Uncertainty Quantification (ACM41000)
Mini-project 1

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Instructions

- Assignment handed out: Friday, Week 3.
- Assignment due: Friday, Week 7.
- Instructions for hand-in: Hand in a hard copy of the report in class in Friday of Week 7.
- The report should be typeset in Latex. For maximum marks, the report should be clearly structured and the different steps in the calculations explained / summarized as appropriate. Diagrams should be included as required, and should be captioned and referred to in the text.
- The framework for marking this project is the 'UCD modular grades explained' document[1]
- Include all codes in an appendix – this can be done very quickly using the 'listings' package in Latex.

Background

This project is based on **epidemic modelling**. A simple deterministic model is proposed for the spread of an infectious disease throughout a population of constant size N. As such, te population is split into three groups: susceptible, infected, and recovered (SIR). These are denoted by $x(t)$, $y(t)$, and $z(t)$ respectively, with

$$x(t) + y(t) + z(t) = N. \tag{1}$$

Here, the total population N is assumed to be constant. The reason for this is that the epidemic is fast-moving, such that the more slow-moving changes in the population arising from natural births and natural deaths do not matter on the timescale of interest.

[1] https://www.ucd.ie/registry/assessment/staff_info/modular%20grades%20explained%20staff.pdf

A simple set of equations to model the dynamics of the different population subgroups is given here:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kxy, \tag{2a}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = kxy - \ell y, \tag{2b}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \ell y, \tag{2c}$$

where k and ℓ are positive rate constants. Equation (2) is the simplest possible SIR model possible. The idea of the model is that individuals leave the susceptible group by contact with an infected individual, at a rate kxy. As such, the susceptible group increases in size by an equal and opposite amount, but this growth is diminished by individuals leaving the infected group and entering the recovered group at a rate ℓy. The aim of this section of the project is to characterize Equation (2) fully – both analytically and numerically.

Tasks

1. Consider the x- and y-equations alone:

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = -kxy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = kxy - \ell y.$$

   In the y-equation, notice that $y = (\mathrm{d}z/\mathrm{d}t)/\ell$. Hence, by substitution into the x-equation, one obtains
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = -(k/\ell)x\frac{\mathrm{d}z}{\mathrm{d}t}.$$
   You should now formally cancel the $\mathrm{d}t$'s on both sides, separate the variables, integrate your result, to arrive at the expression

   $$x(t) = x_0\mathrm{e}^{-kz/\ell}, \tag{3}$$

   where the time-dependence on the right-hand-side is contained in z, which is a function of t.

2. Argue that
   $$y(t) = N - x_0\mathrm{e}^{-kz/\ell} - z. \tag{4}$$

3. Substitute Equation (4) into the equation for $\mathrm{d}z/\mathrm{d}t = \ell y$ to obtain

   $$\frac{\mathrm{d}z}{\mathrm{d}t} = \left[N - x_0\mathrm{e}^{-kz/\ell} - z\right]. \tag{5}$$

   Comment about the amenability of this equation to the kind of analysis we did in Chapter 1 of the lecture notes.

4. Before analysing Equation (5), it is useful to rescale it. Show that the equation can be rescaled as
   $$\frac{\mathrm{d}u}{\mathrm{d}\tau} = a - bu - \mathrm{e}^{-u}, \tag{6}$$

where

$$u = (k/\ell)z, \qquad a = N/x_0, \qquad b = \ell/(kx_0),$$

and where $\tau = kx_0 t$ is a rescaled time variable.

Notice that $a = N/x_0 = (x_0 + y_0)/x_0 > 1$ while $b = \ell/(kx_0) > 0$. The initial condition with $y_0 > 0$ is related to the idea of 'patient zero' in epidemiology – the epidemic can't start without at least one sick person in the population.

5. We now look at the Equation (6) in more detail, and for brevity we rewrite it as $\mathrm{d}u/\mathrm{d}\tau = f(u)$, where $f(u) = a - bu - \mathrm{e}^{-u}$. We have $f(0) = a - 1 > 0$ and $f \sim -bu$ as $u \to \infty$. Thus, the graph of $f(u)$ crosses the positive horizontal axis at least once (actually, only once), so the equation has a fixed point, which we call u_*.

   - Comment on whether u_* is a stable or unstable fixed point.
   - Using a rootfinding algorithm in Matlab / Python / Mathematica / ..., provide a 3D plot showing the dependence of u_* on a range of a- and b-values.

6. The parameter b is special – show graphically that $f(u)$ has a single maximum for $b > 1$ and is a strictly decreasing function for $b < 1$. It goes without saying that we should take $u \geq 0$ in these plots, since u is a scaled population – hence a non-negative number. Thus, the parameter value $b = 1$ is called the **threshold** of the epidemic.

   This an also be shown analytically fairly straightforwardly – one looks at $f'(u_0) = 0$. If this equation can be solved, the function has a a critical point, and the second-derivative test can then be used to establish that this is indeed a maximum.

7. The model has now been more-or-less completely characterized analytically – we can now characterize it numerically also. The idea is to solve Equation (6) numerically, recalled here (with appropriate initial condition) as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = a - bu - \mathrm{e}^{-u}, \qquad u(0) = 0. \tag{7a}$$

   This could be solved in Matlab / Python / Mathematica / ... (for instance, using ODE45 in Matlab). Then, the different sub-populations can be recovered (up to scaling) via the relations

$$x/x_0 = \mathrm{e}^{-u(\tau)}, \tag{7b}$$
$$y/y_0 = a - bu - \mathrm{e}^{-u(\tau)}, \tag{7c}$$
$$z/z_0 = bu. \tag{7d}$$

   Choose a few representative values of a and b, and solve Equation (7) numerically. Plot x/x_0, y/y_0, and z/z_0 as a function of τ for the different values of a and b. Pay close attention to what happens if $b < 1$ and if $b > 1$.

8. For the last question part, we look at an SIR model that includes an extra effect – the idea that a portion of the recovered population will lose immunity and re-enter the susceptible class:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kxy + mz, \tag{8a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = kxy - \ell y, \tag{8b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \ell y - mz, \tag{8c}$$

where $m > 0$ is constant.

- Show that $N = x + y + z$ is constant.
- Show that $x = N$, $y = z = 0$ is a fixed point.
- Compute the Jacobian associated with the fixed point. Compute the eigenvalues of the Jacobian.
- Say whether the fixed point is stable or unstable.

Note: This example is relatively straightforward. When you try to compare it to the basic SIR model (not required for the homework) your head can go into a spin. Ask me questions about this if you are interested.