Applied Differential Equation

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Chapter 1

Introduction

General introduction:

- What are DE?
- Where do they come from?
- How many types of them?
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  - Systems of 1st order linear DEs
  - Numerical approximation – Soln
- The Laplace transform
- Numerical approximation – Soln

1.1 Mathematical Modeling

Mathematical Model: a differential equation that describes some physical process.

Problem 1 Find the indefinite integral of function e^{2x}.

Everyone knows the soln $= \dfrac{1}{2}e^{2x} + C = \displaystyle\int e^{2x}dx$

Let $u = u(x)$ the indefinite integral (unknown function)

Definition: The indefinite integral of e^{2x} is the function whose derivative equals e^{2x}.

$$\boxed{\frac{du}{dx} = e^{2x}}$$

This is the mathematical model of the problem.

Soln:

$$u = \int e^{2x} dx = \frac{1}{2} e^{2x} + C$$

(Note: an infinite number of such solns)

Problem 2: The law of natural growth or decay

"The change rate of an amount of a radioactive substance, such as radium, is proportional to the amount at the current time."

$R(t)$: the amount of the substance at time t

$$\boxed{\frac{dR}{dt}} \quad \text{this is called Differential Equation}$$

where R is a constant depending on the material property of the substance.

Example: $k = 2$
$\frac{dR}{dt} = 2R$ then $R(t) = Ce^{2t}$, C: any number

Problem 3: A falling object

Consider an object that is falling in the atmosphere near sea level. Formulate a DE that describes the motion.

Newton's Second Law: $F = ma$.

F: force, m: mass, a: acceleration.
g: acceleration due to gravity $= 9.8m/sec^2$.
γ: drag coefficient (e.g. $\gamma = 2kg/sec$)
v: velocity.
$F = mg - \gamma v$, $ma = m\frac{dv}{dt}$

$$\boxed{m\frac{dv}{dt} = mg - \gamma v}$$

Example: $\gamma = 2kg/sec$, $m = 10kg$

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Soln=?

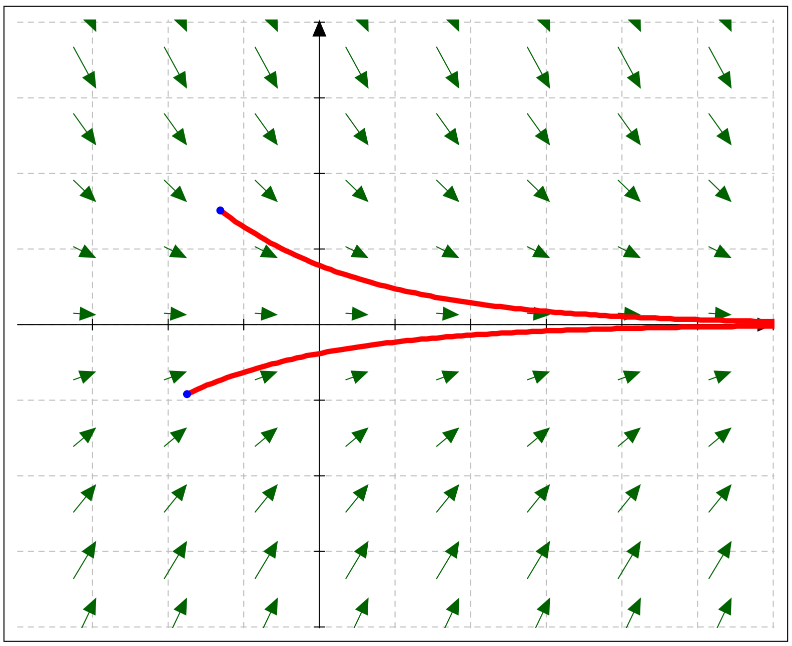
1.2 Direction fields

Example 1:

$$\frac{dy}{dt} = -y$$

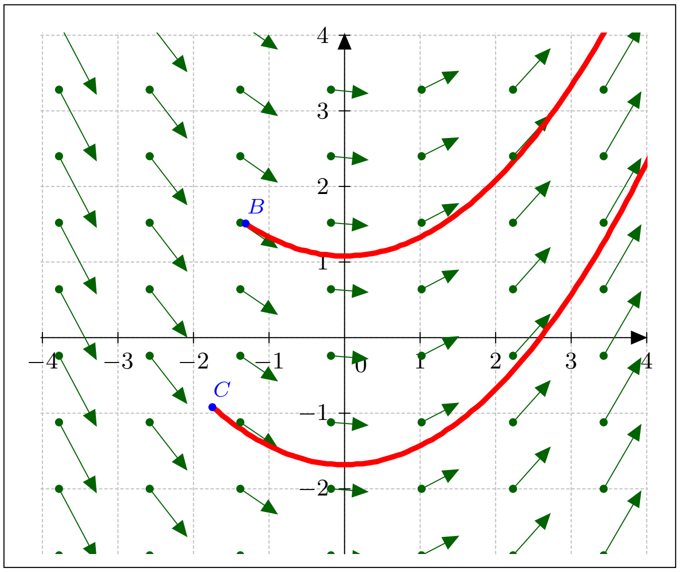
If $y = y(t)$ be the solution, then at the point (t_0, y_0), the slope of the tangent line to $y(t)$ is $-y_0$. Direction filed is used to described the slope of the tangent line to $y(t)$.

For any point on the t-y plane there will be a solution curve passing through it. Given a point on the plane, we can roughly draw the solution curve according to the slope field, for examples:



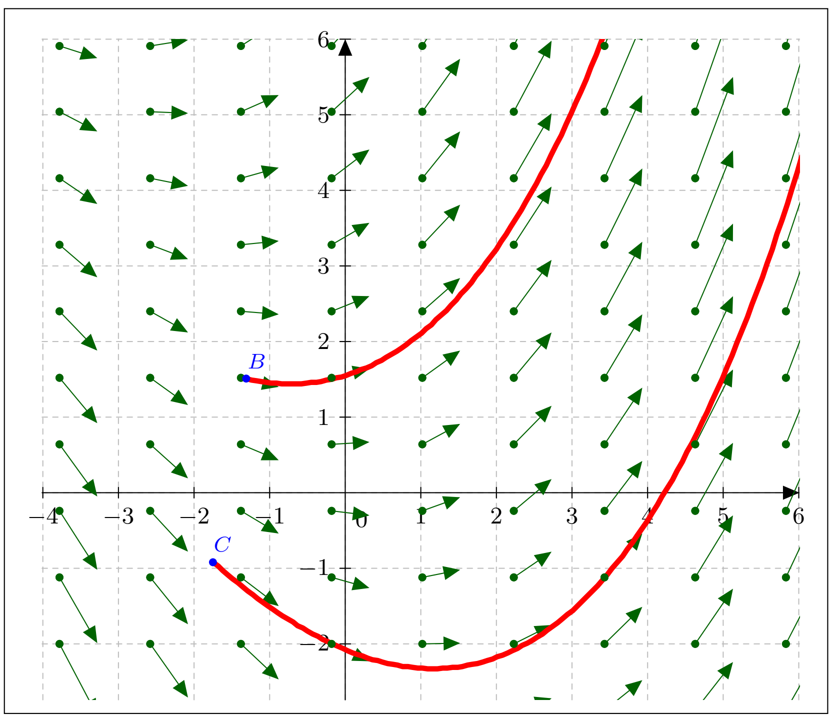
Example 2: $\dfrac{dy}{dx} = -\dfrac{x}{2}$.

We draw the slope field according to the following rule. At point (x_0, y_0), we draw a ray of slope $-\dfrac{x_0}{2}$.

