Stretching revisited

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(Dated: February 9, 2010)

Abstract here

I. INTRODUCTION

In this paper, we derive and study an equation for the gradients of the concentration of a passive tracer in two dimensions:

$$\boldsymbol{\mathcal{B}} = \hat{\boldsymbol{z}} \times \nabla\theta, \tag{1}$$

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{\mathcal{B}}\cdot\nabla\boldsymbol{u} + Pe^{-1}\Delta\boldsymbol{\mathcal{B}}, \tag{2}$$

where $\boldsymbol{u}$ is the advecting velocity field.

II. NUMERICAL METHOD

We solve the two-dimensional vorticity equation

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \nu\Delta\omega, \tag{3}$$

on the bi-periodic domain $\mathbb{T}^2 = [0, L]^2$. The velocity $\boldsymbol{u}$ is incompressible, hence

$$\boldsymbol{u} = \left(\psi_y, -\psi_x, 0\right),$$

and

$$\omega = \hat{\boldsymbol{z}}\cdot\left(\nabla\times\boldsymbol{u}\right) = -\Delta\psi. \tag{4}$$

Equation (3) possesses the following dissipation law,

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 = -\nu\|\nabla\omega\|_2^2,$$

a result that follows from the incompressibility of $\boldsymbol{u}$. The quantity $\|\omega\|_2^2$ is called the *enstrophy*.

We discretize Eq. (3) on a regular spatial grid $\boldsymbol{x}_{ij} = (i, j)\,\Delta x$, where $i, j \in \{0, .., N-1\}$ and $\Delta x = L/N$. Hence, the discrete Fourier transoform of $\omega\left(\boldsymbol{x}_{ij}\right)$ is

$$\hat{\omega}_{\boldsymbol{k}} = \sum_{i,j}\omega\left(\boldsymbol{x}_{ij}\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{ij}},$$

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where

$$k_x, k_y \in \frac{2\pi}{L}\left\{\tfrac{1}{2}N, \tfrac{1}{2}\left(N-1\right), \cdots, 0, \tfrac{1}{2}, \cdots, \tfrac{1}{2}\left(N-1\right)\right\}.$$

In the transformed space, Eq. (3) reads

$$\frac{\partial \hat{\omega}_{\boldsymbol{k}}}{\partial t} = -\nu \boldsymbol{k}^2 \hat{\omega}_{\boldsymbol{k}} + \hat{C}_{\boldsymbol{k}}, \tag{5}$$

where $\hat{C}_{\boldsymbol{k}}$ is the convective term

$$\hat{C}_{\boldsymbol{k}} = \sum_{i,j} \left(\boldsymbol{u}\cdot\nabla\omega\right)\left(\boldsymbol{x}_{ij}\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{ij}}. \tag{6}$$

Moreover, Eq. (4) is inverted according to the scheme

$$\hat{\psi}_{\boldsymbol{k}} = \begin{cases} \text{Arbitrary constant} & \text{if } \boldsymbol{k}=0, \\ -\frac{\hat{\omega}_{\boldsymbol{k}}}{k^2} & \text{otherwise,} \end{cases} \tag{7}$$

hence

$$\hat{\boldsymbol{u}}_{\boldsymbol{k}} = \left(ik_y\hat{\psi}_{\boldsymbol{k}}, -ik_x\hat{\psi}_{\boldsymbol{k}}\right). \tag{8}$$

Using Eqs. (6)–(8), Eq. (5) can now be integrated forward in time, using a standard Euler–Crank–Nicholson scheme:

$$\hat{\omega}_{\boldsymbol{k}}^{n+1} = \left(\frac{1-\left(1-r\right)\Delta t\nu k^2}{1+r\Delta t\nu k^2}\right)\left(\hat{\omega}_{\boldsymbol{k}}^{n} - \Delta t\hat{C}_{\boldsymbol{k}}^{\mathrm{f},n}\right), \qquad 0<r\leq 1,$$

where, in order to avoid aliasing errors, we have filtered $\hat{C}_{\boldsymbol{k}}$:

$$\hat{C}_{\boldsymbol{k}}^{\mathrm{f},n} = \begin{cases} \hat{C}_{\boldsymbol{k}}^{n} & \text{if } |k_x| \text{ and } |k_y| \leq \frac{2}{3}\frac{N\pi}{L}, \\ 0 & \text{otherwise.} \end{cases}$$

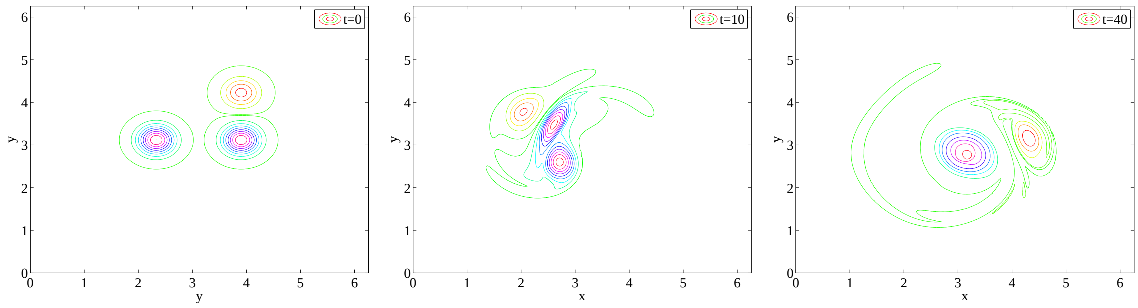
To verify our numerical scheme, we test it against a simulation described in the work of Kevlahan and Farge [1], for which the initial condition is

$$\omega_0\left(\boldsymbol{x}\right) = \frac{\Gamma}{\pi r_0^2}\left[e^{-|\boldsymbol{x}-\boldsymbol{x}_1|^2/r_0^2} + e^{-|\boldsymbol{x}-\boldsymbol{x}_2|^2/r_0^2} - \tfrac{1}{2}e^{-|\boldsymbol{x}-\boldsymbol{x}_3|^2/r_0^2}\right]. \tag{9}$$

We work in non-dimensional units where $L=2\pi$ and $\Gamma=1$, which necessitates the replacement $\nu \rightarrow Re^{-1}$, $Re = \Gamma/\nu$. In these units,

$$\boldsymbol{x}_1 = \left(\tfrac{3}{4}, 1\right), \qquad \boldsymbol{x}_2 = \left(\tfrac{5}{4}, 1\right), \qquad \boldsymbol{x}_3 = \left(\tfrac{5}{4}, 1+\tfrac{1}{2\sqrt{2}}\right).$$

We take $N=256$, $\Delta t = 1\times 10^{-3}$ and $Re = 10^4$. The results of the simulation are shown in Fig. 1



(a) (b) (c)

Figure 1: Comparison with Fig. (1) (a) of Kevlahan and Farge [1]. The contours are at $\pm\pi\left(1+11j\right)/100$, and $j=0,\cdots,9$. Exact agreement is obtained with that benchmark simulation, confirming the validity of our numerical model.

