Solution Sheet 1— Brownian motion and Quadratic Variation

1.1: For $(W_t)_{t\geq 0}$ a standard Brownian motion and times $0 \leq s \leq t$ we have

$$\begin{aligned}\mathbb{C}ov(W_s, W_t) &= \mathbb{E}\left[(W_s - \mathbb{E}[W_s])(W_t - \mathbb{E}[W_t])\right] \\ &= \mathbb{E}[W_s W_t] \\ &= \mathbb{E}\left[W_s(W_t - W_s) + W_s^2\right] \\ &= \mathbb{E}[W_s]\,\mathbb{E}[W_t - W_s] + \mathbb{E}\left[W_s^2\right] \qquad \text{(independent increments)} \\ &= 0 + s.\end{aligned}$$

Similarly, if $0 \leq t \leq s$ then we have $\mathbb{C}ov(W_s, W_t) = t$ and so

$$\mathbb{C}ov(W_s, W_t) = \min\{s, t\}.$$

1.2: By Taylor's theorem we can write

$$f(W_{s+h}) = f(W_s) + f'(W_s)(W_{s+h} - W_s) + \frac{1}{2}f''(W_s)(W_{s+h} - W_s)^2 + g(W_{s+h})(W_{s+h} - W_s)^2,$$

where $g : \mathbb{R} \to \mathbb{R}$ is such that $g(y) \to 0$ as $y \to x$. Then

$$\begin{aligned}\frac{P_h f(x) - f(x)}{h} &= \mathbb{E}\left[\frac{W_{s+h} - W_s}{h} f'(W_s) + \frac{(W_{s+h} - W_s)^2}{2h} f''(W_s) \middle| W_s = x\right] \\ &\qquad + \mathbb{E}\left[\frac{(W_{s+h} - W_s)^2}{h} g(W_{s+h}) | W_s = x\right] \\ &= \frac{1}{2}f''(x) + \mathbb{E}\left[\frac{(W_{s+h} - W_s)^2}{h} g(W_{s+h}) | W_s = x\right]\end{aligned}$$

But $g(W_{s+h}) \to 0$ as $h \to 0$ and so we can argue that this final term disappears in the limit. For example, (although in this courses, we will not worry too much about such details), by assuming that f is sufficiently nice, we must have g bounded, by M say. Therefore we know that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $g(W_{s+h}) < \varepsilon$ for any $h \in [0, \delta)$, with probability at least $1 - \varepsilon$. Then

$$\mathbb{E}\left[\frac{(W_{s+h} - W_s)^2}{h} g(W_{s+h}) | W_s = x\right] < \varepsilon \mathbb{E}\left[\frac{(W_{s+h} - W_s)^2}{h} | W_s = x\right] + M\varepsilon \leq \varepsilon(1 + M).$$

Therefore, this whole term tends to zero and we have that

$$\lim_{h\to 0} \frac{P_h f(x) - f(x)}{h} = \frac{1}{2}f''(x).$$

1.3: (a) We want to use Kolmogorov's Continuity Criterion, and so need to find constants $\alpha, \beta, \gamma > 0$ such that

$$\mathbb{E}\left[|X_{t+h} - X_t|^\alpha\right] \leq \gamma h^{1+\beta}.$$

We have

$$\begin{aligned}\mathbb{E}\left[|X_{t+h}-X_t|^\alpha\right] &= \int_{\mathbb{R}^2} |x_2-x_1|^\alpha \frac{p(t,x,x_1)p(h,x_1,x_2)p(1-t-h,x_2,y)}{p(1,x,y)} \mathrm{d}x_2\mathrm{d}x_1\\ &\le \frac{p(1-t,0,0)}{p(1,x,y)}\int_{\mathbb{R}^2}|x_2-x_1|^\alpha p(t,x,x_1)p(h,x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1,\end{aligned}$$

