Tutorial on turbulence (1):

Turbulent flows

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Independent Junior Research Group:

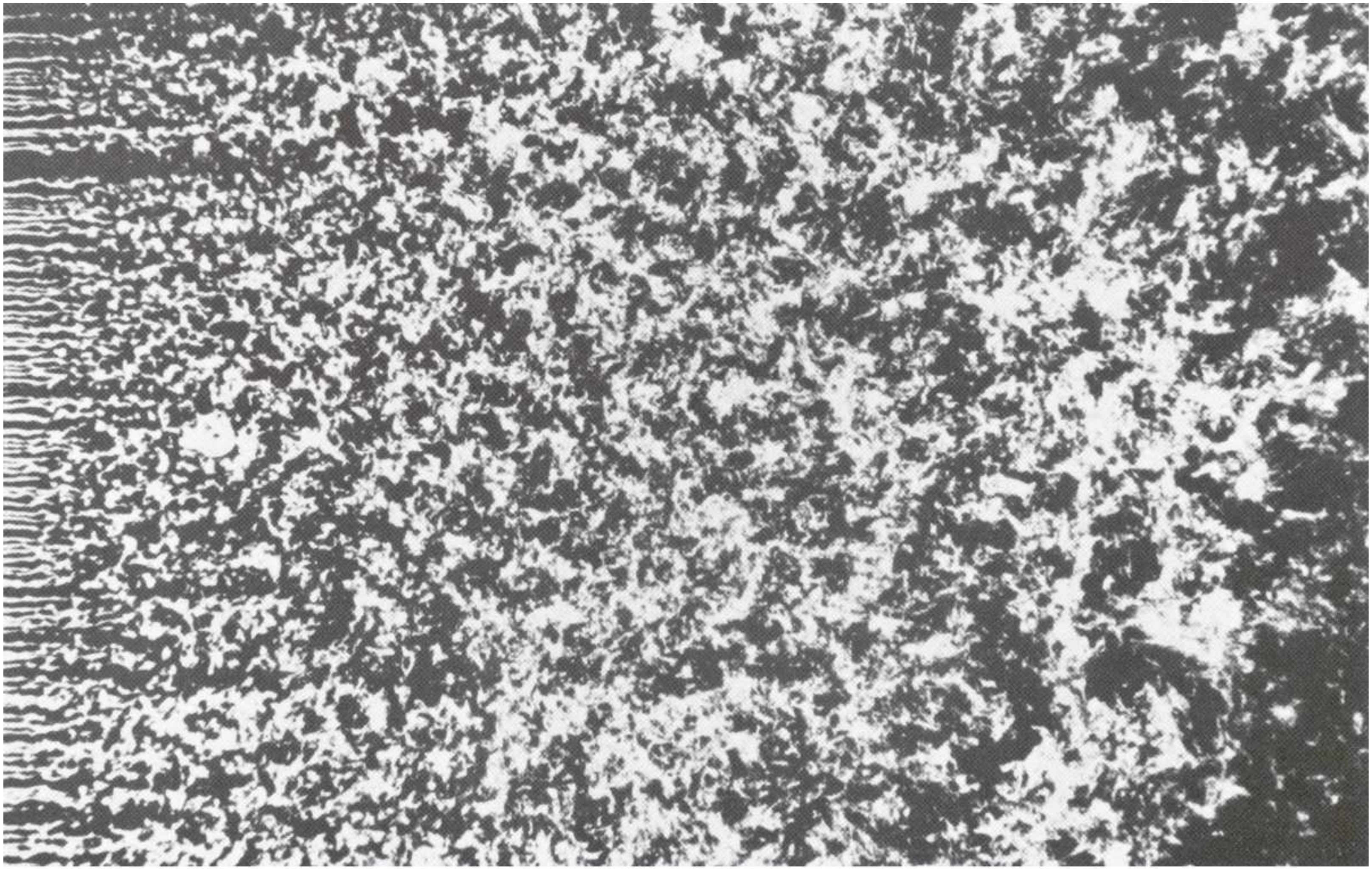
Computational Studies of Turbulence in Magnetized Plasma

IPP, Garching

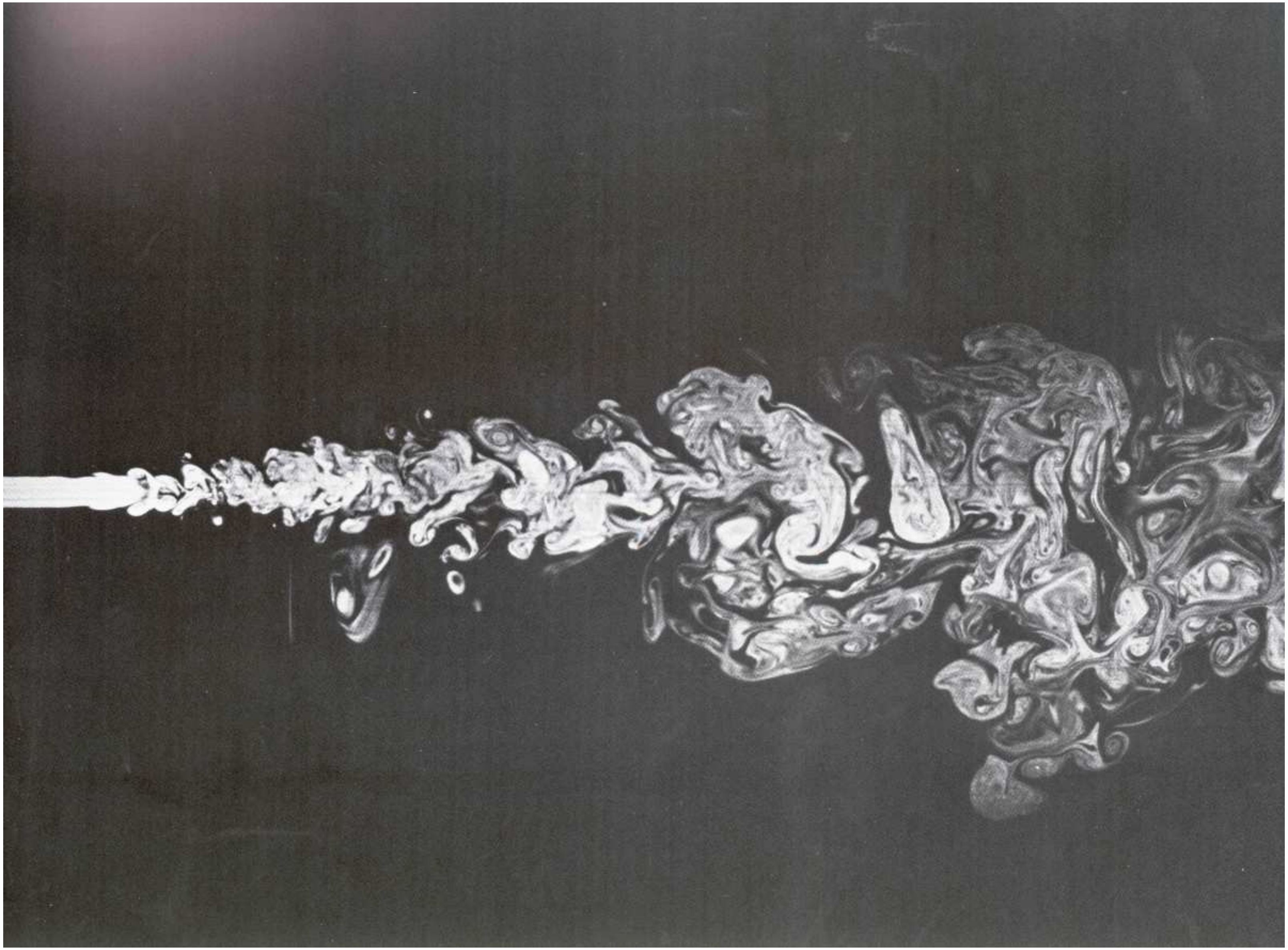
Outline

1. Examples of turbulence
2. Definitions and basic equations
3. Length scales and Richardson cascade picture of turbulence
4. Observations, facts
5. Kolmogorov 1941 theory
6. Two-point closure theory
7. Intermittency
8. Generalization to MHD