Hyperviscosity

In studies of turbulence, we are interested in the spectrum of the kinetic energy,

$$E\left(k\right)=\frac{1}{|\Omega_d|}\int_{\Omega_d}\mathrm{d}\Omega_d\tfrac{1}{2}|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2_{|\boldsymbol{k}|=k}. \tag{10}$$

This possesses definite scaling laws in particular parts of Fourier space. In particular, forced two-dimsional turbulence is thought to possess the scaling laws

$$E\left(k\right)\sim\begin{cases}\alpha\left(\frac{\mathrm{d}\overline{E}}{\mathrm{d}t}\right)^{2/3}k^{-5/3}, & k\leq k_{\mathrm{e}},\\ \beta\eta^{2/3}k^{-3}, & k\geq k_{\mathrm{e}},\end{cases} \tag{11}$$

where $d\overline{E}/dt$ is the total energy $\int_0^\infty \mathrm{d}kE\left(k\right)$, k_{e} is the forcing scale, η is the dissipation rate of enstrophy, and α and β are dimensionless constants. Equation (11) breaks down on very small scales where molecular viscosity dominates, $k\geq 2\pi/\lambda_K$, where

$$\lambda_K\sim\left(\nu^3/\chi\right)^{1/6},\qquad \chi\sim\nu\langle\|\omega\|_2^2\rangle/L^2.$$

The scale λ_K is called the *Kraichnan* scale. The range $k_{\mathrm{e}}\leq k\leq 2\pi/\lambda_K$ is called the *inertial range*. In numerical studies, the inertial range is broadened (so facilitating the extraction of power laws from the spectrum) by replacing the ordinary molecular viscosity in the Navier–Stokes equations by a hyperviscosity. Thus, we consider the vorticity equation

$$\frac{\partial\omega}{\partial t}+\boldsymbol{u}\cdot\nabla\omega=-\nu_p\left(-1\right)^p\Delta^p\omega. \tag{12}$$

We verify the inclusion of hyperviscosity in our numerical code by solving Eq. (12) with the initial conditions (9), together with the following parameters:

$$N=256,\qquad \nu_p=5.9\times10^{-30},\qquad \Delta t=10^{-3}.$$

Forcing

Because the vorticity equation (3) is dissipative, a necessary (but not sufficient) condition for the achievement of a steady state is that the equation be forced:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega = \nu \Delta \omega + Q, \tag{13}$$

where

$$Q = \hat{\boldsymbol{z}} \cdot (\nabla \times \boldsymbol{f}),$$

and $\boldsymbol{f}(\boldsymbol{x}, t)$ is a force. To obtain a stochastic driving force on a particular scale k_{e}, we use the method developed by Lilly [2], and deployed elsewhere by Molenaar *et al.* [3]:

$$\begin{aligned} \hat{Q}_{\boldsymbol{k}}^{n+1} &= R\hat{Q}_{\boldsymbol{k}}^{n} + \sqrt{1-R^2} B_0 e^{2\pi i \theta_{\boldsymbol{k}}^{n+1}}, \\ Q(\boldsymbol{x}) &= \frac{1}{N^2} \Re \left(\sum_{\boldsymbol{k}} \hat{Q}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right). \end{aligned} \tag{14}$$

where B_0 is an amplitude and R is a correlation coefficient that correlates the n- and $n+1$-level values of the forcing function Q. The phase θ_k is a random number between zero and one that is uncorrelated in space and time. To give the forcing a characteristic lengthscale, we apply a filter to the quantity

$$\begin{aligned} \hat{F}_{\mathrm{k}}^{n+1} &= \sqrt{1-R^2} B_0 e^{2\pi i \theta_{\boldsymbol{k}}^{n+1}}, \\ F(\boldsymbol{x}) &= \frac{1}{N^2} \Re \left(\sum_{\boldsymbol{k}} \hat{F}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right). \end{aligned}$$

That is, we make the replacement $\hat{F}_{\mathrm{k}}^{n+1} \rightarrow \hat{F}_{\mathrm{k}}^{\mathrm{f},n+1}$, where

$$\hat{F}_{\mathrm{k}}^{\mathrm{f}} = \begin{cases} \sqrt{1-R^2} B_0 e^{2\pi i \theta_{\boldsymbol{k}}^{n+1}} & \text{if } k_{\mathrm{e}1} \leq |\boldsymbol{k}| \leq k_{\mathrm{e}2}, \\ 0 & \text{otherwise.} \end{cases}$$

Some results of such a protocol are shown in Fig. 2, where the simulation is carried out for $\Delta t = 10^{-3}$ and $N = 256$ (the other parameter-values are given in the caption). Note that if we take

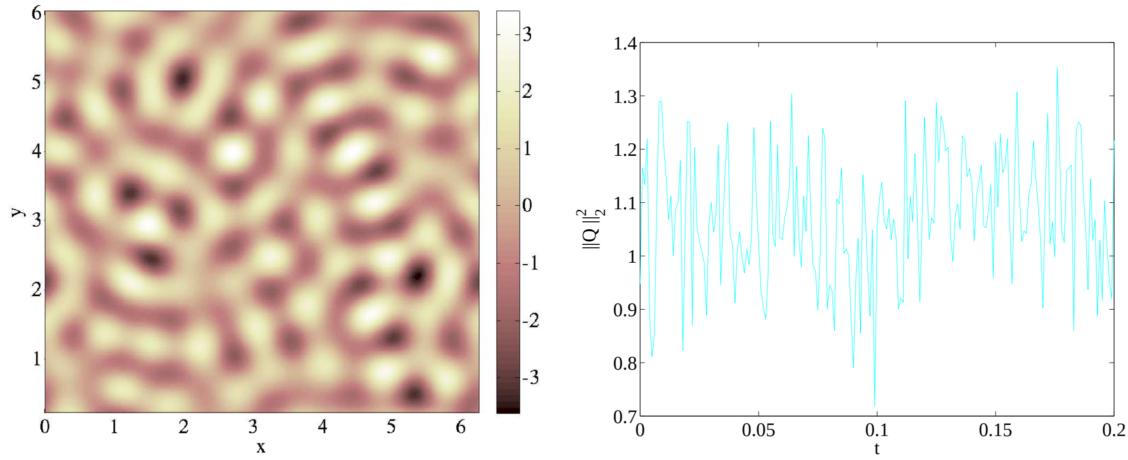
$$B_0 = \frac{1}{\sqrt{2}} \frac{A_0 N^2}{\sqrt{\pi \left(k_{\mathrm{e}2}^2 - k_{\mathrm{e}1}^2\right)}}$$

then

$$\langle \|F\|_2^2 \rangle := \left\langle \frac{1}{L^2} \int_{[0,L]^2} \mathrm{d}^d x |f(\boldsymbol{x})|^2 \right\rangle \approx \left\langle \frac{1}{N^2} \sum_{i,j} F^2(\boldsymbol{x}_{ij}) \right\rangle \approx A_0^2.$$

