Assignment
Statistical Inference for ODEs
50% of Final Grade
Due date 4^{th} May 2018 at 23.59

1 ODE with constant coefficients (10%)

Much of the analysis and pricing activity that takes place in the bond markets revolves around the yield curve. The yield curve describes the relationship between a particular redemption yield and a bond's maturity. The main measure of return associated with holding bonds, is the yield to maturity or redemption yield .

```
intall.package(YieldCurve)
library(YieldCurve)
```

The Nelson-Siegel model, is a parametric model which specifies the yield curve as:

$$f_t(\tau) = \beta_{0,t} + \beta_{1,t}\frac{1-\exp(-\lambda\tau)}{\lambda\tau} + \beta_{2,t}\left(\frac{1-\exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau)\right)$$

```
data(ECBYieldCurve)
rate.ECB = first(ECBYieldCurve,'2 day')
maturity.ECB = c(0.25,0.5,seq(1,30,by=1))
NSParameters <- Nelson.Siegel(rate.ECB, maturity.ECB)
NS.rate <- NSrates(NSParameters,maturity.ECB)
plot(maturity.ECB, last(rate.ECB,'1 day'),main="Fitting Nelso Siegel yield curve",
xlab=c("Pillars in years"), ylab=c("Rates"),type="o")
lines(maturity.ECB, last(NS.rate,'1 day'), col=2)
legend("topleft",legend=c("observed yield curve","fitted yield curve"),
col=c(1,2),lty=1)
grid()
```

The Svensson's model is a parametric model which specifies the yield curve as:

$$f_t(\tau) = \beta_{0,t} + \beta_{1,t}\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} + \beta_{2,t}\left(\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} - \exp(-\frac{\tau}{\lambda_1})\right) + \beta_3\left(\frac{1-\exp(-\frac{\tau}{\lambda_2})}{\frac{\tau}{\lambda_2}} - \exp(-\frac{\tau}{\lambda_2})\right)$$

```
data(ECBYieldCurve)
rate.ECB = first(ECBYieldCurve,'2 day')
maturity.ECB = c(0.25,0.5,seq(1,30,by=1))
SvenssonParameters <- Svensson(rate.ECB, maturity.ECB)
Svensson.rate <- Srates( SvenssonParameters ,maturity.ECB,"Spot")
plot(maturity.ECB, last(rate.ECB,'1 day'),main="Fitting Svensson yield curve",
xlab=c("Pillars in years"), ylab=c("Rates"),type="o")
lines(maturity.ECB, last(Svensson.rate,'1 day'), col=2)
legend("topleft",legend=c("observed yield curve","fitted yield curve"),
col=c(1,2),lty=1)
grid()
```

The Nelson-Siegel model is the solution of the differential equation

$$\frac{\mathrm{d}f^3}{\mathrm{d}\tau^3} + \alpha_0 \frac{\mathrm{d}f^2}{\mathrm{d}\tau^2} + \alpha_1 \frac{\mathrm{d}f}{\mathrm{d}\tau} = 0$$

1. Use Data2LD and the data in $rate.ECB$ to:
   (a) estimate the parameters α_0 and α_1 and their 95% confidence intervals.
   (b) estimate $\hat{f}$ and its 95% confidence intervals.
   (c) Comment on the accuracy of the fit to the data (SSE, ISE).
   (d) Compare and contrast the estimate yield curve by Data2LD and the Nelson Siegel and Svensson's model solution given in the above R code.

2 Two ODE models one with estimated forcing function and one time-varying parameter (20%)

Satellite altimeter data collected since 1993 have measured a rise in global mean sea level (GMSL) of $\sim 3\pm0.4$ mm/y, resulting in more than 7 cm of total sea-level rise over the last 25 y. This rate of sea-level rise is expected to accelerate as the melting of the ice sheets and ocean heat content increases as greenhouse gas concentrations rise. Using the altimeter record coupled with careful consideration of interannual and decadal variability as well as potential instrument error. The GMSL cen be modelled using the following model:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \beta_0 f + \alpha_1 u(t) = 0$$

1. Use Data2LD to:
   (a) estimate the parameters β_0 and $\alpha_1 u(t)$ and their 95% confidence intervals.
   (b) estimate $\hat{f}$ and its 95% confidence intervals.
   (c) Comment on the accuracy of the fit to the data (SSE, ISE).
   (d) Interpret the results what does the model say about the GMSL.
   (e) To represent the seasonality in the time-series. Use non-linear regression fit to fit a cylical parametric model $P(t)$ which may be $\sin(wt), \cos(wt), \sin(wt) + \cos(wt)$ to $\alpha_1 u(t)$.
2. Consider a new model:

   $$\frac{\mathrm{d}f}{\mathrm{d}t} = \beta(t) f + P(t) = 0$$

   Use Data2LD to:
   (a) estimate the parameters $\beta(t)$ its 95% confidence intervals.
   (b) estimate $\hat{f}$ and its 95% confidence intervals.
   (c) Comment on the accuracy of the fit to the data (SSE, ISE).
   (d) Interpret the results what does the model say about the GMSL.