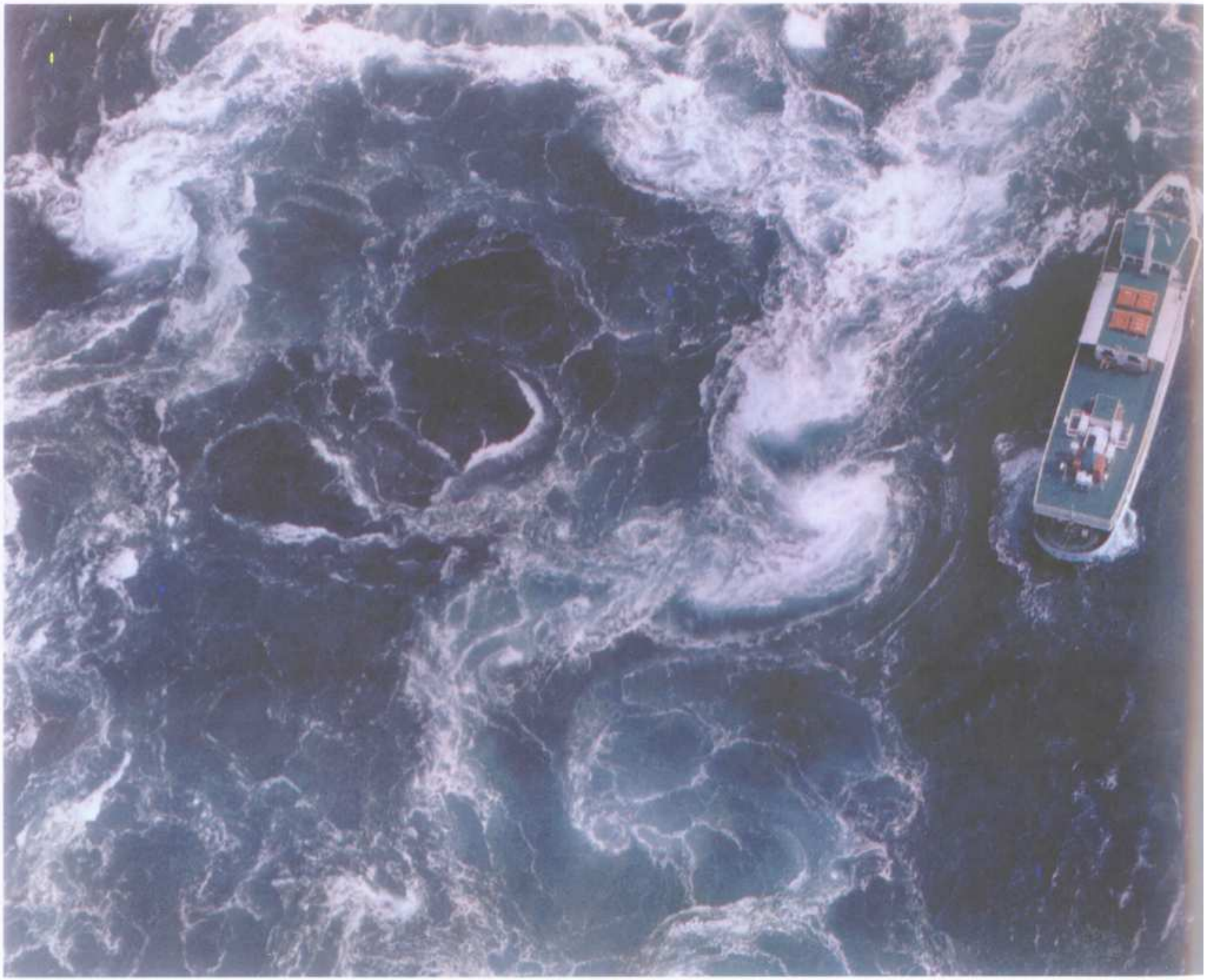
Examples: Grid turbulence



Examples: Turbulent jet



Examples: Vortices



Examples of turbulent systems

- smoke from a chimney, cigarette
- water in a river, canals
- oceanic currents
- fumes from a rocket motor
- processes in pumps, boilers
- flows in pipelines, tunnels
- flows around vehicles
- mixing in reactors
- combustion processes
- acceleration of chemical reactions
- fusion in tokamaks, stellarators
- strong wind, hurricans
- cumulus clouds
- pollutants in the atmosphere
- disturbances in the atmosphere
- Earth's outer core
- wake of the Earth
- Solar convective zone
- solar wind
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Big part of the Nature is in turbulent state!

To be turbulent is natural!

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The best description is with a set of properties:

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Important: Turbulence is not a feature of fluids but of fluid flows!

