CHAPTER 4

FOURIER SERIES

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4.1 Periodic Function

A function $f(t)$ is **periodic** with period of T if

$$F(t + T) = f(t),\ T \neq 0$$

Example 4.1.1

Show that $f(x) = \cos 2\pi x$ is a periodic function and find its period.

Solution:

Since $f(x) = \cos 2\pi x$, then

$$f(x+T) = \cos 2\pi(x+T).$$

What is the value of T, such that $\cos 2\pi(x+T) = \cos 2\pi x$?

Note that :

$$\cos 2\pi(x+T) = \cos 2\pi x \cos 2\pi T - \sin 2\pi x \sin 2\pi T$$

$$= \cos 2\pi x \text{ if } \begin{cases} \cos 2\pi T = 1 \\ \sin 2\pi T = 0 \end{cases}$$

This is true for $T = 1, 2, 3, \ldots$

The smallest value for T is 1, known as **fundamental period**. Therefore, $f(x) = \cos 2\pi x$ is a periodic function with period of 1.

Example 4.1.2

Determine whether $f(x) = \sin 5x$ is a periodic function. If yes, find the period.

Solution:

$$
\begin{aligned}
f(x+T) &= \sin 5(x+T) \\
&= \sin 5x \cos 5T + \cos 5x \sin 5T \\
&= \sin 5x \quad \text{if} \begin{cases} \cos 5T = 1 \\ \sin 5T = 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\cos 5T = \cos 2\pi = 1 \\
&\Rightarrow\ 5T = 2\pi \\
&\quad \therefore T = \frac{2\pi}{5}
\end{aligned}
$$

Therefore, $f(x) = \sin 5x$ is a periodic function with the period of $\frac{2\pi}{5}$.

4.2 Even and Odd Function

$f(x)$ is an **<u>even</u>** function if $f(-x)=f(x)$, for all x. (I)

$f(x)$ is an **<u>odd</u>** function if $f(-x)=-f(x)$, for all x. (II)

Geometrically,

Even The graph is **symmetrical** about the **y-axis.**

Odd The graph is **symmetrical** about the **origin.**

NOTE: If **<u>none</u>** of (I) and (II) are satisfied, then $f(x)$ is called **<u>neither even nor odd.</u>**

Example 4.2.1

Determine whether the following functions are even, odd, or neither.

(i) $f(x)=x^2$

(ii) $h(x)=\sin x$

(iii) $g(x)=x^{1/3}-\sin x$

(iv) $f(t)=e^t$

Solutions:

(i) $f(-x) = (-x)^2 = x^2 = f(x)$.

$\Rightarrow$ $f(x) = x^2$ is an **even** function.

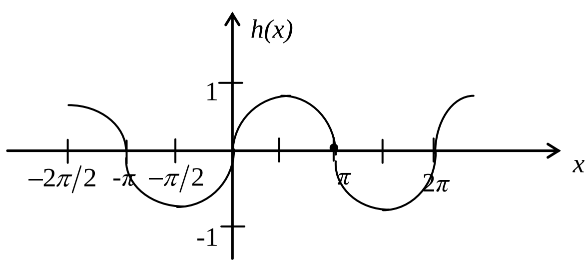
(ii) $h(-x) = \sin(-x)$

$= \sin(0 - x)$

$= -\sin(x)$ (by trigonometric identity)

$= -h(x)$

Or, refer to the graph.



Symmetrical about the origin.

$\Rightarrow$ $h(x) = \sin x$ is an **odd** function.

(iii) $g(-x) = (-x)^{1/3} - \sin(-x)$

$= -x^{1/3} + \sin x$

From (ii), $\sin(-x) = -\sin x$

$= -(x^{1/3} - \sin x)$

$= -g(x)$

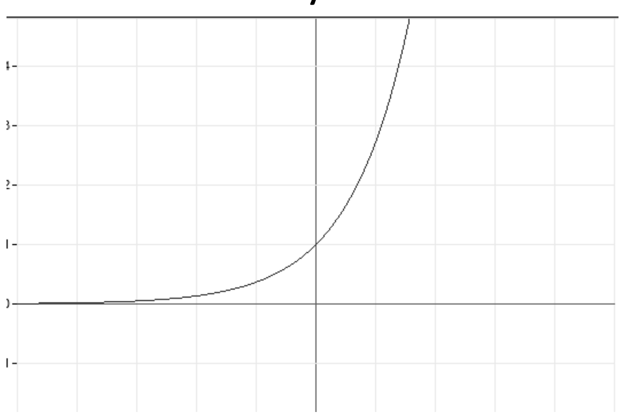
$\Rightarrow$ $g(x) = x^{1/3} - \sin x$ is an **odd** function.

