# Uncertainty Quantification (ACM41000)
# Exercises – Set 1

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January 30, 2018

1. Solve the following first-order **elementary** ODEs and sketch the family of solution curves.
$$\frac{dy}{dx} = \mathrm{e}^x \sin x, \qquad \frac{dy}{dx} = \frac{1}{1+x^2}.$$

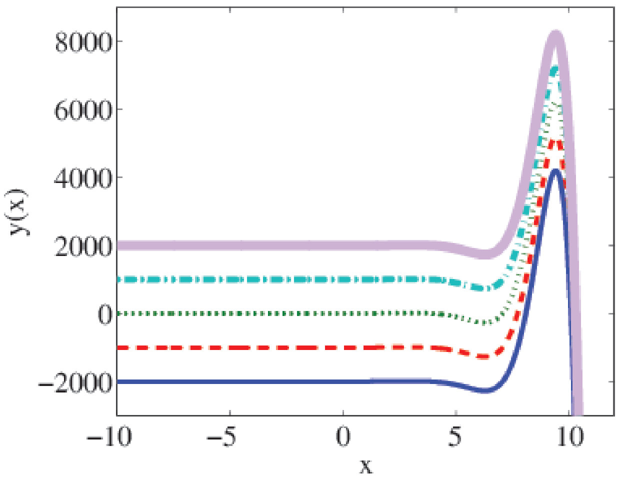
First, take $dy/dx = \mathrm{e}^x \sin x$. This is an elementary ODE. Formally multiply up by $\mathrm{d}x$ and integrate:
$$y = C + \int \mathrm{d}x\, \mathrm{e}^x \sin x.$$
We need to do IBP here:
$$
\begin{aligned}
I &:= \int \mathrm{d}x\, \underbrace{\mathrm{e}^x}_{=\mathrm{d}v} \underbrace{\sin x}_{=u}, \\
&= uv - \int v\mathrm{d}u, \\
&= \mathrm{e}^x \sin x - \int \mathrm{d}x\, \underbrace{\mathrm{e}^x}_{=\mathrm{d}v} \underbrace{\cos x}_{=u}, \\
&= \mathrm{e}^x \sin x - \left( uv - \int v\mathrm{d}u \right), \\
&= \mathrm{e}^x \sin x - \left( \mathrm{e}^x \cos x + \int \mathrm{d}x\, \mathrm{e}^x \cos x \right), \\
&= \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - I, \\
2I &= \mathrm{e}^x \left( \sin x - \cos x \right), \\
I &= \tfrac{1}{2}\mathrm{e}^x \left( \sin x - \cos x \right).
\end{aligned}
$$
Hence,
$$y = C + \tfrac{1}{2}\mathrm{e}^x \left( \sin x - \cos x \right).$$
The solution for various $C$-values is plotted in Fig. 1(a).

Next, take $dy/dx = 1/(1+x^2)$. Again, this is an elementary ODE, and the solution is
$$y = C + \int \frac{1}{1+x^2}.$$

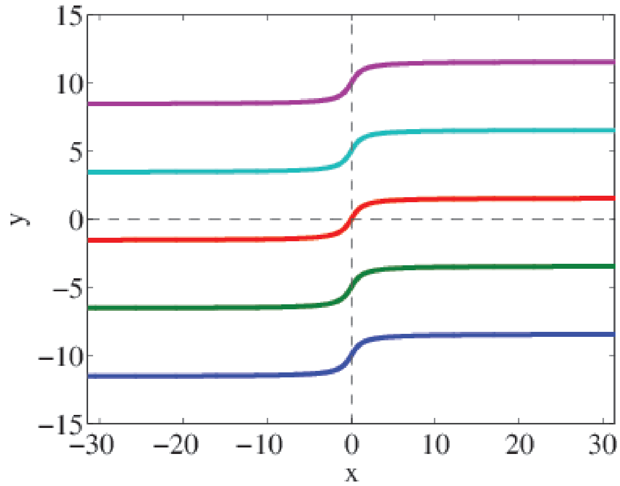
This is a standard integral with solution

$$y = C + \tan^{-1} x.$$

The solution for various $C$-values is plotted in Fig. 1(b).




