Uncertainty Quantification (ACM41000)
Mini Project 2

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April 8, 2018

Abstract

Analytic and numerical analysis of the heat / diffusion equation.

Contents

1 Introduction

2 Mixed Inhomogenous Boundary Conditions

Breaking up the equation into two parts

$$u(x,t) = u_{hom}(x,t) + u_{PI}(x,t) \tag{1}$$

2.1 $u_{hom}(x,t)$ solution with mixed homogenous boundary conditions

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad\qquad x \in (0,L) \tag{2}$$

Initial Conditions

$$u(x, t=0) = f(x) \tag{3}$$

Mixed homogenous boundary conditions

$$u(x=0, t>0) = u_x(x=L, t>0) = 0 \tag{4}$$

Using the separation of variables method $u(x,t) = X(x)T(t)$

$$\begin{aligned} \frac{\partial X(x)T(t)}{\partial t} &= D\frac{\partial^2 X(x)T(t)}{\partial x^2} \\ \frac{1}{T(t)}\frac{\partial T(t)}{\partial t} &= D\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} \end{aligned}$$

LHS is a function of $T(t)$ only and the RHS is a function of $X(x)$ only. This means LHS = RHS = Constants

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = D\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda D$$

This produces the two following equations which can solved separately and recombined for the final solution

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -\lambda D \tag{5}$$

$$D\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda D \tag{6}$$

2.1.1 Solve for $X(x)$

Rearranging equation 6

$$\cancel{D}\frac{\partial^2 X(x)}{\partial x^2} + \lambda X(x)\cancel{D} = 0$$

This is a second order linear homogenous ODE with constant coefficients. This can be seen as the solution of a quadratic of the form

$$\Psi^2 + \lambda\Psi = 0$$

Solutions to this quadratic are

$$\Psi = 0$$
$$\Psi = \lambda$$

Ignoring the $\Psi = 0$ solution and focusing on the $\Psi = \lambda$

- $\lambda = 0$:

  $X(x) = Ax + B$
  $X'(x) = A$

  Applying boundary conditions

  $X(0) = 0 = A(0) + B$
  $B = 0$

  Trivial Solution, no further analysis on this solution

- $\lambda < 0$:
  Trivial Solution, no further analysis on this solution

- $\lambda > 0$:

  The form of the solution is

  $X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

Applying boundary conditions

$$\begin{aligned} X(0) &= A\sin(\sqrt{\lambda}0) + B\cos(\sqrt{\lambda}0) = 0 \\ X(0) &= A\sin(0) + B\cos(0) = 0 \\ X(0) &= A \cdot 0 + B \cdot 1 = 0 \\ B &= 0 \end{aligned}$$

$$\begin{aligned} X'(x) &= A\cancel{\sqrt{\lambda}}\cos(\sqrt{\lambda}x) - B\cancel{\sqrt{\lambda}}\sin(\sqrt{\lambda}x) = 0 \\ X'(L) &= A\cos(\sqrt{\lambda}L) - 0 \cdot \sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \\ X(0) &= A\cos(\sqrt{\lambda}L) = 0 \end{aligned}$$

Ignoring the trivial solution $A = 0$, yields the period solution

$$\begin{aligned} \cos(\sqrt{\lambda}L) &= 0 = \cos((n + \frac{1}{2})\pi) \\ \sqrt{\lambda}L &= (n + \frac{1}{2})\pi \\ \lambda_n &= \frac{(n + \frac{1}{2})^2\pi^2}{L^2} \end{aligned}$$

Plugging this value back into $X(x)$ yields

$$X(x) = A_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \tag{7}$$

where $n \in \mathbb{Z}_{\geq 0}$

2.1.2 Solve for $T(t)$

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -\lambda D \tag{8}$$

Using the periodic value of λ_n

$$\frac{1}{T_n(t)}\frac{\partial T_n(t)}{\partial t} = -\lambda_n D \tag{9}$$

Integrating both side with respect to t and solving for $T(t)$

$$\begin{aligned}
\int \frac{1}{T_n(t)}\frac{\partial T_n(t)}{\partial t}\mathsf{d}t &= -\int \lambda_n D\mathsf{d}t \\
log(T_n(t)) &= -\lambda_n Dt + C \\
T_n(t) &= \mathsf{e}^{-\lambda_n Dt}\mathsf{e}^{C} \\
T_n(0) &= \mathsf{e}^(C) \\
T_n(t) &= T_n(0)\mathsf{e}^{-\lambda_n Dt}
\end{aligned}$$