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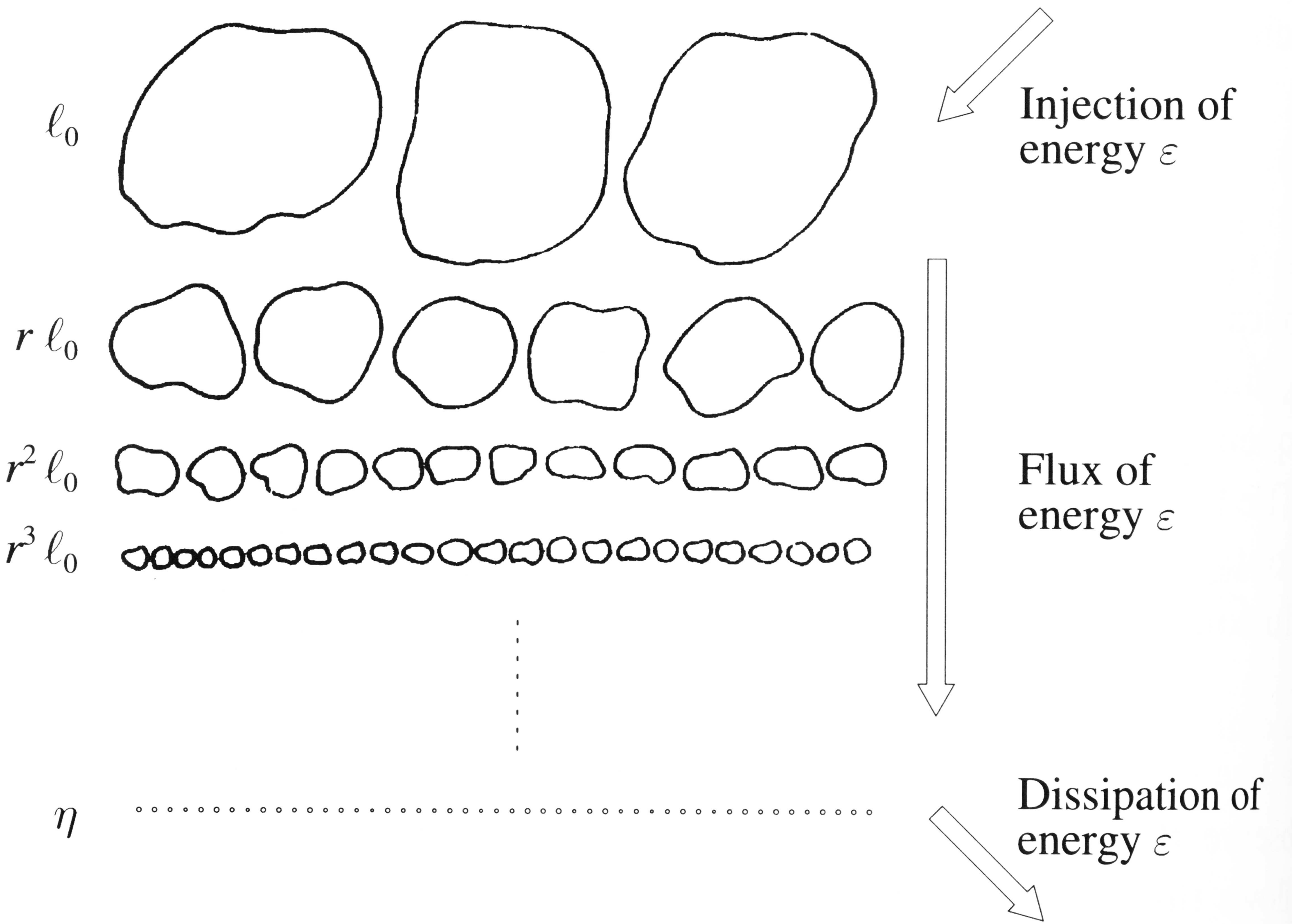
Limited validity:

- only hydrodynamical turbulence, without kinetic effects
- continuum approach, smallest scales larger than mean-free path

$$L_{integral} \gg l_{intermediate} \gg l_{smallscales} \gg l_{meanfreepath}$$

- without large velocities introducing compressibility
- without relativistic effects
- etc...

Richardson cascade



Large scale separation: many different scales

Basic picture: Turbulence composed of *eddies* of different size.

eddy size l, characteristic velocity $v(l)$ and timescale $\tau(l) \equiv l/v(l)$

Different scales:

* Energy input at integral scales $\sim l_0$
* Transfer of this energy to smaller and smaller scales $\sim l$
* At smallest scales, the energy is dissipated by viscosity $\sim l_d$

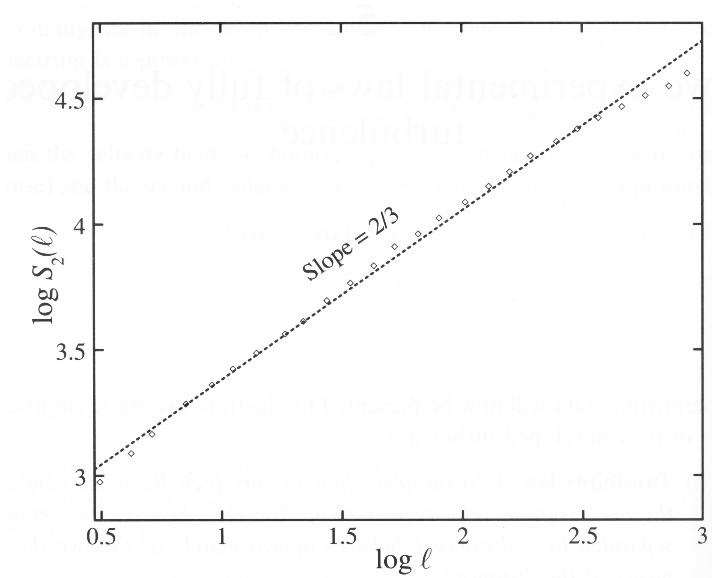
Large eddies are unstable, break up, transferring their energy to somewhat smaller eddies. This process - *energy cascade* - continues until the $Re(l) \equiv v(l)l/\nu$ is sufficiently small.

energy is transfered with the rate: $\epsilon = (v_l)^2/\tau_l = (v_l)^3/l \sim \varepsilon$

Observations: 2/3 empirical law

Second order structure function:

$$S_2(l) \equiv < (\delta v(l))^2 > \qquad \delta v(\mathbf{r}, \mathbf{l}) \equiv [\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l}$$



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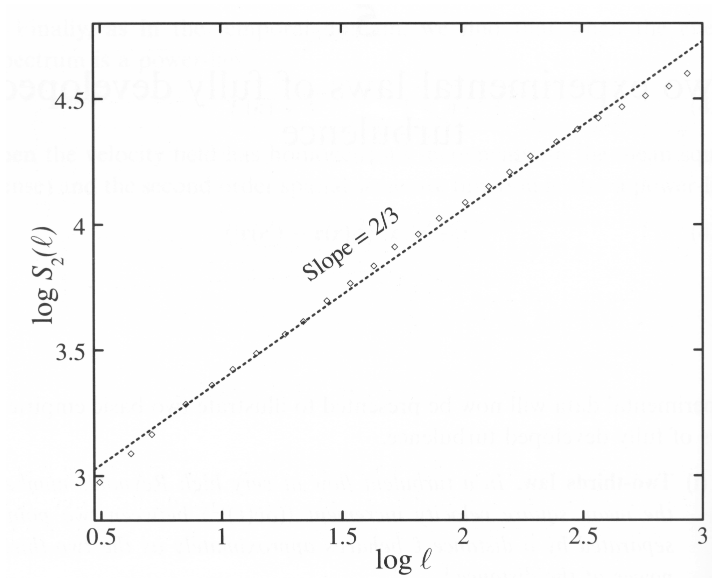
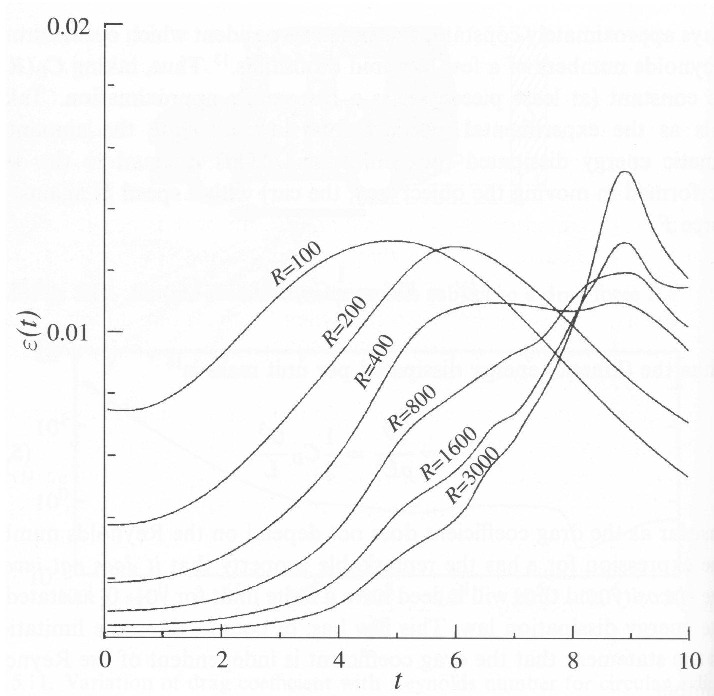


1) Two-thirds law: In a turbulent flow at very high Reynolds number, the mean square velocity increment $<(\delta v_{||}(l))^2>$ between two points separated by a distance l behaves approximately as the two-thirds power of the distance.