(iv) $f(-t) = e^{-t}$

Argument 1: Test for even.
Note that $f(-t) = f(t)$ only when $t = 0$. (recall definition: for a function to be an even function, it must satisfies $f(-t) = f(t)$ **for all** t).

Argument 2: Test for odd.
e^{-t} is never equal to $-e^{t}$, $\therefore$ $f(-t) \neq -f(t)$.

Argument 3: From the graph, $f(t) = e^{t}$ is neither symmetrical about the y-axis nor about the origin.



Conclusion: $f(t) = e^{t}$ is neither even nor odd.

4.2.1 Properties of Even and Odd Functions

Even $\pm$ Even $\rightarrow$ Even

Odd $\pm$ Odd $\rightarrow$ Odd

Even $\pm$ Odd $\rightarrow$ Neither

Even **×/÷** Even $\rightarrow$ Even

Odd **×/÷** Odd $\rightarrow$ Even

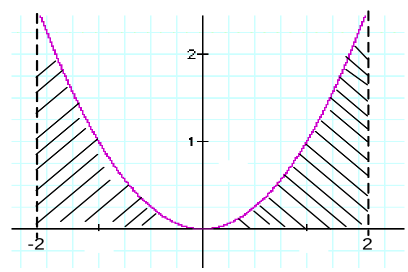
Odd **×/÷** Even $\rightarrow$ Odd

Integral properties

i) For even function: $$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

Example: $f(x) = x^2$

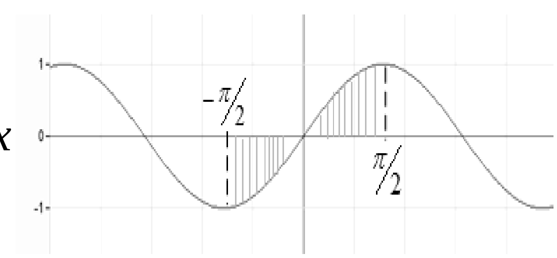
$$\int_{-2}^{2} x^2 dx = 2\int_{0}^{2} x^2 dx$$



ii) For odd function: $$\int_{-l}^{l} f(x)\ dx = 0$$

Example: $f(x) = \sin x$

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = \int_{-\pi/2}^{0} \sin x\,dx + \int_{0}^{\pi/2} \sin x\,dx$$
$$= 0.$$



NOTE: It is important to understand the integral properties of even and odd function before you can proceed to the next section.

Example:

If $f(x)$ is **<u>even</u>**, then

- $\int_{-l}^{l} \underbrace{f(x)} \cdot \underbrace{\sin nx}\, dx = 0.$

  Even X Odd ---> Odd

- $\int_{-l}^{l} \underbrace{f(x)} \cdot \underbrace{\cos nx}\, dx = 2\int_{0}^{l} f(x) \cdot \cos nx\, dx$

  Even X Even ---> Even

If $f(x)$ is **<u>odd</u>**, then

- $\int_{-l}^{l} \underbrace{f(x)} \cdot \underbrace{\sin nx}\, dx = 2\int_{0}^{l} f(x) \cdot \sin nx\, dx.$

  Odd X Odd ---> Even

- $\int_{-l}^{l} \underbrace{f(x)} \cdot \underbrace{\cos nx}\, dx = 0.$

  Odd X Even ---> Odd

4.3 Fourier Series for Periodic Function $f(x)$, consisting of Sine and Cos.

Fourier series for $f(x)$ given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right]$$

where a_0, a_n and b_n can be expressed as following:

(I)

$$\begin{cases} a_0=\dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\,dx \\ a_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\dfrac{n\pi x}{l}\,dx \\ b_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\dfrac{n\pi x}{l}\,dx \end{cases}$$

where $l=\frac{T}{2}$ and T is period for periodic function.

Even X Odd→Odd.
Recall properties.

If $f(x)$ is an **even function**, then $b_n=0$.

Hence, its Fourier series is given by :

(II)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{l}x$$

where

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx$$

Even properties

Recall properties

If $f(x)$ is an **odd function**, then $a_0 = 0,\ a_n = 0$.

Hence, its Fourier series is given by:

(III)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

and $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

Odd X Odd → Even

<u>Summary:</u>

$f(x)$	Formula
Neither	I
Even	II
Odd	III

Example 4.3.1

Find the Fourier series for

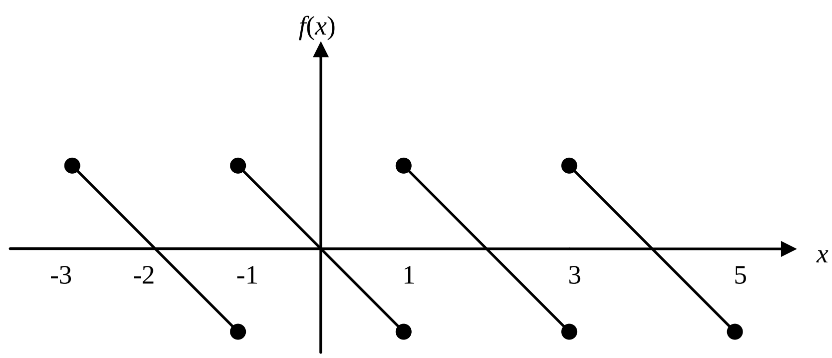
$$f(x) = -x \,, \qquad -1 < x < 1$$
$$f(x+2) = f(x).$$

Solution:

i) Determine T and l.