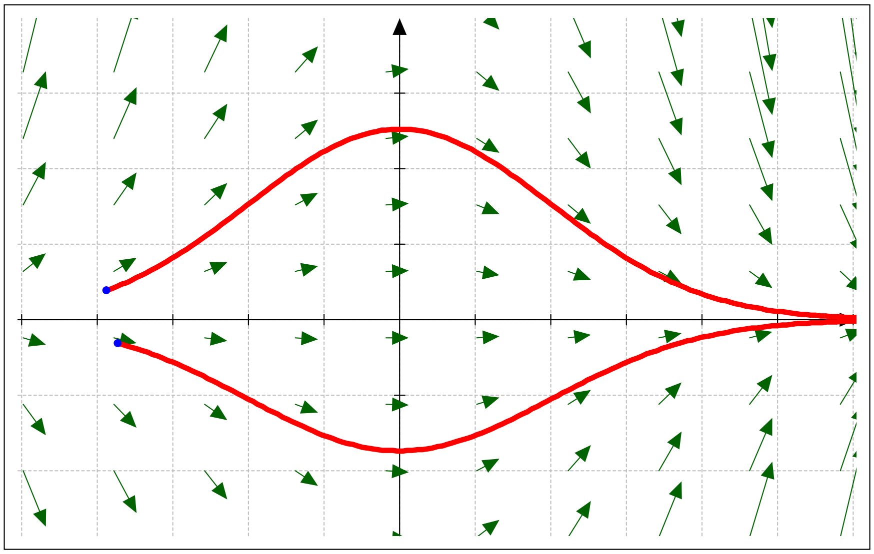


Example 3: $\dfrac{dy}{dx} = 0.4x + 0.2y$.

At point (x_0, y_0), we draw a ray of slope $0.4x_0 + 0.2y_0$.



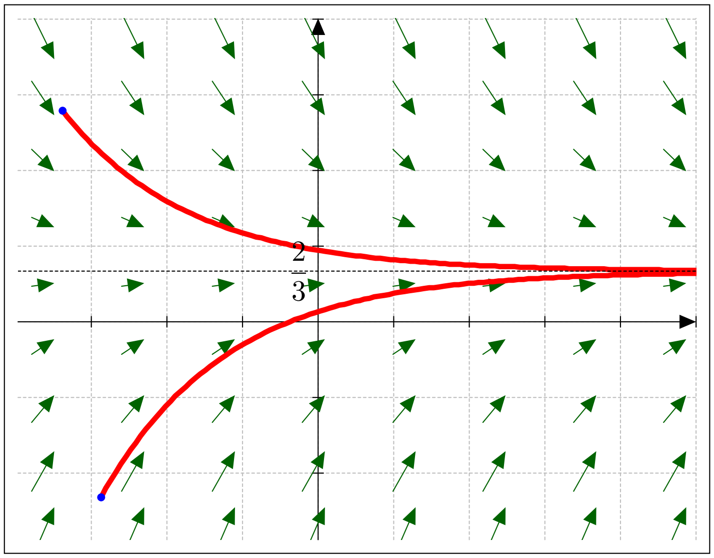
Example 4: $\dfrac{dy}{dx} = -\dfrac{xy}{4}$.



Example 5: For a DE which can be written in a form $\frac{dy}{dt} = ay + b$, for example $\frac{dy}{dt} = -3y + 2$,whose solution have the required behavior as $t \to \infty$, all solutions approach $y = \frac{2}{3}$.

$$\begin{aligned}
&\frac{dy}{dt} = -3y + 2 \\
&\frac{dy}{-3y+2} = dt \\
&\int \frac{1}{-3y+2} dy = \int 1 dt \\
&\int \frac{1}{y} dy = \ln|y| \ \Rightarrow \int \frac{1}{-3y+2} dy = -\frac{1}{3} \ln|-3y+2| \\
&-\frac{1}{3} \ln|-3y+2| = t + C, \quad \ln|-3y+2| = -3t + C \\
&|-3y+2| = e^{-3t+C} = Ce^{-3t}, \quad -3y+2 = Ce^{-3t} \\
&y = Ce^{-3t} + \frac{2}{3}
\end{aligned}$$

Since $\lim_{t\to\infty} e^{-3t} = 0$, $\lim_{t\to\infty} y = \frac{2}{3}$.



1.3 Classification of DEs

Definition of DE: Suppose that there is an independent variable (say t) and there is a dependent variable that is an unknown function of t (say $y(t)$). Then a DE is an identity that relates the independent variable, dependent variable, and its derivative.

Examples

- $\dfrac{dy}{dt} = y$, (1st order)
- $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = t$, (2nd order)
- $\dfrac{dy}{dt} = \sqrt{y^2+1}$, (1st order)

Orders: order of the highest derivative appearing in the DE.

Solution of a DE: a function satisfying the equation identically.

General Solution vs particular Solution: The general solution is a form of function such that every solution of the DE can be cast in the form.

Example $\dfrac{dy}{dt} = -y$

- (particular) Solutions: $y = e^{-x}$ (verify)

- (General solution) $y = Ce^{-x}$, C any real number.

Integral Curve=graph of a (particular) solution

General solution=a family of integral curves.

<u>**Linear and non-linear eqns**</u>:

$$F(t, y, y', ..., y^{(n)}) = 0$$

is said to be linear if F is a linear function of the variables y,y',...,y^n. Otherwise, it is a non-linear equation.

The general linear ordinary DE of order n is

$$\boxed{a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)}$$

Example: 1st order linear DE:

$$y' + p(t)y = g(t)$$

Example 1: Order and linearity

- $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$, 2nd order, non-linear
- $\dfrac{dy}{dt} + ty^2 = 0$, first order, non-linear.
- $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$, 3rd order, linear.
- $\sqrt{1 + y'} = t$, first order, non-linear.
- $\dfrac{1 + t^2y'}{1 + t^3y''} = 2t$, second order, linear.
- $\dfrac{1 + t^2y'}{1 + t^3y''} = 2t + y$, second order, nonlinear.

Example 2: Determine the value of r such that $y = e^{rt}$ is a soln of

$$y'' + y' - 6y = 0$$

Soln: Replace y by e^{rt} on the right hand-side,

$$(e^{rt})'' + (e^{rt})' - 6e^{rt} = r^2(e^{rt}) + r(e^{rt}) - 6e^{rt} = (r^2 + r - 6)e^{rt} = 0.$$

Therefore, it is equivalent to solve

$$\begin{aligned}
&r^2 + r - 6 = 0\\
&(r + 3)(r - 2) = 0\\
&r = -3,\ r = 2
\end{aligned}$$

Chapter 2

First Order Differential Equations

$$y' = f(t, y) \text{ or } \frac{dy}{dt} = f(t, y)$$

Main task: find the general solution.

2.1 Linear equations

The simplest case: $f(t, y)$ is independent of y

$$\boxed{y' = f(t)} \quad \boxed{y = \int f(t)dt}$$

Example 1 $y' = \cos t$

$$y = \int \cos t dt = \sin t + C$$

Linear 1st order DEs:

$f(t, y)$ is a linear function about y. For example

$$f(t, y) = -p(t)y + g(t)$$

where $p(t)$ and $g(t)$ are given functions.

$$\boxed{y' + p(t)y = g(t)}$$

Solution method – integration factor

Example 2: $y' - 2ty = t$, so $p(t) = -2t$, $g(t) = t$

Solution: key – rewrite the DE into a form that can be solved easily

$\mu = \mu(t)$: A function to be determined

$\mu y' - 2t\mu y = \mu t$

$\underbrace{\mu y' + \mu' y}_{[\mu y]'} - \mu' y - 2t\mu y = \mu t$

$[\mu y]' + \underbrace{[-\mu' - 2t\mu]}_{\text{choose } \mu \text{ such that this is zero}} y = \mu t$

$[\mu y]' = \mu t$

$\mu y = \int \mu(t) t dt + C$

$y = \dfrac{\int \mu(t) t dt + C}{\mu(t)}$

<u>**Find** $\mu(t)$</u>:

$-\mu' - 2t\mu = 0$

$\mu' = -2t\mu$ or $\dfrac{d\mu}{dt} = -2t\mu$

$\dfrac{d\mu}{\mu} = -2tdt \quad \int \dfrac{1}{\mu} d\mu = \int -2tdt$

$\ln \mu = -t^2 + C$

$\mu(t) = e^{-t^2} e^C$

$\mu(t) = e^{-t^2}$ (Choose a simple one!)

