Uncertainty Quantification (ACM41000) Exercises – Set 3

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1. Solve the diffusion equation
$$u_t = Du_{xx}, \qquad 0 < x < L, \qquad t > 0,$$
with homogeneous Neumann boundary conditions
$$u_x(x=0, t>0) = u_x(x=L, t>0) = 0,$$
subject to the initial condition
$$u(x, t=0) = u_0 = \text{Const.}$$

Linearity, constant coefficients, **homogeneous PDE**: attempt separation of variables:
$$u(x,t) = X(x)T(t).$$
Substitute into the PDE:
$$X(x)\frac{dT}{dt} = DT(t)\frac{d^2X}{dx^2}.$$
Divide out by $X(x)T(t)$:
$$\frac{1}{T}\frac{dT}{dt} = \frac{D}{X}\frac{d^2X}{dx^2}.$$
But now the LHS is a function of t alone and the RHS is a function of x alone. Hence,
$$\text{LHS} = \text{RHS} = \text{Const.} := -\lambda D.$$
Also, substitute the trial solution into the BCs and the ICs:

$$\begin{aligned} \text{Initial condition:} &\quad u(x,t=0) = X(x)T(0) = f(x), \qquad 0 < x < L, \\ \text{Boundary condition:} &\quad T(t)X'(0) = T(t)X'(L) = 0 \end{aligned}$$

Focusing on the $X(x)$-equations, we have:

$$\begin{aligned} \frac{X''}{X} &= -\lambda, \qquad 0 < x < L, \\ X'(0) &= X'(L) = 0. \end{aligned}$$

Equation in the **bulk** $0 < x < L$:

$$X'' + \lambda X = 0, \tag{1}$$

Different possibilities for λ:

(a) $\lambda = 0$. Then, the solution is $X(x) = Ax + B$, with $X'(x) = A$. However, the BCs specify $X'(0) = 0$, hence $A = 0$. Thus, only a constant solution remains, in which we have no interest.

(b) $\lambda < 0$. Then, the solution is $X(x) = Ae^{\mu x} + Be^{-\mu x}$, where $\mu = \sqrt{-\lambda}$. The BCs give

$$\mu A - \mu B = \mu A e^{\mu L} - \mu B e^{-\mu L} = 0.$$

In matrix terms,

$$\begin{pmatrix} \mu & -\mu, \\ \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

The determinant of the matrix is

$$-\mu^2 e^{-\mu L} + \mu^2 e^{\mu L} = 2\mu^2 \sinh(\mu L),$$

which is not identically zero, hence $(A, B)^T = 0$ and only the trivial solution exists.

(c) Thus, we are forced into the third option: $\lambda > 0$.

Solving Equation (1) with $\lambda > 0$ gives

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

and

$$X'(x) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x),$$

with boundary conditions

$$-\sqrt{\lambda}A \cdot 0 + \sqrt{\lambda}B \cdot 1 = -\sqrt{\lambda}A\sin(\sqrt{\lambda}L) + \sqrt{\lambda}B\cos(\sqrt{\lambda}L) = 0.$$

Hence, $B = 0$. Grouping the second and third equations in this string together therefore gives

$$A\sin(\sqrt{\lambda}L) = 0.$$

Of course, $A = 0$ is a solution, but this is the trivial one. Therefore, we must try to solve

$$\sin(\sqrt{\lambda}L) = 0.$$

This is possible, provided

$$\sqrt{\lambda}L = n\pi, \qquad n \in \{1, 2, \cdots\}.$$

Thus,

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2},$$

and

$$X(x) = A_n \cos\left(\frac{n\pi x}{L}\right),$$

where A_n labels the constant of integration.

Now substitute $\lambda_n = n^2\pi^2/L^2$ back into the $T(t)$-equation:

$$\frac{1}{T}\frac{dT}{dt} = -\lambda D = -\lambda_n D.$$

Solving gives

$$T(t) = T(0)e^{-\lambda_n Dt},$$

or

$$T(t) = T(0)e^{-n^2\pi^2 Dt/L^2}.$$

Now recall the ansatz:

$$u(x,t) = X(x)T(t).$$

Thus, we have a solution

$$X(x)T(t) = T(0)A_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}.$$

Calling $T(0)B_n := C_n$, this is

$$X_n(x)T_n(t) = C_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}.$$

The label n is just a label on the solution. However, each $n = 0, 1, 2, \cdots$ produces a different solution, linearly independent of all the others (NOTE THE STARTING-VALUE OF THE INDEX). We can add all of these solutions together to obtain a **general solution** of the PDE:

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} X_n(x)T_n(t), \\
&= \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}, \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}.
\end{aligned}$$

We are almost there. However, we still need to take care of the initial condition,

$$\begin{aligned}
u(x,t=0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right), \\
&= f(x).
\end{aligned}$$

Let's do C_0 first. Integrate both sides of this equation over x. The cosine terms all vanish, leaving

$$LC_0 = \int_0^L f(x)\mathrm{d}x \implies C_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x.$$

Thus, C_0 is the average value of the initial condition.

