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SSCE 1793 DIFFERENTIAL EQUATIONS **TUTORIAL 2**

1. Solve the following homogeneous equations.

a. $2y'' + 7y' - 4y = 0$ **b.** $4y'' - 4y' + y = 0$
c. $y'' + y = 0$ **d.** $y'' - 4y' + 7y = 0$

2. Solve the given initial value problems.

a. $y'' + y' = 0;$ $y(0) = 2, \quad y'(0) = 1$
b. $y'' - 4y' + 3y = 0;$ $y(0) = 1, \quad y'(0) = 1/3$
c. $w'' - 4w' + 2w = 0;$ $w(0) = 0, \quad w'(0) = 1$
d. $y'' - 2y' + y = 0;$ $y(0) = 1, \quad y'(0) = -2$
e. $2y'' - 2y' + y = 0;$ $y(0) = -1, \quad y'(0) = 0$

3. Solve the following boundary value problems.

a. $y'' - 10y' + 25y = 0;$ $y(0) = 1, \quad y(1) = 0$
b. $y'' + 4y = 0;$ $y(0) = 0, \quad y'(\pi) = 1$
c. $2y'' + y' - 6y = 0;$ $y(0) = 0, \quad y'(1) = 1$

4. Solve the following nonhomogeneous equations using the method of undetermined coefficients.

a. $2x' + x = 3t^2 + 10t$ **b.** $y'' - y' + 9y = 3\sin 3x$
c. $y'' + y' + y = 2\cos 2x - 3\sin 2x$ **d.** $\theta'' - 5\theta' + 6\theta = re^r$
e. $\theta''(t) - \theta(t) = t\sin t$ **f.** $y'' - 2y' + y = 8e^x$
g. $y'' - y = -11x + 1$ **h.** $y'' + 4y = \sin 2\theta - \cos\theta$
i. $y'' + 2y' + 2y = e^{-\theta}\cos\theta$ **j.** $y'' - 3y = x^2 - e^x$
k. $x'' - 4x' + 4x = te^{2t}$ **l.** $y'' - y = (5x + 1)e^{3x}$

5. Solve the following initial value problems using the method of undetermined coefficients.

a. $y' - y = 1;$ $y(0) = 1$
b. $z'' + z = 2e^{-x};$ $z(0) = 0, \quad z'(0) = 0$
c. $y'' + y' - 12y = e^x + e^{2x} - 1;$ $y(0) = 1, \quad y'(0) = 3$
d. $y'' - y = \sin\theta - e^{2\theta};$ $y(0) = 1, \quad y'(0) = -1$

6. Determine the form of a particular solution to the differential equations.

a. $y'' - y = e^{2x} + xe^{2x} + x^2e^{2x}$
b. $y'' + 5y' + 6y = \sin x - \cos 2x$
c. $y'' - 4y' + 5y = e^{5x} + x\sin 3x - \cos 3x$
d. $y'' - 4y' + 4y = x^2e^{2x} - e^{2x}$
e. $y'' + y' - 6yy = e^{-3x} + x\cos 3x - \sin 2x$

7. Solve the following nonhomogeneous equations using the method of variation of parameters.

a. $y'' + y = \sec x$ **b.** $y'' + y = \cos^2 x$

c. $y'' - y = \cosh x$ **d.** $y'' + y = \sec x \tan x$

e. $y'' + 4y = 2\tan x - e^x$ **f.** $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

g. $y'' + 3y' + 2y = \sin e^x$ **h.** $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$

i. $y'' + 2y' + y = e^{-x}\ln 2x$ **j.** $y'' + y = 3\sec x - x^2 + 1$

k. $2y'' - 2y' - 4y = 2e^{3t}$ **l.** $y'' + 4y = \csc^2 2x$

8. A vibrating string without damping can be modelled by the differential equation

$$my'' + ky = 0.$$

a. If $m = 10$kg, $k = 250$kg/sec^2, $y(0) = 0.3$m and $y'(0) = -0.1$m/sec, find the equation of motion for this system.

b. When the equation of motion is of the form displayed in (a), the motion is said to be oscillatory with frequency $\beta/2\pi$. Find the frequency of the oscillation for the spring system of part (a).

9. A vibrating string with damping can be modelled by the differential equation

$$my'' + by' + ky = 0.$$

a. If $m = 10$kg, $k = 250$kg/sec^2, $b = 60$kg/sec, $y(0) = 0.3$m and $y'(0) = -0.1$m/sec, find the equation of motion for this system.

b. Find the frequency of the oscillation.

c. Compare the results of this problem to Question 8 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution?