so

$y = \dfrac{\int e^{-t^2} \cdot t dt + C}{e^{-t^2}} = e^{t^2} \left[-\dfrac{1}{2} e^{-t^2} + C \right]$

$\boxed{y = -\dfrac{1}{2} + Ce^{t^2}}$

<u>**General Case**</u>:

$$\boxed{\begin{gathered} y' + p(t)y = g(t) \\ y = \frac{\int \mu(t) g(t) dt + C}{\mu(t)} \\ \mu(t) = e^{\int p(t) dt} \end{gathered}}$$

Proof: $y' + p(t)y = g(t)$. Multiply $\mu = \mu(t)$ to both sides.

$$\mu y' + \mu p(t)y = \mu g(t) \quad \mu \text{ to be determined}$$
$$\mu y' + \mu' y - \mu' y + \mu p(t)y = \mu g(t)$$
$$\underbrace{\mu y' + \mu' y}_{=(\mu y)'} - \underbrace{\mu' y - \mu p(t)y}_{\text{choose } \mu \text{ such that this is zero}} = \mu g(t)$$
$$\mu' y - \mu p(t)y = 0 \;\Rightarrow\; \mu' = \mu p(t) \quad \frac{\mu'}{\mu} = p(t)$$
$$\int \frac{\mu' dt}{\mu} = \int p(t)dt \quad \int \frac{d\mu}{u} = \int p(t)dt \quad \ln \mu = \int p(t)dt$$

$\boxed{\mu = e^{\int p(t)dt}}$ we just need to pick one particular μ, so we do not put C, the constant term here.

$$(\mu y)' = \mu g(t)$$
$$\mu y = \int \mu(t)g(t)dt + C$$
$$\boxed{y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}}$$

Examples:

1. $y' + \frac{1}{2}y = 2\cos t$

   $p(t) = \frac{1}{2}$, $g(t) = 2\cos t$

   $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{2}dt} = e^{t/2}$

   $$\int \mu(t)g(t)dt = \int e^{t/2} \cdot 2\cos t dt = \frac{4e^{t/2}\cos t + 8e^{t/2}\sin t}{5}$$

   $$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{2e^{t/2}\sin t - 4e^{t/2}\cos t + C}{5e^{t/2}} = \frac{4\cos t + 8\sin t}{5} + Ce^{-\frac{t}{2}}$$

   (Note: $Ce^{-\frac{t}{2}}$ is a general solution to $y' + \frac{1}{2}y = 0$.)

   Verification:

   $$\frac{d}{dt}\left(\frac{4\cos t + 8\sin t}{5}\right) + \frac{1}{2}\left(\frac{4\cos t + 8\sin t}{5}\right) = \frac{-4\sin t + 8\cos t}{5} + \frac{2\cos t + 4\sin t}{5} = 2\cos t$$

2. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$

   $p(t) = \frac{2}{t}$, $g(t) = \frac{\cos t}{t^2}$

   $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$

$$\int \mu(t)g(t)dt = \int t^2 \cdot \frac{\cos t}{t^2}dt = \int \cos t dt = \sin t$$

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\sin t + C}{t^2}$$

Initial Value Problem (IVP):

Sometimes it is important to pick out one particular solution. This is done by specifying and auxiliary condition (initial condition)

$$y(t_0) = y_0$$

or specifying that the solution curve should pass through (t_0, y_0)

DE + initial condition = IVP

Example 2 Solve

$$y' - 2y = 4 - t$$

Sketch the graphs of several solutions. Find the initial point on the y-axis that separates solutions that grow large positively from those that grow large negatively as $t \to \infty$

Solution $p(t) = -2$, $g(t) = 4 - t$

$$\mu(t) = e^{\int p(t)dt} = e^{-2t}$$

$$\begin{aligned} y &= \frac{\int \mu(t)g(t)dt + C}{\mu(t)} \\ &= \frac{\int e^{-2t}(4-t)dt + C}{e^{-2t}} \\ &= e^{2t}[\int e^{-2t}(4-t)dt - \int e^{-2t}tdt + C] \\ &= e^{2t}\left[-2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + C\right] \\ &= -2 + \frac{1}{2}t + \frac{1}{4} + Ce^{2t} \end{aligned}$$

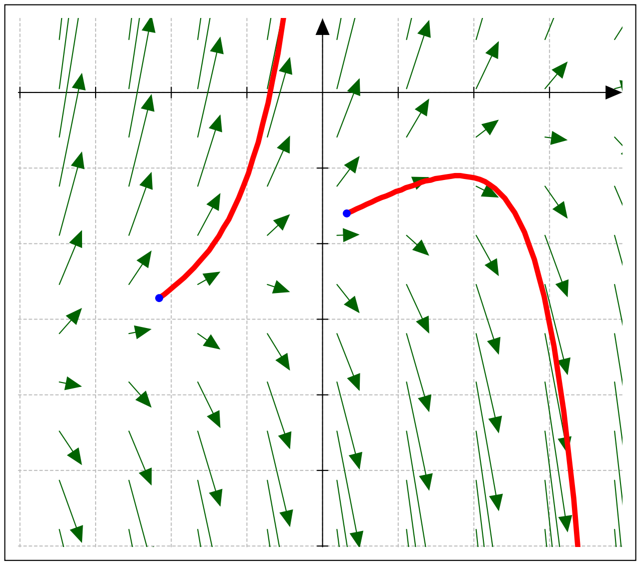
$$\begin{aligned} \int e^{-2t}tdt &= -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt \\ &= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{2t} \end{aligned}$$

$$\boxed{y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}}$$

- $C > 0$: $\lim_{t\to\infty} y = \infty$
- $C < 0$: $\lim_{t\to\infty} y = -\infty$

$$C = \frac{y + \frac{7}{4} - \frac{1}{2}t}{e^{2t}}$$

The threshold is the straight line $y = -\frac{1}{2}t - \frac{7}{4}$



Example 3: Find the solution of IVP $ty' + 2y = \sin t, \quad y\left(\frac{\pi}{2}\right) = 1$

Soln: $y' + \dfrac{2y}{t} = \dfrac{\sin t}{t}$.

$p(t) = \dfrac{2}{t}, \quad g(t) = \dfrac{\sin t}{t}$

$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2$

$$y = \frac{\int t^2 \frac{\sin t}{t} dt + C}{t^2}$$
$$= t^{-2}\left(\int t \sin t dt + C\right), \quad \sin t dt = d(-\cos t)$$
$$= t^{-2}\left(\int t d(-\cos t) + C\right), \quad \int u dv = uv - \int v du$$
$$= t^{-2}(-t\cos t + \int \cos t dt + C)$$

$y = t^{-2}(-t\cos t + \sin t + C)$

$y\left(\frac{\pi}{2}\right) = 1, \quad 1 = \dfrac{1}{\left(\frac{\pi}{2}\right)^2}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C\right)$

$\left(\frac{\pi}{2}\right)^2 = 1 + C$

$C = \left(\frac{\pi}{2}\right)^2 - 1$

$$\boxed{y = t^{-2}\left[-t\cos t + \sin t + \left(\frac{\pi}{2}\right)^2 - 1\right]}$$

Methods of Variation of parameters:

$$y' + p(t)y = g(t) \quad \text{inhomogeneous eqn.}$$

homogeneous equation (every term involves either y or y')

$$\begin{aligned}
&y' + p(t)y = 0\\
&\frac{y'}{y} = -p(t)\\
&[\ln y]' = -p(t)\\
&\ln y = -\int p(t)dt + C_1\\
&\boxed{y = Ce^{-\int p(t)dt}} \quad \text{Solution of the homogeneous eqn.}
\end{aligned}$$