Next, note that the functions

$$\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

are orthogonal on $[0, L]$:

$$\begin{aligned} I_{n,m} &= \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x, \\ &= \frac{L}{\pi}\int_0^\pi \cos(ny)\cos(my)\mathrm{d}y, \\ &= \frac{L}{2}\delta_{mn}. \end{aligned}$$

Thus, consider the IC again:

$$\begin{aligned} u(x, t = 0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right), \\ &= f(x). \end{aligned}$$

Multiply both sides by $\cos(m\pi x/L)$ and integrate:

$$\begin{aligned} \int_0^\pi f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= \int_0^L C_0\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x + \int_0^\pi \sum_{n=1}^{\infty} C_n\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \\ &= 0 + \sum_{n=1}^{\infty} C_n \int_0^\pi \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \\ &= \sum_{n=1}^{\infty} C_n \frac{L}{2}\delta_{m,n}, \\ &= \frac{C_m L}{2}. \end{aligned}$$

Hence,

$$C_m = \frac{L}{2}\int_0^\pi f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x.$$

In conclusion,

$$\begin{aligned} u(x,t) &= \frac{1}{L}\int_0^L f(x)\mathrm{d}x + \sum_{n=1}^{\infty} C_n\cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2 Dt/L^2}, \\ C_n &= \frac{L}{2}\int_0^\pi f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \end{aligned}$$

2. Prove that the solution to the diffusion equation with Neumann boundary conditions and arbitrary initial conditions conserves the quantity

$$\int_0^L u(x,t)\mathrm{d}x.$$

To what kind of system might these boundary conditions correspond?

Differentiate w.r.t. time:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(x,t)\mathrm{d}x &= \int_\Omega \frac{\partial u}{\partial t}\mathrm{d}x, \\
&= D\int_\Omega \frac{\partial^2 u}{\partial x^2}\mathrm{d}x, \\
&= D\left[\frac{\partial u}{\partial x}\right]_0^L, \qquad \Omega = [0, L], \\
&= 0.
\end{aligned}$$

Alternative proof: Take the solution in Question 1,

$$u(x,t) = \frac{1}{L}\int_0^L f(x)\mathrm{d}x + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}.$$

Integrating over space gives

$$\int_0^L u(x,t) = \int_0^L f(x)\mathrm{d}x + 0 = \text{Const.};$$

hence, the total mass is conserved.

Note: This would not work if the BCs in Question 1 were Dirichlet, since then we would have

$$u_{D-BC}(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2},$$

whose space integral is

$$\begin{aligned}
\int_0^L u_{D-BC}(x,t)\mathrm{d}x &= \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 Dt/L^2} \int_0^L \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \\
&= \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 Dt/L^2} \frac{L}{n\pi}\left[\cos(n\pi) - 1\right], \\
&= \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 Dt/L^2} \frac{L}{n\pi}\left[(-1)^n - 1\right], \\
&= -\frac{2L}{\pi}\sum_{n=0}^{\infty} C_{2n+1} e^{-(2n+1)^2\pi^2 Dt/L^2} \neq 0.
\end{aligned}$$

The physical systems modelled by these boundary conditions include

- The concentration $u(x,t)$ of particles undergoing Brownian motion. The conservation law $\partial_t \int_0^L u(x,t)\mathrm{d}x = 0$ means that particles are neither created nor destroyed;
- The temperature $u(x,t)$ of a medal rod with insulating boundary conditions. The Neumann boundary conditions means that energy does not leave or enter the system through the physical boundaries.

3. Find the Fourier series of

$$F(x) = x^4,$$

on the interval $[-\pi,\pi]$. By setting $x=\pi$, evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Note the hints at the end of the exam.

We pose a Fourier series in a standard fashion, letting $F(x) = x^4$,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]. \tag{2}$$

Integrate both sides:

$$\begin{aligned} \int_{-\pi}^{\pi} x^4 \mathrm{d}x &= \frac{a_0}{2} 2\pi + 0, \\ \tfrac{2}{5}\pi^4 &= a_0. \end{aligned}$$

Multiply both sides by $\sin(px)$ and integrate:

$$\int_{-\pi}^{\pi} \sin(px) x^4 \mathrm{d}x = b_p \int_{-\pi}^{\pi} \sin^2(px)\mathrm{d}x.$$

By the oddness of $\sin(px)x^4$, we obtain $b_p = 0$. Lastly, multiply both sides by $\cos(px)$ and integrate:

$$\int_{-\pi}^{\pi} \cos(px) x^4 \mathrm{d}x = b_p \int_{-\pi}^{\pi} \cos^2(px)\mathrm{d}x = \pi b_p.$$

Hence,

$$\begin{aligned} \pi b_p &= \left[\frac{4x(p^2x^2-6)\cos(px)}{p^4} + \frac{(p^4x^4 - 12p^2x^2 + 24)\sin(px)}{p^5}\right]_{-\pi}^{\pi}, \\ &= 2\left[\frac{4x(p^2x^2-6)\cos(px)}{p^4}\right]_0^{\pi}, \\ &= \frac{8\pi(p^2\pi^2-6)\cos(p\pi)}{p^4}, \\ b_p &= \frac{8\pi^2}{p^2}\cos(p\pi) - \frac{48}{p^4}\cos(p\pi). \end{aligned}$$

Putting it all together,

$$x^4 = \frac{\pi^4}{5} + 8\pi^2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} \cos(px) - 48 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^4} \cos(px).$$

This is the first required answer.

Now set $x = \pi$ in the sum:

$$\begin{aligned}
\pi^4 &= \frac{\pi^4}{5} + 8\pi^2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2}(-1)^p - 48 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^4}(-1)^p, \\
48 \sum_{p=1}^{\infty} \frac{1}{p^4} &= \frac{\pi^4}{5} + 8\pi^2 \sum_{p=1}^{\infty} \frac{1}{p^2} - \pi^4, \\
&= \frac{\pi^4}{5} + 8\pi^2 \frac{\pi^2}{6} - \pi^4, \\
&= \pi^4 \frac{6 + 40 - 30}{30} = \tfrac{16}{30}\pi^4, \\
\sum_{p=1}^{\infty} \frac{1}{p^4} &= \tfrac{16}{30.\cdot 48}\pi^4 = \tfrac{1}{90}\pi^4.
\end{aligned}$$