5.1 Basic Concept of PDE

5.1.1 Definition and Terminologies

Let $u = u(x, y)$ where x and y are the independent variables. A PDE is an equation containing at least one partial derivative of u.

5.1.2 Notations

Followings are some partial derivatives of u:

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_y = \frac{\partial u}{\partial y},$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

Examples of PDE:

1-Dimensional wave equations:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad \alpha \text{ constant}$$

1-Dimensional heat equations:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

2-Dimensional Laplace equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2-Dimensional Wave equations:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right]$$

3-Dimensional heat equations:

$$\frac{\partial u}{\partial t} = \partial^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

5.1.3 Solution of a Partial differential Equations

For a given PDE (eg: $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$), a function $u(x, t)$ is called a solution.

How to show that $u(x, t)$ is a solution for a given PDE?

Step 1: find the appropriate derivatives of u according to the given PDE

Step 2: substitute the derivatives of u into the given PDE

- if it satisfies the given PDE, then $u(x, t)$ is a solution of the given PDE

<u>Example 1:</u>

Show that the given function of $u = \sin \alpha t \sin x$ is the solution for the following partial differential equations

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

<u>Solution:</u>

Given that $u(x, t) = \sin \alpha t \sin x$ (1)

STEP 1: Find $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$

From (1), $$\frac{\partial u}{\partial t} = \alpha(\cos \alpha t)(\sin x)$$

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2(\sin \alpha t)(\sin x) \qquad (2)$$

$$\frac{\partial u}{\partial x} = (\sin \alpha t\ )(\cos x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\ (\sin \alpha t\ )(\sin x) \qquad (3)$$

STEP 2: Subtitute $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$ into the PDE

From (2)

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2(\sin \alpha t)(\sin x)$$

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2(-\sin \alpha t \sin x)$$

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Therefore, $u(x,t)$ is a solution of the given PDE.

Example 2:

Show that the given function $u(x,t) = 2\sin 3\pi x \cos 3\alpha\pi t$ is a solution to this wave's equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Solution:

$$\frac{\partial u}{\partial t} = (2\sin 3\pi x)(-3\alpha\pi \sin 3\alpha\pi t) \tag{2a}$$

$$\frac{\partial^2 u}{\partial t^2} = (-18\alpha^2\pi^2)(\sin 3\pi x)(\sin 3\alpha\pi t) \tag{2b}$$

$$\frac{\partial u}{\partial x} = (3\pi)(2\sin 3\pi x)(\cos 3\alpha\pi t)$$

$$\frac{\partial^2 u}{\partial x^2} = (-18\pi^2)(\sin 3\pi x)(\cos 3\alpha\pi t) \tag{3}$$

From (2b): $\frac{\partial^2 u}{\partial t^2} = -18\alpha^2\pi^2(\sin 3\pi x)(\cos 3\alpha\pi t)$

$$= \alpha^2(-18\pi^2)(\sin 3\pi x)(\cos 3\alpha\pi t)$$

$$= \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Therefore, $u(x,t)$ is a solution to the given wave's equation.

Exercise:

Show that

- $u(x,t) = x + e^{-t}\sin \mathrm{x}$ is a solution of $u_t = u_{xx}$
- $u(x,y,t) = \cos x \cos y \cos 2t$ is a solution of

$$u_{tt} = 2(u_{xx} + u_{yy})$$

5.2 Method of separation of variables

Consider the following PDE

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Step 1: Assume that the PDE have the following solution

$$u(x,t) = X(x)T(t) \qquad (2)$$

Step 2: Find the corresponding derivatives of u

For equation (1), we have to find $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial u}{\partial t} = XT' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = X''T \qquad (3)$$

Step 3: Substitute the derivatives in equation (3) into equation (1)

Equation (1) becomes

$$XT' = k^2 X''T \qquad (4)$$

Step 4: Separate the variables $X(x)$ and $T(t)$ respectively on the LHS and RHS

Equation (4) becomes $\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T}$ (5)

Step 5: Introduce a separation constant, λ in Equation (5)

$$\frac{X''}{X} = \frac{1}{k^2}\frac{T'}{T} = \lambda$$

This leads to two ordinary differential equations

$$\frac{X''}{X} = \lambda \quad \Rightarrow \quad X'' - \lambda X = 0 \qquad (6)$$

and $\frac{1}{k^2}\frac{T'}{T} = \lambda \quad \Rightarrow \quad T' - k^2\lambda T = 0$ (7)

The above steps are the basic steps in the method of separation of variables.

In the next section, we will discuss on how to apply this method specifically to the heat equations, wave equations and Laplace's equations.