To find the solution of the inhomogeneous eqn, let

$$\begin{aligned}
&y = A(t)e^{-\int p(t)dt}\\
&A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt}[-p(t)] + p(t)A(t)e^{-\int p(t)dt} = g(t)\\
&A'(t) = g(t)e^{\int p(t)dt}
\end{aligned}$$

Denote

$$\mu(t) = e^{\int p(t)dt}$$

$$A'(t) = g(t)\mu(t) \text{ or } A(t) = \int \mu(t)g(t)dt + C$$

$$\boxed{y = \frac{A(t)}{\mu(t)} = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}}$$

2.2 Separable equations

$$y' = f(t,y), \quad f(t,y) = -\frac{M(t)}{N(y)}$$

$f(t,y)$ is a product of two functions, one of them is a function of only t and the other is a function of only

y.

$$
\begin{aligned}
&\frac{dy}{dt} = -\frac{M(t)}{N(y)} \\
&N(y)dy = -M(t)dt \quad \text{or} \quad \boxed{M(y)dy + N(t)dt = 0} \\
&\underbrace{\int N(y)dy}_{H_2(y)} = -\underbrace{\int M(t)dt}_{H_1(t)} \\
&H_2(y) = -H_1(t) + C \\
&H_1(t) + H_2(y) = C \quad \text{implicit solution}
\end{aligned}
$$

Solution: if you can simplify it, simplify. If not, this is the solution – implicit solution.

Example 1: $\dfrac{dy}{dt} = -3y + 2$

$$
\begin{aligned}
&\frac{dy}{dt} = -3y + 2 \\
&\frac{dy}{-3y+2} = dt \\
&\int \frac{1}{-3y+2} dy = \int 1 dt \\
&\int \frac{1}{y} dy = \ln y \;\Rightarrow \int \frac{1}{-3y+2} dy = -\frac{1}{3} \ln -3y + 2 \\
&-\frac{1}{3} \ln(-3y+2) = t + C, \quad \ln(-3y+2) = -3t + C \\
&|-3y+2| = e^{-3t+C} = Ce^{-3t}, \quad -3y + 2 = Ce^{-3t} \\
&y = Ce^{-3t} + \frac{2}{3}
\end{aligned}
$$

Since $\lim_{t\to\infty} e^{-3t} = 0$, $\lim_{t\to\infty} y = \dfrac{2}{3}$.

$$
\boxed{\begin{gathered} y' = ay + b \text{ or } \frac{dy}{dt} = ay + b \\[1em] y = Ce^{at} - \frac{b}{a} \end{gathered}}
$$

Example 1: $\dfrac{dy}{dt} = \dfrac{t - e^{-t}}{y + e^{y}}, \quad y(0) = 1$

$$\begin{aligned}
&(y + e^{y})dy = (t - e^{-t})dt \\
&\int (y + e^{y})dy = \int (t - e^{-t})dt \\
&\frac{y^2}{2} + e^{y} = \frac{t^2}{2} + e^{-t} + C \\
&\frac{y^2}{2} + e^{y} - \frac{t^2}{2} - e^{-t} = C
\end{aligned}$$

$y(0) = 1, \; \dfrac{1^2}{2} + e^{1} - \dfrac{0^2}{2} - e^{0} = C$

$C = \dfrac{1}{2} + e - 1 = e - \dfrac{1}{2}.$

Example 2: $\begin{cases} y' = 1 + x + y^2 + xy^2 = (1 + x) + y^2(1 + x) = (1 + y^2)(1 + x) \\ y(0) = 0 \end{cases}$

Fact: $\displaystyle\int \frac{1}{1 + y^2} dy = \arctan y$

$$\int \frac{1}{1 + y^2} dy = \int 1 + x dx$$

$$\arctan y = x + \frac{x^2}{2} + C$$

find C: $\arctan(0) = 0 + \dfrac{0^2}{2} + C$

$C = 0$

$\arctan y = x + \dfrac{x^2}{2}$ or $\boxed{y = \tan(x + \frac{x^2}{2})}$

Example 3: $\begin{cases} y' = \dfrac{1 + 3x^2}{3y^2 - 6y} \\ y(0) = 1 \end{cases}$

Example 4: $y' = \dfrac{2y + 1}{y - 1}$

2.3 Mathematical Modeling with 1st Order equations

Recall The general solution of $y' = ay + b$, $a \neq 0$.

$$y' = ay + b \text{ or } \frac{dy}{dt} = ay + b$$

$$y = Ce^{at} - \frac{b}{a}$$

Example 1(Water tank problem)

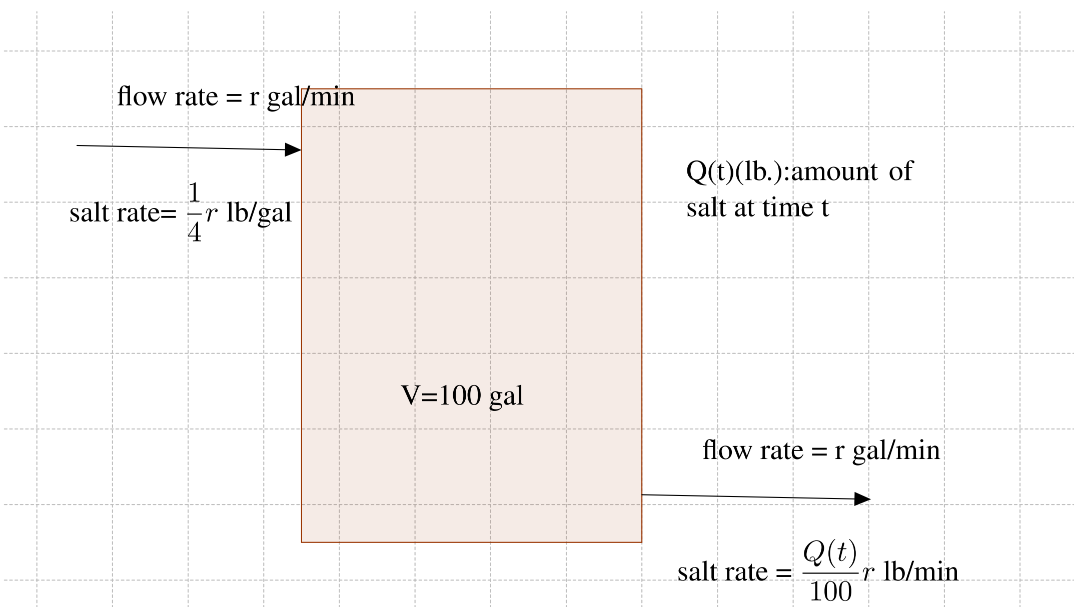
At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ (lb of salt)/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate.

(a)Set up the IVP describing this flow process.

(b)Find the amount of salt $Q(t)$ in the tank at any time; and also find the limiting amount Q_L that is present after a very long time.

(c)If $r = 3$, and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L.

(d)Find the flow rate that is required if the value of T is not to exceed 45 min.