noting that our integrand is positive and $p(1-t-h,x_2,y)\le p(1-t,0,0)$. This integral can then be calculated directly, but it is instead easier to write this is an expectation of a standard Brownian motion. Note that $p(s,w_1,w_2)$ is the transition probability of a standard Brownian motion, or equivalently the probability that a Brownian motion at w_1 moves to w_2 at time s. Then

$$\int_{\mathbb{R}} |x_2-x_1|^\alpha p(h,x_1,x_2)\,\mathrm{d}x_2 = \mathbb{E}\left[|W_{t+h}-W_t|^\alpha | W_t = x_1\right],$$

and so

$$\int_{\mathbb{R}^2} |x_2-x_1|^\alpha p(t,x,x_1)p(h,x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 = \mathbb{E}\left[\mathbb{E}\left[|W_{t+h}-W_t|^\alpha|W_t\right]|W_0=x\right].$$

We know, from facts about the Normal distribution, that since $W_{t+h}-W_t\sim N(0,h)$,

$$\mathbb{E}\left[|W_{t+h}-W_t|^4|W_t=x_1\right]=3h^2.$$

Then, setting $\alpha=4$, we have

$$\begin{aligned}\mathbb{E}\left[|X_{t+h}-X_t|^4\right] &\le \frac{p(1-t,0,0)}{p(1,x,y)}\int_{\mathbb{R}^2}|x_2-x_1|^4 p(t,x,x_1)p(h,x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1\\ &= \frac{p(1-t,0,0)}{p(1,x,y)}\mathbb{E}\left[3h^2|W_0=x\right]\\ &= \frac{3p(1-t,0,0)}{p(1,x,y)}h^2,\end{aligned}$$

so $\alpha=4$, $\gamma=\frac{3p(1-t,0,0)}{p(1,x,y)}$, and $\beta=1$ will do.

(b) We need to show that the conditional transition probabilities are the same in each case. When we condition on the previous $n-1$ positions we have

$$\begin{aligned}&\mathbb{P}\left(X_{t_n}\in \mathrm{d}z|X_{t_1},\dots,X_{t_{n-1}}\right)\\ &\quad= \frac{\mathbb{P}\left(X_{t_n}\in\mathrm{d}z, X_{t_1},\dots,X_{t_{n-1}}\right)}{\mathbb{P}\left(X_{t_1},\dots,X_{t_{n-1}}\right)}\\ &\quad= \frac{p(t_1,x,X_1)\prod_{i=2}^{n-1}p(t_i-t_{i-1},X_{t_{i-1}},X_{t_i})p(t_n-t_{n-1},X_{t_n},z)p(1-t_n,z,y)}{p(1,x,y)}\\ &\qquad\times\frac{p(1,x,y)}{p(t_1,x,x_1)\prod_{i=2}^{n-1}p(t_i-t_{i-1},x_{i-1},x_i)p(1-t_{n-1},X_{t_{n-1}},y)}\mathrm{d}z\\ &\quad= \frac{p(t_n-t_{n-1},X_{t_n},z)p(1-t_n,z,y)}{p(1-t_{n-1},X_{t_{n-1}},y)}\mathrm{d}z.\end{aligned}$$

But in the other case,

$$\begin{aligned}
\mathbb{P}\left(X_{t_n} \in \mathrm{d}z | X_{t_{n-1}}\right) &= \frac{\mathbb{P}\left(X_{t_n} \in \mathrm{d}z, X_{t_{n-1}}\right)}{\mathbb{P}\left(X_{t_{n-1}}\right)}\mathrm{d}z \\
&= \frac{p(t_{n-1}, x, X_{t_{n-1}})p(t_n - t_{n-1}, X_{t_{n-1}}, z)p(1-t_n, z, y)}{p(1,x,y)} \\
&\qquad \times \frac{p(1,x,y)}{p(t_{n-1}, x, X_{t_{n-1}})p(1-t_{n-1}, X_{t_{n-1}}, y)}\mathrm{d}z \\
&= \frac{p(t_n - t_{n-1}, X_{t_n}, z)p(1-t_n, z, y)}{p(1-t_{n-1}, X_{t_{n-1}}, y)}\mathrm{d}z.
\end{aligned}$$

We can see that these probabilities are the same, and then the result follows.