Plugging in the value of $\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}$ yields the solution

$$T_n(t) = T_n(0)\mathsf{e}^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2}Dt} \tag{10}$$

2.1.3 Recombine $u(x,t) = X(x)T(t)$

Since we used the method of separation of variables to solve $u(x,t)$

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
u_n(x,t) &= A_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)T_n(0)\mathsf{e}^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2}Dt}
\end{aligned}$$

Letting $C_n = A_nT_n(0)$ give final form of solution for $u_n(x,t)$

$$u_n(x,t) = C_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)\mathsf{e}^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2}Dt} \tag{11}$$

2.1.4 Linearity of solutions $u_n(x,t), n \in \mathbb{Z}_{\geq 0}$

Each $u_n(x,t), n \in \mathbb{Z}_{\geq 0}$ is a solution to the PDE in Equation (2) with homogenous boundary conditions , $u_{hom}(x,t)$. The linearity property of homogenous PDE's states that if $u_1(x,t)$ and $u_2(x,t)$ are solutions of a linear homogenous PDE on some region $\mathcal{R}(= (0,L))$, then $u_*(x,t) = C_1u_1(x,t) + C_2u_2(x,t)$ is also a solution of $u_{hom}(x,t)$. Where $C_1, C_2 \in \mathbb{R}$ are constants.

This implies that any number of linear combinations of solutions to $u_{hom}(x,t)$ are also solutions.

$$u_{hom}(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{12}$$

2.1.5 Form of C_n's

Since the PDE is linear and to solve we split it into homogenouss and Inhomogenous equations under the condition that

$$u_{hom}(x,t=0) + u_{PI}(x) = 0$$

And we have found that $u_{PI}(x) = u_0$, therefore $u_{hom}(x,t=0) = -u_0$

Using the orthogonality porperty of the basis functions on $(0,L)$

$$\left\{ sin\left[\frac{(n+\frac{1}{2})\pi x}{L}\right]\right\}_{n=0}^{\infty}$$

$$I_{n,m} = \int sin\left[\frac{(n+\frac{1}{2})\pi x}{L}\right] sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right] \mathsf{d}x$$

where $y = \frac{\pi x}{L}$ and $\mathsf{d}y = \frac{L}{\pi} = \mathsf{d}x$

$$I_{n,m} = \frac{L}{\pi}\int_0^{\pi} sin\left[(n+\frac{1}{2})y\right] sin\left[(m+\frac{1}{2})y\right] \mathsf{d}y$$

$$I_{n,m} = \frac{L}{\pi}\left[\frac{sin((m-n)y)}{2(m-n)} - \frac{sin((m+n+)y)}{2(m+n+1)}\right]_0^{\pi}$$

$$I_{n,m} = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

$$I_{n,m} = \frac{L}{2}\delta_{n,m}$$

where

$$\delta_{n,m} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

From Equation (11-12) it can be seen and the initial condition derived above

$$u_{hom}(x,t=0) = -u_0 = \sum_{n=0}^{\infty} C_n sin\left[\frac{(n+\frac{1}{2})\pi x}{L}\right] = -u_0$$

Multiplying both sides by $sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]$ and integrating

$$-\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x = \int_0^L \sum_{n=0}^{\infty} C_n sin\left[\frac{(n+\frac{1}{2})\pi x}{L}\right] sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x$$
$$-\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x = \sum_{n=0}^{\infty} C_n \int_0^L sin\left[\frac{(n+\frac{1}{2})\pi x}{L}\right] sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x$$

Using the orthogonality property this simplifies to

$$-\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x = \sum_{n=0}^{\infty} C_n \frac{L}{2}\delta_{n,m}$$
$$-\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x = C_n \frac{L}{2}$$

Solving for C_n

$$C_n = -\frac{2}{L}\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi x}{L}\right]\mathsf{d}x \tag{13}$$

Pluggin the experssion for C_n in Equation (12)

$$u_{hom}(x,t) = \sum_{n=0}^{\infty} -\frac{2u_0}{L}\int_0^L u_0 sin\left[\frac{(m+\frac{1}{2})\pi\phi}{L}\right]\mathsf{d}\phi \sin\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)\mathsf{e}^{-\frac{(n+\frac{1}{2})^2\pi^2}{L^2}Dt} \tag{14}$$