- t: in min

- $Q(t)$: amount of salt in the tank
- r_{in}: rate of salt poured in the tank.

  r_{in} = incoming density of salt $\times$ incoming rate of solution $= \dfrac{1}{4}r$
- $\dfrac{Q(t)}{100}$: density of salt in the tank at time t.
- $r_{out} = \dfrac{Q}{100} \cdot r$: rate of salt following out.
- $Q(0) = Q_0$: initial condition
- (a) $\begin{cases} \dfrac{dQ}{dt} = r_{in} - r_{out} = \dfrac{r}{4} - \dfrac{Qr}{100} \\ Q(0) = Q_0 \end{cases}$
- $a = -\dfrac{r}{100}, \quad b = \dfrac{r}{4}, \quad \dfrac{b}{a} = -25 \; Q(t) = Ce^{-(r/100)t} + 25$
- (b) $Q(0) = Q_0, \quad C + 25 = Q_0, \quad \boxed{Q(t) = (Q_0 - 25)e^{-(r/100)t} + 25}$

  $Q_L = \lim_{t\to\infty} (Q_0 - 25)e^{-(r/100)t} + 25 = 25$
- (c) $Q(t) = (2Q_L - Q_L)e^{-3/100t} + Q_L = 1.02Q_L$

  $e^{-0.03t} + 1 = 1.02, \quad -0.03t = \ln 0.02, \quad t = 130.4$ min
- (d) $Q(45) = (2Q_L - Q_L)e^{-r/100\times 45} + Q_L = 1.02Q_L$

  $e^{-0.45r} + 1 = 1.02, \quad -0.45r = \ln 0.02, \quad r = 8.69$ lb/gal

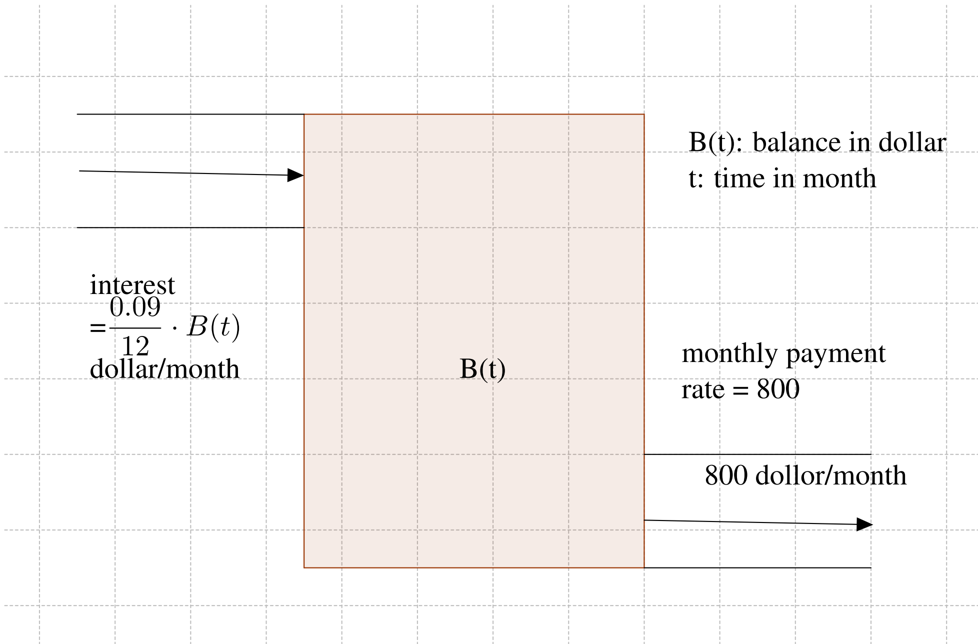
Example 2(Mortgage Problem)

A home buyer can afford to spend no more than $\$800/month$ on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 30 years.

Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow.

(b) Determine the total interest paid during the term of the mortgage.



- t: in month
- $B(t)$: balance (money owed to the bank)
- monthly interest rate=$\frac{0.09}{12} = 0.0075$
- $r_{in} = 0.0075B(t)$: increment of balance per month = interest per month
- $r_{out} = 800$: decrement of balance per month = monthly payment.
- $B(0) = B_0$: initial amount of money borrowed from the bank.
- $\begin{cases} \frac{dB}{dt} = r_{in} - r_{out} = 0.0075B - 800 \\ B(0) = B_0 \end{cases}$
- $B(t) = Ce^{0.0075t} + \frac{800}{0.0075}$
- $B(0) = C + \frac{800}{0.0075} = B_0$, $\boxed{B(t) = \left(B_0 - \frac{800}{0.0075}\right) e^{0.0075t} + \frac{800}{0.0075}}$
- (a) Find B_0. $B(30 \times 12) = 0$, $\left(B_0 - \frac{800}{0.0075}\right) e^{0.0075\times 30\times 12} + \frac{800}{0.0075} = 0$

  $B_0 = 99,498$
- (b) Find the total interest

Method 1: $TI = \int_0^{30\times 12} 0.0075 B(t)dt$

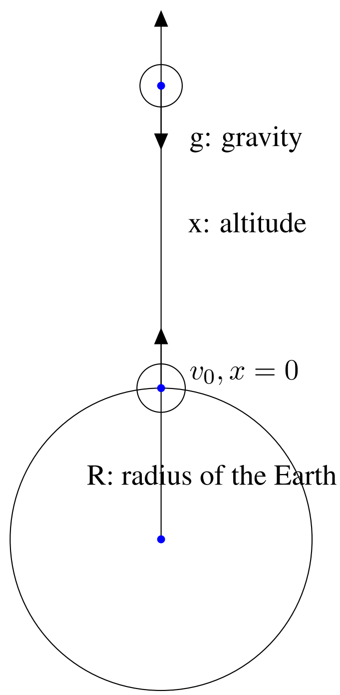
Method 2: $TI =$ total payment $- Q_0 = 800 \times 30 \times 12 - Q_0 = 288,000 - 99,498$

Example 3 (Escape Velocity)

A body of constant mass m is projected away from the earth in a direction perpendicular to the Earth's surface with an initial velocity v_0. The gravitational force acting on the body is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = k/(x+R)^2$, where R is the radius of the earth, and x is the distance between the body and the surface of the earth Assuming that there is no air resistance, but taking into account the variation of the Earth's gravitational field with distance,

(a)find an expression for the velocity during the ensuing motion,

(b)find the escape velocity.



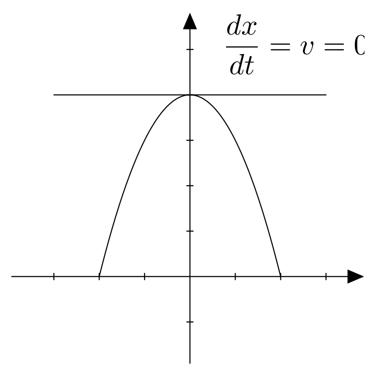
Soln:

- $F = ma, \quad a = \dfrac{dv}{dt}.$
- t in second

- $x(t)$: distance between the body and the surface of the earth at time t.
- R: radius of the earth.
- $m\frac{dv}{dt} = -w = -\frac{k}{(x+R)^2}$: negative sign signifies that w is directed in the negative v direction (starts from the surface and goes up).
- $w = mg = \frac{k}{(0+R)^2}, \quad k = mgR^2$: on the surface of earth ($x = 0$), $w = mg$.
- $\frac{dv}{dt} = \frac{dv}{dx} \cdot \underbrace{\frac{dx}{dt}}_{=v} = v\frac{dv}{dx}$: reduce variable t. From now on, v is a function of x.
- $v(0) = v_0$: on the surface of the earth ($x = 0$), velocity is v_0.
- (a) $\begin{cases} mv\frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2} \\ v(0) = v_0 \end{cases}$

$$vdv = -\frac{gR^2}{(x+R)^2}dx$$
$$\int vdv = -gR^2 \int \frac{1}{(x+R)^2}dx$$
$$\frac{1}{2}v^2 = \frac{gR^2}{x+R} + C$$
$$v(0) = v_0, \quad \frac{1}{2}v_0^2 = \frac{gR^2}{R} + C, \quad C = \frac{1}{2}v_0^2 - gR$$
$$\boxed{v(t) = \pm\sqrt{\frac{2gR^2}{x+R} + v_0^2 - 2gR}}$$

- (b) Find the escape velocity.



Set $v = 0$, maximize altitude $x_{max} = \frac{v_0^2 R}{2gR - v_0^2}$ or $v_0 = \sqrt{2gR \cdot \frac{x_{max}}{R + x_{max}}}$.
Let $x_{max} \to \infty$ (escape !) $\Rightarrow v_0 = \sqrt{2gR}$ (escape velocity).
$g = 9.8m/s^2$, $R = 6,286$ km, $\boxed{v_\infty \sim 11.1km/sec}$

2.5 Autonomous Equations and Population Dynamics

Autonomous equations: the independent variable (in the following case t) does not appear explicitly.

$$\frac{dy}{dt} = f(y)$$

has an important application in population dynamics.

objective: study a geometric method to obtain important qualitative information directly from the differential equation, without solving the equation — qualitative analysis.