10. The motion of a certain mass-spring system with damping is governed by

$$y''(t) + 6y'(t) + 16y(t) = 0$$
$$y(0) = 1, \quad y'(0) = 0.$$

Find the equation of motion.

11. Determine the equation of motion for an undamped system at resonance governed by

$$\frac{d^2y}{dt^2} + 9y = 2\cos 3t$$
$$y(0) = 1, \quad y'(0) = 0.$$

Sketch the solution.

12. An undamped system is governed by

$$m\frac{d^2y}{dt^2} + ky = F_0 \cos \gamma t;$$
$$y(0) = y'(0) = 0.$$

where $\gamma \neq \omega = \sqrt{\frac{k}{m}}$. Find the equation of motion of the system.

13. Consider the vibrations of a mass-spring system when a periodic force is applied. The system is governed by the differential equation

$$mx'' + bx' + kx = F_0 \cos \gamma t$$

where F_0 and γ are nonnegative constants, and $0 < b^2 < 4mk$.

a. Show that the general solution to the corresponding homogeneous equation is

$$x_h(t) = Ae^{(-b/m)t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right).$$

b. Show that the general solution to the nonhomogeneous problem is given by

$$x(t) = Ae^{(-b/m)t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta).$$

14. **RLC Series Circuit.** In the study of an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force, we are led to an initial value problem of the form

$$\begin{aligned} L\frac{dI}{dt} + RI + \frac{q}{C} &= E(t) \\ q(0) = q_0, \quad I(0) &= I_0. \end{aligned} \tag{1}$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, $q(t)$ is the charge in coulombs on the capacitor at time t, and $I = \dfrac{dq}{dt}$ is the current in amperes.

Find the current at time t if the charge on the capacitor is initially zero, the initial current is 0, $L = 10$henrys, $R = 20$ohms, $C = 6260^{-1}$farads and $E(t) = 100$volts.

Hint: Differentiate both sides of the differential equation to obtain a homogeneous linear second order equation for $I(t)$. Then use equation (1) to determine $\frac{dI}{dt}$ at $t = 0$.

15. An RLC series circuit has an elctromotive force given by $E(t) = \sin 100t$ volts, a resistor of 0.02 ohms, an inductor of 0.001 henrys, and a capacitor of 2 farads. If the initial current and the initial charge on the capacitor are both zero, determine the current in the circuit for $t > 0$.

SOLUTIONS TO TUTORIAL 2

1. **a.** $y = c_1 e^{x/2} + c_2 e^{-4x}$. **b.** $y = (c_1 + c_2 x)e^{x/2}$.
c. $y = c_1 \cos x + c_2 \sin x$. **d.** $y = e^{2x}(c_1 \cos 2\sqrt{3}x + c_2 \sin 2\sqrt{3}x)$.

2. **a.** $y = 3 - e^{-x}$. **b.** $y = \dfrac{4}{3}\, e^{x} - \dfrac{1}{3}\, e^{3x}$.
c. $y = \dfrac{1}{2\sqrt{2}}\left(e^{(2+\sqrt{2})x} - e^{(2-\sqrt{2})x}\right)$. **d.** $y = (1-3x)e^{x}$.
e. $y = e^{x/2}(\sin \dfrac{x}{2} - \cos \dfrac{x}{2})$.

3. **a.** $y = e^{5x} - xe^{5x}$. **b.** $y = \dfrac{1}{2}\sin 2x$.
c. $y = \dfrac{e^{2x} - e^{-3x}}{3e^{-3/2} + 2e^{-2}}$. **d.** $y = \dfrac{2}{7}e^{3x/2} - \dfrac{2}{7}e^{-2x}$.