Moreover, if $\langle \|F\|_2^2 \rangle = A_0^2$, the $\langle \|Q\|_2^2 \rangle = A_0^2$ too. To see this, we square both sides of Eq. (14) and time-average:

$$\begin{aligned} \hat{Q}_{\boldsymbol{k}}^{n+1} \left(\hat{Q}_{\boldsymbol{k}}^{n+1}\right)^* &= R^2 \hat{Q}_{\boldsymbol{k}}^{n} \left(\hat{Q}_{\boldsymbol{k}}^{n}\right)^* + \left(1-R^2\right) B_0^2 \\ &+ \sqrt{1-R^2} B_0^2 \left[\hat{Q}_{\boldsymbol{k}}^{n} e^{-2\pi i \theta_{\boldsymbol{k}}^{n+1}} + \left(\hat{Q}_{\boldsymbol{k}}^{n}\right)^* e^{2\pi i \theta_{\boldsymbol{k}}^{n+1}} \right], \\ \langle |\hat{Q}_{\boldsymbol{k}}^{n+1}|^2 \rangle &= R^2 \langle |\hat{Q}_{\boldsymbol{k}}^{n}| \rangle + \left(1-R^2\right) B_0^2, \end{aligned}$$



(a) (b)

Figure 2: (a) Force field $Q(\boldsymbol{x}, t = 10)$, $A_0 = 1$, $R = 0.5$, $k_{e1} = 7$, k_{e2}; (b) L^2 norm of the force over a short time interval.

where we have used the fact that $\hat{Q}^n_{\boldsymbol{k}}$ and $\hat{F}^{n+1}_{\boldsymbol{k}}$ are uncorrelated. Next, we use Parseval's identity

$$\sum_{i,j} |f_{ij}|^2 = \frac{1}{N^d} \sum_{\boldsymbol{k}} |\hat{f}_{\boldsymbol{k}}|^2,$$

for any function f on $[0, 2\pi]^2$, and $d = 2$. In an alternative form, this is

$$\|f\|_2^2 \approx \frac{1}{N^d} \sum_{i,j} |f_{ij}|^2 = \frac{1}{N^{2d}} \sum_{\boldsymbol{k}} |\hat{f}_{\boldsymbol{k}}|^2.$$

Hence,

$$\langle \|Q^{n+1}\|_2^2 \rangle = R^2 \langle \|Q^n\|_2^2 \rangle + \left(1 - R^2\right) B_0^2 N^{-2d} \left(\sum_{\boldsymbol{k}, k_{e1} \le |\boldsymbol{k}| \le k_{e2}} 1 \right).$$

Now the density of points in discrete Fourier space is

$$\rho_P = \Delta k^{-2}, \qquad \Delta k = \pi / L.$$

The area defined by the ring $k_{e1} \le |\boldsymbol{k}| \le k_{e2}$ is $\pi\left(k_{e2}^2 - k_{e1}^2\right)$. Hence, the sum to be evaluated is

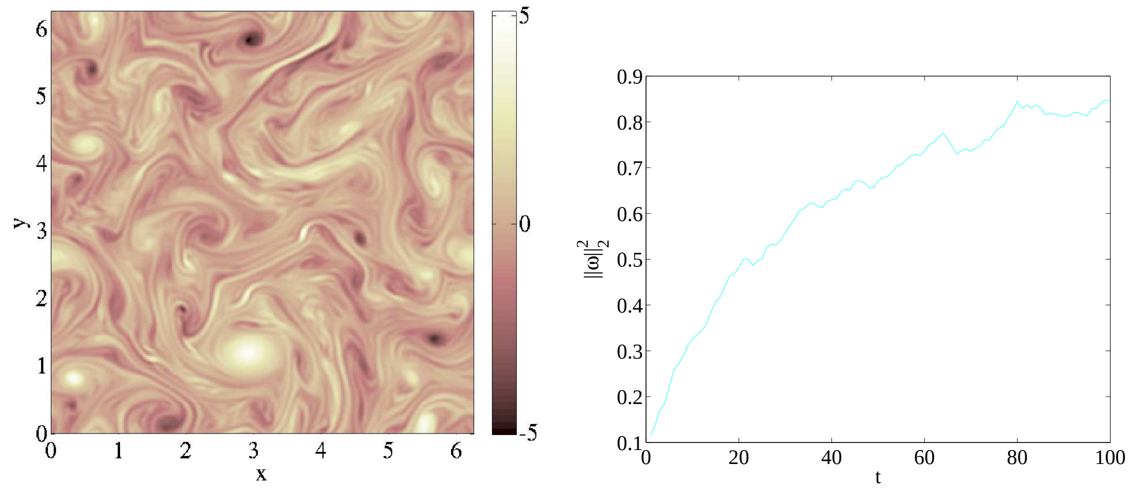
$$\sum_{\boldsymbol{k}, k_{e1} \le |\boldsymbol{k}| \le k_{e2}} 1 \approx \left(k_{e2}^2 - k_{e1}^2\right) L^2/\pi \;=\; 4\pi \left(k_{e2}^2 - k_{e1}^2\right).$$

Hence,

$$\langle \|Q^{n+1}\|_2^2 \rangle = R^2 \langle \|Q^n\|_2^2 \rangle + \left(1 - R^2\right) 2A_0^2.$$

Because we take only the real part of Q into consideration, we must separate out the contributions made by the random force to the real and imaginary parts of Q. Thus, we have

$$\langle \|Q^{n+1}\|_2^2 \rangle = R^2 \langle \|Q^n\|_2^2 \rangle + \left(1 - R^2\right) A_0^2.$$



(a) (b)

Figure 3: (a) Vorticity field $\omega\left(\boldsymbol{x}, t=100\right)$; (b) L^2 norm of the vorticity over the total runtime of the simulation.

This is a difference equation for $\langle \|Q^n\|_2^2 \rangle$, with solution

$$\langle \|Q^n\|_2^2 \rangle = R^{2n} \langle \|Q^0\|_2^2 \rangle + \left(1 - R^{2n}\right) A_0^2.$$

As $n \to \infty$,

$$\langle \|Q^n\|_2^2 \rangle \to A_0^2,$$

as confirmed by Fig. 2.

