DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITI TEKNOLOGI MALAYSIA

SSCE 1793 DIFFERENTIAL EQUATIONS **TUTORIAL 4**

1. Determine whether the given function is periodic. If it is periodic, give it's period.

a. $f(x) = \sin 5x$. **b.** $f(x) = x^2$.

c. $f(x) = \cos 2\pi x$. **d.** $f(x) = e^x$.

e. $f(x) = \cosh x$. **f.** $f(x) = x + \cos x$.

g. $f(x) = 1 + \sin x$ **h.** $f(x) = \tan 2x$.

2. Determine whether the given function is even, odd or neither.

a. $f(x) = x^3$. **b.** $f(x) = |x|$.

c. $f(x) = x^2 - 2x$. **d.** $f(x) = x + |x|$.

e. $f(x) = x \sin n\pi x$. **f.** $f(x) = \sin 2x \cos 3x$.

g. $f(x) = e^{-x} \cos 3x$ **h.** $f(x) = \sqrt{x^3 + x}$.

3. Evaluate the following integral using the properties of odd and even functions.

a. $\displaystyle\int_{-1}^{1} \sin m\pi x \, \cos m\pi x \, dx$. **b.** $\displaystyle\int_{-\pi}^{\pi} x^2 \cos x \, dx$.

c. $\displaystyle\int_{-\pi}^{\pi} x \sin x \, dx$. **d.** $\displaystyle\int_{-1}^{1} x^4 \, dx$.

e. $\displaystyle\int_{-\pi}^{\pi} x \cos 2x \, dx$. **f.** $\displaystyle\int_{-\pi}^{\pi} x^6 \sin 6x \, dx$.

g. $\displaystyle\int_{-1}^{1} (x^3 \cos 3\pi x + \sin \pi x) \, dx$

4. For each of the given function

i. Sketch the graph of $y = f(x)$.
ii. Determine if the function is even, odd or neither.
iii. Then, find the Fourier series of $f(x)$.

a. $f(x) = \begin{cases} 2, & -\pi < x < 0 \\ -2, & 0 < x < \pi \end{cases}$; $f(x) = f(x + 2\pi)$.

b. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & \pi < x < 2\pi \end{cases}$; $f(x) = f(x + 2\pi)$.

c. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{x}{4}, & 0 < x < \pi \end{cases}$; $f(x) = f(x + 2\pi)$.

d. $f(x) = x^2, \quad -1 < x < 1$; $f(x) = f(x + 2)$.

5. For each of the given function

i. Determine the odd 2π periodic extension f_o and compute the Fourier sine series.
ii. Determine the even 2π periodic extension f_e and compute the Fourier cosine series.
iii. Determine the π periodic extension.

a. $f(x) = x^2, \quad 0 < x < \pi$ **b.** $f(x) = \pi - x, \quad 0 < x < \pi$

c. $f(x) = 1, \quad 0 < x < \pi$ **d.** $f(x) = \sin x, \quad 0 < x < \pi$

6. For each of the given function, find the Fourier cosine and the Fourier sine series respectively.
   a. $f(x) = 1 - x, \quad 0 < x < 1$ **b.** $f(x) = x - x^2, \quad 0 < x < 1$

   c. $f(x) = e^x, \quad 0 < x < 1$

7. Show that the function $f(x) = x^2$ has the Fourier series, on $-\pi < x < \pi$,

$$f(x) \approx \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

   Then by choosing an appropriate value of x, show that

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

8. Find the Fourier coefficient b_n in the Fourier sine series expansion of

$$\begin{aligned} f(x) &= x + \tfrac{\pi}{4}, \quad 0 < x < \pi \\ f(x) &= f(x + 2\pi). \end{aligned}$$

9. Show that the Fourier series of

$$\begin{aligned} f(x) &= \begin{cases} 0, & -2 < x < 0 \\ 2, & 0 < x < 2 \end{cases} \\ f(x) &= f(x + 4). \end{aligned}$$

   is given by $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left[\frac{(2n-1)\pi x}{2}\right].$

   Then, show that $\displaystyle \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$

10. Given $f(x) = -x^2, \quad -\pi < x < \pi \;\; ; \; f(x) = f(x + \pi).$

    (a) Sketch the graph of $f(x)$ such that $-3\pi < x < 3\pi$.
    (b) Determine if $f(x)$ is even, odd or neither.
    (c) Show that

$$\int_0^{\pi} x^2 \cos nx \, dx = \frac{2\pi(-1)^n}{n^2}, \; n = 1, 2, 3, \ldots.$$

    Then, compute the Fourier series of $f(x)$.