4. **a.** $y = c_1 \cos \dfrac{t}{\sqrt{2}} + c_2 \sin \dfrac{t}{\sqrt{2}} + 3t^2 + 10t - 12$.
b. $y = e^{x/2}\left(c_1 \cos \dfrac{\sqrt{35}}{2}x + c_2 \sin \dfrac{\sqrt{35}}{2}x\right) + \cos 3x$.
c. $y = e^{-x/2}\left(c_1 \cos \dfrac{\sqrt{3}}{2}x + c_2 \sin \dfrac{\sqrt{3}}{2}x\right) + \sin 2x$.
d. $\theta = c_1 e^{2r} + c_2 e^{3r} + \dfrac{3}{4}e^{r} + \dfrac{1}{2}re^{r}$.
e. $\theta = c_1 e^{-t} + c_2 e^{t} - \dfrac{1}{2}t \sin t - \dfrac{1}{2}t \cos t$.
f. $y = (c_1 + c_2 x)e^{x} + 4x^2 e^{x}$.
g. $y = c_1 e^{x} + c_2 e^{-x} + 11x - 1$.
h. $y = c_1 \cos 2\theta + c_2 \sin 2\theta - \dfrac{1}{4}\theta \cos 2\theta - \dfrac{1}{3}\cos \theta$.
i. $y = e^{-\theta}(c_1 \cos \theta + c_2 \sin \theta) + \theta e^{-\theta} \cos \theta$.
j. $y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x} - \dfrac{1}{3}x^2 - \dfrac{2}{5} + \dfrac{1}{2}e^{x}$.
k. $y = (c_1 + c_2 t)e^{2t} + \dfrac{1}{6}t^3 e^{2t}$.
l. $y = c_1 e^{x} + c_2 e^{-x} + \left(\dfrac{5}{8}x - \dfrac{11}{32}\right)e^{3x}$.

5. **a.** $y = e^{x} - 1$. **b.** $y = -\cos x + \sin x + e^{-x}$.
c. $y = \dfrac{1}{60}e^{-4x} + \dfrac{7}{6}e^{3x} - \dfrac{1}{10}e^{x} - \dfrac{1}{6}e^{2x} + \dfrac{1}{12}$. **d.** $y = \dfrac{7}{12}e^{\theta} + \dfrac{3}{4}e^{-\theta} - \dfrac{1}{2}\sin \theta - \dfrac{1}{3}e^{2\theta}$.

6. **a.** $y_p = (Ax^2 + Bx + C)e^{2x}$
b. $y_p = A\cos x + B \sin x + C \cos 2x + D \sin 2x$.
c. $y_p = Ae^{5x} + (Bx + C)(D \cos 3x + E \sin 3x)$.
d. $y_p = x^2(Ax^2 + Bx + C)e^{2x}$.
e. $y_p = Axe^{-3x} + (Bx + C)(D \sin 3x + E \cos 3x) + F \sin 2x + G \cos 2x$.

7. **a.** $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln |\cos x|$.

b. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} - \dfrac{1}{6} \cos 2x$.

c. $y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{4} x e^x - \dfrac{1}{4} x e^{-x}$.

d. $y = c_1 \cos x + c_2 \sin x + x \cos x + \sin x \ln |\sec x|$.

e. $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{e^x}{5} - \dfrac{1}{2} \cos 2x \ln |\sec 2x + \tan 2x|$.

f. $y = c_1 e^{-x} + c_2 e^{-2x} + (e^{-x} + e^{-2x}) \ln(1 + e^x)$.

g. $y = c_1 e^{2x} + c_2 e^{-x} - e^{-2x} \sin e^{2x}$.

h. $y = c_1 e^x + c_2 x e^x - \dfrac{1}{2}[e^x \ln(x^2 + 1) + x e^x \tan^{-1} x]$.

i. $y = c_1 e^{-x} + c_2 x e^{-x} + \dfrac{1}{2} x^2 e^{-x} \ln 2x - \dfrac{3}{4} x^2 e^{-x}$.

j. $y = c_1 \cos x + c_2 \sin x - x^2 + 3 + 3x \sin x + 3 \cos x \ln |\cos x|$.

k. $y = c_1 e^{2t} + c_2 e^{-t} + \dfrac{1}{4} e^{3t}$.

l. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{4}(\cos 2x \ln |\csc 2x + \cot 2x| - 1)$.

8. **a.** $y = 0.3 \cos 5t - 0.02 \sin 5t$. **b.** $0.8Hz$

9. **a.** $y = e^{-3t}(0.3 \cos 4t + 0.2 \sin 4t)$. **b.** $5/2\pi$.

c. Exponential factor (damping factor)

10. $y = e^{-3x} \left(\cos \sqrt{7} x + \dfrac{3}{\sqrt{7}} \sin \sqrt{7} x \right)$.

11. $y = \cos 3t + \dfrac{1}{2} t \sin t$.

12. $y = \dfrac{-F_0}{m(\omega^2 - \gamma^2)} \cos \omega t + \dfrac{F_0}{m(\omega^2 - \gamma^2)} \sin \gamma t$.

14. $I = \dfrac{2}{5} e^{-t} \sin 25t$.

15. $I = e^{-10t} \left(\dfrac{95}{9.425} \cos 20t - \dfrac{105}{18.85} \sin 20t \right) - \dfrac{95}{9.425} \cos 100t + \dfrac{20}{9.425} \sin 100t$.