Observations: finite dissipation empirical law

Mean energy dissipation: $\varepsilon = \nu \int_V \omega^2 dV$



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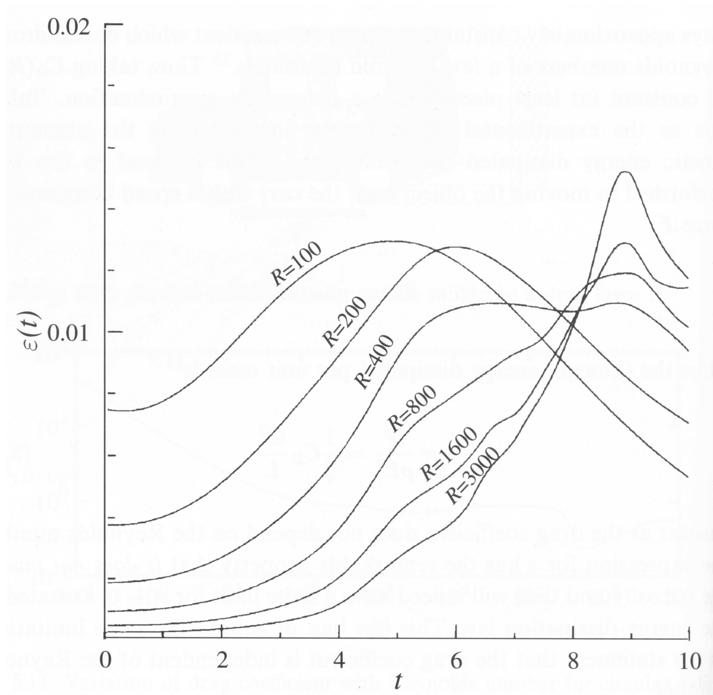

2) The law of finite energy dissipation: If in an experiment on turbulent flow all the control parameters are kept the same, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass ε behaves in a way consistent with a finite positive limit.

Exact result - exception in the turbulence theory

Derived from Navier-Stokes equation: **Karman-Horwath-Monin relation**

$$-\frac{1}{4}\nabla_l \cdot < |\delta\mathbf{v}(\mathbf{l})|^2\delta\mathbf{v}(\mathbf{l}) >= -\partial_t\frac{1}{2} < \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}+\mathbf{l}) >$$

$$+ < \mathbf{v}(\mathbf{r})\cdot\frac{\mathbf{f}(\mathbf{r}+\mathbf{l})+\mathbf{f}(\mathbf{r}-\mathbf{l})}{2} > +\nu\nabla_l^2 < \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}+\mathbf{l}) >$$

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The four-fifth law:

$$< (\delta v_{\|}(\mathbf{r}, \mathbf{l})^3 >= -\frac{4}{5} \varepsilon l$$

- in the limit of infinite Reynolds number
- for homogeneous and isotropic case
- for increments smaller than the integral scale

Kolmogorov 1941 theory I

Kolmogorov's hypothesis of local isotropy: At sufficiently high Reynolds number, the small-scale turbulent motions $(l \ll l_0)$ are statistically isotropic.

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Kolmogorov (dissipation) scales:

$$\eta \equiv (\nu^3/\varepsilon)^{1/4}, \quad v_\eta \equiv (\varepsilon\nu)^{1/4}, \quad \tau_\eta \equiv (\nu/\varepsilon)^{1/2}; \qquad Re(\eta) = \eta v_\eta/\nu = 1$$

$$l_0/\eta \sim Re^{3/4}, \quad v_0/v_\eta \sim Re^{1/4}, \quad \tau_0/\tau_\eta \sim Re^{1/2}$$

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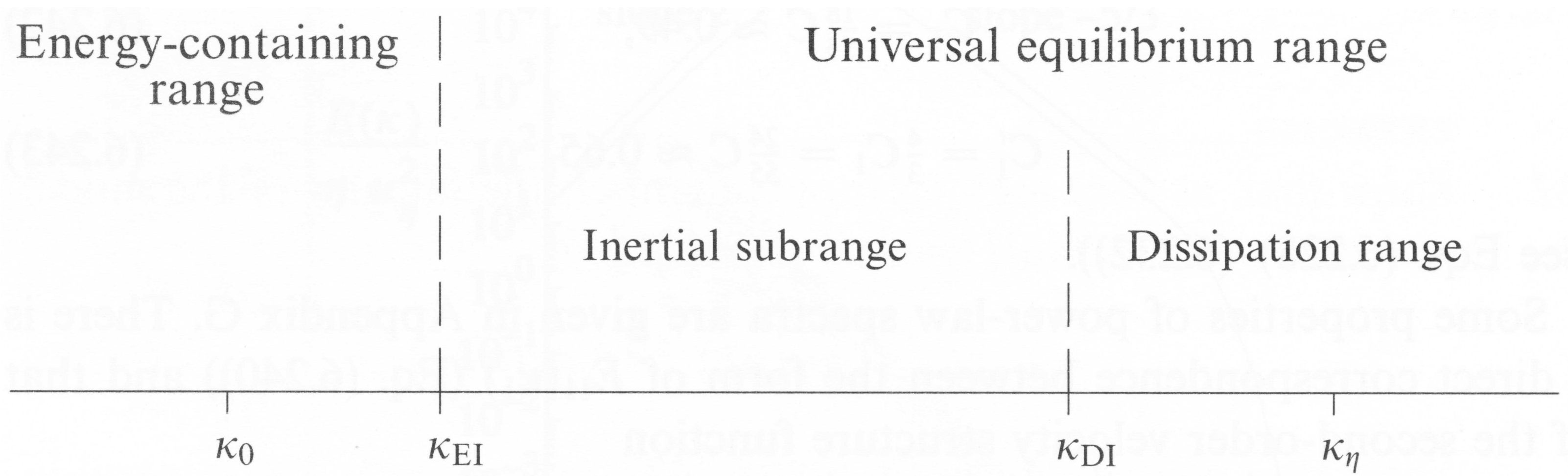
Kolmogorov's second similarity hypothesis: In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale l in the range $l_0 \gg l \gg \eta$ have a universal form that is uniquely determined by ε, independent of ν.