The next steps are as follows:

After we obtain equation for X and T (equations (6) and (7)), we have to

$\Downarrow$

find the boundary conditions for X

$\Downarrow$

solve the equation with three cases of λ:

$\lambda = 0$, $\lambda > 0$ and $\lambda < 0$

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- Solve for X first and apply the boundary condition for X
- if the solution for X is $X(x) = 0$,
  no need to solve for T.
  This is called a trivial solution (it's of no interest).
- If there is a solution for X,
  we have to solve for T and
  substitute X and T in the equation $u = XT$

$\Downarrow$
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Sum all solutions from case 1-3

(Principle of superposition)

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Substitute the initial condition, $u(x,0)$

- we will obtain the equation in the form of Fourier series. – apply the formula in Fourier series to find the constants.

$\Downarrow$

Substitute the obtained constants and we will get the final answer, $u(x,t) = \cdots$.

5.2.1 Heat Equations

The evolution of temperature inside a rod with length L is given by the following equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ \ t > 0$$

where α is constant and the initial condition is

$$u(x,0) = f(x), \quad 0 < x < L.$$
~~~~~~~~~~~~~~~~~~~~~~

We will discuss two types of heat equation here, which depend on different boundary conditions:

Types	Boundary conditions
Zero temperature at endpoints	$u(0,t)=0,\ \ u(L,t)=0,\ \ t>0$
Insulated endpoints	$u_x(x,0)=0,\ \ u_x(L,t)=0,\ \ t>0$

5.2.1a Zero temperature at endpoints

Example:

Use the method of separation of variables to solve the

heat equation

$$\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2},\quad 0<x<1,\ \ t>0$$

subject to the boundary conditions

$$u(0,t)=0,\quad u(1,t)=0,\quad t>0$$

and initial condition

$$u(x,0)=\frac{x}{4},\quad 0<x<1.$$

Solution:

Stage 1:

Let $u(x,t) = X(x)T(t)$

Find derivatives of u: $\dfrac{\partial u}{\partial t} = XT'$, $\dfrac{\partial^2 u}{\partial x^2} = X''T$.

Substitute in the given equation:

$$XT' = 4X''T$$

Separate the variables and introduce λ:

$$\frac{X''}{X} = \frac{T'}{4T} = \lambda$$

We obtain two equations as follows:

$$X'' - \lambda X = 0 \quad \text{and} \quad T' - 4\lambda T = 0.$$

Stage 2:

Consider the boundary conditions

$$u(0,t) = 0\text{:}\quad X(0)T(t) = 0$$

$$u(1,t) = 0\text{:}\quad X(1)T(t) = 0$$

If $T(t) = 0$, then $u(x,t) = 0$ and the solution is of no interest.

Therefore,

$$X(0) = 0 \text{ and } X(1) = 0.$$

Stage 3:

Consider three cases of λ ($\lambda = 0, \lambda > 0$ and $\lambda < 0$).

Case 1: $\lambda = 0$

Equation for X becomes $X'' = 0$.

The solution for X is $X(x) = Ax + B$.

Apply the boundary conditions for X:

$$X(0) = 0: \quad B = 0 \quad \Rightarrow \quad X(x) = Ax.$$

$$X(1) = 0: \quad A = 0 \quad \Rightarrow \quad X(x) = 0$$

Therefore, $u(x,t) = 0$ and the solution is of no interest.

Case 2: $\lambda > 0$

Let $\lambda = p^2$

The equation for X becomes: $X'' - p^2 X = 0$

and its solution is: $X(x) = Ae^{px} + Be^{-px}$

Apply the boundary conditions for X:

$$X(0) = 0: \;\; A + B = 0 \quad \Rightarrow \quad B = -A$$

$$X(x) = A(e^{px} - e^{-px})$$

$$X(1) = 0: \;\; A(e^{p} + e^{-p}) = 0 \quad \Rightarrow \quad A = 0$$

$$B = 0$$

$$X(x) = 0$$

The solution is of no interest.

Case 3: $\lambda < 0$

Let $\lambda = -p^2$

The equation for X becomes: $X'' + p^2 X = 0$

And its solution is: $X(x) = A\cos px + B\sin px$

Apply the boundary conditions for X:

$$X(0) = 0: \;\; A = 0 \quad \Rightarrow \quad X(x) = B\sin px$$

$$X(1) = 0: \;\; B\sin p = 0$$

if $B = 0$, $X(x) = 0$ and the solution is of no interest.

Let $\sin p = 0 \quad \Rightarrow \quad p = n\pi, \;\; n = 1,2,3, \ldots$