III. PRELIMINARY RESULTS

We present preliminar results for the solution of the vorticity equation

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega = -\left(-1\right)^p \nu_p \nabla^p \omega + Q,$$

where the forcing Q is fixed as in Sec. II. The intial conditions, although not important, are the same as in Sec. II. Moreover, the numerical parameters are given in Tab. I. The

Order of viscosity, $p = 8$	Correlation coefficient, $R = .9$
$\nu_p = 5.9 \times 10^{-30}$	Forcing amplitude, $A_0 = 1$
Resolution $N^2 = 256^2$	Upper forcing scale $k_{e1} = 7$
$\Delta t = 10^{-3}$	Lower forcing scale $k_{e1} = 9$

Table I: Parameters used in the simulation

instantaneous vorticity field at $t = 100$ is shown in Fig. 3 (a), and the L^2 norm of the vorticity over the entire simulation runtime is shwon in (b). Fig. (a) has the characteristic

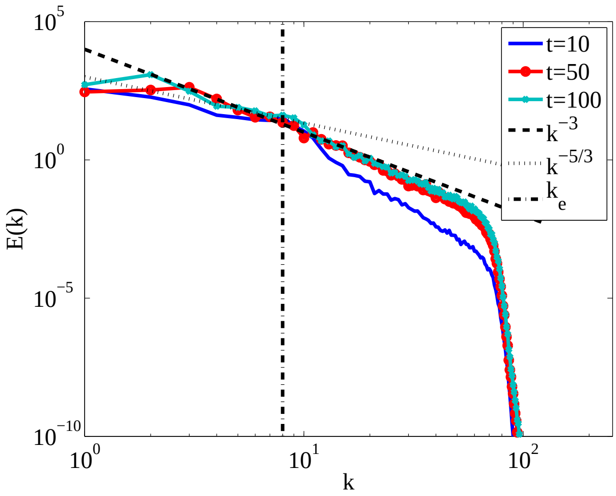


Figure 4: Instantaneous power spectra at $t = 10, 50, 100$.

large-scale vortices of two-dimensional turbulence, and Fig. (b) suggests that the vorticity has reached a quasi-steady value, in the sense that its average value fluctuates around a mean value. Howeever, to obtain reliable statistics, a longer runtime is clearly necessary, perhaps up to $t = 10^3$. To verify rigorously that large-scale vortices are forming, we study the power spectrum of the velocity field, defined previously in Eq. (16). Several snaphots of this quantity are shown in Fig. 4. The power spectrum exhibts the scaling laws proposed by Kraichnan and summarised in our Eq. (11). That is, on scales shorter than the scale of energy injection $k_{\rm e}$, $E\left(k\right) \sim k^{-3}$, while on larger scales, $E\left(k\right) \sim k^{-5/3}$. Furthermore, because $E\left(k,t\right)$ increases in time in the range of large scales, our power spectra show evidence of the inverse cascade, wherein energy is transferred from the injection scale to the largest scales in the problem. However, a longer integration time is necessary to demonstrate this conclusively. In any case, we have subjected our numerical method to some rigorous tests, and it has passed them. We are therefore confident in its application to the problem in hand, namely the advection of a passive tracer by a turbulent flow.

We investigate the advection of a passive tracer in a preliminary way as follows. The equation for the passive tracer is taken to be

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = -\left(-1\right)^{p}\nu_{\theta,p}\nabla^{p}\theta, \tag{15}$$

where changes in θ and its gradients do not feed back into the vorticity. As in the vorticity equation, we study a hyperdiffusive tracer, to observe inertial range and so draw firm conclusions about the spectrum.

The instantaneous concentration level θ at $t = 100$ is shown in Fig. 5, where we have set $\nu_{\theta,8} = 10\nu_8$. The distribution of the tracer looks similar to what has been observed in other studies of passive-tracer mixing. To verify this rigorouosly, we plot a power spectrum of the scalar energy,

$$E_{\theta}\left(k\right) = \frac{1}{|\Omega_d|}\int_{\Omega_d} \mathrm{d}\Omega_d \tfrac{1}{2}|\hat{\theta}_{\boldsymbol{k}}|^2_{|\boldsymbol{k}|=k}. \tag{16}$$

Several snaphots of this quantity are shown in Fig. 6. The power spectrum exhibts the

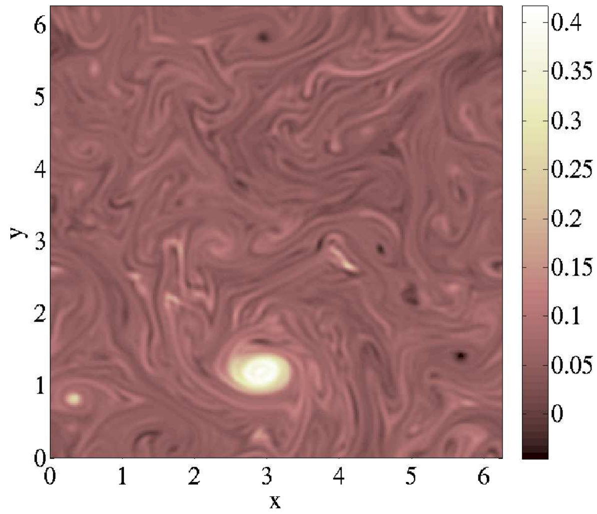


Figure 5: Concentration level field $\theta(\boldsymbol{x}, t = 100)$; $\nu_{\theta,8} = 5\nu_8$.

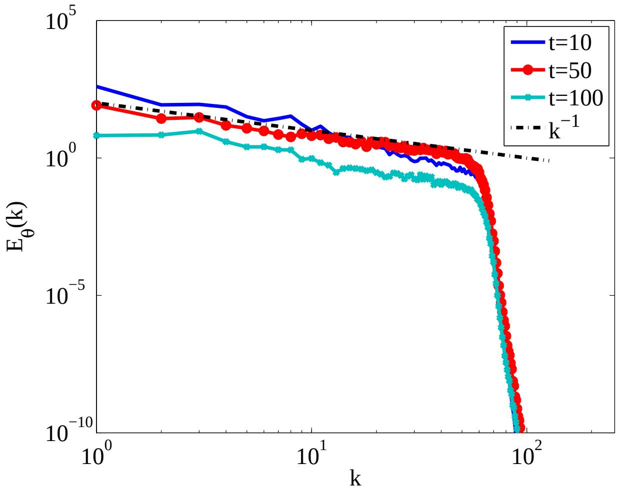


Figure 6: Instantaneous power spectra of the passive scalar at $t = 10, 50, 100$.

well-known scaling law $E_\theta(k) \sim k^{-1}$ in the inertial range, a further confirmation of our numerical scheme.