11. Given $f(x) = |x|, \quad -1 < x < 1$
    $f(x) = f(x + 2).$

    (a) Draw the graph of $f(x)$ such that $-3 < x < 3$.
    (b) Determine if $f(x)$ is even, odd or neither.
    (c) Compute the Fourier series of $f(x)$.
    (d) Use the Fourier series obtained in (c) to approximate the value of π.

12. Given $f(x) = x, \; 0 \le x \le \pi$.

    (a) Determine the even 2π periodic extension of $f(x)$ and draw the graph of the function obtained for $-3\pi \le x \le 3\pi$.
    (b) Find the Fourier cosine series of $f(x)$ which is defined in (a).

(c) By considering the convergence of the Fourier series in (b) at $x = 0$ and $x = \pi$, derive two series and its corresponding sum.

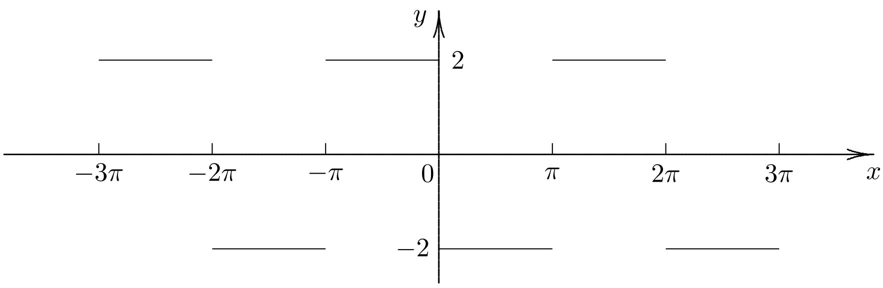
SOLUTIONS TO TUTORIAL 4

1. **a.** periodic, $T = 2\pi$. **b.** non periodic.
c. periodic, $T = 2\pi$. **d.** non periodic.
e. non periodic. **f.** non periodic.
g. periodic, $T = 2\pi$. **h.** periodic, $T = 2\pi$.

2. **a.** odd. **b.** even.
c. neither. **d.** neither.
e. even. **f.** odd.
g. neither. **h.** neither.

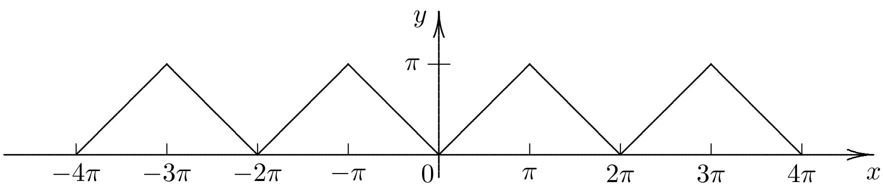
3. **a.** 0. **b.** 0.
c. 2π. **d.** $\frac{2}{5}$.
e. 0. **f.** 0.
g. 0. **h.** 0.

4. **a.** (i)



(ii) $f(x)$ odd function. (iii) $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx.$

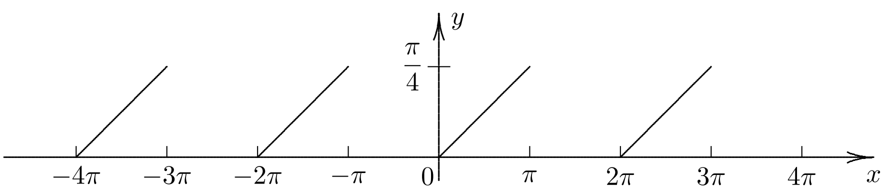
b. (i)



(ii) $f(x)$ even function.