2.2 $u_{PI}(x)$ Inhomogenous boundary conditions

$$\frac{\mathsf{d}^2 u_{PI}}{\mathsf{d}x^2} = 0 \tag{15}$$

Inhomogenous boundary conditions

$$u_{PI}(x=0) = u_0$$
$$u'_{PI}(x=L) = 0$$

Solving for u_{PI}

$$\int\int \frac{\mathsf{d}^2 u_{PI}}{\mathsf{d}x^2}\mathsf{d}x\mathsf{d}x = 0$$
$$\int [\frac{\mathsf{d}u_{PI}}{\mathsf{d}x} + A]\mathsf{d}x = 0$$
$$u_{PI}(x) + Ax + B = 0$$

Using the boundary conditions

$$
\begin{aligned}
u'_{PI}(L) + A &= 0 \\
0 + A &= 0 \\
A &= 0
\end{aligned}
$$

$$
\begin{aligned}
u_{PI}(0) + A(0) + B &= 0 \\
u_0 + (0)(0) + B &= 0 \\
B &= -u_0
\end{aligned}
$$

Yields the expected solution

$$
u_{PI}(x) - u_0 = 0 \tag{16}
$$

$$
u_{PI}(x) = u_0 \tag{17}
$$

2.3 Full solution $u(x,t) = u_{hom}(x,t) + u_{PI}(x)$ at $t \to \infty$

Using only the first term in the series

$$
\begin{aligned}
u(x,t) &= u_{hom}(x,t) + u_{PI}(x) \\
&= -\frac{2u_0}{L}\int_0^L u_0 sin\left[\frac{\pi\phi}{2L}\right]\mathsf{d}\phi \sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - u_0 \\
&= -\frac{2u_0}{L}\left[\frac{2L}{\pi}cos\left[\frac{\pi\phi}{2L}\right]_0^L\right]\sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - u_0
\end{aligned}
$$

$$
cos\left[\frac{\pi x}{2L}\right]_0^L = cos\left[\frac{\pi L}{2L}\right] - cos(0) = -1
$$

$$
\begin{aligned}
&= -\frac{2u_0}{\cancel{L}}\left[-\frac{2\cancel{L}}{\pi}\right]\sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - u_0 \\
&= \frac{4u_0}{\pi}\sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - u_0 \\
&= u_0\left[\frac{4}{\pi}\sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - 1\right]
\end{aligned}
$$

Using the first term of the series

$$
u(x,t) = u_0\left[\frac{4}{\pi}\sin\left(\frac{\pi x}{2L}\right)\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} - 1\right] \tag{18}
$$

Finding an estimate for the time at which the tempature at the end of the bar $x = L$ is 99% of its final value.

At $t \to \infty$ and $x = L$

$$
\sin\left(\frac{\pi x}{2L}\right) = \sin\left(\frac{\pi}{2}\right) = 1
$$

Equation (18) simplifies to

$$u(x=L,t=\infty)=u_0\left[\frac{4}{\pi}\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt}-1\right] \tag{19}$$

The final temperature at the point L as $t\to\infty$ with be approximately u_0, therefore

$$\begin{aligned}
u_0 &= u_0\left[\frac{4}{\pi}\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt}-1\right]\\
1 &= \frac{4}{\pi}\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt}-1\\
\frac{4}{\pi}\mathsf{e}^{-\frac{\pi^2}{4L^2}Dt} &= 2\\
-\frac{\pi^2}{4L^2}Dt &= log\left(\frac{2}{\pi}\right)\\
-\frac{\pi^2 D}{4L^2}t &= log\left(\frac{2}{\pi}\right)\\
t &= -\frac{4L^2}{\pi^2 D}log\left(\frac{2}{\pi}\right)
\end{aligned}$$

The time at which the point L on the bar is close to its final value is

$$t\approx\frac{4L^2}{5\pi^2 D} \tag{20}$$

3 Uniqueness of Solutions for the Heat Equation

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions to Equation (2) that are subject to the Inhomogenous Dirchlet boundary conditions

$$\begin{aligned}
u(x=0,t) &= b_L(t)\\
u(x=L,t) &= b_R(t)
\end{aligned}$$

Using the linearity property then $\phi(x,t)=u_1(x,t)-u_2(x,t)$ is also a solution to Equation (2)

$$\frac{\partial\phi}{\partial t}=D\frac{\partial^2\phi}{\partial x^2} \qquad x\in(0,L) \tag{21}$$