T = 2, l = 1.

ii) Draw the graph of $f(x)$.



iii) Check whether $f(x)$ is an odd or even function (or neither). Identify the corresponding formulae.

From the graph, $f(x)$ is an odd function. Hence, the Fourier series formulae are given by (III):

iv) Calculate the Fourier series of f(x).

$$f(x)=\sum_{n=1}^{\infty} b_n \sin n\pi x \,.$$

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin \frac{n\pi x}{l}\,dx \\
&= 2\int_0^1 -x\sin n\pi x\,dx \\
&= -2\int_0^1 x\sin n\pi x\,dx \\
&= -2\left[-x\frac{\cos n\pi x}{n\pi}+\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 \\
&= -2\left[\left(-\frac{\cos n\pi}{n\pi}+\frac{\sin n\pi}{n^2\pi^2}\right)-(0+0)\right] \\
&= 2\frac{\cos n\pi}{n\pi} \\
&= \frac{2}{n\pi}(-1)^n
\end{aligned}$$

Remarks:

$$\sin n\pi = 0$$

$$\cos n\pi = (-1)^n$$

$$\text{for } n = 0,1,2,\ldots$$

Therefore,

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x \,.$$

Note: It is **not wrong** to use the full range of [-1,1] when computing b_n. The result will be the same; it's just that you may encounter a longer calculation.

Example 4.3.2

Find the Fourier series for

$$f(x)=\begin{cases}-x & -\pi \le x < 0 \\ x & 0 \le x < \pi\end{cases}$$

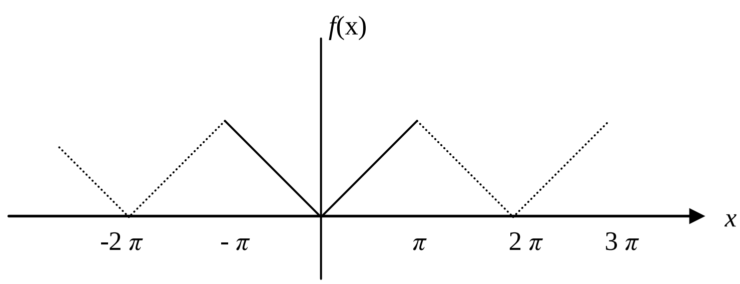
$$f(x+2\pi)=f(x)$$

Solution:

i) Determine T and l.

$T = 2\pi,\ l = \pi.$

ii) Draw the graph of $f(x)$.



iii) Check whether $f(x)$ is an odd or even function (or neither). Identify the corresponding formulae.

From the graph, $f(x)$ is an **even** function. Hence, the Fourier series formulae are given by (II):

iv) Calculate the Fourier series of f(x).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l} x$$

$$a_0 = \frac{2}{l}\int_0^l x\,dx$$

Can also use

$$\int_{-\pi}^{0} -x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \frac{1}{\pi}(\pi^2 - 0) = \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos\frac{n\pi x}{\pi}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1), \quad n = 1, 2, 3, \ldots.$$

$$= \begin{cases} \dfrac{-4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_n = \begin{cases} \dfrac{-4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Example 4.3.3

Show that the Fourier series of

$$f(x) = |\sin x|, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = f(x+\pi)$$

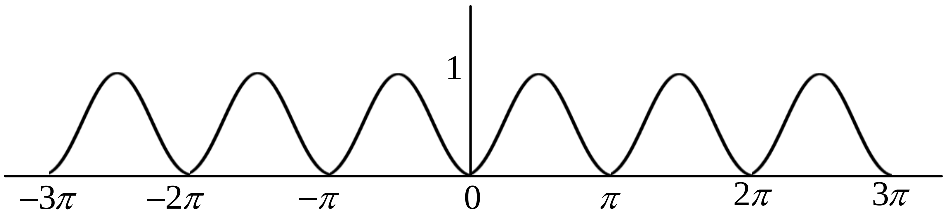
is given by $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{4n^2-1}\cos 2nx.$

Solution:

i) Determine T and l.

$$T = \pi,\ l = \frac{\pi}{2}.$$

ii) Draw the graph of $f(x)$.



iii) Check whether $f(x)$ is an odd or even function (or neither). Identify the corresponding formulae.