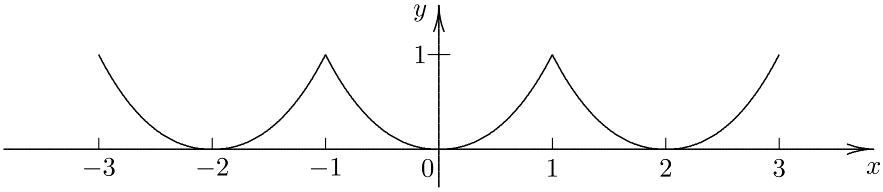
(iii) $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx.$

c. (i)



(ii) $f(x)$ neither even nor odd. (iii) $f(x) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$

d. (i)



(ii) $f(x)$ even function. (iii) $f(x) = \dfrac{1}{3} + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \sin n\pi x.$

5.

b.(i) $2\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin nx}{n}$; (ii) $\dfrac{\pi}{2} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(2n-1)x}{(2n-1)^2}$

c.(i) $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)x}{(2n-1)}.$

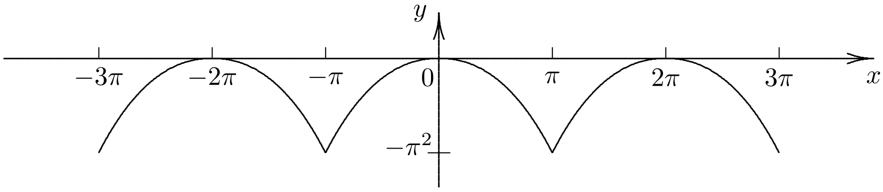
d.(ii) $\dfrac{2}{\pi} + \dfrac{4}{\pi}\left[\dfrac{2}{3}\cos 2x + \dfrac{4}{15}\cos 4x + \dfrac{6}{35}\cos 6x + \cdots\right].$

6.

a. $1 + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(2n-1)\pi x}{(2n-1)^2}$; $\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin n\pi x}{n}.$

8. $b_n = \dfrac{1-5(-1)^n}{2n}.$

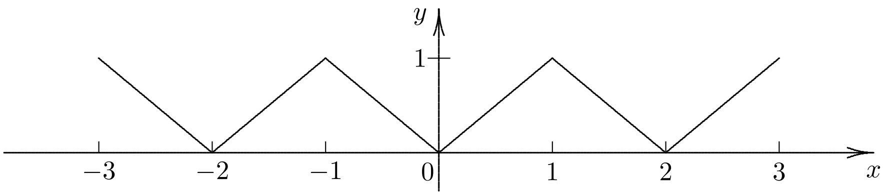
10. (a)



(b) $f(x)$ even function.

(c) $f(x) = -\dfrac{1}{3}\pi^2 + 2\pi\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} \cos nx.$

11. (a)

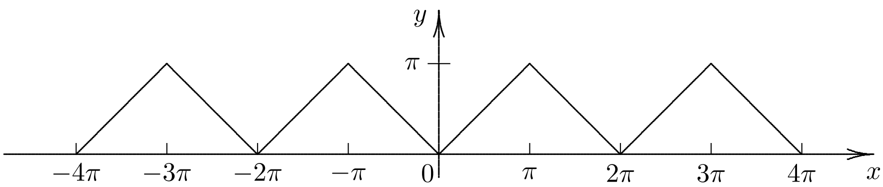


(b) $f(x)$ even function.

(c) $f(x) = -\dfrac{1}{3}\pi^2 + 2\pi\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} \cos nx.$

(d) $\pi^2 = 8\left[1 + \dfrac{1}{9} + \dfrac{1}{25} + \cdots\right].$

12. (a)



(b) $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \dfrac{((-1)^n - 1)}{n^2} \cos n\pi x.$

(c) $0 = \dfrac{1}{2} - \dfrac{4}{\pi^2} \left[1 + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} + \cdots\right].$

$1 = \dfrac{1}{2} + \dfrac{4}{\pi^2} \left[1 + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} + \cdots\right].$