (x - y + 5)^2.$

14. Use the method discussed under "Bernoulli Equations" to solve:

a. $\frac{dy}{dx} - y = e^{2x}y^3.$ **b.** $\frac{dy}{dx} = \frac{2y}{x} - x^2y^2.$

c. $\frac{dy}{dx} + \frac{y}{x - 2} = 5(x - 2)y^{\frac{1}{2}}.$ **d.** $\frac{dx}{dt} + tx^3 + \frac{x}{t} = 0.$

15. **Newton's Law of Cooling.** According to Newton's Law of Cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M-T$. This gives the differential equation

$$\frac{dT}{dt} = k(M-T).$$

(a) Solve the equation for T.

(b) A thermometer reading 100^o is placed in a medium having the constant temperature of 70^o. After 6 minutes, the thermometer reads 80^o. What is the reading after 20 minutes?

(c) It was noon on a cold December day in Cameron Highland; 16^oC. Detective Ismail arrived at the crime scene to find Sergeant Normah leaning over a body. Sergeant Normah said that there were several suspects. If only they knew the exact time of death, then they could narrow down the list. Detective Ismail took out a thermometer and measured the temperature of the body; 34.5^oC. He then left for lunch. Upon returning at 1:00 pm, he found the body temperature to be 33.7^oC. When did the murder occur? Hint: Normal body temperature is 37^oC.

(d) Just before midday, the body of an apparent homicide victim was found in a room that was kept at a constant temperature of 70^oF. At 12 noon, the temperature of the body was 80^oF and at 1 pm it was 75^oF. Assume that the temperature of the body at the time of death was 98.6^oF and that it had cooled according to Newton's law of cooling. What was the time of death?

16. **Free Fall.** An object falls through the air towards earth. Assuming that only air resistance and gravity are acting on the object, then the velocity v satisfies the equation

$$m\frac{dv}{dt} = mg - bv$$

where m is the mass, g is the acceleration due to gravity, and $b > 0$ is a constant. If $m = 100$ kg, $g = 9.8$ m/sec^2, $b = 5$ kg/sec, and $v(0) = 10$m/sec, solve for $v(t)$. What is the limiting (i.e., terminal) velocity of the object?.

17. **Vertical Motion.** A particle moves vertically under the force of gravity against air resistance kv^2, where k is a constant. The velocity v at any time t is given by the differential equation

$$\frac{dv}{dt} = g - kv^2.$$

If the particle starts off from rest show that

$$v = \frac{\lambda(e^{2\lambda kt}-1)}{(e^{2\lambda kt}+1)}$$

where $\lambda = \sqrt{\frac{g}{k}}$. Then find the velocity as the time approaches infinity.

18. **Electric Circuit.** The simplest electric circuit shown in Figure 1 contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohm Ω and an inductor with an inductance of L henries (H). Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L\frac{dI}{dt}$. One of Kirchhoff's law states that the sum of voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$L\frac{dI}{dt} + RI = E(t)$$

which is a first order linear differential equation. The solution gives the current I at time t.

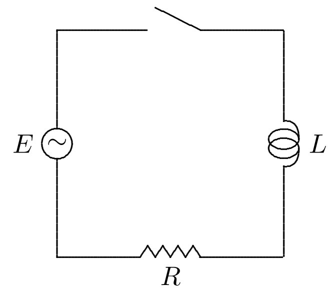

Figure 1

(a) The simple circuit is shown in Figure 1. Given that the resistance is 12Ω and the inductance is 4H. The battery gives a constant voltage of 60V and the switch is turned off when $t = 0$, so the current starts with $I(0) = 0$. Find

i. $I(t)$

ii. the current after 1 sec

iii. the limiting value of the current.

(b) Suppose that the resistance and inductance remain as in part **(a)** but, instead of a battery, we use a generator that produces a variable voltage of $E(t) = 60 \sin 30t$ volts. Find $I(t)$.

19. **Drug Concentration.** The rate at which a drug is absorbed into the blood system is given by

$$\frac{dx}{dt} = \alpha - \beta x$$

where $x(t)$ is the concentration of the drug in the blood stream at time t. Find $x(t)$. What does $x(t)$ approach in the long run (that is, as $t \to \infty$)? At what time is $x(t)$ equal to half this limiting value? Assume that $x(0) = 0$.

20. **Bernoulli Equations.** The equation

$$\frac{dy}{dx} + 2y = xy^{-2} \tag{1}$$

is an example of a Bernoulli equation.

(a) Show that the substitution $v = y^3$ reduces Equation (1) to

$$\frac{dv}{dx} + 6v = 3x. \tag{2}$$

(b) Solve Equation (2) for v. Then make the substitution $v = y^3$ to obtain the solution to Equation (1).

SOLUTIONS TO TUTORIAL 1

1. **(a)** ODE, 2^{nd} order, ind.var.t, dep.var.x, linear.
(b) ODE, 1^{st} order, ind.var.x, dep.var.y, nonlinear.
(c) PDE, 2^{nd} order, dept.var.u, indep.x,y, linear.
(d) ODE, 1^{st} order, ind.var.t, dep.var.p, nonlinear.
(e) ODE, 1^{st} order, dept.var.x, indep.t, nonlinear.
(f) ODE, 2^{nd} order, dept.y, ind.x, linear.
(g) ODE, 4^{th} order, dept.y, ind.x, nonlinear.