From the graph, $f(x)$ is an **even** function. Hence, the Fourier series formulae are given by (II):

iv) Calculate the Fourier series of f(x).

$$a_0 = \frac{2}{\pi/2}\int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \frac{4}{\pi}\left[-\cos x\right]_0^{\frac{\pi}{2}} = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin x \cos(2nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\sin(x+2nx) + \sin(x-2nx))\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos(x+2nx)}{1+2n} - \frac{\cos(x-2nx)}{1-2n}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2n} + \frac{1}{1-2n}\right] = \frac{4}{\pi(1-4n^2)}$$

Hence, $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx.$

Example 4.3.4

Find the Fourier series of

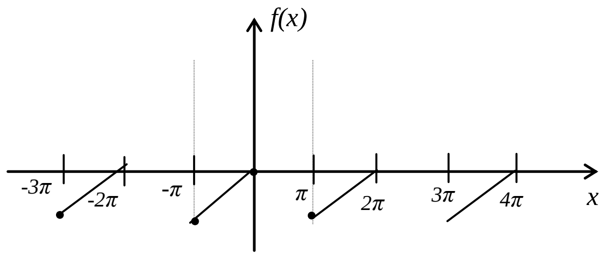
$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

$$f(x+2\pi) = f(x)$$

Solution:

i) Determine T and l.
$T = 2\pi,\ l = \pi.$

ii) Draw the graph of $f(x)$.



iii) Check whether $f(x)$ is an odd or even function (or neither). Identify the corresponding formulae.

$f(x)$ is neither even nor odd function. Hence, the Fourier series formulae are given by (I):

iv) Calculate the Fourier series of f(x).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x \right]$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\, dx + \int_{0}^{\pi} 0\, dx \right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} \right]_{-\pi}^{0}$$

$$= \frac{1}{2\pi} - \pi^2 = \frac{-\pi}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi}{l} x \right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n} sin(nx) + \frac{1}{n^2}\cos(nx) \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{ 0 + \frac{1}{n^2} - \left[\frac{(-\pi)}{n} \sin n(-\pi) + \frac{1}{n^2} \cos\ n\left(-\pi^2\right) \right] \right.$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2} - \frac{1}{n^2}\cos\ n\pi\right\}$$

$$= \frac{1}{\pi n^2}\left[1 - \cos\ n\pi\right]$$

$$= \frac{1}{\pi n^2}\left[1 - (-1)^n\right], n = 1, 2, 3, \ldots$$

Remarks: $\sin(-n\pi) = -\sin n\pi = 0,\ n = 1, 2, 3, ..$

$\cos(-n\pi) = \cos\ n\pi = (-1)^n,\ n = 1, 2, 3, ..$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx\ dx$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos\ nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{0 + \sin 0 - \left[-\frac{(-\pi)}{n}\cos(-n\pi) + \frac{1}{n^2}\sin(-n\pi)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{n}\cos\ n\pi\right\}$$

$$= -\frac{1}{n}\cos(n\pi)$$

$$= -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\ \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

$$= -\frac{\pi}{4} + \sum_{m=1}^{\infty}\left[\frac{2}{\pi(2m-1)^2}\cos(2m-1)x + \frac{(-1)^{m+1}}{m}\sin mx\right]$$

4.4 Fourier Series for Half Range Expansions.

Sometimes, if we only needs a Fourier series for a function defined on the interval [0, l], it may be preferable to use a sine or cosine Fourier series instead of a regular Fourier series.

This can be accomplished by extending the definition of the function in question to the interval [−l, 0] so that the extended function is either **even (if we wants a cosine series)** or **odd (if we wants a sine series)**.

Such Fourier series are called **half-range expansions**.

Fourier Cosine series form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos \frac{n\pi x}{l}\, dx, \quad n = 0,1,0...$$

Fourier Sine series form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin \frac{n\pi x}{l}\, dx, \qquad n = 1,2,3...$

Example 4.4.1 (Final Exam Sem II 2004/05)

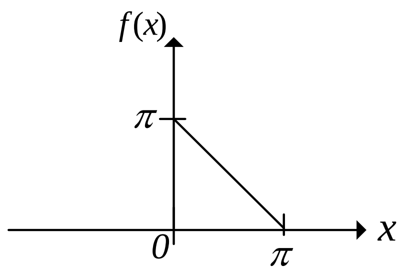
Consider the function
$f(x) = \pi - x; \quad 0 < x < \pi.$

a) Sketch the appropriate graphs in the interval
$-3\pi < x < 3\pi$ for each of the following cases:

i) The π periodic extension of $f(x)$.
ii) The odd 2π-periodic extension of $f(x)$.
iii) The even 2π-periodic extension of $f(x)$.