Initial Conditions

$$\phi(x,t=0)=f(x) \tag{22}$$

Homogenous boundary conditions

$$\phi(x=0,t>0)=\phi(x=L,t>0)=0 \tag{23}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial(u_1 - u_2)}{\partial t} \\
&= \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \\
&= D\frac{\partial^2 u_1}{\partial x^2} - D\frac{\partial^2 u_2}{\partial x^2} \\
&= \frac{\partial^2 u_1 - u_2}{\partial x^2} \\
&= \frac{\partial^2 \phi}{\partial x^2}
\end{aligned}$$

and the difference of the initial and boundary conditions

$$\begin{aligned}
u_1(x=0,t) - u_2(x=0,t) = b_L(t) - b_L(t) = 0 \\
u_1(x=L,t) - u_2(x=L,t) = b_R(t) - b_R(t) = 0 \\
u_1(x,t=0) - u_2(x,t=0) = 0 - 0 = 0
\end{aligned}$$

Multpling Equation (21) by $\phi(x,t)$ and integrating with respect to x

$$\begin{aligned}
\int_0^L \phi(x,t)\frac{\partial \phi}{\partial t}\mathsf{d}x &= D\int_0^L \phi(x,t)\frac{\partial^2 \phi}{\partial x^2}\mathsf{d}x \\
\frac{1}{2}\frac{\mathsf{d}}{\mathsf{d}t}\int_0^L [\phi(x,t)]^2\mathsf{d}x &= D\int_0^L \phi(x,t)\frac{\partial^2 \phi}{\partial x^2}\mathsf{d}x \\
&= D\int_0^L \left[\frac{\partial}{\partial x}\left(\phi(x,t)\frac{\partial \phi}{\partial x}\right) - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\mathsf{d}x \\
&= D\left[\phi(x)\frac{\partial \phi}{\partial x}\right]_{x=0}^{x=L} - D\int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2\mathsf{d}x \\
&= D\left[\phi(L)\frac{\partial \phi}{\partial x}(L) - \phi(0)\frac{\partial \phi}{\partial x}(0)\right] - D\int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2\mathsf{d}x
\end{aligned}$$

From the boundary conditions $\phi(x=0,t) = \phi(x=L,t) = 0$, so

$$\frac{1}{2}\frac{\mathsf{d}}{\mathsf{d}t}\int_0^L [\phi(x,t)]^2\mathsf{d}x = -D\int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2\mathsf{d}x$$

Defining the L^2 norm of ϕ

$$\parallel \phi \parallel_2^2 = \int_0^L [\phi(x,t)]^2\mathsf{d}x$$

$$\frac{1}{2}\frac{\mathsf{d}}{\mathsf{d}t} \parallel \phi \parallel_2^2 = -D\int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2\mathsf{d}x \tag{24}$$

Equation (24) can be integrated with respeect to t

$$\int_0^t \frac{1}{2}\frac{\mathsf{d}}{\mathsf{d}t} \parallel \phi \parallel_2^2 \mathsf{d}t = -D\int_0^t \int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2 \mathsf{d}x\mathsf{d}t$$

$$\parallel \phi \parallel_2^2 \Big|_0^t = -2D\int_0^t \parallel \phi \parallel_2^2 (s)\mathsf{d}s$$

$$\parallel \phi \parallel_2^2 (t) = \parallel \phi \parallel_2^2 (0) - 2D\int_0^t \parallel \phi \parallel_2^2 (s)\mathsf{d}s$$

$2D\int_0^t \parallel \phi \parallel_2^2 (s)\mathsf{d}s \geq 0$, therefore

$$\parallel \phi \parallel_2^2 (t) \leq \parallel \phi \parallel_2^2 (0) = 0 \implies \parallel \phi \parallel_2^2 (t) = 0$$

SO $\phi(x,t) = 0 = u_1(x,t) - u_2(x,t)$, therefore $u_1(x,t) - u_2(x,t)$, the solution to the PDE is unique on $(0, L)$

4 Numerical Methods for the Heat Equation

The code in appendix A is a python class which solves the heat equation , with parameters $L, D, N, u0, t_min, t_max$

The class **HeatEquation1DFFTSolve** has three methods

- solve ; solves the PDE using python numpy methods numpy fast fourier transfirm methods
- plot_solution ; plots the distribution of heat over time
- animate_solution ; animation of the distribution evolve over time

Figure (1) shows the distribution of heat over time. Initially it starts as a box shaped function but as time progresses, the heat flows from areas of higher temperature to areas of lower temperature.