Therefore, $X_n(x) = B_n \sin n\pi x$

Now solve for T:-

Using $\lambda = -p^2$, the equation for T becomes:

$$T' + 4p^2T = 0$$

And its solution is:

$$T(t) = Ce^{-4p^2t}$$

$$T_n(t) = C_n e^{-4n^2\pi^2t}$$

The solution for $u(x,t)$ is:

$$u_n(x,t) = B_n \sin n\pi x\, C_n e^{-4n^2\pi^2t}$$

$$= b_n e^{-4n^2\pi^2t} \sin n\pi x$$

where $b_n = B_n \times C_n$.

Using the superposition principle (sum all solutions from case 1-3):

$$u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t} \sin n\pi x$$

Apply the initial conditions:-

$$u(x,0) = \frac{x}{4}: \qquad \frac{x}{4} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \tfrac{2}{1}\int_0^1 \tfrac{x}{4} \sin n\pi x \, dx$$

$$= -\frac{\cos n\pi}{2n\pi}$$

$$= -\frac{(-1)^n}{2n\pi}$$

Therefore, the final solution is:

$$u(x,t) = -\tfrac{1}{2\pi}\sum \tfrac{(-1)^n}{n} e^{-4n^2\pi^2 t} \sin n\pi x.$$

Exercise: Solve the following heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \ \ t > 0$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(3,t) = 0, \quad t > 0$$

and initial condition

$$u(x,0) = 2x, \quad 0 < x < 2.$$

5.2.1b Insulated endpoints

Example:

Use the method of separation of variables the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions

$$u_x(0,t) = 0, \;\; u_x(L,t) = 0, \quad t > 0$$

and the initial condition

$$u(x,0) = x, \quad 0 < x < L.$$

Solution:

Stage 1:

Follow the steps as shown in the previous example.

You will obtain

$X'' - \lambda X = 0$ and $T' - \alpha^2 \lambda T = 0.$

Stage 2:

Consider the boundary conditions:

$$u_x(0,t) = 0: \quad X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0$$

$$u_x(L,t) = 0: \quad X'(L)T(t) = 0 \quad \Rightarrow \quad X'(L) = 0$$

Stage 3:

Case 1: $\lambda = 0$

As shown earlier, the solution for X is

$$X(x) = Ax + B$$

Differentiate $X(x)$ to obtain $X'(x)$:

$$X'(x) = A$$

Apply the boundary conditions:

$$X'(0) = 0: \; A = 0 \quad \Rightarrow \quad X'(x) = 0$$

$X'(L) = 0$: this condition does no affect B.

Therefore, $X(x) = B$

Find the solution for T, we have

$$T' = 0$$

The solution is: $T(t) = C$

The solution for u is: $u(x,t) = B \times C = D$

Case 2: $\lambda > 0$

For this case, you will obtain $X(x) = 0$.

Therefore, the solution is of no interest.

Case 3: $\lambda < 0$

The solution for X is: $X(x) = A\cos px + B\sin px$

Differentiate $X(x)$: $X'(x) = -pA\sin px + pB\cos px$

Apply the boundary conditions for X:

$$X'(0) = 0: \ \ pB = 0 \quad \Rightarrow \quad B = 0$$

$$X'(x) = -pA\sin px$$

$$X'(L) = 0: \ -pA\sin pL = 0 \quad \Rightarrow \quad pL = n\pi$$

$$p = \frac{n\pi}{L}, \ \ n = 1,2,3, \dots$$

$$X_n(x) = A_n \cos\frac{n\pi x}{L}$$

Solve the equation for T, the solution is:

$$T(t) = Ce^{-\alpha^2 p^2 t}$$

$$T_n(t) = C_n e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

The solution for u is:

$$u(x,t) = A_n \cos\frac{n\pi x}{L} C_n e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

$$= a_n \cos\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

Stage 4:

Sum all solutions from case 1-3:

$$u(x,t) = D + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

Apply the initial condition:

$$u(x,0) = x: \quad x = D + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L x\,dx = L$$

$$D = \frac{a_0}{2} = \frac{L}{2}$$

$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2L}{\pi^2}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2L}{\pi^2}\left[\frac{(-1)^n - 1}{n^2}\right]$$

The final answer is:

$$u(x,t) = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{L}\,e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}$$

Exercise:

Use the method of separation of variables the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions

$$u_x(0,t) = 0,\;\; u_x(5,t) = 0, \quad t > 0$$

and the initial condition

$$u(x,0) = x^2, \quad 0 < x < 5.$$