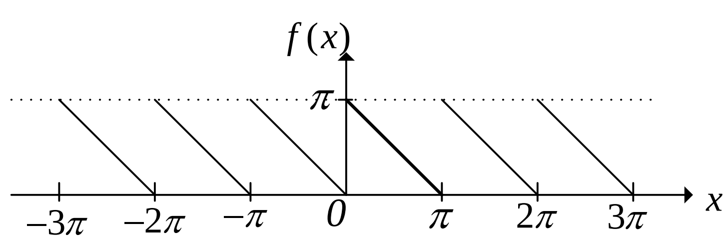
b) Find the Fourier cosine series of $f(x)$.

Solution:
a) The original graph of $f(x)$ is given by



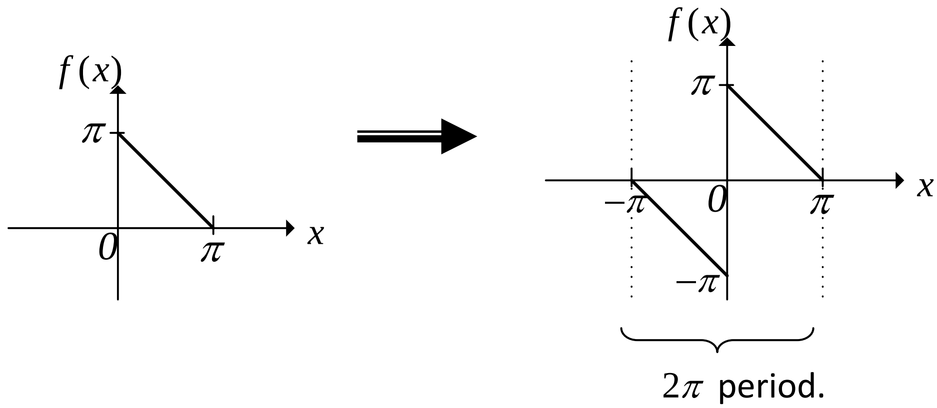
Therefore, the solution to a) is just an extension to the original graph.

i) The π periodic, range $-3\pi < x < 3\pi$.

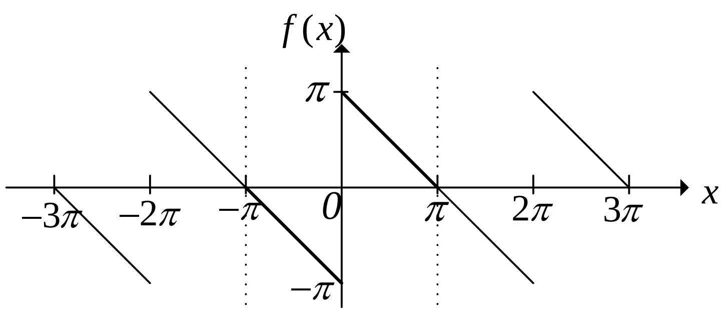


ii) The odd 2π-periodic, range $-3\pi < x < 3\pi$.

1 - Transform the original graph into an odd graph.
(Remember, the odd graph is symmetry about the origin.)

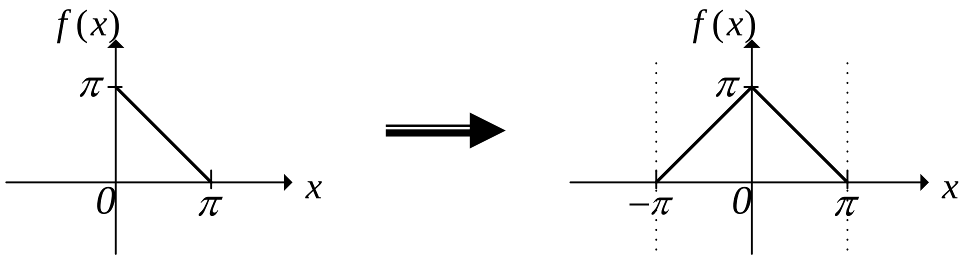


2 – Extend the odd 2π-periodic graph to the range $-3\pi < x < 3\pi$. (Repeat the shape).

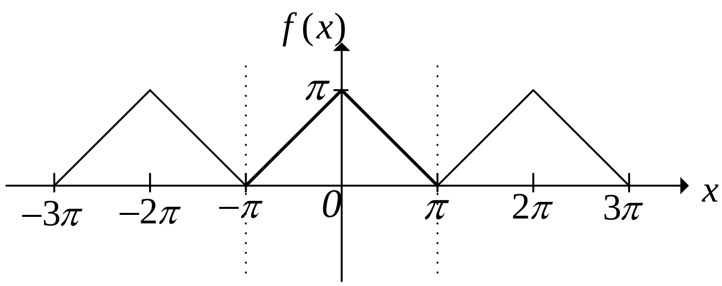


iii) The even 2π-periodic, range $-3\pi < x < 3\pi$.