2. **a.**linear **b.**separable **c.**not linear, not separable
d.linear **e.**separable and linear

3. **(a)** $2y + \sin 2y = 4\arctan x + C$ **(b)** $-\frac{3}{8}\ln\left(1-4v^2\right) = \ln x + C$
(c) $x = \dfrac{Ce^t}{Ce^t+1}$, **(d)** $\tan^{-1} y = x^3 + C$
(e) $y = \dfrac{1}{C - e^{\cos x}}$ **(f)** $ln\left(1+y^2\right) = e^{-x^2} + C$

4. **(a)** $y = 1 - 4e^{\frac{-x^4}{4}}$ **(b)** $\tan^{-1} = -\ln\cos x + \tan^{-1}\left(\sqrt{3}\right)$
(c) $y = \sin^2 x + 2\sin x$ **(d)** $y = \arctan\left(1+x^2\right)$

5. **(a)** $r = \sin\theta + C\cos\theta$ **(b)** $y = -t - 1 + Ce^t$
(c) $y = 1 + C\left(x^2+1\right)^{\frac{-1}{2}}$ **(d)** $\left(x^2+1\right)^2 y = \dfrac{x^5}{5} + \dfrac{x^4}{2} + x^2 - x + \mathrm{C}$

6. **(a)** $y = \frac{1}{3}e^{-x} + e^{-4x}$ **(b)** $x = \dfrac{t^{-1}}{2} - 2t^{-3}$
(c) $y = \frac{3}{5}x^2 - \frac{x}{2} + \dfrac{9}{10x^3}$ **(d)** $y = 2x\cot x + \cos ecx$

7. **(a)** linear with y as dependent variable
(b) linear with y as dependent variable
(c) separable
(d) exact, linear with x as dep. var
(e) linear, r as dep. var

8. **(a)** $y = \frac{(C-3x)}{(x^2-1)}$
(b) $sinx\cos y + x^2 - y^2 = C$
(c) $t\ln y + t = C$
(d) $y + e^t\left(y+1-t\right) = \mathrm{C}$
(e) $x^2 - y^2 + \arctan\left(xy\right) = C$

9. **(a)** $e^{xy} - \frac{x}{y} = e - 1$
(b) $\sin x - x\cos x = \ln y + \frac{1}{y} + \pi - 1$. (equation is separable, not exact.)

10. **(a)** $-\ln|y| + f(x)$
(b) $x\cos xy + f\left(y\right)$
(c) $y = \theta\sin^{-1}\left(\ln\theta + \mathrm{C}\right)$
(d) $y = xe^{Ax}$

11. **a.** $y = \dfrac{x^2}{C-x}$. **d.** yes, y=0.

12. **(a)** $\ln\left(\frac{y^2}{x^6}\right) - \frac{y^2}{x^2} = C$

(b) $x^3 + 3xy^2 = \mathrm{C}$

(c) $y = xe^{Ax}$

13. **(a)** $y = \dfrac{(x+C)^2}{4} - x$ and $y = -x$.

(b) $y = x + \dfrac{(6+4Ce^{2x})}{(1+Ce^{2x})}$ and $y = x + 4$.

14. **(a)** $\frac{1}{y^2} = \mathrm{C}e^{-2x} - \frac{1}{2}e^{2x}$

(b) $y = \dfrac{5x^2}{x^5 + C}$ and $y = 0$

(c) $y^{\frac{1}{2}} = (x-2)^2 + \frac{C}{\sqrt{x-2}}$

(d) $x^{-2} = 2t \ln|t| + Ct^2$ and $x = 0$

15. **(a)** $T = M + \mathrm{A}e^{-kt}$

(b) $70.77^0 C$

(c) 9.08 am

(d) 10.30 am

16. $v(t) = 196 - 186e^{-0.05t}$

17.
$$y(t) = -\left[\frac{F_0}{m-k\gamma^2}\right]\cos\left(\sqrt{\frac{k}{mt}}\right) + \left[\frac{F_0}{k-m\gamma^2}\right]\cos\gamma t$$
$$= \left[\frac{F_0}{m(w^2-\gamma^2)}\right](\cos\gamma t - \cos \mathrm{cot})$$

18.

(a) **i.**$I(t) = 5 - 5e^{-3t}$ **ii.**$\sim 4.751\mathrm{A}$ **iii.**5

(b) $I(t) = \frac{E_0}{R^2+w^2L^2}\left(R\mathrm{Sin}wt - w\mathrm{LCos}wt\right) + \frac{E_0}{R^2+w^2L^2}e^{-\frac{Rt}{L}}$

19. $x(t) = (\alpha - Ae^{-\beta t})/\beta;\ x(t) \to \alpha/\beta;\ t = \ln 2/\beta$

20. **(b)** $y^3 = \frac{x}{2} - \frac{1}{12} + \mathrm{A}e^{-6x}$