In the excel file GMSL.csv you are given the GMSL data.

3 Dynamical Systems (20%)

In the excel file Measles.csv you are given the weekly case reports on measles and chickenpox from various regions: London, Birmingham, Manchester, Liverpool, Sheffield, Bristol, Newcastle. The Measles incidences can be modelled as a dynamic system

$$\begin{aligned}
\frac{\mathrm{d}London(t)}{\mathrm{d}t} &= \beta_{1,1}London(t) + \beta_{1,2}Liverpool(t) + \beta_{1,3}Birmingham(t) + \beta_{1,4}Manchester(t) + \\
&\quad \beta_{1,5}Sheffield(t) + \beta_{1,6}Bristol(t) + \beta_{1,7}Newcastle \\
\frac{\mathrm{d}Liverpool(t)}{\mathrm{d}t} &= \beta_{2,1}London(t) + \beta_{2,2}Liverpool(t) + \beta_{2,3}Birmingham(t) + \beta_{2,4}Manchester(t) + \\
&\quad \beta_{2,5}Sheffield(t) + \beta_{2,6}Bristol(t) + \beta_{2,7}Newcastle \\
\frac{\mathrm{d}Birmingham(t)}{\mathrm{d}t} &= \beta_{3,1}London(t) + \beta_{3,2}Liverpool(t) + \beta_{3,3}Birmingham(t) + \beta_{3,4}Manchester(t) + \\
&\quad \beta_{3,5}Sheffield(t) + \beta_{3,6}Bristol(t) + \beta_{3,7}Newcastle \\
\frac{\mathrm{d}Manchester(t)}{\mathrm{d}t} &= \beta_{4,1}London(t) + \beta_{4,2}Liverpool(t) + \beta_{4,3}Birmingham(t) + \beta_{4,4}Manchester(t) + \\
&\quad \beta_{4,5}Sheffield(t) + \beta_{4,6}Bristol(t) + \beta_{4,7}Newcastle \\
\frac{\mathrm{d}Sheffield(t)}{\mathrm{d}t} &= \beta_{5,1}London(t) + \beta_{5,2}Liverpool(t) + \beta_{5,3}Birmingham(t) + \beta_{5,4}Manchester(t) + \\
&\quad \beta_{5,5}Sheffield(t) + \beta_{5,6}Bristol(t) + \beta_{5,7}Newcastle \\
\frac{\mathrm{d}Bristol(t)}{\mathrm{d}t} &= \beta_{6,1}London(t) + \beta_{6,2}Liverpool(t) + \beta_{6,3}Birmingham(t) + \beta_{6,4}Manchester(t) + \\
&\quad \beta_{6,5}Sheffield(t) + \beta_{6,6}Bristol(t) + \beta_{6,7}Newcastle \\
\frac{\mathrm{d}Newcastle(t)}{\mathrm{d}t} &= \beta_{7,1}London(t) + \beta_{7,2}Liverpool(t) + \beta_{7,3}Birmingham(t) + \beta_{7,4}Manchester(t) + \\
&\quad \beta_{7,5}Sheffield(t) + \beta_{7,6}Bristol(t) + \beta_{7,7}Newcastle
\end{aligned}$$

The population in London, Bristol, Liverpool, Manchester, Newcastle, Birmingham and Sheffield is 8171000, 436000, 747500, 661000, 269400, 1105000 and 494000 respectively.

1. Use Data2LD to:
    (a) Estimate the incidences of measles and chickenpox relative to the popuation from various regions: London, Birmingham, Manchester, Liverpool, Sheffield, Bristol, Newcastle and their 95% confidence intervals.
    (b) estimate the parameters $\beta_{1,1}, \ldots, \beta_{7,7}$ and their 95% confidence intervals.
    (c) Comment on the accuracy of the fit to the data (SSE, ISE).
    (d) Interpret the results what does the model say about the incidences of measles and chickenpox from various regions: London, Birmingham, Manchester, Liverpool, Sheffield, Bristol, Newcastle.