5.2.2 Wave equation

5.2.2a The vibrating string with an initial velocity

The motion of an elastic string with length L is given by the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ \ t > 0$$

With boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$

and initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Example:

Use the method of separation of variables to find the solution of the wave equation (c is constant)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0$$

with the boundary conditions

$$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0$$

and initial conditions

$$u(x,0) = 2 - x, \quad 0 < x < 2$$

$$u_t(x,0) = 1, \quad 0 < x < 2.$$

Solution:

Stage 1:

Let $u(x,t) = X(x)T(t)$

Find the corresponding derivatives of u:

$$\frac{\partial^2 u}{\partial t^2} = XT'', \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substitute in the equation:

$$XT'' = c^2 X''T$$

Separate the variables and introduce λ:

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda$$

We obtain two equations as follows:

$X'' - \lambda X = 0 \quad$ and $\quad T'' - c^2 \lambda T = 0$

Stage 2:

Consider the boundary conditions

$$u(0,t) = 0: \quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$u(2,t) = 0: \quad X(2)T(t) = 0 \quad \Rightarrow \quad X(2) = 0$$

Stage 3:

Consider three cases of λ.

In this problem, the solution for case 1 and case 2 is of no interest.

Only case 3 ($\lambda < 0$) gives the solution

Let $\lambda = -p^2$

Solve for X first.

The equation for X becomes:

$$X'' + p^2 X = 0$$

Its solution is:

$$X(x) = A\cos px + B\sin px$$

Apply the boundary conditions for X:

$$X(0) = 0: \quad A = 0 \quad \Rightarrow \quad X(x) = B \sin px$$

$$X(2) = 0: \quad B \sin 2p = 0 \quad \Rightarrow \quad 2p = n\pi$$

$$p = \frac{n\pi}{2}$$

Therefore, $X_n(x) = B_n \sin \frac{n\pi x}{2}$

Now solve for T:-

Using $\lambda = -p^2$, the equation for T becomes

$$T'' + c^2 p^2 T = 0$$

and its solution is:

$$T(t) = C \cos cpt + D \sin cpt$$

since $p = \frac{n\pi}{2}$, $\quad T_n(t) = C_n \cos \frac{cn\pi t}{2} + D_n \sin \frac{cn\pi t}{2}$

and the solution for u is:

$$u_n(x,t) = B_n \sin \frac{n\pi x}{2} \left(C_n \cos \frac{cn\pi t}{2} + D_n \sin \frac{cn\pi t}{2} \right)$$

$$= \sin\frac{n\pi x}{2}\left(F_n \cos\frac{cn\pi t}{2} + G_n \sin\frac{cn\pi t}{2}\right)$$

where $F_n = B_n \times C_n$ and $G_n = B_n \times D_n$.

Stage 4:

By superposition principle,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{2}\left(F_n \cos\frac{cn\pi t}{2} + G_n \sin\frac{cn\pi t}{2}\right)$$

Apply the initial condition:-

$u(x,0) = 2 - x$:

$$2 - x = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{2}$$

$$F_n = \tfrac{2}{2}\int_0^2 (2-x)\sin\tfrac{n\pi x}{2}\,dx$$

$$= \left[(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\tfrac{4}{n\pi}.$$

Differentiate $u(x,t)$ w.r.t t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{2}\left(\frac{-F_n cn\pi}{2}\sin\frac{cn\pi t}{2} + \frac{G_n cn\pi}{2}\cos\frac{cn\pi t}{2}\right)$$

Apply the initial condition $u_t(x,0) = 1$:

$$1 = \sum_{n=1}^{\infty} \frac{G_n cn\pi}{2}\sin\frac{n\pi x}{2}$$

$$\frac{G_n cn\pi}{2} = \frac{2}{2}\int_0^2 (1)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \left[-\frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\right]$$

$$= \frac{2}{n\pi}[1-(-1)^n]$$

$$G_n = \frac{4}{cn^2\pi^2}[1-(-1)^n]$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{2}\left(\frac{4}{n\pi}\cos\frac{cn\pi t}{2} + \frac{4}{cn^2\pi^2}[1-(-1)^n]\sin\frac{cn\pi t}{2}\right)$$

Exercise:

Solve the following wave equation by the method of separation of variables

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

with the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0$$

and the initial conditions