1 - Transform the original graph into an even graph.
(Remember, the even graph is symmetry about y-axis.)



2 – Extend the even 2π-periodic graph to the range $-3\pi < x < 3\pi$. (Repeat the shape).



b) Fourier cosine series.

$$f\left(x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi} \quad \left\langle \text{by parts or tabular} \right.$$

$$= \frac{2}{\pi}\left[0 - \frac{\cos n\pi}{n^2} - \left(\frac{\pi \sin 0}{n} - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] = \begin{cases} \dfrac{4}{n^2\pi} & , n \text{ odd} \\ 0 \quad , & n \text{ even} \end{cases}$$

$$= \frac{4}{\pi(2m-1)^2}\ ;\ m = 1, 2, 3, ..$$

Therefore $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2}$.

Example 4.4.2

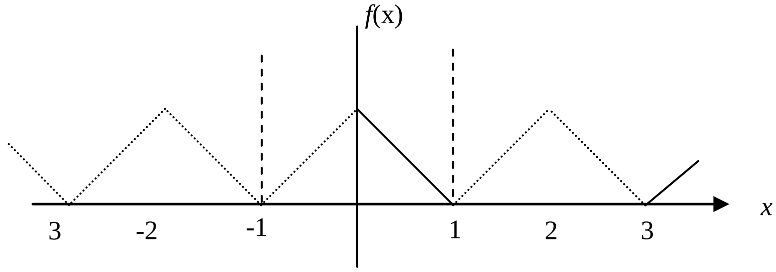
Find the half range Fourier series for

$f(x) = 1 - x \qquad 0 < x < 1$.

Solution:

i) <u>Fourier Cosine.</u>

To get the Fourier cosine series, we need to extend $f(x)$ to be an **even function** as shown below:



Hence, the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad , \quad l = 1$$

where

$$a_0 = 2\int_0^1 (1-x)\,dx$$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1 = 1$$

and

$$a_n = 2\int_0^1 (1-x)\cos\ n\pi x\ dx$$

$$= 2\left[\frac{1}{n\pi}(1-x)\sin n\pi x - \frac{1}{n^2\pi^2}\cos\ n\pi x\right]_0^1$$

$$= 2\left[-\frac{1}{n^2\pi^2}\cos\ n\pi + \frac{1}{n^2\pi^2}\right]$$

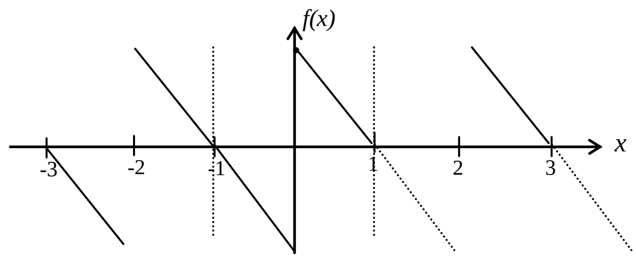
$$= \frac{2}{n^2\pi^2}\left[(-1)^{n+1} + 1\right] = \begin{cases} \dfrac{4}{n^2\pi} & , n \text{ odd} \\ 0 \quad , & n \text{ even} \end{cases}$$

$$= \frac{4}{\pi(2m-1)^2}\ ;\ m = 1, 2, 3, ..$$

$$\therefore\ f(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi^2}\ \cos(2m-1)\pi x$$

ii) <u>Fourier Sine.</u>

To get the Fourier sine series, we need to extend $f(x)$ to be an **odd function** as shown below:



<u>Remember!</u>
Odd function
$\downarrow$
Symmetry about the origin.

Hence, the Fourier sine series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \qquad l = 1$$

$$= 2\int_0^1 (1-x) \sin n\pi x \; dx$$

$$= \left[-2\frac{(1-x)}{n\pi} \cos \; n\pi x - \frac{2}{n^2\pi^2} \sin n\pi x \right]_0^1$$

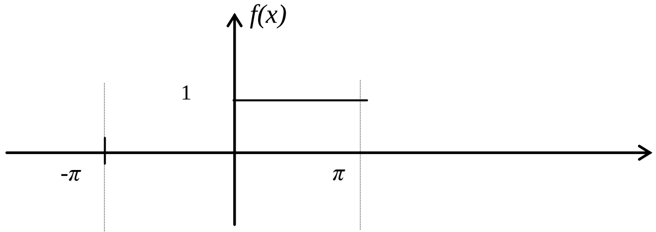
$$= \left[\frac{-2}{n\pi} \cdot 0 - \frac{2}{n^2\pi^2} \sin n\pi + \frac{2}{n\pi} \cos \; 0 + 0 \right]$$

$$= \frac{2}{n\pi}$$

$$\therefore \; f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x$$

Example 4.4.3

Given that $f(x)=1 \quad , 0 < x < \pi$.

