UNIVERSITI TEKNOLOGI MALAYSIA
SSE1793 DIFFERENTIAL EQUATIONS
TUTORIAL 5

1. A uniform rod of length 20 meters is held at both ends. The initial temperature at any point on the rod is $f(x)$. The temperature at both ends is held fixed at a constant temperature of $0°C$. The temperature $u(x,t)$ at any time t of a point P which is x distance away from any of of the two endpoints is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}, \quad 0 < x < 20, t > 0, \tag{1}$$

where c is a constant.

(a) Use the method of separation of variables to show that the general solution is given by

$$u(x,t) = (A\cos px + B\sin px)e^{-c^2p^2t} \tag{2}$$

such that A, B and p are constants.

(b) State the boundary and initial conditions of the problem.

(c) By substituting the boundary condition $u(0,t) = 0$, into (2) and then letting $\lambda = cp$, show that

$$u(x,t) = Be^{-\lambda^2 t}\sin\left(\frac{\lambda x}{c}\right).$$

(d) By applying the boundary condition $u(20,t) = 0$, show that

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda^2 t}\sin\left(\frac{n\pi x}{20}\right). \tag{3}$$

(e) Compute B_n in (3) by applying respectively the initial conditions,

i. $f(x) = 100$

ii. $f(x) = \sin 2x\cos x$

2. Use the method of separation of variables to solve the heat conduction problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions and initial conditions given by,

(a) $u(0,t) = 1,\ u(2,t) = 3,\ u(x,0) = \sin\frac{\pi x}{2}.$

Solution: $u(x,t) = 1 + x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2[3(-1)^n - 1]}{n}e^{\frac{-n^2\pi^2 t}{4}}\sin\frac{n\pi x}{2}$

(b) $u(0,t) = 0,\ u(1,t) = 1,\ u(x,0) = 1 - x.$

Solution: $u(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-4n^2\pi^2 t}\sin 2n\pi x$

3. Use the method of separation of variables to show that the general solution of the heat equation
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$$
is given by
$$u(x,t) = Ax + B + (C\cos px + D\sin px)e^{-\alpha^2 p^2 t}.$$

(a) Then, use this results to solve the equation with the boundary and initial conditions given as following:

i. $u(0,t) = 0 \qquad u(10,t) = 0, \qquad t > 0,$
$u(x,0) = 100, \qquad 0 < x < 10.$

Solution: $u(x,t) = \dfrac{400}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{10}\right) e^{\frac{-\alpha^2 t}{100}((2n-1)\pi)^2}$

ii. $u(0,t) = 70, \qquad u(10,t) = 120, \qquad t > 0,$
$u(x,0) = 8x + 100 \qquad 0 < x < 10.$
Solution: $u(x,t) = 5x + 70 + \displaystyle\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{10}x\right) e^{-\frac{\alpha(n\pi)^2 t}{100}}$

where $\qquad D_n = 2\displaystyle\int_0^{10} (3x+30)\sin\frac{n\pi x}{10}\,dx.$

(b) Let $\alpha^2 = 2$. The heat conduction equation of a uniform rod is now given by,
$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0$$

i. Show that the general solution is given by
$$u(x,t) = Ax + B + (C\cos px + D\sin px)e^{-2p^2 t}$$
where p is a constant.

ii. Hence, using the above result find the solution to the heat equation with boundary conditions;
$$u_x(0,t) = 0, \; u(\pi,t) = 100, \qquad t > 0$$
and initial condition
$$u(x,0) = \begin{cases} 60, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$
(left end of rod insulated, right end held at a constant temperature)

iii. Find the solution for the heat equation in (b)i with the following boundary and initial conditions.
$$u_x(0,t) = 2 = u_x(\pi,t), \qquad t > 0$$

$$u(x,0) = 4x + 3, \qquad 0 < x < \pi$$

Solution of (b)ii:

$$u(x,t) = 100 + \sum_{n=1}^{\infty} A_{2n-1} \cos\left(\frac{(2n-1)x}{2}\right) e^{\frac{-(2n-1)^2 t}{2}}$$

$$A_{2n-1} = \frac{80}{(2n-1)\pi}\left[3\sin\left(\frac{(2n-1)\pi}{4}\right) - 5\sin\left(\frac{(2n-1)\pi}{2}\right)\right]$$

4. A uniform bar of length 2 meters, is held at $0°C$ at both its end. The bar is first heated with an initial temperature distribution given by,

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

At any time t, the temperature u at the point x on the bar satisfies the heat conduction equation,

$$2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ 0 < x < 2, \ t > 0.$$

(a) By letting $u(x,t) = X(x)T(t)$, show that

$$X(x) = A\cos px + B\sin px \ \text{ and } \ T(t) = Ce^{-2p^2 t}$$

such that A, B, C and p a constant.

(b) Show that $p = \dfrac{n\pi}{2}$, $n = 1, 2, 3, \ldots$.

(c) Find the solution $u(x,t)$.

5. Given the wave equation

$$\frac{\partial^2 u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < 2, \ t > 0.$$

(a) Given the boundary conditions

$$u(0,t) = 0 = u(2,t), \ t > 0$$

and the initial condition

$$u(x,0) = f(x), \quad \left[\frac{\partial u}{\partial t}\right]_{t=0} = 0$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2} \cos\frac{n\pi t}{2}$$

such that

$$A_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}$$

(b) Find the solution if

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

6. Use the method of separation of variables to solve the wave equation given by

a. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0,\ u(\pi,t) = 0,$

$u(x,0) = \dfrac{x}{10}(\pi^2 - x^2),\ u_t(x,0) = 0.$

b. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0,\ u(1,t) = 0,$

$u(x,0) = 0,\ u_t(x,0) = x.$

c. $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0,\ u(\pi,t) = 0,$

$u(x,0) = x^2(\pi - x),\ u_t(x,0) = 1.$

7. Use the method of separation of variables to solve the Laplace Equation

a. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u(0,y) = 0,\ u(1,y) = 0,\ \ 0 < y < 1$

$u(x,1) = 0,\ u(x,0) = x(x-1),\ \ 0 < x < 1.$

b. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u(0,y) = 0,\ u(1,y) = 0,\ \ 0 < y < 1$

$u(x,0) = 0,\ u(x,1) = x,\ \ 0 < x < 1.$

c. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u(0,y) = \sin y + \sin 4y,\ u(\pi,y) = 0,\ \ 0 < y < \pi$

$u(x,0) = 0,\ \ u(x,\pi) = 0,\ 0 < x < \pi.$

d. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u(0,y) = 0,\ u(1,y) = 10\sin\dfrac{3\pi y}{2},\ \ 0 < y < 2$

$u(x,0) = 0,\ u(x,2) = 0,\ \ 0 < x < 1.$

Answer No(7):

a. $u(x,y) = \sum_{n=1}^{\infty} D_n \sinh n\pi(y-1)\sin n\pi x,$ where $D_n = \dfrac{1-(-1)^n}{(n\pi)^3 \sinh n\pi}$.

c. $u(x,y) = \dfrac{\sinh(x-\pi)\sin y}{\sinh(-\pi)} + \dfrac{\sinh 4(x-\pi)\sin 4y}{\sinh(-4\pi)}$.

d. $u(x,y) = \dfrac{10\sinh\frac{3\pi}{2}x \sin\frac{3\pi}{2}y}{\sinh\frac{3\pi}{2}}$.