(c) We act as in **1.2** in that we Taylor expand our function, but in this case we do not have a nice expectation in terms of a Brownian motion and we need to compute the integrals directly. We know from b) of this question that we can write, for $t \in [0,1)$,

$$\mathbb{E}\left[g(X_{t+h}, X_t)|X_t\right] = \int g(X_{t+h}, X_t)\frac{p(h, X_t, z)p(1-t-h, z, y)}{p(1-t, X_t, y)}\mathrm{d}z,$$

so we hope to rewrite this as an integral of a Brownian transition probability, and then we can compute it directly, or use our knowledge of the normal distribution to calculate it. If we expand the brackets in the exponentials then we can regroup the terms to find that

$$\begin{aligned}
p(h, X_t, z)p(1-t-h, z, y) &= \frac{1}{\sqrt{2\pi h}}\frac{1}{\sqrt{2\pi(1-t-h)}}\mathrm{e}^{-\frac{(X_t - z)^2}{2h}}\mathrm{e}^{-\frac{(z-y)^2}{2(1-t-h)}} \\
&= \frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-\frac{(z-\mu)^2}{2T}}\frac{1}{\sqrt{2\pi(1-t)}}\mathrm{e}^{-\frac{(y-X_t)^2}{2(1-t)}},
\end{aligned}$$

where $\mu = \frac{yh + X_t(1-t-h)}{1-t}$ and $T = \frac{h(1-t-h)}{1-t}$. So then,

$$\frac{p(h, X_t, z)p(1-t-h, z, y)}{p(1-t, X_t, y)} = \frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-\frac{(z-\mu)^2}{2T}},$$

which is exactly $p(T, \mu, z)$, or the probability density of a Gaussian r.v. Z, with mean μ and variance T. If we write

$$f(X_{h+t}) = f(X_t) + f'(X_t)(X_{h+t} - X_t) + \frac{1}{2}f''(X_t)(X_{h+t} - X_t)^2 + g(X_{h+t})(X_{h+t} - X_t)^2,$$

then

$$\begin{aligned}
\frac{\mathbb{E}\left[f(X_{t+h})|X_t\right] - f(X_t)}{h} &= \frac{1}{h}\mathbb{E}\left[f'(X_t)(X_{h+t} - X_t)|X_t = x\right] \\
&\quad + \frac{1}{2h}\mathbb{E}\left[f''(X_t)(X_{h+t} - X_t)^2|X_t = x\right] \\
&\qquad + \frac{1}{h}\mathbb{E}\left[g(X_{h+t})(X_{h+t} - X_t)^2\right].
\end{aligned}$$

Using the above, if Z is Gaussian, with mean μ and variance T, independent of X_t, then

$$\begin{aligned}
\frac{1}{h}\mathbb{E}\left[f'(X_t)(X_{h+t}-X_t)|X_t=x\right] &= \int \frac{1}{h}f'(X_t)(z-X_t)\frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-\frac{(z-\mu)^2}{2T}}\mathrm{d}z \\
&= \mathbb{E}\left[\frac{Z-X_t}{h}\middle|X_t\right]f'(X_t) \\
&= \frac{\mu-X_t}{h}f'(X_t) \\
&= \frac{y-X_t}{1-t}f'(X_t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{2h}\mathbb{E}\left[f''(X_t)(X_{h+t}-X_t)^2|X_t=x\right] &= \mathbb{E}\left[\frac{(Z-X_t)^2}{2h}\middle|X_t\right]f''(X_t) \\
&= \left(\frac{(\mu-X_t)^2}{2h}+\frac{T}{2h}\right)f''(X_t) \\
&= \left(\frac{h(y-X_t)^2}{2(1-t)^2}+\frac{1-t-h}{2(1-t)}\right)f''(X_t) \\
&\to \frac{1}{2}f''(X_t) \quad \text{as } h\to 0.
\end{aligned}$$