- $y(t)$: population at time t.
- r: growth rate of the population.

1. Exponential Growth: (e.g.,when the resource is unlimited, such that the growth is roughly constant.)

Ex 1. In average, one rabbit reproduces r new baby rabbits per year. Totally, the population of rabbit increases by ry per year.

$$\begin{cases} \dfrac{dy}{dt} = ry, \quad r > 0 : \text{ growth rate} \\ y(0) = y_0 \end{cases}$$

Soln:

$$\frac{dy}{y} = rdt$$

$$\int \frac{dy}{y} = \int rdt + C$$

$$\ln y = rt + C$$

$$y = e^{rt+C} = e^C \cdot e^{rt} = Ce^{rt}, \quad \text{general solution}$$

$$y(0) = Ce^0 = y_0, \quad C = y_0$$

$$\boxed{y = y_0 e^{rt}}$$

Ex 2. In average, one rabbit reproduces r new baby rabbits per year. And at the same time, 10% rabbit dies per year (mortality rate is 0.1 per year). The population of rabbits changes by $ry - 0.1y$ per year.

$$\begin{cases} \dfrac{dy}{dt} = (r - 0.1)y \\ y(0) = y_0 \end{cases}$$

Soln: $\boxed{y = y_0 e^{(r-0.1)t}}$

Summary: When the growth rate is constant, then it's a exponential growth model. $\lim_{t\to\infty} y(t) = \infty$.

2. Logistic Growth: (e.g., when resource is limited) grow rate depends on the population or $r = h(y)$, such that

- $h(y) \simeq r$, when y is small.
- $h(y) \searrow$, when $y \nearrow$.
- $h(y) < 0$, when y is large.

$\Rightarrow\ h(y) = r - ay = r\left(1 - \frac{y}{k}\right)$.

$\Rightarrow$ Logistic equation:

$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right) y}$$

r: intrinsic growth rate.

k: saturation level.

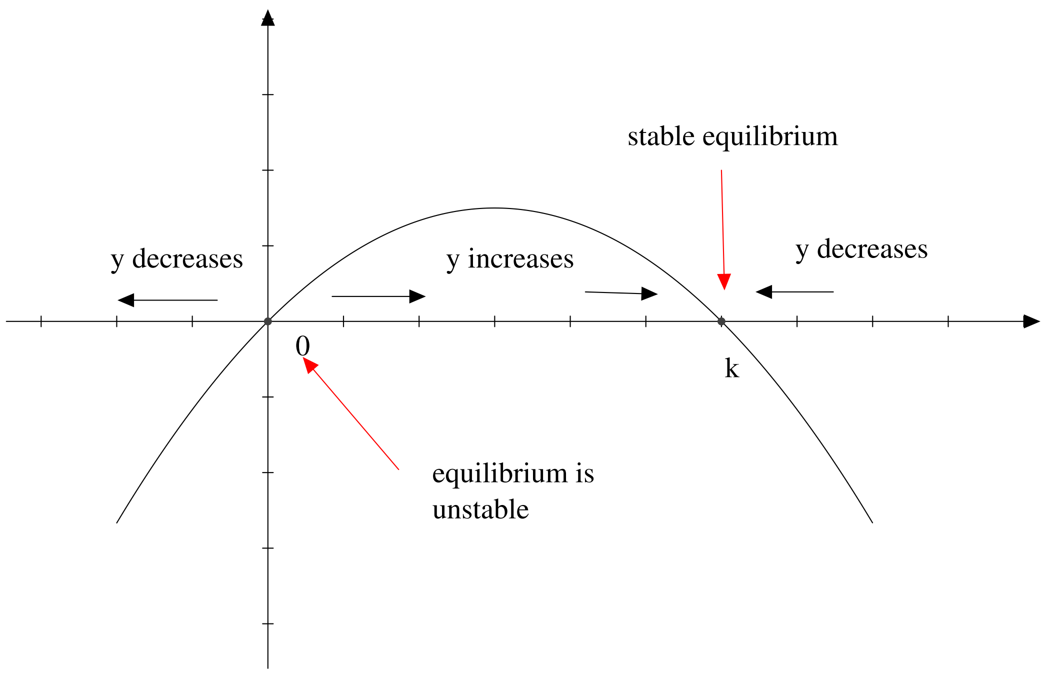
Qualitative analysis of the logistic equation:

Let $f(y) \equiv r\left(1 - \frac{y}{k}\right) y$

- Step 1: Set $f(y) = 0$, obtain zeros: $y = 0$ and $y = k \Rightarrow$ two constant solutions:

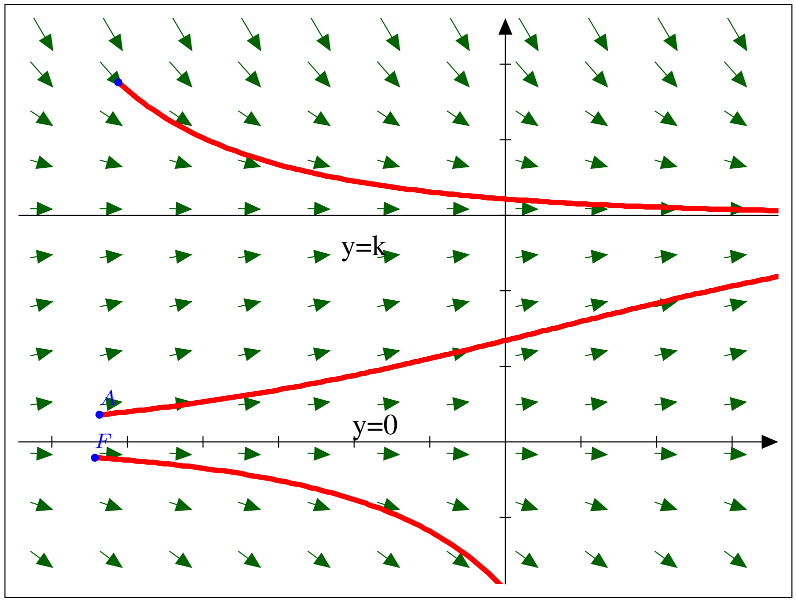
$$y = \phi_1(t) = 0, \quad y = \phi_2(t) = k.$$

- Step 2: sketch the graph of $f(y)$:



- $\frac{dy}{dt} > 0$, for $0 < y < k$
- $\frac{dy}{dt} < 0$, for $y > k$
- $\frac{dy}{dt} < 0$, for $y < 0$

• Step 3: sketch integral curves



$\phi_1(t) = k$: asymptotically stable because every nearby integral curve is converging to k as $t \to \infty$.

$\phi_2(t) = 0$: asymptotically unstable because every nearby integral curve is leaving. A little disturbing will drag away the curve.

<u>**Verifying**</u> (Separable equation) $\frac{dy}{\left(1-\frac{y}{k}\right)y} = rdt$

partial fraction $\frac{1}{\left(1-\frac{y}{k}\right)y} = \frac{1}{y} + \frac{1}{k-y}$

$$\int \frac{1}{y} + \frac{1}{k-y} dy = \int rdt + C$$

$$\ln y - \ln(k-y) = rt + C, \quad \underbrace{\boxed{\ln\left(\frac{y}{k-y}\right) = rt + C}}_{\text{Logistic Model}}, \quad \frac{y}{k-y} = C_2 e^{rt}$$

$$y(0) = y_0, \quad \frac{y_0}{k-y_0} = C_2$$

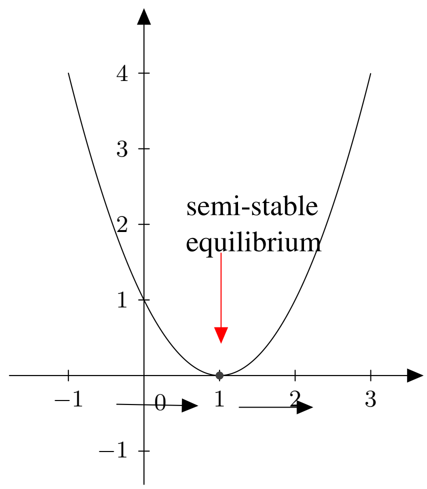
$\Rightarrow y = \dfrac{y_0 k}{y_0 + (k - y_0)e^{-rt}}$

$t \to \infty, y \to k.$

Example 1: (Semi-stable) $\dfrac{dy}{dt} = (1-y)^2$

Zeros: $y = 1$.