(a) $dy/dx = \mathrm{e}^x \cos x$




(b) $dy/dx = 1/(1 + x^2)$

Figure 1: Family of solution curves for Question 1. Panel (a), from bottom to top $C = -2000, -1000, 0, 1000, 2000$; Panel (b), from bottom to top: $C = -10, -5, 0, 5, 10$

2. Let $T(t)$ be the temperature of a hot object at time $t$. The temperature cools to the background temperature $T_0 < T(t)$ according to Newton's Law of cooling:
$$\frac{dT}{dt} = -k\left(T - T_0\right), \qquad k \in \mathbb{R}^{+}.$$
Notice that this ODE is separable – hence or otherwise, solve for $T(t)$. Leave your answer in terms of $T(0)$, the initial temperature of the object.

We use separation of variables – we formally multiply both sides by $\mathrm{d}t$ and divide both sides by $T - T_0$. The result is

$$\frac{\mathrm{d}T}{T - T_0} = -k\mathrm{d}t.$$

Integrate:
$$\int \frac{\mathrm{d}T}{T - T_0} = C - kt,$$
where $C$ is a constant of integration. The integral on the left-hand side is standard:

$$\log\left(T - T_0\right) = C - kt,$$

where $T > T_0$ always because the body is cooling towards $T_0$. Exponentiate both sides here:
$$T - T_0 = D\mathrm{e}^{-kt}, \qquad D = \mathrm{e}^{C}.$$

Hence,
$$T = T_0 + D\mathrm{e}^{-kt}.$$

We eliminate the constant $D$ as follows:

$$T(0) = T_0 + D \implies D = T(0) - T_0,$$

hence
$$T = T_0 + \left(T(0) - T_0\right)\mathrm{e}^{-kt},$$

or
$$T = T(0)\mathrm{e}^{-kt} + T_0\left(1 - \mathrm{e}^{-kt}\right).$$

3. Solve the following **separable** initial-value problem:

$$\frac{du}{dt} = u^2 t, \qquad u(0) = 1.$$

Solution – this is a separable problem, with

$$\frac{\mathrm{d}u}{u^2} = t\mathrm{d}t.$$

Integrate:

$$\int \mathrm{d}u\, u^{-2} = \int t\, \mathrm{d}t,$$

or

$$-\frac{1}{u} = C + \tfrac{1}{2}t^2.$$

Re-arrange:

$$\frac{1}{u} = D - \tfrac{1}{2}t^2, \qquad D = -C,$$

and

$$u = \frac{1}{D - (t^2/2)}.$$

The initial condition is $u(t=0) = 1$, hence $1/D = 1$ and $D = 1$. Finally, the solution is

$$u = \frac{1}{1 - (t^2/2)}.$$

Note that the solution is valid only for $0 \leq t < \sqrt{2}$.

4. Sketch the one-dimensional vector field for the following autonomous ODEs:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\cos(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)\left(1 - \tfrac{1}{2}y\right).$$

- First ODE – Figure 2
- Second ODE – Figure 3

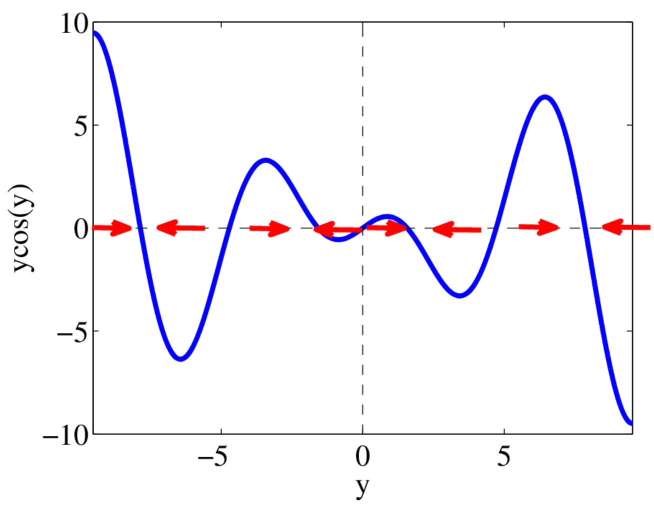



Figure 2: One-dimensional vector field of $dy/dt = y\cos(y)$.