Large-scale damping

In the work that is to follow, we are intersted in driving the passive-tracer (15) with a quasi-steady flow. However, the two-dimensional turbulence we have studied is not quasi-steady, because energy is cascading continually to larger scales. Therefore, throughout the rest of the study, we work with the following vorticity-scalar equation pair:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega \ = \ -(-1)^p \, \Delta^p \omega + Q - K\omega, \tag{17}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta \ = \ -(-1)^p \, \Delta^p \theta. \tag{18}$$

IV. PROPERTIES OF THE $\boldsymbol{\mathcal{B}}$-EQUATION

We consider the $\boldsymbol{\mathcal{B}}$-equation (2) again. For now, let us ignore diffusion. Taking the dot product of the equation with $\boldsymbol{\mathcal{B}}$ itself, we obtain

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) |\boldsymbol{\mathcal{B}}|^2 = (\boldsymbol{\mathcal{B}}, S\boldsymbol{\mathcal{B}}),$$

where $S_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$ is the symmetric rate-of-strain tensor. We write the matrix as

$$S = \begin{pmatrix} s & d \\ d & -s \end{pmatrix}.$$

Hence, its eigenvalues are

$$\mathrm{spec}\,(S) = \{\lambda, -\lambda\}, \qquad \lambda = \sqrt{s^2 + d^2}.$$

The eigenvectors are orthognonal:

$$\boldsymbol{X}_{(+)} = \frac{1}{\mathcal{N}} \begin{pmatrix} 1 \\ -\alpha + \sqrt{\alpha^2 + 1} \end{pmatrix}, \qquad \alpha = s/d,$$

where

$$\mathcal{N}^2 = 2\sqrt{\alpha^2+1}\left(-\alpha + \sqrt{\alpha^2+1}\right)$$

is the normalization. Similarly,

$$\boldsymbol{X}_{(-)} = \frac{1}{\mathcal{N}} \begin{pmatrix} \alpha - \sqrt{\alpha^2+1} \\ 1 \end{pmatrix}$$

and

$$\boldsymbol{X}_{(-)} \cdot \boldsymbol{X}_{(+)} = 0, \qquad \boldsymbol{X}_{(-)} \cdot \boldsymbol{X}_{(-)} = \boldsymbol{X}_{(+)} \cdot \boldsymbol{X}_{(+)} = 1.$$

Thus, we introduce an angle

$$\cos\varphi = \mathcal{N}^{-1}, \qquad \sin\varphi = \frac{-\alpha + \sqrt{\alpha^2+1}}{\mathcal{N}},$$

where it is readily shown that $\cos^2\varphi \leq 1$, $\sin^2\varphi \leq 1$, and $\cos^2\varphi + \sin^2\varphi = 1$. Thus, the eigenvectors can be re-expressed as

$$\boldsymbol{X}_{(+)} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad \boldsymbol{X}_{(-)} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix},$$

or

$$\begin{pmatrix} \boldsymbol{X}_{(+)} \\ \boldsymbol{X}_{(-)} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix},$$

and the strain matrix can be written as

$$S = \lambda \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}.$$

Thus, φ is the angle between the x-axis and the expanding direction of a straining flow. Hence, if $\boldsymbol{x} = a\boldsymbol{X}_{(+)} + b\boldsymbol{X}_{(-)}$, then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and $\boldsymbol{x} \cdot (S\boldsymbol{x}) = \lambda\left(a^2 - b^2\right)$. This is

$$\boldsymbol{x} \cdot (S\boldsymbol{x}) = \lambda\left[\cos 2\varphi\left(x^2 - y^2\right) + 2\sin 2\varphi xy\right].$$

Substituting back into the $\boldsymbol{\mathcal{B}}$-equation,

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = \lambda\left(\boldsymbol{x}, t\right)\left[\cos 2\varphi\left(\theta_y^2 - \theta_x^2\right) - 2\sin 2\varphi\,\theta_x\theta_y\right],$$

Thus, let us define $\cos\beta = \theta_x/|\nabla\theta| = \theta_x/|\boldsymbol{\mathcal{B}}|$, $\sin\beta = \theta_y/|\nabla\theta|$. Now the stretching equation becomes

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\left(\boldsymbol{x}, t\right)|\boldsymbol{\mathcal{B}}|^2\left[\cos 2\varphi\left(\cos^2\beta - \sin^2\beta\right) + 2\sin 2\varphi\sin\beta\cos\beta\right],$$

or,

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\left(\boldsymbol{x}, t\right)|\boldsymbol{\mathcal{B}}|^2\left[\cos 2\varphi\cos 2\beta + 2\sin\varphi\sin 2\beta\right].$$

Using the sum formula,

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\left(\boldsymbol{x}, t\right)|\boldsymbol{\mathcal{B}}|^2\cos\left[2\left(\varphi - \beta\right)\right].$$

To agree with notation introduced elsewhere, let us define a new angle through the relation

$$\varphi = \frac{\pi}{4} - \psi.$$

Hence,

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\left(\boldsymbol{x}, t\right)|\boldsymbol{\mathcal{B}}|^2\sin\left[2\left(\psi + \beta\right)\right].$$

Thus, for exponential stretching to occur, we must have

$$\tfrac{1}{2}\pi + \beta \le \psi \le \pi + \beta.$$

We must now find an advection equation for β. We use $\tan\beta = \theta_y/\theta_x$, hence

$$\begin{aligned}\partial_t \tan\beta &= \frac{1}{\theta_x}\partial_t\theta_y - \frac{\theta_y}{\theta_x^2}\partial_t\theta_x, \\ &= \frac{\partial_y}{\partial_x}\left[\frac{1}{\theta_x}\partial_t\theta_y - \frac{1}{\theta_x}\partial_t\theta_x\right].\end{aligned}$$

Similarly,

$$\boldsymbol{u}\cdot\nabla\tan\beta = \frac{\partial_y}{\partial_x}\left[\frac{1}{\theta_x}\boldsymbol{u}\cdot\nabla\theta_y - \frac{1}{\theta_x}\boldsymbol{u}\cdot\nabla\theta_x\right].$$