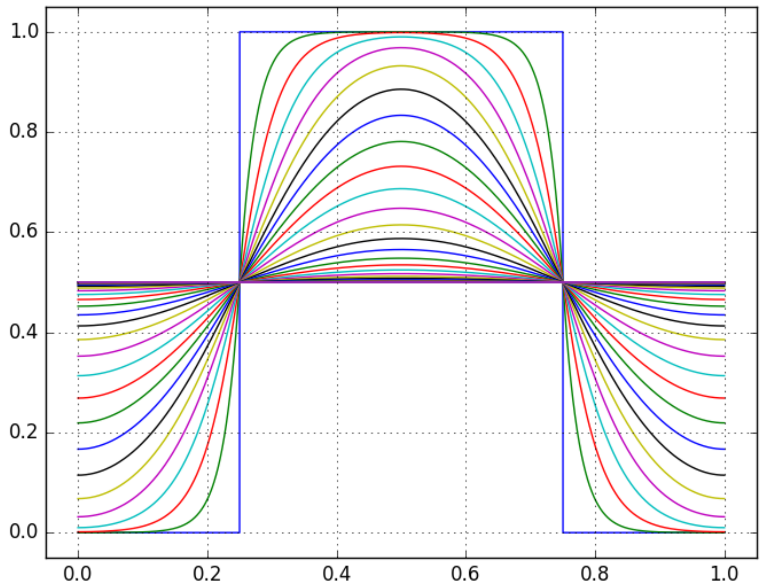

Figure 1: Distribution of heat over time

Appendices

A Python Code

Python solve Heat Equation PDE (1)

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation

class HeatEquation1DFFTSolve:

    def __init__(self, L, D, N, u0, t_min, t_max):
        self.L = L
        self.D = D
        self.N = N
        self.u0 = u0
        self.t_min = t_min
        self.t_max = t_max
        self.x_vec = np.arange(0, L, float(L)/N)
        self.delta_t = float(t_max) / N
        self.k_vec = (2*np.pi/L) * np.arange(-N/2, N/2)
        self.un = [self.u0]
        self.solution_epsilon = 0.000001

    def solve(self):
        uhat = np.fft.fftshift(np.fft.fft(self.u0))
        for i in range(1,self.N+1):
            uhat_new = uhat/(1+self.D*(i*self.delta_t)*self.
                k_vec*self.k_vec)
            uhat = uhat_new
            self.un.append(np.fft.ifft(np.fft.ifftshift(uhat)))

    def plot_solution(self):
        plt.axes().set_xlim(-0.05, self.L +.05)
        plt.axes().set_ylim(-0.05, self.L +.05)
        plt.axes().grid()
        for i in range(0, len(self.un)):
            plt.plot(self.x_vec,self.un[i])
            if (max(abs(self.un[i]))-min(abs(self.un[i]))) <
                self.solution_epsilon:
                break

    def animate_solution(self):
        fig, ax = plt.subplots()
        line, = ax.plot(self.x_vec, self.u0, lw=2)
        ax.grid()
        xdata, ydata = [], []
        time_label = ax.text(0.05, 0.90, 'time = 0.0', transform
            =ax.transAxes) # initialize the time label for the
            graph

        def solution_data():
            for i in range(0, len(self.un)):
                yield i*self.delta_t, self.x_vec, self.un[i]

        def init():
            ax.set_ylim(-0.05, self.L+.05)
            ax.set_xlim(-0.05, self.L+.05)
            del xdata[:]
            del ydata[:]
```

```
            line.set_data(xdata, ydata)
            return line,

        def run(data):
            t, x, u = data
            time_label.set_text('time = %.3f' % t) # Display the
                 current time to the accuracy of your liking.
            line.set_data(x, u)
            return line,time_label

        ani = animation.FuncAnimation(
            fig,
            run,
            solution_data,
            blit=False,
            interval=1000,
            repeat=True,
            init_func=init)
        return ani

########################
########################
##Init class
########################
########################

NN=2**14
he = HeatEquation1DFFTSolve(
    L = 1,
    D = 1,
    N = NN,
    u0 = np.array([1 if float(i)/NN >= .25 and float(i)/NN <=
        .75 else 0 for i in range(0,NN) ]),
    t_min = 0,
    t_max = 10)

########################
#Solve
########################
he.solve()

########################
#Solution Plot
########################
he.plot_solution()

########################
#Solution Animation
########################
he_ani = he.animate_solution()
```