For the final term we can argue exactly as before to show that it tends to zero. Putting all of these together, we see that for $t\in[0,1)$

$$\lim_{h\searrow 0}\frac{\mathbb{E}\left[f(X_{t+h})|X_t\right]-f(X_t)}{h}=\frac{y-X_t}{1-t}f'(X_t)+\frac{1}{2}f''(X_t).$$

(d) To make things a little clearer, wlog we set $x=0$ here (otherwise we could write $\hat{y}=y-x$), and we let $t_0=0$, $x_0=x=0$. Then we have

$$\begin{aligned}
&\mathbb{E}\left[f(X'_{t_1},X'_{t_2},\dots,X'_{t_n})\right] \\
&= \int f(x_1+t_1(y-z),\dots,x_n+t_n(y-z))\prod_{i=1}^n p(t_i-t_{i-1},x_{i-1},x_i) \\
&\qquad\qquad p(1-t_n,x_n,z)\mathrm{d}x_1\cdots\mathrm{d}x_n\mathrm{d}z \\
&= \int f(\hat{x}_1,\dots,\hat{x}_n)\prod_{i=1}^n p(t_i-t_{i-1},\hat{x}_{i-1}-t_{i-1}(y-z),\hat{x}_i-t_i(y-z)) \\
&\qquad\qquad p(1-t_n,\hat{x}_n-t_n(y-z),z)\mathrm{d}\hat{x}_1\cdots\mathrm{d}\hat{x}_n\mathrm{d}z,
\end{aligned}$$

where we have set $\hat{x}_i=x_i+t_i(y-z)$. Now we notice that

$$\begin{aligned}
&p(t_i-t_{i-1},\hat{x}_{i-1}-t_{i-1}(y-z),\hat{x}_i-t_i(y-z)) \\
&= \frac{1}{\sqrt{2\pi(t_i-t_{i-1})}}\exp\left(-\frac{(\hat{x}_i-\hat{x}_{i-1}-(t_i-t_{i-1})(y-z))^2}{2(t_i-t_{i-1})}\right) \\
&= \frac{1}{\sqrt{2\pi(t_i-t_{i-1})}}\exp\left(-\frac{(\hat{x}_i-\hat{x}_{i-1})^2}{2(t_i-t_{i-1})}\right)\exp\left(-\frac{(t_i-t_{i-1})(y-z)^2}{2}\right) \\
&\qquad\qquad \exp\left((\hat{x}_i-\hat{x}_{i-1})(y-z)\right).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&p(1-t_n, \hat{x}_n - t_n(y-z), z) \\
&\qquad = \frac{1}{\sqrt{2\pi(1-t_n)}} \exp\left(-\frac{((\hat{x}_n - y) + (1-t_n)(y-z))^2}{2(1-t_n)}\right) \\
&\qquad = p(1-t_n, \hat{x}_n, y) \exp\left(-\frac{(1-t_n)(y-z)^2}{2}\right) \exp\left(-(\hat{x}_n - y)(y-z)\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\prod_{i=1}^n p(t_i - t_{i-1}, \hat{x}_{i-1} - t_{i-1}(y-z), \hat{x}_i - t_i(y-z)) p(1-t_n, \hat{x}_n - t_n(y-z), z) \\
&\qquad = \prod_{i=1}^n p(t_i - t_{i-1}, \hat{x}_{i-1}, \hat{x}_i) p(1-t_n, \hat{x}_n, y) \exp\left(-\frac{z^2}{2}\right) \exp\left(\frac{y^2}{2}\right).
\end{aligned}
$$

We now only have one term depending on z, so we can do the integral with respect to z to find that

$$\int \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z = \sqrt{2\pi},$$

and then $\frac{\sqrt{2\pi}}{\exp\left(\frac{y^2}{2}\right)} = \frac{1}{p(1,0,y)}$, and the result follows.