Hence,

$$\begin{aligned}(\partial_t + \boldsymbol{u}\cdot\nabla)\tan\beta &= \frac{\theta_y}{\theta_x}\left[\frac{1}{\theta_y}(\partial_t + \boldsymbol{u}\cdot\nabla)\theta_y - \frac{1}{\theta_x}(\partial_t + \boldsymbol{u}\cdot\nabla)\theta_x\right], \\ &= \frac{\theta_y}{\theta_x}\left\{\frac{1}{\theta_y}\left[\partial_y(\partial_t + \boldsymbol{u}\cdot\nabla)\theta - \boldsymbol{u}_y\cdot\nabla\theta\right] - \frac{1}{\theta_x}\left[\partial_y(\partial_t + \boldsymbol{u}\cdot\nabla)\theta - \boldsymbol{u}_x\cdot\nabla\theta\right]\right\}, \\ &= \frac{1}{\theta_x^2}\left[-\theta_x(\boldsymbol{u}_y\cdot\nabla\theta) + \theta_y(\boldsymbol{u}_x\cdot\nabla\theta)\right].\end{aligned}$$

But this is

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\tan\beta = \frac{1}{\theta_x^2}\left[-\theta_x(\boldsymbol{\mathcal{B}}\cdot\nabla)u - \theta_y(\boldsymbol{\mathcal{B}}\cdot\nabla)v\right].$$

Using the chain rule, this is

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\beta = \frac{1}{|\boldsymbol{\mathcal{B}}|^2}\left[-\theta_x(\boldsymbol{\mathcal{B}}\cdot\nabla)u - \theta_y(\boldsymbol{\mathcal{B}}\cdot\nabla)v\right]. \tag{19}$$

If we identify the vector $\boldsymbol{\mathcal{A}} = (\theta_x, \theta_y)$, $\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{B}} = 0$, then

$$-\theta_x(\boldsymbol{\mathcal{B}}\cdot\nabla)u - \theta_y(\boldsymbol{\mathcal{B}}\cdot\nabla)v = -\boldsymbol{\mathcal{A}}^T\left[(\nabla\boldsymbol{u})\boldsymbol{\mathcal{B}}\right],$$

where

$$\begin{aligned}\nabla\boldsymbol{u} &= \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}, \\ &= \begin{pmatrix} u_x & \frac{1}{2}(u_y + v_x) \\ \frac{1}{2}(u_y + v_x) & v_y \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2}(v_x - u_y) \\ -\frac{1}{2}(v_2 - u_y) & 0 \end{pmatrix}, \\ &= S - \tfrac{1}{2}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}|\boldsymbol{\mathcal{B}}|^2\dot{\beta} &= -\boldsymbol{\mathcal{A}}^T\left[S - \tfrac{1}{2}\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\boldsymbol{\mathcal{B}}, \\ &= -\boldsymbol{\mathcal{A}}^T S\boldsymbol{\mathcal{B}} + \tfrac{1}{2}\omega\boldsymbol{\mathcal{A}}^T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{\mathcal{B}},\end{aligned}$$

or

$$|\boldsymbol{\mathcal{B}}|^2\dot{\beta} \;=\; -\boldsymbol{\mathcal{A}}^T S\boldsymbol{\mathcal{B}} + \tfrac{1}{2}\omega|\boldsymbol{\mathcal{B}}|^2.$$

Going back to the eigendecomposition, we have $\boldsymbol{\mathcal{A}} = (A_x, A_y) = a\boldsymbol{X}_{(+)} + b\boldsymbol{X}_{(-)}$, and $\boldsymbol{\mathcal{B}} = (-A_y, A_x) = \tilde{a}\boldsymbol{X}_{(+)} + \tilde{b}\boldsymbol{X}_{(-)}$, where

$$\begin{aligned}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \cos\varphi A_x + \sin\varphi A_y \\ -\sin\varphi A_x + \cos\varphi A_y \end{pmatrix}, \\ \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} &= \begin{pmatrix} -\cos\varphi A_y + \sin\varphi A_x \\ \sin\varphi A_y + \cos\varphi A_x \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}.\end{aligned}$$

Thus,

$$-\boldsymbol{\mathcal{A}}^T S\boldsymbol{\mathcal{B}} = -\left(-2\lambda ab\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\sin\left[2\left(\varphi-\beta\right)\right]|\boldsymbol{\mathcal{B}}|^2 = -\lambda\cos\left[2\left(\psi+\beta\right)\right]|\boldsymbol{\mathcal{B}}|^2.$$

Hence,

$$\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\beta = \tfrac{1}{2}\omega - \lambda\cos\left[2\left(\psi+\beta\right)\right], \tag{20}$$

which supplements the equation

$$\tfrac{1}{2}\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\sin\left[2\left(\psi+\beta\right)\right]|\boldsymbol{\mathcal{B}}|^2. \tag{21}$$

for the magnitude of the gradient vector.

Note that if diffusion were included, we would have the following equation pair:

$$\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\beta = \tfrac{1}{2}\omega - \lambda\cos\left[2\left(\psi+\beta\right)\right] + \frac{1}{Pe}\cos^2\beta\,\nabla\cdot\left(\theta_x\nabla\theta - \theta_y\nabla\theta\right), \tag{22}$$

together with

$$\tfrac{1}{2}\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -\lambda\sin\left[2\left(\varphi+\beta\right)\right]|\boldsymbol{\mathcal{B}}|^2 + \frac{1}{Pe}\boldsymbol{\mathcal{B}}\cdot\left(\Delta\boldsymbol{\mathcal{B}}\right). \tag{23}$$

These equations are not that different from those studied by Lapeyre *et al.* [4], so we will need to do better than this.

Next, we develop an advection equation for the angle φ. Since $\cos\varphi = \mathcal{N}^{-1}$, it follows that

$$\dot{\varphi} = \frac{\cos^2\varphi}{\sin\varphi}\dot{\mathcal{N}},$$

where

$$\begin{aligned}\dot{\mathcal{N}} &= \frac{1}{\mathcal{N}}\frac{d}{dt}\left[\sqrt{1+\alpha^2}\left(-\alpha+\sqrt{\alpha^2+1}\right)\right], \\ &= \frac{1}{\mathcal{N}}\left[\frac{\alpha\dot{\alpha}}{\sqrt{1+\alpha^2}}\left(-\alpha+\sqrt{\alpha^2+1}\right) + \sqrt{1+\alpha^2}\left(-\dot{\alpha}+\frac{\alpha\dot{\alpha}}{\sqrt{1+\alpha^2}}\right)\right], \\ &= \frac{1}{\mathcal{N}}\frac{\dot{\alpha}}{\sqrt{\alpha^2+1}}\left[2\alpha\left(-\alpha+\sqrt{\alpha^2+1}\right)-1\right].\end{aligned}$$