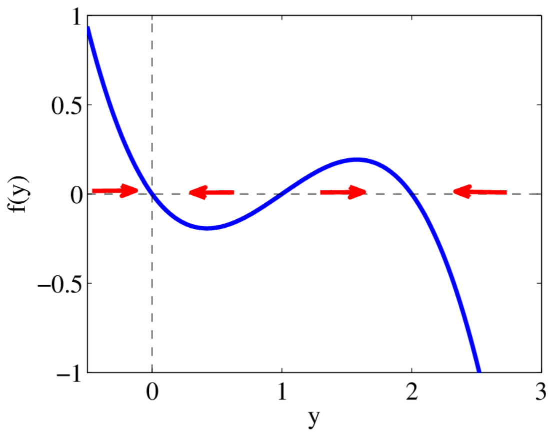



Figure 3: One-dimensional vector field of $dy/dt = y(y-1)[y-(1/2)y]$.

5. Solve the following ODEs using the integrating-factor technique:

$$\frac{dy}{dx} + \frac{y}{x+1} = 1, \qquad x\frac{dy}{dx} + 2y = x\mathrm{e}^x.$$

In the first case, $P(x) = 1/(x+1)$ and $Q(x) = 1$. The integrating factor is

$$\mu(x) = \mathrm{e}^{\int P(x)\mathrm{d}x} = \exp\left(\int \frac{\mathrm{d}x}{1+x}\right) = \exp\log(1+x) = 1+x.$$

Thus, the ODE can be recast as

$$\frac{d}{dx}(\mu y) = \mu Q(x),$$

or

$$\frac{d}{dx}\left[(1+x)\,y\right] = (1+x)\,.$$

Integrate once:

$$(1+x)\,y = C + x + \tfrac{1}{2}x^2$$

or

$$y = \frac{C}{1+x} + \frac{x(1+\frac{1}{2}x)}{1+x}.$$

In the second case, we need first of all to re-write the equation as

$$\frac{dy}{dx} + \frac{2}{x}y = \mathrm{e}^x,$$

since this brings it to standard form, with $P(x) = 2/x$ and $Q(x) = \mathrm{e}^x$. The integrating factor is

$$\mu(x) = \mathrm{e}^{\int P(x)\mathrm{d}x} = \exp\left(\int \frac{2}{x}\mathrm{d}x\right) = \exp\left(2\log x\right) = \exp\left(\log x^2\right) = x^2.$$

Thus, the ODE can be recast as

$$\frac{d}{dx}(\mu y) = \mu Q(x),$$

or

$$\frac{d}{dx}\left(x^2 y\right) = x^2 \mathrm{e}^x.$$

Integrate once:

$$x^2 y = C + \int x^2 \mathrm{e}^x \,\mathrm{d}x,$$

Introduce

$$\begin{aligned} I &= \int x^2 \mathrm{e}^x \,\mathrm{d}x, \\ &= x^2 \mathrm{e}^x - 2\int x\mathrm{e}^x \,\mathrm{d}x, \\ &= x^2 \mathrm{e}^x - 2\left(x\mathrm{e}^x - \int \mathrm{e}^x \,\mathrm{d}x\right), \\ &= x^2 \mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x \end{aligned}$$

(note that this is exactly the same as the integral in Question 1). Hence,

$$y = \frac{C}{x^2} + \mathrm{e}^x\left(1 - \frac{2}{x} + \frac{2}{x^2}\right).$$