Graph of $(1-y)^2$



Example 2 $\dfrac{dy}{dt} = e^y - 1$

zeros: $y = 0$

signs of $f(y) = e^y - 1$.

- $y < 0$, $f(y) < 0$
- $y > 0$, $f(y) > 0$

$y = 0$ is a non-stable equilibrium.

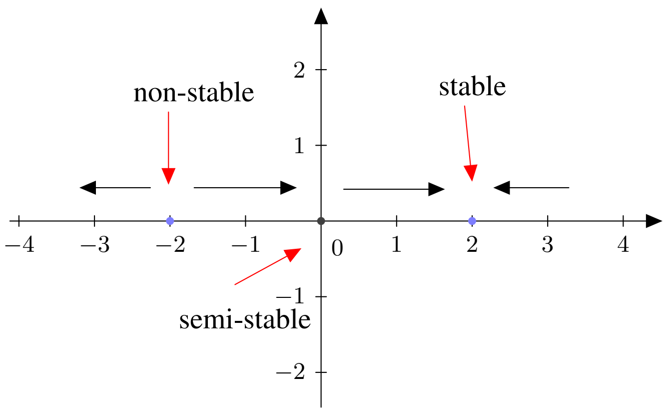
Example 3 $\dfrac{dy}{dt} = y^2(4-y^2)$

zeros: $y = 0$, $y = -2$, $y = 2$

signs of $f(y) = y^2(4-y^2)$.

- $y < -2$, for example $f(-3) = 9(4-9) < 0$

- $-2 < y < 0$, $f(-1) = 1(4-1) > 0$
- $0 < y < 2$, $f(1) = 1(4-1) > 0$
- $y > 2$, $f(3) = 9(4-9) < 0$



2.6 Exact Equations

$$\frac{dy}{dt} = f(t, y) \equiv -\frac{M(t, y)}{N(t, y)} \quad (*)$$

1. Rewrite the DE into

$$\boxed{M(t, y)dt + N(t, y)dy = 0}$$

2. Consider the total variation of $\psi(t, y)$:

$$\boxed{d\psi(t, y) \equiv \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial y} dy}$$

which measures the change of ψ when t and y undergo a small change.

If $\dfrac{\partial \psi}{\partial t} = M(t, y)$, $\dfrac{\partial \psi}{\partial y} = N(t, y)$ then

$d\psi(t, y) = \underbrace{M(t, y)}_{= \frac{\partial \psi}{\partial t}} dt + \underbrace{N(t, y)}_{\frac{\partial \psi}{\partial y}} dt = 0$

$\Rightarrow d\psi(t, y) = 0, \quad \Rightarrow \quad \psi(t, y) = C$ (constant)

Therefore, $\boxed{\psi(t, y) = C}$ is a general solution of (*) of an implicit form.

3. Condition for the existence of such a $\psi(t,y)$ satisfying $\dfrac{\partial\psi}{\partial t} = M(t,y)$, $\dfrac{\partial\psi}{\partial y} = N(t,y)$:

$$\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial y}\right)$$

or

$$\boxed{\frac{\partial}{\partial y}M(t,y) = \frac{\partial}{\partial t}N(t,y)}, \quad \text{Definition of exact equation}$$

<u>**Theorem**</u> Rewrite the DE into

$$\underbrace{M(t,y)}_{\frac{\partial\psi}{\partial t}}\,dt + \underbrace{N(t,y)}_{\frac{\partial\psi}{\partial y}}\,dy = 0$$

If M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

then there exists a function $\psi = \psi(t,y)$ such that

$$\frac{\partial\psi}{\partial t} = M(t,y), \quad \frac{\partial\psi}{\partial y} = N(t,y).$$

4 How to find ψ.

Examples Determine whether each of the equations is exact. If it is exact, find the solution.

(a) $(y+4t) + (t-y)\dfrac{dy}{dt} = 0.$

$$(y+4t)dt + (t-y)dy = 0$$

$$\begin{cases} \dfrac{\partial}{\partial y}(y+4t) = 1 \\ \dfrac{\partial}{\partial t}(t-y) = 1 \end{cases}$$

So it is an exact equation.

$$\begin{cases} \psi(t,y) = \displaystyle\int M(t,y)dt = \int (y+4t)dt = yt + 2t^2 + C_1(y) \quad (1) \\ \psi(t,y) = \displaystyle\int N(t,y)dy = \int t - ydy = ty - \frac{y^2}{2} + C_2(t) \quad (2) \end{cases}$$

So $yt + 2t^2 + C_1(y) = ty - \dfrac{y^2}{2} + C_2(t)$.

$$C_2(t) = 2t^2, \quad C_1(y) = -\frac{y^2}{2}$$

So $\psi(t,y) = yt + 2t^2 - \dfrac{y^2}{2}$. The solution of the DE is

$$\psi(t,y) = yt + 2t^2 - \frac{y^2}{2} = C$$

which is a implicit solution of $y(t)$. If given initial condition $y(1) = 0$, then

$$\psi(1,0) = 0 + 2 = C, \Rightarrow yt + 2t^2 - \frac{y^2}{2} = 2.$$

verifying the solution:

$$\begin{cases} \dfrac{\partial}{\partial y}(yt + 2t^2 - \dfrac{y^2}{2}) = t - y = N(t,y) \\ \dfrac{\partial}{\partial t}(yt + 2t^2 - \dfrac{y^2}{2}) = y + 4t = M(t,y) \end{cases}$$

(b) $e^x \sin y - 2y \sin x + (e^x \cos y + 2\cos x)y' = 0$

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{dy}{dx} = 0,$$

$$(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0$$

$$\begin{cases} \dfrac{\partial}{\partial y}(e^x \sin y - 2y \sin x) = e^x \cos y - 2\sin x \\ \dfrac{\partial}{\partial x}(e^x \cos y + 2\cos x) = e^x \cos y - 2\sin x \end{cases}$$

It is an exact equation.

$$\begin{cases} \psi(t,y) = \displaystyle\int (e^x \sin y - 2y \sin x)dx = e^x \sin y + 2y\cos x + C_1(y) \\ \psi(t,y) = \displaystyle\int (e^x \cos y + 2\cos x)dy = e^x \sin x + 2y \cos x + C_2(x) \end{cases}$$

$\Rightarrow$

$$C_1(y) = 0, \quad C_2(x) = 0$$

Solution:

$$\psi(x,y) = e^x \sin y + 2y\cos x = C$$

(c) $(x \ln y + xy)dx + (y \ln x + xy)dy = 0$

$$\begin{cases} \dfrac{\partial}{\partial y}(x \ln y + xy) = \dfrac{x}{y} + x \\ \dfrac{\partial}{\partial x}(y \ln x + xy) = \dfrac{y}{x} + x \end{cases}$$

Not exact.