Now

$$\cos\varphi\sin\varphi = \mathcal{N}^{-2}\left(-\alpha+\sqrt{\alpha^2+1}\right) = \tfrac{1}{2}\frac{1}{\sqrt{\alpha^2+1}},$$

hence $\sin 2\varphi = 1/\sqrt{\alpha^2+1}$, similarly $\cos 2\varphi = \alpha/\sqrt{\alpha^2+1}$, hence

$$\begin{aligned}
\dot\varphi &= \frac{\cos^3\varphi}{\sin\varphi}\frac{1}{\sqrt{\alpha^2+1}}\left[2\alpha\left(-\alpha+\sqrt{\alpha^2+1}\right)-1\right]\dot\alpha,\\
&= \frac{\cos^3\varphi}{\sin\varphi}\frac{1}{\sqrt{\alpha^2+1}}\left(2\alpha\mathcal{N}\sin\varphi-1\right)\dot\alpha,\\
&= \left(2\cos^2\varphi\frac{\alpha}{\sqrt{\alpha^2+1}}-\frac{\cos^3\varphi}{\sin\varphi}\frac{1}{\sqrt{\alpha^2+1}}\right)\dot\alpha,\\
&= \left(2\cos^2\varphi\cos 2\varphi-\frac{\cos^3\varphi}{\sin\varphi}2\sin\varphi\cos\varphi\right)\dot\alpha,\\
&= \left(2\cos^2\varphi\cos 2\varphi-2\cos^4\varphi\right)\dot\alpha.
\end{aligned}$$

Finally, this is

$$\dot\varphi = -2\cos^2\varphi\sin^2\varphi\,\dot\alpha.$$

Thus, the advection of the angle φ is proportional to the advection of the ratio $\alpha = s/d$. Therefore, we must compute the advection rate of α. This is simply

$$\dot\alpha = \frac{s}{d}\left(\frac{\dot s}{s}-\frac{\dot d}{d}\right).$$

Now $s = u_x$ and, assuming Euler dynamics for now,

$$\dot s = -p_{xx} - \boldsymbol{u}_x\cdot\nabla u.$$

But $\boldsymbol{u}_x\cdot\nabla u = u_xu_x + v_xu_y = s^2 + v_xu_y$. Furthermore, $d = (u_y+v_x)/2$ and $\omega = v_x - u_y$. Hence,

$$u_y = d - \tfrac{1}{2}\omega, \qquad v_x = d + \tfrac{1}{2}\omega,$$

$v_xu_y = d^2 - (\omega^2/4)$, and

$$\begin{aligned}
\dot s &= -p_{xx} - s^2 - d^2 + \tfrac{1}{4}\omega^2,\\
&= -p_{xx} - \lambda^2 + \tfrac{1}{4}\omega^2.
\end{aligned}$$

Similary,

$$\dot d = -p_{xy} - \tfrac{1}{2}\left(\boldsymbol{u}_y\cdot\nabla u + \boldsymbol{u}_x\cdot\nabla v\right),$$

but here the straining terms cancel, and $\dot d = -p_{xy}$. Putting it all together,

$$\dot\alpha = \frac{1}{d^2}\left(-dp_{xx} + sp_{xy} - d\lambda^2 + \tfrac{1}{4}d\omega^2\right).$$

Hence,

$$\dot\varphi = 2\cos^2\varphi\sin^2\varphi\,d^{-2}\left(d\lambda^2 - \tfrac{1}{4}d\omega^2 + dp_{xx} - sp_{xy}\right).$$

Thus, we obtain a new equation pair for the stretching, in terms of the angle $\gamma := 2(\psi+\beta)$:

$$\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)|\boldsymbol{\mathcal{B}}|^2 = -2\sin\gamma|\boldsymbol{\mathcal{B}}|^2, \tag{24}$$

$$\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\gamma = 2\lambda\left(r-\cos\gamma\right), \tag{25}$$

where

$$r = \frac{\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\psi + \frac{1}{2}\omega}{\lambda} = \tfrac{1}{4}\cos^2 2\psi\frac{1}{\lambda d^2}\left(d\lambda^2 - \tfrac{1}{4}d\omega^2 + dp_{xx} - sp_{xy}\right) + \tfrac{1}{2}\frac{\omega}{\lambda}.$$

Note:If we introduce a timescale $\mathrm{d}t = 2\mathrm{d}\tau$, a potential $V = \sin\gamma$, and a force $f(\tau) = r$, then Eq. (25) reduces to the equation of motion for a forced, overdamped particle experiencing a periodic potential:

$$\frac{\mathrm{d}\gamma}{d\tau} = -\frac{\partial V}{\partial\gamma} + f(\tau).$$

Example: Random-phase sine flow

$$\begin{array}{lll} u = A_0\sin\left(ky+\phi_j\right), & v = 0\ , & j\tau < t \le \frac{1}{2}\left(j+1\right)\tau, \\ u = 0, & v = A_0\sin\left(kx+\psi_j\right)\ , & \frac{1}{2}\left(j+1\right)\tau < t < \left(j+1\right)\tau. \end{array}$$

Here $s = 0$ during each half-period. In the first half-period, $d = \frac{1}{2}A_0k\cos\left(ky+\phi_j\right)$; in the second half-period $d = \frac{1}{2}A_0k\cos\left(kx+\psi_j\right)$, summed up in the formula

$$d = \tfrac{1}{2}A_0k\left[H_\tau(t)\cos\left(ky+\phi_j\right) + \left(1-H_\tau(t)\right)\cos\left(kx+\psi_j\right)\right],$$

where

$$H_\tau(t) = \begin{cases} 1, & j\tau < t \le \frac{1}{2}\left(j+1\right)\tau, \\ 0, & \frac{1}{2}\left(j+1\right)\tau < t \le \left(j+1\right)\tau. \end{cases}$$

Similarly,

$$\omega = A_0k\left[-H_\tau(t)\cos\left(ky+\phi_j\right) + \left(1-H_\tau(t)\right)\cos\left(kx+\psi_j\right)\right]$$

Since $\alpha = s/d$=0, the eigenvalues are $\pm d$, and the eigenvectors are

$$\boldsymbol{X}_+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \boldsymbol{X}_- = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Hence, $\cos\varphi = \hat{\boldsymbol{x}}\cdot\boldsymbol{X}_+ = 1/\sqrt{2}$, and $\varphi = \pi/4 = \text{Const}$. Thus, the angle $\psi = (\pi/4) - \varphi = 0$ is constant too. Note that $\boldsymbol{X}_+$ is no longer necessarily the direction of expansion: it is the expnasive direction if $d > 0$, and it becomes the compressive direction if $d < 0$. Now

$$\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\beta = \tfrac{1}{2}\omega - d\cos\left[2\left(\psi+\beta\right)\right],$$

and $\gamma = 2(\psi+\beta) = 2\beta$, hence

$$\begin{aligned}(\partial_t + \boldsymbol{u}\cdot\nabla)\gamma &= 2(\partial_t + \boldsymbol{u}\cdot\nabla)\beta, \\ &= \omega - 2d\cos\gamma. \end{aligned} \tag{26}$$