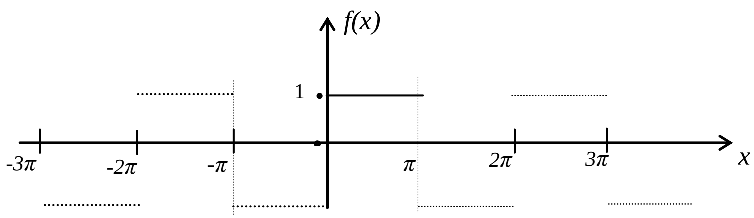


Find the odd and even extension of $f(x)$.

Solution:

i) Odd extension ≡ Fourier sine series.

To get the Fourier sine series, we need to extend $f(x)$ to be an **odd function** as shown below:



$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin nx \; dx = -\frac{2}{\pi n}\cos nx\Big|_0^{\pi}$$

$$= -\frac{2}{\pi n}\left[\cos \; n\pi - 1\right] = -\frac{2}{\pi n}\left[(-1)^n - 1\right]$$

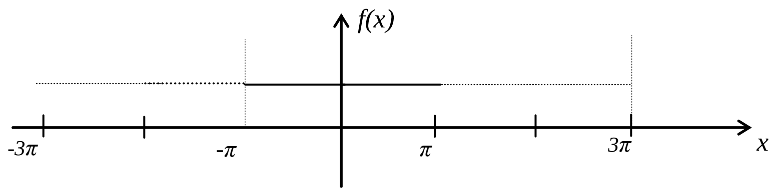
$$= \begin{cases} \dfrac{4}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$= \frac{4}{\pi(2m-1)}, m = 1, 2, 3, \ldots$$

Therefore, $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin(2m-1)x$

ii) Even extension ≡ Fourier cosine series.

To get the Fourier cosine series, we need to extend $f(x)$ to be an **even function** as shown below:



$$a_0 = \frac{2}{\pi} \int_0^{\pi} 1 \, dx = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos nx \, dx$$

$$= \frac{2}{\pi} \frac{1}{n} \sin nx \Big|_0^{\pi}$$

$$= 0$$

$$\therefore f(x) = \frac{a_0}{2} = 1$$

4.5 Approximate Sum of the Infinite Series

We can find the approximate sum of the infinite series by using Fourier series for $f(x)$.

NOTE:

If $f(x)=\begin{cases} A; & -l<x<0 \\ B; & 0<x<l \end{cases}$, then $f(0)=\dfrac{A+B}{2}$.

$x=0$ is not in the interval.

Example 4.5.1

If the Fourier series for

$$f(x)=\begin{cases} 0, & -1<x<0 \\ 1, & 0<x<1 \end{cases} \qquad , f(x+2)=f(x)$$

is given by

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x,$$

show that the approximate sum of this infinite series is

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$$

Solution:

i) Expand $f(x)$.

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x$$
$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \dots\right]$$

ii) Choose an appropriate value of x.

Let $x = \frac{1}{2}$, then we obtain

$$f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \dots\right]$$

Recall:
$f(x) = 1$
if $0 < x < 1$

$$1 = \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \dots\right]$$

or $$\frac{1}{2} = \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \dots\right]$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$$

Example 5.5.2

i) Show that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, -\pi < x < \pi$

ii) By choosing an appropriate value of x, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution:

i) Since $f(x) = x^2$ is an even function, its Fourier series is govern by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_o = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[\pi^3\right] = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right]$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}\left(-1\right)^n$$

Therefore, $f(x) = x^2 = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty} \dfrac{4}{n^2}(-1)^n \cos nx$.

ii) Choose an appropriate value of x.

Let $x = \pi$.

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3.4} = \frac{\pi^2}{6} \ .$$

Let $x = 0$.

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n = -\frac{\pi^2}{12} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} = \frac{\pi^2}{12} \ .$$

Example 5.5.3 (continuation of Eg. 4.4.1)

If the Fourier cosine series of function

$$f(x)=\begin{cases}\pi+x; & -\pi<x<0\\ \pi-x; & 0<x<\pi\end{cases}$$

is given by $f(x)=\dfrac{\pi}{2}+\dfrac{4}{\pi}\sum_{m=1}^{\infty}\dfrac{\cos(2m-1)x}{(2m-1)^2}$, find the sum of

$1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots.$

Solution:

1 – Expand $f(x)$.

$$f(x)=\frac{\pi}{2}+\frac{4}{\pi}\left[\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\ldots\right]$$

2 – Choose an appropriate value of x.