(d) $y^2 dx + (2xy - ye^y)dy = 0, \quad y(1) = 2$

- Check exactness. $\begin{cases} \dfrac{\partial}{\partial y} y^2 = 2y \\ \dfrac{\partial}{\partial x}(2xy - ye^y) = 2y \end{cases}$
- Exact.
- $\begin{cases} \psi(x,y) = \displaystyle\int y^2 dx = xy^2 + C_1(y) \\ \psi(x,y) = \displaystyle\int (2xy - ye^y) dy = xy^2 + e^y - ye^y + C_2(x) \end{cases}$
- $\displaystyle\int ye^y dy = \int y de^y = ye^y - \int e^y dy = ye^y - e^y$
- $C_1(y) = e^y - ye^y, \quad C_2(x) = 0$
- $\psi(x,y) = xy^2 + e^y - ye^y$
- General solution: $xy^2 + e^y - ye^y = C$
- Initial value: $1 \to x,\ 2 \to y, \quad 4 + e^2 - 2e^2 = C = 4 - e^2$
- Solution: $xy^2 + e^y - ye^y = 4 - e^2$

2.7 Theory of 1st Order DEs and difference between linear and non-linear DEs

For a given IVP

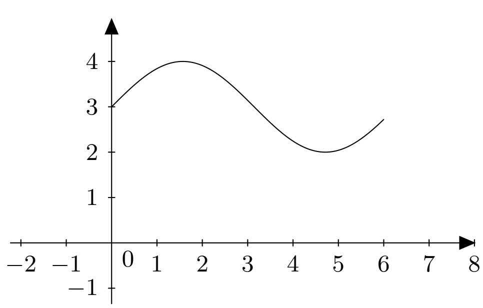
$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

most concerned:existence and uniqueness of the solutions.

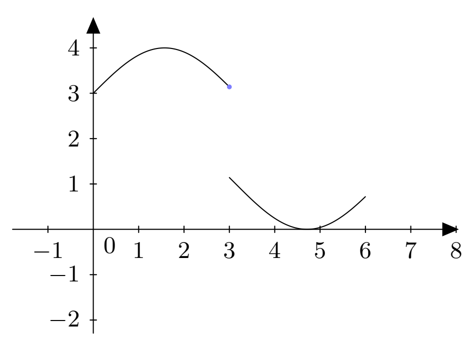
Theorem If functions $p(t)$ and $g(t)$ are continuous on an open interval $I : \alpha < t < \beta$ containing point $t = t_0$, then there exists a unique solution $y = \psi(t)$ of the IVP

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

$y = \psi(t)$ is valid for all $t \in I$.



Solution exists and be unique.



not sure

Example: $y' + \frac{2}{t}y = \frac{\sin t}{t}, \; y\left(\frac{\pi}{2}\right) = 1$

$p(t) = \frac{2}{t}$, $g(t) = \frac{\sin t}{t}$ are continuous in $(-\infty, 0) \cup (0, \infty)$. But $t = \frac{\pi}{2}$ is in $(0, \infty)$.

Therefore there exists a unique solution in $t \in (0, \infty)$.

Theorem 2 (General case including nonlinear DEs)

If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R: \; |t| \le a, \; |y| \le b$, then there is some interval $|t| \le h \le a$ (Note: h can be very small) in which there exists a unique solution $y = \psi(t)$ of the IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Example 2 $\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases}$

$f(t, y) = y^2$, $\frac{\partial f}{\partial y} = 2y$ very smooth. But

$$y(t) = -\frac{1}{(t - t_0) - \frac{1}{y_0}} = \frac{1}{\frac{1}{y_0} - (t - t_0)}$$

If $y_0 > 0$, the solution is valid only

$$\frac{1}{y_0} - (t - t_0) > 0, \text{ or } t < t_0 + \frac{1}{y_0}$$

Comparison:

	linear	nonlinear
Problem	$y' + p(t)y = g(t)$ $y(t_0) = y_0$	$y' = f(t, y)$ $y(t_0) = y_0$
General Soln formula	$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$ $\mu = e^{\int p(t)dt}$	non-available
Existence and Uniqueness	Theorem 1	Theorem 2
Interval of definition	global: determine by $p(t)$ and $g(t)$	local, not indicated in $f(t, y)$
Examples	$y' + \frac{2}{t}y = 4t$ $y = t^2 + \frac{C}{t^2}$	$y' = y^2$ $y = -\frac{1}{t + C}$

Review of Chapters 1 and 2

Chapter 1: Introduction

- basic concepts: DE, order, linearity, solution, the general solution, integral curve.
- Direction field.
- homogeneous equation: $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$
  - One particular solution is $y = e^{rt}$, where r is the solution of $r^n + a_1 r^{n-1} + \cdot a_{n-1} r + a_n = 0$

Chapter 2: 1st order DEs $\dfrac{dy}{dt} = f(t, y)$

- $y' = f(t): \ y = \displaystyle\int f(t)dt.$
- Linear equation $y' + p(t)y = g(t)$:
  - Every first order linear equation can be written as $y' + p(t)y = g(t)$.

$$a(t)y' + b(t)y + d(t) = 0, \quad y' + \frac{b(t)}{a(t)}y = -\frac{d(t)}{a(t)}$$

  - general solution:

$$\boxed{\begin{array}{c} y' + p(t)y = g(t) \\[1em] y = \dfrac{\int \mu(t)g(t)dt + C}{\mu(t)} \\[1em] \mu(t) = e^{\int p(t)dt} \end{array}}$$

- separate equation $y' \equiv \dfrac{dy}{dt} = -\dfrac{M(t)}{N(y)}$.
  - general solution $\displaystyle\int M(t)dt + \int N(y)dy = C$
  -

$$\boxed{\begin{array}{c} y' = ay + b \text{ or } \dfrac{dy}{dt} = ay + b \\[1em] y = Ce^{at} - \dfrac{b}{a} \end{array}}$$

– Logistic model

$$\boxed{\begin{array}{c} y' = k(y+a)(y+b) \text{ or } \dfrac{dy}{dt} = k(y+a)(y+b) \\[2ex] \ln\left(\dfrac{y+a}{y+b}\right) = k(b-a)t + C, \text{ or } \dfrac{y+a}{y+b} = Ce^{k(b-a)t} \end{array}}$$

- Exact equation $M(t,y)dt + N(t,y)dy = 0$
  – Exact: $\dfrac{\partial}{\partial y}M(t,y) = \dfrac{\partial}{\partial t}N(t,y)$
  – find $\psi(t,y)$ such that

$$\psi(t,y) = \int M(t,y)dt + C_1(y) = \int N(t,y)dy + C_2(t)$$

  – General solution :$\psi(t,y) = C$.
- Qualitative analysis for autonomous equation $y' = f(y)$: equilibrium solution $f(y) = 0$, stability of equilibrium solutions.
- Theory (existence and uniqueness)
- Application: Water tank problems, loan problems, population dynamics (logistic model).

Examples:

1. $\dfrac{dy}{dt} = \dfrac{2t+y}{3+3y^2-t}$
2. $y' = \dfrac{t^2-1}{y^2+1}$
3. $\dfrac{dy}{dt} = k(y+a)(y+b)$
   - $\dfrac{1}{k(y+a)(y+b)}dy = dt$
   - $\dfrac{1}{(y+a)(y+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{y+a} - \dfrac{1}{y+b}\right)$
   - $\displaystyle\int \frac{1}{k(y+a)(y+b)}dy = \frac{1}{k(b-a)}(\ln(y+a) - \ln(y+b))$
   - $\displaystyle\int \frac{1}{k(y+a)(y+b)}dy = \int 1dt + C, \quad \frac{1}{k(b-a)}(\ln(y+a) - \ln(y+b)) = t + C$

- General solution: $\ln\left(\frac{y+a}{y+b}\right) = k(b-a)t + C$, or $\frac{y+a}{y+b} = Ce^{k(b-a)t}$

$$\boxed{\begin{array}{c} y' = k(y+a)(y+b) \text{ or } \dfrac{dy}{dt} = k(y+a)(y+b) \\[2ex] \ln\left(\dfrac{y+a}{y+b}\right) = k(b-a)t + C, \text{ or } \dfrac{y+a}{y+b} = Ce^{k(b-a)t} \end{array}}$$

4. $\frac{dy}{dt} = y(1-y)$
5. $\frac{dy}{dt} = y^2 - 3y - 4$
6. $y' = \cos t - \frac{y}{t}$