$$u(x,0) = x(\pi - x), \quad 0 < x < \pi$$

$u_t(x,0) = 1, \quad 0 < x < \pi.$

5.2.3 Laplace's Equations

The laplace equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

There are two types of Laplace equations

Type	Boundary conditions
Type A	$u(x,0) = f(x),\ \ u(x,b) = 0, \quad 0 < x < a$ $u(0,y) = 0, \qquad u(a,y) = 0, \quad 0 < y < b$ or $u(x,0) = \ ,\ \ u(x,b) = f(x), \quad 0 < x < a$ $u(0,y) = 0, \qquad u(a,y) = 0, \quad 0 < y < b$
Type B	$u(x,0) = 0,\ \ u(x,b) = 0, \quad 0 < x < a$ $u(0,y) = 0, \qquad u(a,y) = g(y), \quad 0 < y < b$ or $u(x,0) = \ ,\ \ u(x,b) = 0, \quad 0 < x < a$ $u(0,y) = g(y), \qquad u(a,y) = 0, \quad 0 < y < b$

5.2.3a Type A

Example:

A square plate is bounded by $x = 0$, $x = a, y = 0$ and $y = a$. Apply the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

To determine the potential distribution $u(x, y)$ over the plate, subject to the following boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \qquad 0 < y < a$$

$$u(x, 0) = 0, \quad u(x, a) = u_0\left(\sin\frac{\pi x}{a} + 2\sin\frac{2\pi x}{a}\right), \quad 0 < x < a,$$

where u_0 is a constant.

Solution:

Stage 1:

Let $u(x, y) = X(x)Y(y)$

Find the corresponding derivatives of u:

$$\frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Substitute in the given equation:

$$X''Y + XY'' = 0$$

Separate the variables and introduce λ:

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

We obtain two equations as follows:

$$X'' - \lambda X = 0 \quad \text{and} \quad Y'' + \lambda Y = 0$$

Stage 2:

Consider the boundary conditions

$$u(0, y) = 0: \quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$u(a, 0) = 0: \quad X(a)T(t) = 0 \quad \Rightarrow \quad X(a) = 0$$

$$u(x, 0) = 0: \quad X(x)T(0) = 0 \quad \Rightarrow \quad T(0) = 0$$

Stage 3:

The solution for case 1 ($\lambda = 0$) and case 2 ($\lambda > 0$) are of no interest.

Consider case 3 ($\lambda < 0$):

Let $\lambda = -p^2$

Solve for X first.

The equation for X becomes:

$$X'' + p^2 X = 0$$

Its solution is:

$$X(x) = A\ \cos px + B \sin px$$

Apply the boundary conditions for X:

$$X(0) = 0: \quad A = 0 \quad \Rightarrow \quad X(x) = B \sin px$$

$$X(a) = 0: \quad B \sin pa = 0 \quad \Rightarrow \quad p = \frac{n\pi}{a}$$

Therefore, $X_n(x) = B_n \sin \frac{n\pi x}{a}$.

Now solve for Y:

Using $\lambda = -p^2$, the equation for Y becomes

$$Y'' - p^2 Y = 0$$

and its solution is:

$$Y(y) = Ce^{py} + De^{-py}$$

or we may write it as $Y(y) = F \cosh py + G \sinh py$

since $p = \frac{n\pi}{a}$, $\quad Y_n(y) = F_n \cosh\frac{n\pi y}{a} + G_n \sinh\frac{n\pi y}{a}$

Apply the boundary conditions for Y:

$$Y(0) = 0: \quad F = 0 \quad \Rightarrow \quad Y_n(y) = G_n \sinh\frac{n\pi y}{a}$$

The solution for u is:

$$u(x,t) = B_n \sin\frac{n\pi x}{a} G_n \sinh\frac{n\pi y}{a}$$

$$= b_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

where $b_n = B_n \times G_n$.

Stage 4:

By superposition principle,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

Given $u(x,a) = u_0\left(\sin\frac{\pi x}{a} + 2\sin\frac{2\pi x}{a}\right)$

Then $u_0\left(\sin\frac{\pi x}{a} + 2\sin\frac{2\pi x}{a}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \sinh n\pi$

$$u_0 \sin\frac{\pi x}{a} + 2u_0 \sin\frac{2\pi x}{a} = \quad b_1 \sinh n\pi \sin\frac{\pi x}{a}$$
$$+b_2 \sinh 2\pi \sin\frac{2\pi x}{a}$$

Comparing the LHS with the RHS, we obtain

$b_1 = u_0/\sinh n\pi$ and $b_2 = 2u_0/\sinh 2\pi$.

Substitute b_1 and b_2 in the general solution to get the final answer.

For type B case, the same procedure is applied but we have to solve equation Y first!

Exercise:

Solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \ \ 0 < y < 2$$

subject to the following conditions:

$$u(x,0) = 0, \quad u(x,2) = 0, \quad 0 < x < 1,$$

$$u(0,y) = 0, \quad u(1,y) = y + 1, \quad 0 < y < 2.$$