Let $x=0$.

$$f(0)=\frac{\pi}{2}+\frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right]$$

Recall function $f(x)$. $x=0$ is not in the interval. Hence, how do we find $f(0)$?

$$f(0)=\frac{(\pi+x)+(\pi-x)}{2}=\pi$$

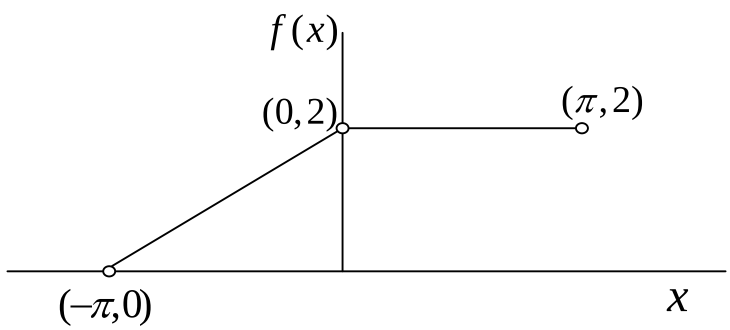
Therefore,

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

$$\therefore \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right] = \frac{\left(\pi - \frac{\pi}{2}\right)\cdot \pi}{4} = \frac{\pi^2}{8}$$

Exercise:

1. The graph of $f(x)$ is shown as below. Find its Fourier series.



Ans. $\frac{3}{2} + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{(n\pi)^2}\cos nx + \frac{(-1)^{n+1}}{n\pi}\sin nx\right]$

2. Write the Fourier series for $f(x) = \sinh x$, $-\pi < x < \pi$.

Ans. $\frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{1+n^2}\sin nx.$

3. Show that if $f(x) = 0$ when $-3 < x < 0$, $f(x) = 1$ when $0 < x < 3$, and $f(0) = \frac{1}{2}$, then

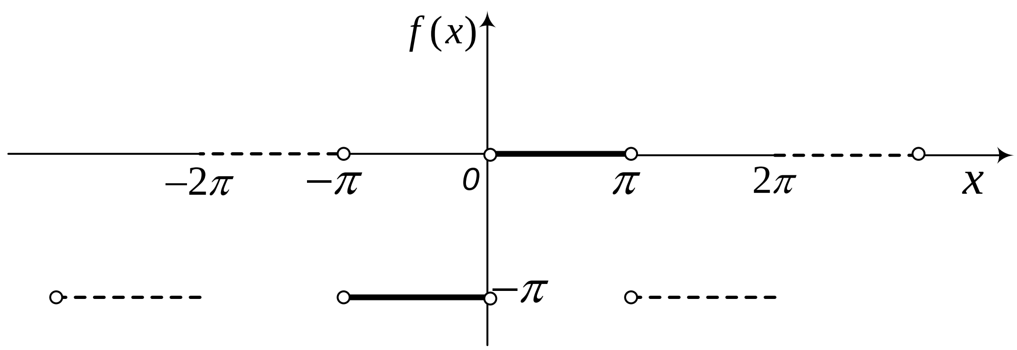
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{3}.$$

4. Recall Example 4.3.3. Using an appropriate value of x, find the sum of the following series

$$-\frac{1}{3}+\frac{1}{15}-\frac{1}{35}+\ldots+\frac{(-1)^n}{4n^2-1}+\ldots$$

Ans. $\frac{1}{2}-\frac{\pi}{4}$.

5. Consider the following graph of $f(x)$.



a) Write the function of $f(x)$.
b) Determine the Fourier series of $f(x)$.

Ans. $-\frac{\pi}{2}+2\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$.

6. Write the Fourier Sine series for $f(x) = \pi - x \ (0 < x < \pi)$.

Ans. $2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

7. Calculate the Fourier sine series of the function $f(x) = x(\pi - x)$ on $(0, \pi)$. Use its Fourier representation to the find the value of the infinite series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \ldots$

Ans. $f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$, the sum of series is $\frac{\pi^3}{32}$ if $x = \pi/2$

8. Let h be a given number in the interval $(0, \pi)$. Find the Fourier cosine series of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < h \\ 0 & \text{if} \quad h < x < \pi \end{cases}$$

Ans. $f(x) = \frac{2h}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin nh}{nh}\cos nx\right\}$

END

REFERENCES

1. Normah Maan et. al., (2008), Differential Equations Module, Jabatan Matematik, UTM.
2. Nagel et. al., (2004), Fundamentals of Differential Equations, 5th ed., Addison Wesley Longman.
3. Ruel V.Churchill and James W.B., (1978), Fourier Series and Boundary Value Problems, McGraw Hill.