> *The process constructed in this question is called the* Brownian Bridge. *It has the interpretation of a Brownian motion started at x, conditioned to be at y at time 1. This can be seen by integrating the first expectation against the probability that the Brownian motion goes from x to y, $p(1,x,y)$.*

1.4: To ensure that our process stays in $(0,1)$ we require that the volatility term σ disappears at these boundary points, and that our drift μ should be bounded, non-positive at 1 and non-negative at 0. See file at end of solutions for an example.

1.5: (a) By *iv)* of Theorem 2.2 we know that for any $\alpha, \beta > 0$, $W_{\frac{\beta^2}{\alpha^2}t}$ is equal in distribution to $\frac{\alpha}{\beta} W_t$. Then, if we let $t' = \frac{\alpha^2}{\beta^2} t$,

$$
\begin{aligned}
\mathbb{P}\left(\frac{|W_t|}{t} > \alpha C, \text{ some } t \in \left[0, \frac{1}{\alpha^2}\right]\right) &= \mathbb{P}\left(\frac{\left|W_{\frac{\beta^2}{\alpha^2}t'}\right|}{\frac{\beta^2}{\alpha^2}t'} > \alpha C, \text{ some } t' \in \left[0, \frac{1}{\beta^2}\right]\right) \\
&= \mathbb{P}\left(\frac{|\frac{\beta}{\alpha} W_{t'}|}{\frac{\beta^2}{\alpha^2}t'} > \alpha C, \text{ some } t' \in \left[0, \frac{1}{\beta^2}\right]\right) \\
&= \mathbb{P}\left(\frac{|W_{t'}|}{t'} > \beta C, \text{ some } t' \in \left[0, \frac{1}{\beta^2}\right]\right),
\end{aligned}
$$

so it is independent of α.

(b) Using the same ideas as above, with $\beta = 1$,

$$\begin{aligned}\mathbb{P}\left(\frac{|W_t|}{t} > \alpha, \text{ some } t \in \left[0, \frac{1}{\alpha^4}\right]\right) &= \mathbb{P}\left(\frac{|W_t|}{t} > \frac{1}{\alpha}, \text{ some } t \in [0,1]\right) \\ &\geq \mathbb{P}\left(|W_1| > \frac{1}{\alpha}\right) \\ &\to 1 \quad \text{as } \alpha \to \infty.\end{aligned}$$

For differentiability at 0 we require the limit of $\frac{|W_t|}{t}$ as $t \to 0$ to exist, and be finite, however we have shown that with probability 1 this is not the case, so Brownian motion is almost surely not differentiable at 0.

(c) For differentiability at t we consider the limit of $\frac{|W_{t+s}-W_t|}{s}$ as $s \to 0$, however we know that $W_{t+s} - W_t$ is equal in distribution to W_s, and so the last part tells us immediately that Brownian motion is almost surely not differentiable at any t.

1.6: (a) We suppose that X is a continuous function such that

$$\mathbb{E}\left[|X_t - X_s|^\gamma\right] \leq c|t-s|^{1+\varepsilon}$$

for some constants $\gamma, c, \varepsilon > 0$. But then we can see that

$$\mathbb{E}\left[K_i^\gamma\right] \leq \sum_{(s,t)\in\Delta_i} \mathbb{E}\left[|X_t - X_s|^\gamma\right] \leq \sum_{(s,t)\in\Delta_i} c2^{-i(1+\varepsilon)} \leq 2^{i+1}c2^{-i(1+\varepsilon)} = \tilde{c}2^{-i\varepsilon}.$$

(b) For $s \in D$ we can find an increasing sequence of points s_m in D such that for each m, $s_m \in D_m$, $s_m \leq s$, and $s_m = s$ for every $m \geq \tilde{m}$, some $\tilde{m}$. Then we can write

$$X_t - X_s = \sum_{i=m}^{\infty}(X_{t_{i+1}} - X_{t_i}) + X_{t_m} - X_{s_m} - \sum_{i=m}^{\infty}(X_{s_{i+1}} - X_{s_i}),$$

where the series are actually finite sums due to our assumptions on the sequences. Then,

$$|X_t - X_s| \leq \sum_{i=m+1}^{\infty} K_i + K_m + \sum_{i=m+1}^{\infty} K_i = 2\sum_{i=m}^{\infty} K_i.$$

(c) From the above we then get

$$
\begin{aligned}
M_\alpha &= \sup\left\{\frac{|X_t - X_s|}{|t-s|^\alpha} : s,t \in D, s \neq t\right\}\\
&\leq \sup_{m\in\mathbb{N}}\left\{\sup_{2^{-(m+1)}\leq|t-s|\leq 2^{-m}} \frac{|X_t - X_s|}{|t-s|^\alpha} : s,t \in D, s\neq t\right\}\\
&\leq \sup_{m\in\mathbb{N}}\left\{2^{(m+1)\alpha}\sup_{2^{-(m+1)}\leq|t-s|\leq 2^{-m}} |X_t - X_s| : s,t \in D, s\neq t\right\}\\
&\leq \sup_{m\in\mathbb{N}}\left\{2^{(m+1)\alpha}\sup_{|t-s|\leq 2^{-m}} |X_t - X_s| : s,t \in D, s\neq t\right\}\\
&\leq \sup_{m\in\mathbb{N}}\left\{2^{(m+1)\alpha} 2\sum_{i=m}^\infty K_i\right\}\\
&\leq \sup_{m\in\mathbb{N}}\left\{2^{\alpha+1}\sum_{i=m}^\infty 2^{i\alpha}K_i\right\}\\
&= 2^{\alpha+1}\sum_{i=0}^\infty 2^{i\alpha}K_i.
\end{aligned}
$$

Now, for $\gamma \geq 1$ and $\alpha < \frac{\varepsilon}{\gamma}$,

$$
\|M_\alpha\|_\gamma \leq 2^{\alpha+1}\sum_{i=0}^\infty 2^{i\alpha}\|K_i\|_\gamma \leq 2^{\alpha+1}\sum_{i=0}^\infty 2^{i\alpha}\tilde{c}^{\frac{1}{\gamma}}2^{-i\frac{\varepsilon}{\gamma}} = \hat{c}\sum_{i=0}^\infty 2^{i\left(\alpha-\frac{\varepsilon}{\gamma}\right)} < \infty.
$$

This tells us that, almost surely, X is uniformly continuous on D, and in fact, by the continuity of X,

$$
\mathbb{E}\left[\left(\sup_{s\neq t}\left(\frac{|X_t - X_s|}{|t-s|^\alpha}\right)\right)^\gamma\right] = \mathbb{E}\left[\left(\sup_{s\neq t, s,t\in D}\left(\frac{|X_t - X_s|}{|t-s|^\alpha}\right)\right)^\gamma\right] < \infty,
$$

so X is α-Hölder continuous for $\alpha < \frac{\varepsilon}{\gamma}$.

For a Brownian motion W, we have, by properties of the Normal distribution, that

$$
\mathbb{E}\left[|W_t - W_s|^{2p}\right] = C_p|t-s|^p,
$$

for some constant C_p. Therefore we can take $\gamma = 2p$ and $\varepsilon = p-1$ to show that Brownian motion is α-Hölder continuous for $\alpha < \frac{p-1}{2p}$. Since p is arbitrary we let $p \to \infty$ to get the result.

Note in fact that a slight modification of this result can be used to show that for any process X which satisfies the conditions of the question *except* for continuity also has a continuous version which is then Hölder continuous. Specifically, the argument down to the proof of uniform continuity (on D) is unchanged, and in fact, this is still true if we take $D\cup\{s_0\}$, for an arbitrary $s_0 \in [0,1]\setminus D$. It follows by Fatou's Lemma that we can define a continuous process $\tilde{X}$ by $\tilde{X}_t = \lim_{s\in D, s\to t} X_s$, and $\tilde{X}_{s_0} = X_{s_0}$ a.s.. Hence $\tilde{X}$ is a modification of X, and the paths of $\tilde{X}$ are α-Hölder continuous by the same argument as above.