Different ranges



Kolmogorov 1941 theory II

Velocity scale, time-scale formed from ϵ with the help of l:

$$v(l) = (\varepsilon l)^{1/3} \sim v_{\mathrm{EI}}(l/l_{\mathrm{EI}})^{1/3}, \quad \tau(l) = (l^2/\varepsilon)^{1/3} \sim \tau_{\mathrm{EI}}(l/l_{\mathrm{EI}})^{2/3}$$

Energy transfer rate $\epsilon = v(l)^2/\tau(l)$ is indepedent of l:

$$\epsilon(l_{\mathrm{EI}}) = \epsilon(l) = \epsilon(l_d) = \varepsilon$$

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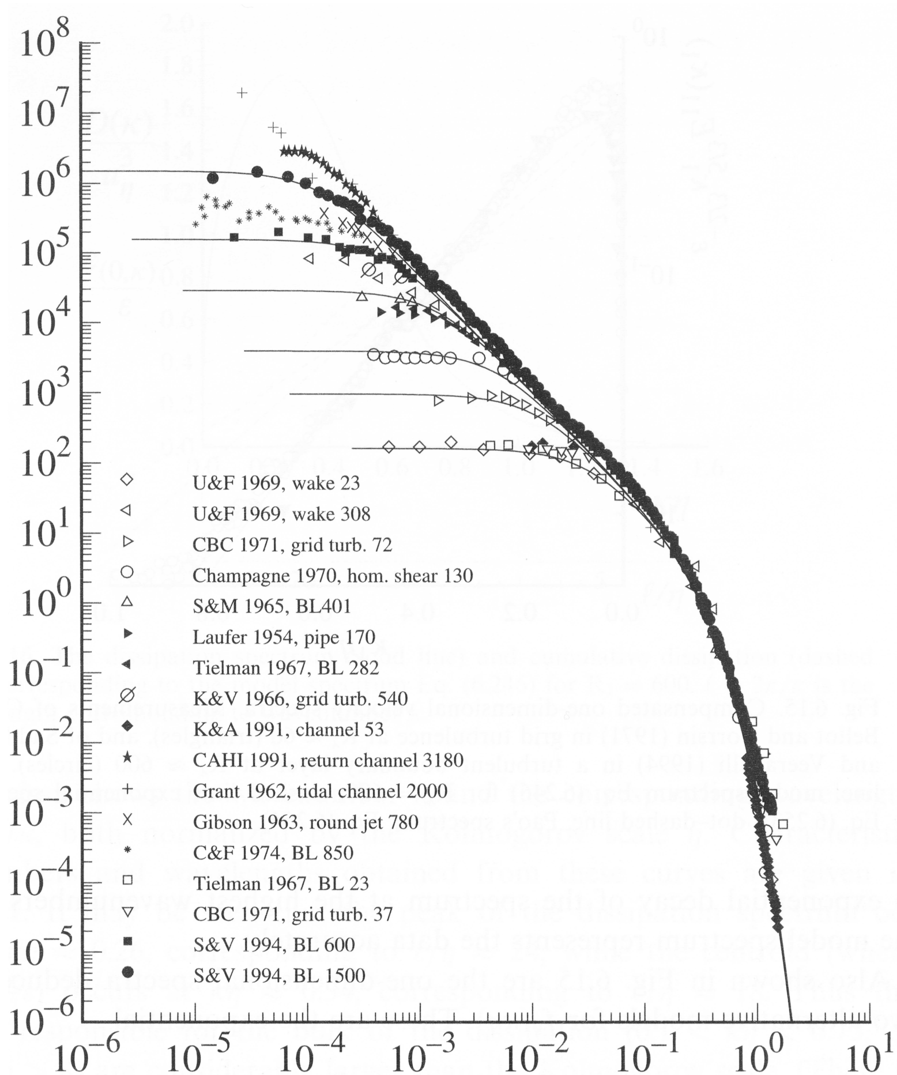
The energy spectrum of homogeneous turbulence:

$$v(l)^2 \sim E_{kin} \sim \varepsilon^{2/3} l^{2/3}$$

in k-space, the energy in the k-band $\sim kE(k)$:

$$E(k) = C\varepsilon^{2/3} k^{-5/3}$$

Typical energy spectrum



The spectral energy balance

For homogeneous turbulence:

$$\frac{\partial}{\partial t}E(k,t) = \ P_k(k,t) \ \ -\tfrac{\partial}{\partial k}T_k(k,t) \ \ -2\nu k^2 E(k,t)$$

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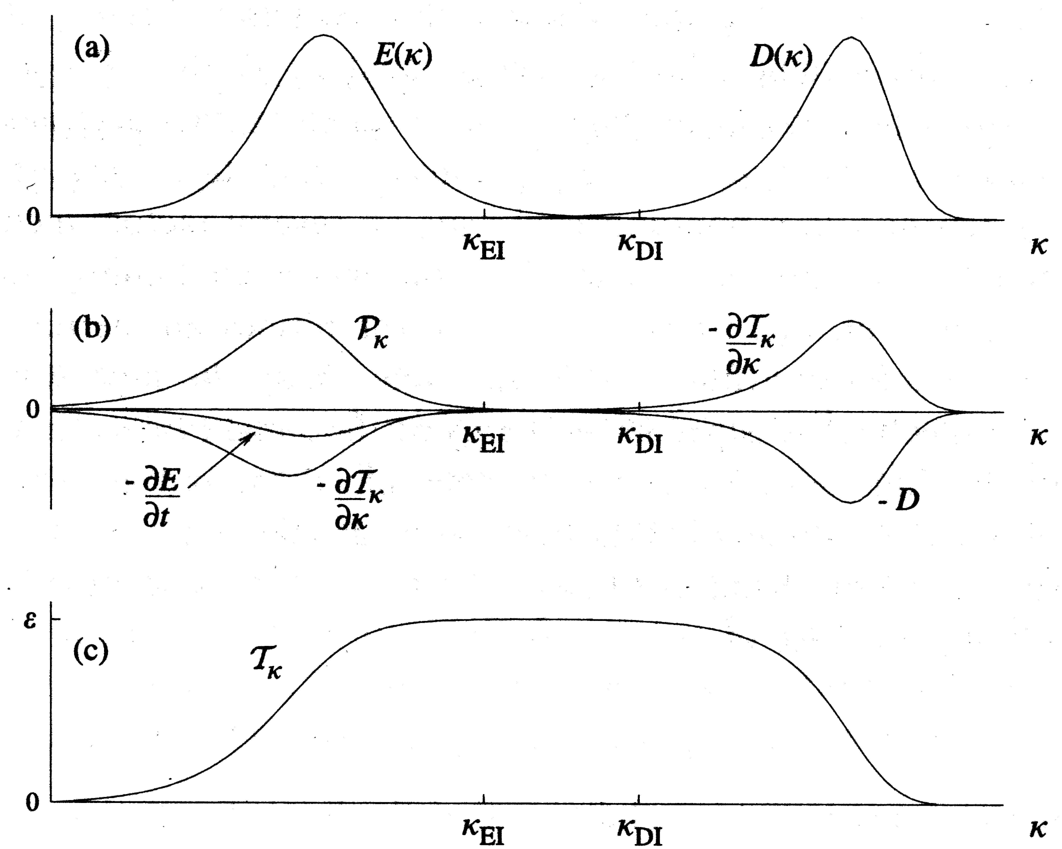

Two-point closure: Quasi-normal approximation

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Several remarks:

- Navier-Stokes equations transformed to infinite hierarchy
- Gaussianity assumption due to Central limit theorem: odd-order moments vanish, even-order expressed with the help of lower orders

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Eddy-Damped Quasi-Normal Markovian approx.

Q.N. approximation:

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- spectrum develops negative values in the inertial range
- triple correlations are not enough damped - large memory time
- additional "eddy-damping" term (no energy damping!) $\Rightarrow$ EDQN

$$\nu(k^2 + p^2 + q^2) \rightarrow \nu(k^2 + p^2 + q^2) + \mu_{kpq}$$

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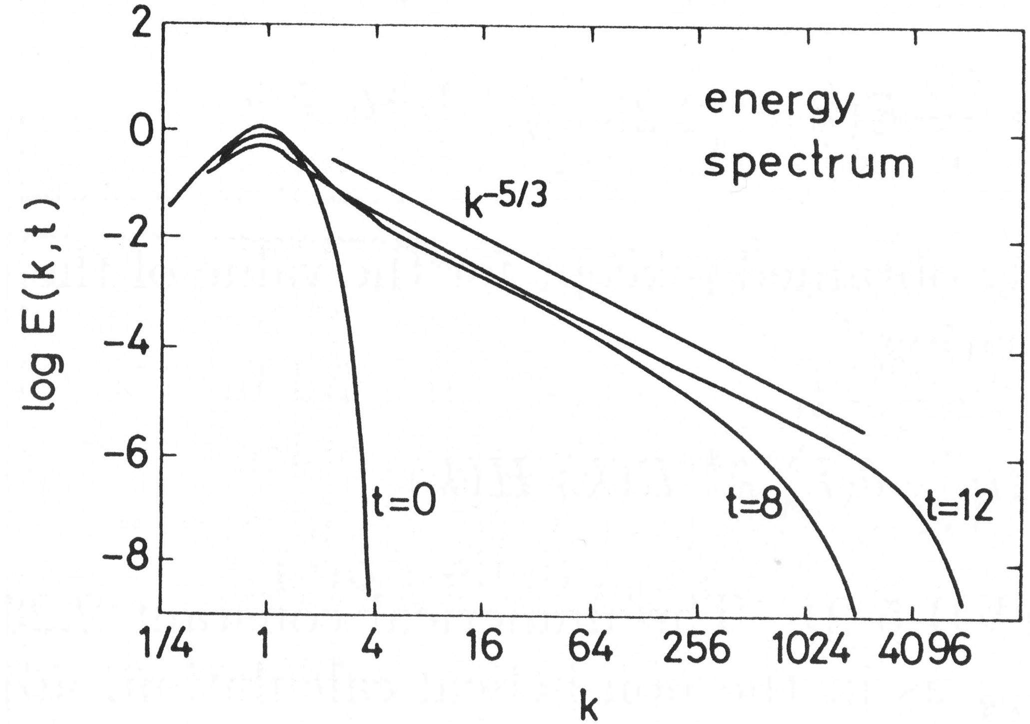
not sufficient for realization $\Rightarrow$ **Markovianization:**

- interaction time much shorter than the global evolution time

$$[\partial_t + 2\nu k^2]\mathbf{V}(\mathbf{k},t) = \int_{t'}^t d\tau \int_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} e^{-[\mu_{kpq}+\nu(k^2+p^2+q^2)](t-\tau)} \quad \Sigma < \mathbf{vv} >< \mathbf{vv} > (t) d\mathbf{p} d\mathbf{q}$$

Application of EDQNM, limitations



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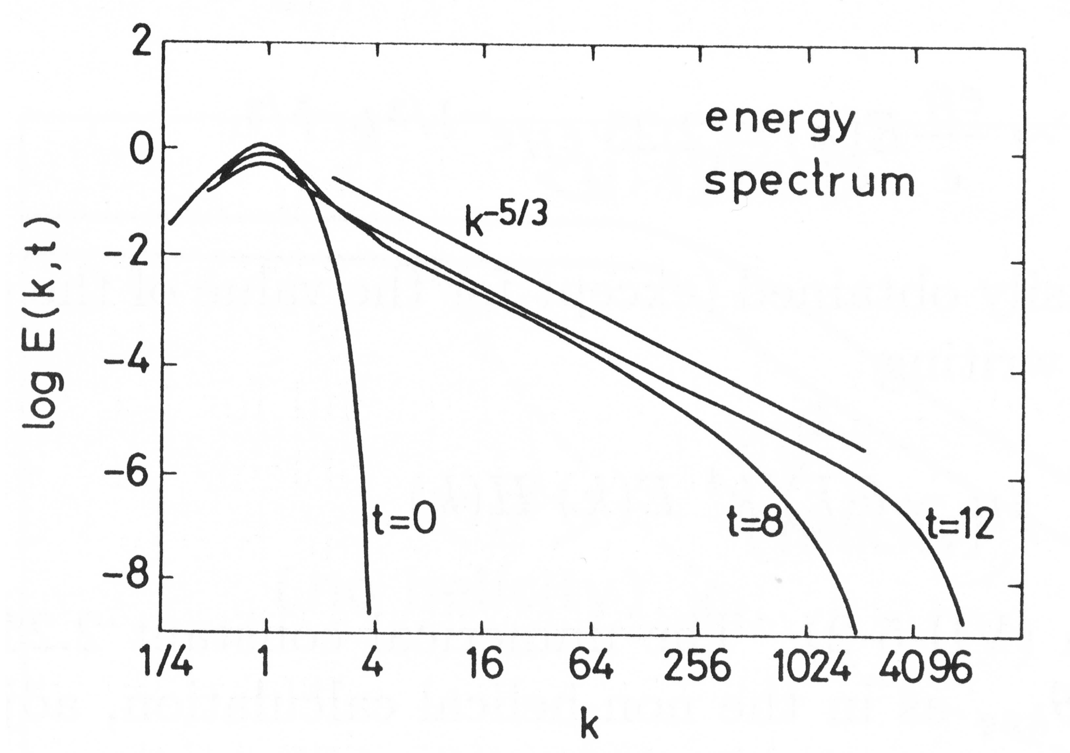
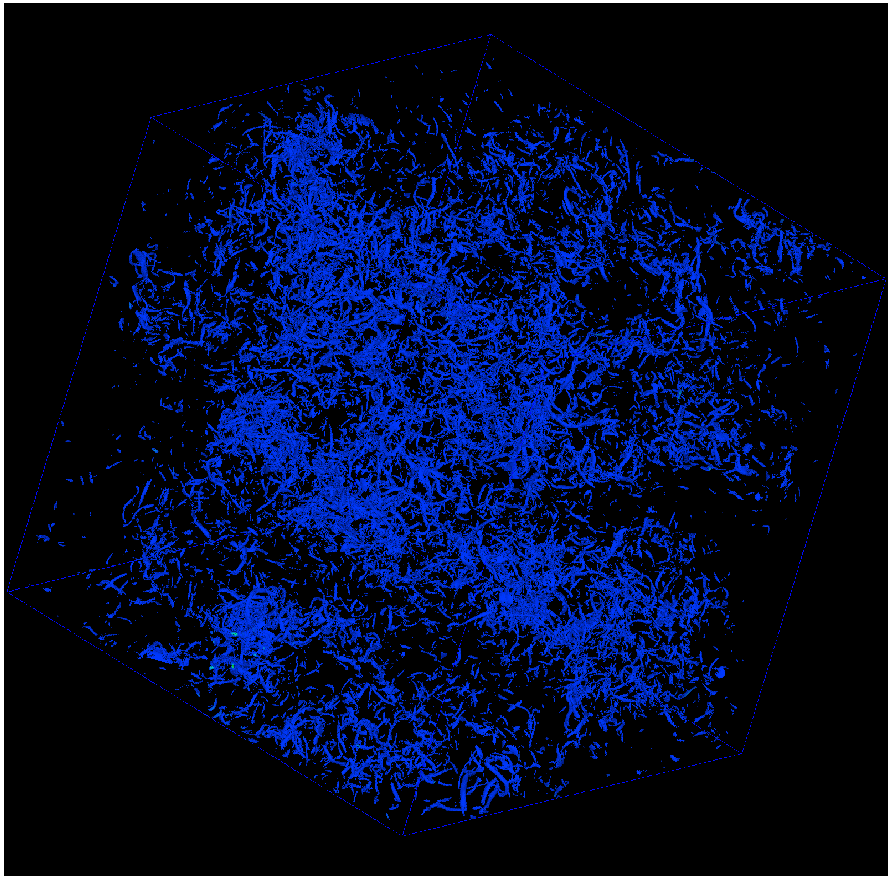
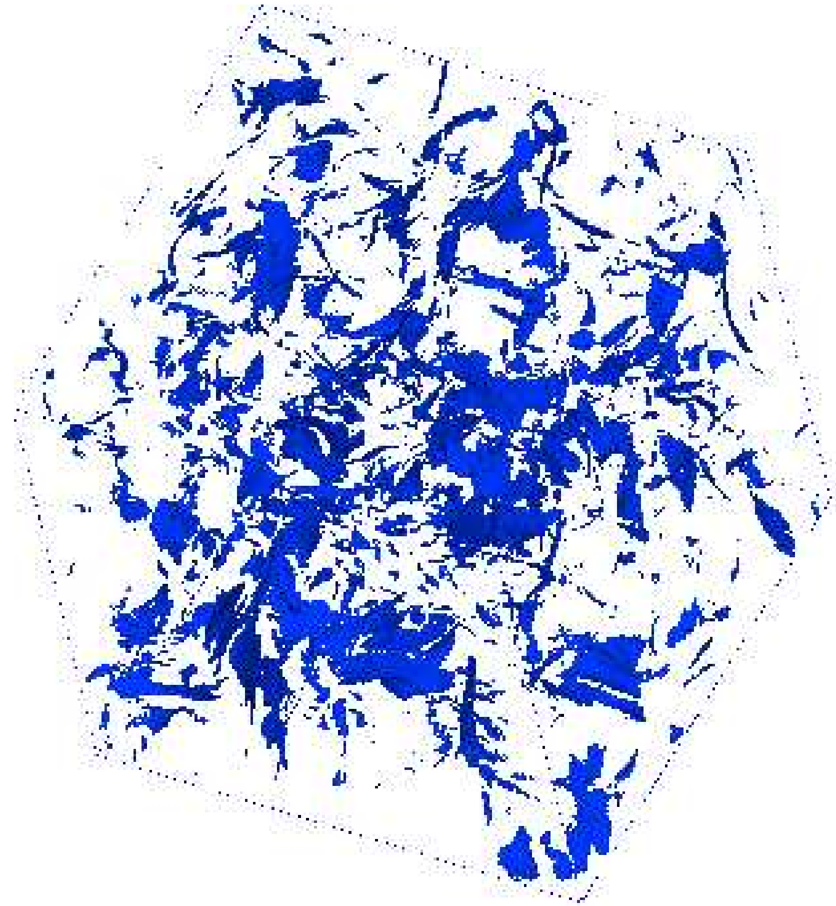


Limitations, shortcomings:

- calculations are not very economic
- problematic simulations of non-local interactions
- does not include some none-Gaussian effects, e.g. intermittency

Intermittency I





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Measure of intermittency: flatness

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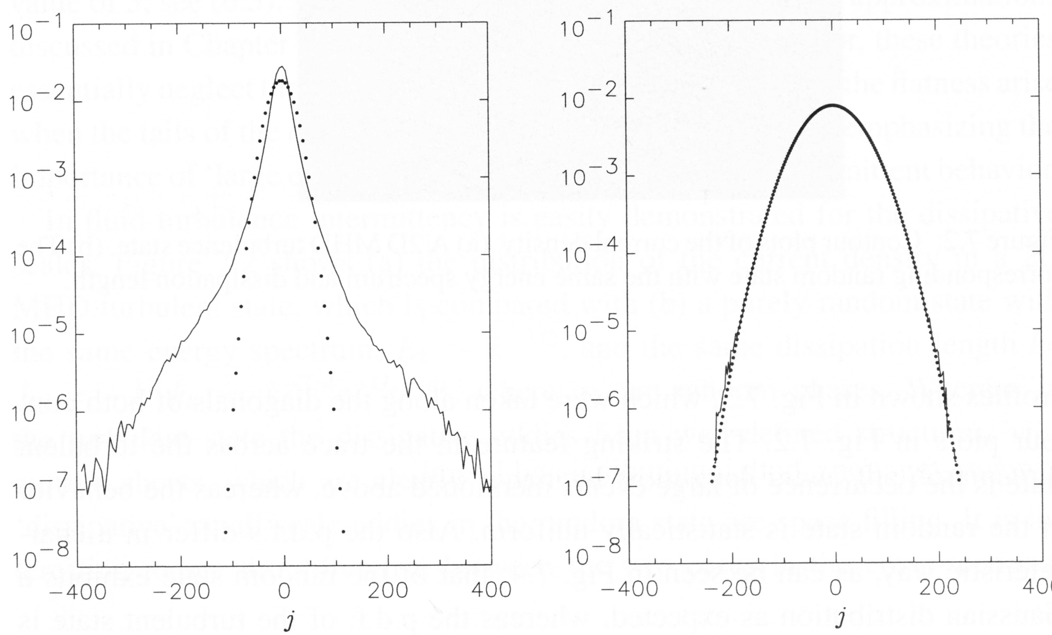

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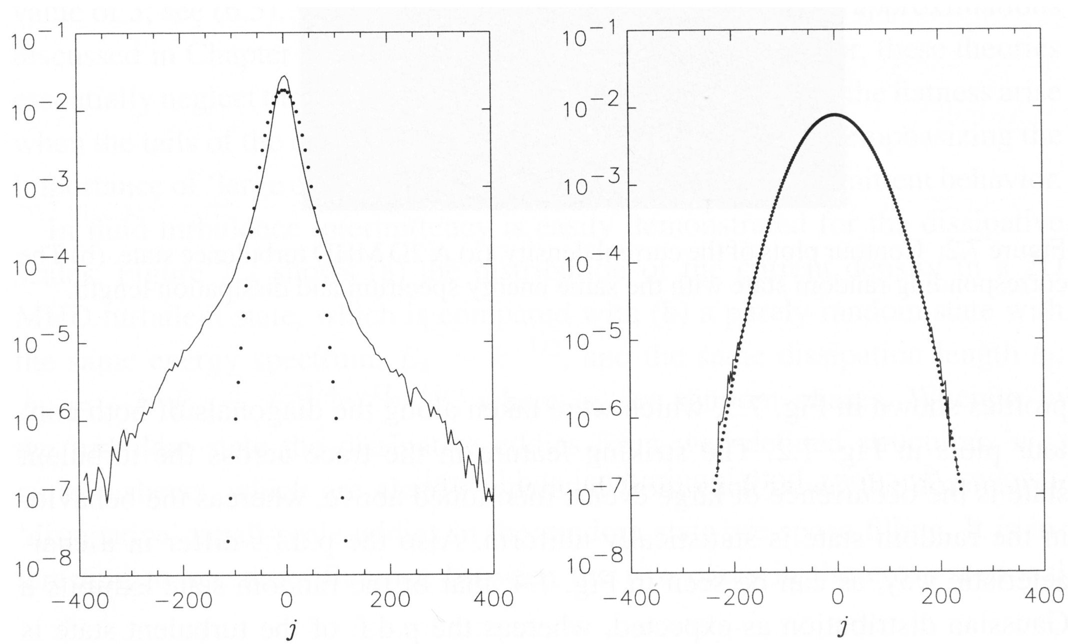

Turbulence is not Gaussian, occurence of events with large amplitudes

Intermittency III

Intermittency of dissipation and the refined similarity hypothesis

Landau objection: dissipation of energy $\varepsilon(\mathbf{x})$ varies rapidly in space and time, neither ε nor C (Kolmogorov constant) cannot be universal.

Spatial average over a volume of scale l: $\varepsilon_l = \frac{1}{V_l}\int_{V_l}\varepsilon(\mathbf{x})dV$

Modification of the scaling law: $\delta v_l \sim \varepsilon_l^{1/3} l^{1/3}$

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Modification of the scaling law: $\delta v_l \sim \varepsilon_l^{1/3} l^{1/3}$

Refined similarity hypothesis:

$$< \varepsilon_l^n > \sim l^{\mu_n} \quad \Rightarrow \quad < [\delta v(l)]^n > = a_n l^{\zeta_n}, \quad \zeta_n = \frac{1}{3}n + \mu_{n/3}$$

- connection between dissipation range and inertial range physics
- p.d.f. $p(\delta v_l/(\varepsilon_l l)^{1/3})$ is universal, independent of l and Reynolds number Re.

Intermittency IV

The log-normal model: Cube of width l_0 and ε_0 is divided into 2^D cubes of width $l_1 = l_0/2$ and so on until the dissipation scale:

$$l_0 \gg l_j = l_0/2^j \gg l_d, \qquad \varepsilon_j = \varepsilon_0 \chi_1 \dots \chi_j, \qquad \chi_i = \varepsilon_i/\varepsilon_{i-1}$$

p.d.f. of $\ln \varepsilon_j$ approaches Gauss $\Rightarrow$ p.d.f. of ε_j approaches Log-normal

$$< \varepsilon_l^n >= \varepsilon_0^n (L/l)^{\mu_n}, \quad \mu_n = \frac{1}{2}\mu n(n-1) \quad \Rightarrow \qquad \zeta_n = \frac{1}{3}n - \mu\frac{n(n-3)}{18}$$

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The log-Poisson model: Assumed relation for orders of $\varepsilon_l^{(n)}$

$$\varepsilon_l^{(n+1)} = A_n (\varepsilon_l^{(n)})^\beta (\varepsilon_l^{(\infty)})^{1-\beta}, \quad 0 < \beta < 1$$

after some algebra gives: $< \varepsilon_l^n > \sim l^{\mu_n}, \quad \mu_n = -nx + x\frac{1-\beta^n}{1-\beta}$
with co-dimension of the dissipative eddies $C_0 = D - d = \frac{x}{1-\beta}$

$$\mu_n = -\frac{2}{3}n + 2[1 - \left(\frac{2}{3}\right)^n], \qquad \zeta_n = \frac{n}{9} + 2[1 - \left(\frac{2}{3}\right)^{n/3}]$$

Generalization to MHD

One of the easiest description of plasma turbulence - incompressible MHD

$$\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{v} \times \omega + S_B \mathbf{j} \times \mathbf{b}) = \tilde{\nu} \Delta \omega, \qquad \frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{b}) = \tilde{\eta} \Delta \mathbf{b}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = 0, \qquad \omega = \nabla \times \mathbf{v}, \qquad \mathbf{j} = \nabla \times \mathbf{b}$$

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$$\nabla \cdot (\mathbf{v} \cdot \mathbf{bb}) - \mathbf{v} \cdot (\nabla \cdot \mathbf{bb}) \qquad \nabla \cdot (\mathbf{v} \cdot \mathbf{bb}) - \mathbf{b} \cdot (\nabla \cdot \mathbf{bv})$$

Selective decay and self-organization

MHD quadratic invariants:

$$\frac{dE}{dt} = -\eta \int j^2 dV - \nu \int \omega^2 dV, \quad \frac{dK}{dt} = -(\nu+\eta) \int \mathbf{j} \cdot \boldsymbol{\omega} dV, \quad \frac{dH}{dt} = -\eta \int \mathbf{j} \cdot \mathbf{b} dV$$

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1. Relaxation to a force-free state: under the constant magnetic helicity

$$\delta \left(\int \frac{1}{2}(v^2 + b^2) dV - \lambda \int \mathbf{A} \cdot \mathbf{b} dV \right) = 0$$

$$\Rightarrow \qquad \mathbf{v} = 0, \quad \nabla \times \mathbf{b} - \lambda \mathbf{b} = 0$$

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2. Relaxation to a pure Alfvenic state: under the constant cross helicity

$$\delta \left(\int \frac{1}{2}(v^2 + b^2) dV - \lambda \int \mathbf{v} \cdot \mathbf{b} dV \right) = 0$$

$$\Rightarrow \qquad \mathbf{v} - \lambda \mathbf{b} = 0, \quad \mathbf{b} - \lambda \mathbf{v} = 0 \qquad \Rightarrow \qquad \mathbf{v} = \pm \mathbf{b}$$

Further reading...

Hydrodynamical turbulence:

Tennekes,H.,Lumley,J.L.: **A First Course in Turbulence**, The MIT Press, 1997.

Pope,S.B.: **Turbulent flows**, CUP, 2000.

Frisch,U.: **Turbulence**, CUP, 1995.

Hinze,J.O.: **Turbulence**, McGraw-Hill, 1987.

Lesieur,M.: **Turbulence in Fluids**, Kluwer, 1997.

Leslie,D.C.: **Developments in the theory of turbulence**, Clarendon, 1983.

MHD turbulence:

Biskamp,D.: **Magnetohydrodynamic turbulence**, CUP, 2003.

Yoshizawa,A.,Itoh,S.,I.,Itoh,K.: **Plasma and Fluid Turbulence**, IoP, 2003.

Zeiler,A.: **Tokamak Edge Turbulence**, IPP Report, 1999.

Biskamp,D.: **Nonlinear Magnetohydrodynamics**, CUP, 1993.