Identify

$$r = \tfrac{1}{2}\frac{\omega}{d},$$

hence

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\gamma = 2d(r+\cos\gamma).$$

Now we compute r: it is

$$r = \frac{-H_\tau(t)\cos(ky+\phi_j) + (1-H_\tau(t))\cos(kx+\psi_j)}{H_\tau(t)\cos(ky+\phi_j) + (1-H_\tau(t))\cos(kx+\psi_j)},$$

or

$$r = -H_\tau(t) + (1-H_\tau(t)).$$

Let us examine (26):

$$\frac{d\gamma}{dt} = 2d(r-\cos\gamma), \qquad r = \pm 1.$$

We can solve this integral because r and d are piecewise constant. In the first half-period, $r=-1$ and $2d = A_0 k\cos(ky_n+\phi_n)$, hence

$$\frac{d\gamma}{1+\cos\gamma} = -A_0 k\cos(ky_n+\phi_n), \tag{27a}$$

and

$$\gamma^{n+1/2} = 2\tan^{-1}\left[\tan\left(\frac{\gamma^n}{2}\right) - A_0 k(\tau/2)\cos(ky_n+\phi_n)\right]. \tag{27b}$$

In the second half-period $r=+1$ and $2d = A_0 k\cos(kx_{n+1}+\psi_n)$, hence

$$\frac{d\gamma}{1-\cos\gamma} = A_0 k\cos(kx_{n+1}+\psi_n), \tag{27c}$$

and

$$\gamma^{n+1} = 2\cot^{-1}\left[\cot\left(\frac{\gamma^{n+1/2}}{2}\right) - A_0 k(\tau/2)\cos(kx_{n+1}+\psi_n)\right]. \tag{27d}$$

Now

$$\frac{d|\boldsymbol{\mathcal{B}}|^2}{dt} = -2d\sin\gamma\,|\boldsymbol{\mathcal{B}}|^2. \tag{27e}$$

Thus,

$$\ln|\boldsymbol{\mathcal{B}}|^2 - \ln|\boldsymbol{\mathcal{B}}|^2(0) = -\int_0^t 2d(s)\sin\gamma\,\mathrm{d}s,$$

or

$$\frac{|\nabla\theta|^2(t)}{|\nabla\theta|^2(0)} = e^{-\int_0^t 2d(s)\sin\gamma\,\mathrm{d}s}.$$

The mean amplification is then given by the spatial average

$$a\left(t\right)=\int\frac{\mathrm{d}^{2}x_{0}}{\left(2\pi\right)^{2}}\exp\left[-\mathcal{I}\left(\boldsymbol{x}\left(s\right);\boldsymbol{x}_{0}\right)\right], \tag{27f}$$

where

$$\mathcal{I}\left(\boldsymbol{x}\left(t\right);\boldsymbol{x}_{0}\right)=\int_{0}^{t}2d\left(\boldsymbol{x}\left(s\right);\boldsymbol{x}_{0}\right)\sin\gamma\left(\boldsymbol{x}\left(s\right);\boldsymbol{x}_{0}\right)\mathrm{d}s. \tag{27g}$$

This integral has a quasi-exact solution in the case of the sine flow:

$$\begin{aligned}&\mathcal{I}^{n+1/2}=\mathcal{I}^{n}+\\&A_{0}k\cos\left(ky_{n}+\phi_{n}\right)\int_{n\tau}^{n\tau+\tau/2}\mathrm{d}s\sin\left[2\tan^{-1}\left(\tan\left(\frac{\gamma^{n}}{2}\right)-A_{0}k\left(s-n\tau\right)\cos\left(ky_{n}+\phi_{n}\right)\right)\right],\end{aligned} \tag{27h}$$

$$\begin{aligned}&\mathcal{I}^{n+1}=\mathcal{I}^{n+1/2}+A_{0}k\cos\left(kx_{n+1}+\psi_{n}\right)\times\\&\int_{n\tau+\tau/2}^{(n+1)\tau}\mathrm{d}s\sin\left[2\cot^{-1}\left(\cot\left(\frac{\gamma^{n+1/2}}{2}\right)-A_{0}k\left(s-\left(n+\tfrac{1}{2}\right)\tau\right)\cos\left(kx_{n+1}+\psi_{n}\right)\right)\right].\end{aligned} \tag{27i}$$

We use the following identity:

$$\int\mathrm{d}s\sin\left[2\tan^{-1}\left(\alpha-\beta(s-t_{0})\right]=-\frac{1}{\beta}\log\left[1+\left(\alpha+\beta\left(t_{0}-s\right)\right)^{2}\right],$$

which is also valid upon replacing $\tan^{-1}$ with $\cot^{-1}$. Note that the logarithm of the time-average of a is proportional to the λ_2-Lyapunov exponent in the notation of Wiggins:

$$\lambda_{2}=\tfrac{1}{2}\left\langle\frac{\log a\left(t\right)}{t}\right\rangle.$$

But this is known exactly for the sine flow:

$$\lambda_{2}=\tfrac{1}{2}\log\left[1+\tfrac{1}{8}A_{0}^{4}+\tfrac{1}{2}A_{0}^{2}\sqrt{16+A_{0}^{2}}\right]. \tag{28}$$

We show some examples of of λ_2 computed both through the method of Eqs. (28), and through the exact equation (28) in Fig. 7. Excellent agreement is obtained, confirming the correctness of our integrations.

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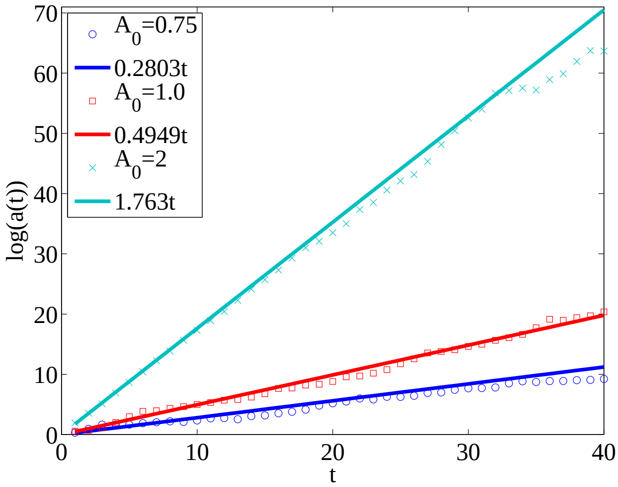

70
60
50
40
30
20
10
0
0
10
20
30
40
t
log(a(t))
A0=0.75
0.2803t
A0=1.0
0.4949t
A0=2
1.763t

Figure 7: