Bondi Accretion

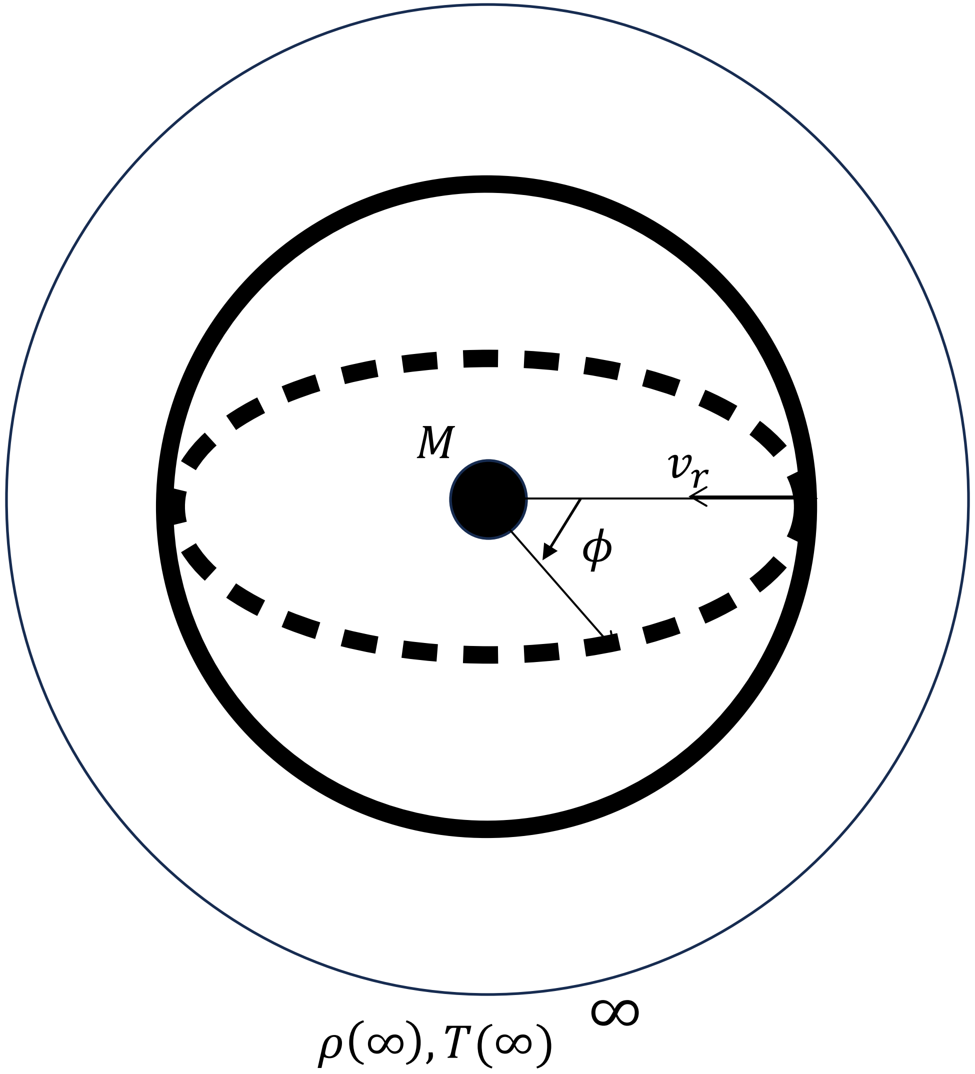
Xiaoyuan Yang

‘*Accretion power in astrophysics*’ by Juhan Frank, Andrew King and Derek Raine

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \mathbf{f} \quad (2)$$

$$\frac{\partial(\frac{1}{2}\rho v^2 + \rho\epsilon)}{\partial t} + \nabla \cdot [(\frac{1}{2}\rho v^2 + \rho\epsilon + P) \cdot \mathbf{v}] = \mathbf{f} \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_{rad} - \nabla \cdot \mathbf{q} \quad (3)$$



Assumptions:

1.spherical symmetric $\rightarrow (r, \theta, \phi)$, $v_r = v$
steady flow $\rightarrow \frac{\partial}{\partial t} = 0$.

2.the star is at rest in an infinite could of gas which is also at rest and uniform at infinity.

3.the increase of the mass of the star is ignored $\rightarrow$ $M = constant$ $\rightarrow$ field of force is unchanging.

4.the angular momentum, magnetic field strength and bulk motion of the gas with respect to the star could be neglected.

5.Pressure effects are neglected since any heat generated would be radiated way rapidly, so that the temperature of the gas is always very low.

Continuity Equation (1) gives :

$$\frac{1}{r^2}\frac{d}{dr}(r^2\rho v) = 0$$

$$r^2\rho v = const$$

The accretion rate :

$$\dot{M} = 4\pi r^2\rho(-v)$$

Conservation of momentum (2) gives :

$$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2}$$

the only external force is the gravitational force:

$$f_r = -\frac{GM\rho}{r^2}$$

$$v \frac{dv}{dr} + \frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} = 0$$

Conservation of energy (3) is replaced by polytropic relation :

$$P = K\rho^{\gamma}$$

$$\begin{cases} \text{adiabatic: } \gamma = \frac{5}{3} \\ \text{isothermal: } \gamma = 1 \end{cases}$$

perfect gas law:

$$P = \frac{\rho k T}{\mu m_H}$$

$$T = \frac{\mu m_H P}{\rho k}$$

For conservation of momentum :

$$v\frac{dv}{dr}+\frac{1}{\rho}\frac{dP}{dr}-\frac{GM}{r^2}=0$$

Define:

$$c_s^2=\frac{dP}{d\rho}$$

Then after some arrangements we have:

$$v\frac{dv}{dt}-\frac{c_s^2}{vr^2}\frac{d}{dr}(vr^2)+\frac{GM}{r^2}=0$$

Before integrating it first, we'd like to find some information through the equation after some complex arrangements:

$$\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{dv^2}{dr}=-\frac{GM}{r}[1-(\frac{2c_s^2r}{GM})]$$

Let's do some analyzations.

$$\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{dv^2}{dr}=-\frac{GM}{r}[1-(\frac{2c_s^2 r}{GM})]$$

For large r, $r \to \infty, c_s^2 \to c_s^2(\infty)$:

$$1-\left(\frac{2c_s^2 r}{GM}\right)<0 \to R.H.S>0$$

Since we want the gas to be at rest at distant and v increases as r decreases:

$$\frac{dv^2}{dr}<0$$

$$v^2<c_s^2 \qquad subsonic$$

As r decreases：

$$1-\left(\frac{2c_s^2 r}{GM}\right)\to 0$$

unless there's a way of increasing c_s^2 sufficiently by heating up the gas.

It gives the ***Bondi Radius***:

$$r_s=\frac{GM}{2c_s^2(r_s)}\ (sonic\ point)$$

where ambient gas is strongly affected by the central object.

For small $r, r \to 0$:

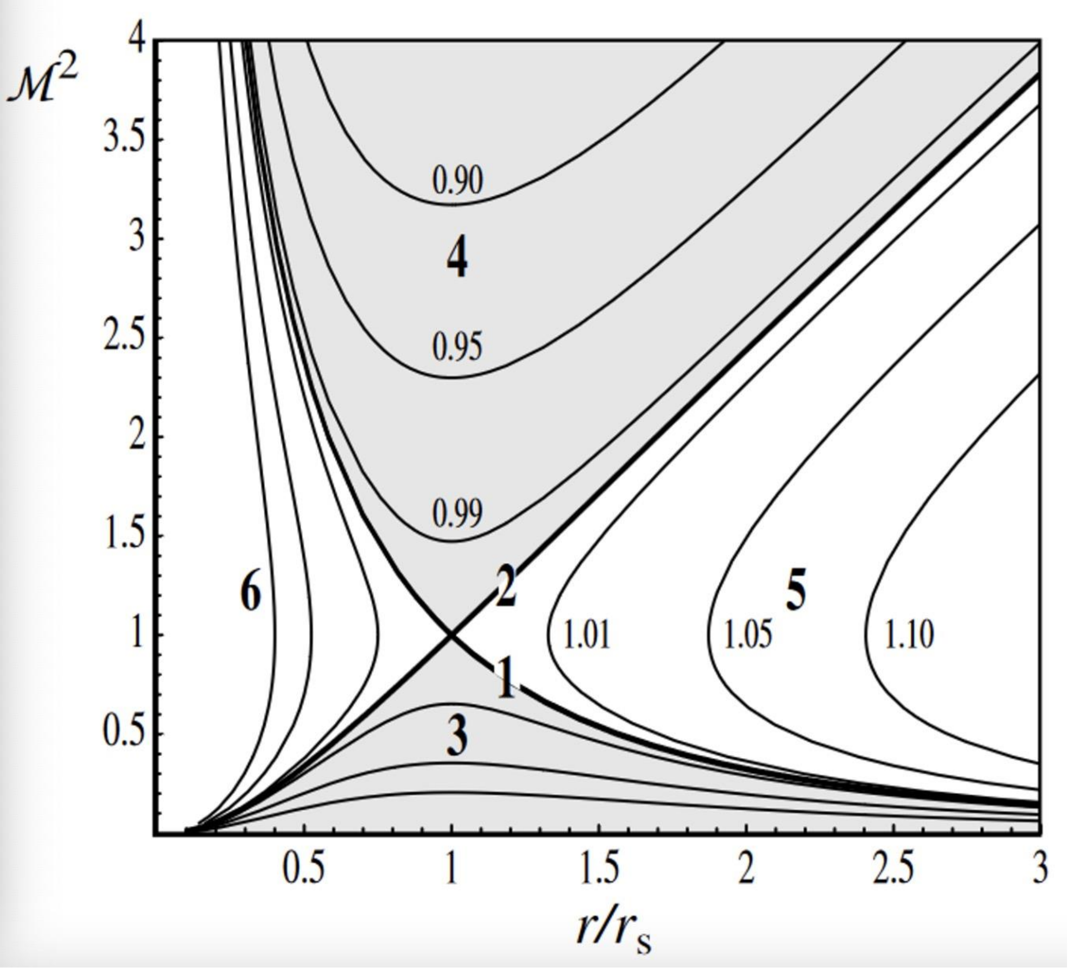
$$1 - \left(\frac{2c_s^2 r}{GM}\right) > 0 \to R.H.S > 0$$

$$v^2 > c_s^2 \qquad supersonic$$

For $r = r_s$, $\mathrm{L.H.S} = 0$:

$$\begin{cases} \dfrac{dv^2}{dt} = 0 \\ v^2 = c_s^2 \end{cases}$$

Plot the figure $\mathcal{M}^2 = \frac{v^2(r)}{c_s^2(r)}$ - $\frac{r}{r_s}$



Type 1 : *Bondi Accretion*

$$v^2(r_s) = c_s^2(r_s), v \rightarrow 0 \; as \; r \rightarrow \infty$$
$$v^2 < c_s^2, r > r_s$$
$$v^2 > c_s^2, r < r_s$$

Type 2 : *Parker Wind*

$$v^2(r_s) = c_s^2(r_s), v \rightarrow 0 \; as \; r \rightarrow 0$$
$$v^2 < c_s^2, r < r_s$$
$$v^2 > c_s^2, r > r_s$$

Type 3 :

$$v^2(r_s) < c_s^2(r_s) \; everywhere$$
$$\frac{dv^2}{dt} = 0 \Big|_{r=r\_s}$$

Type 4 :

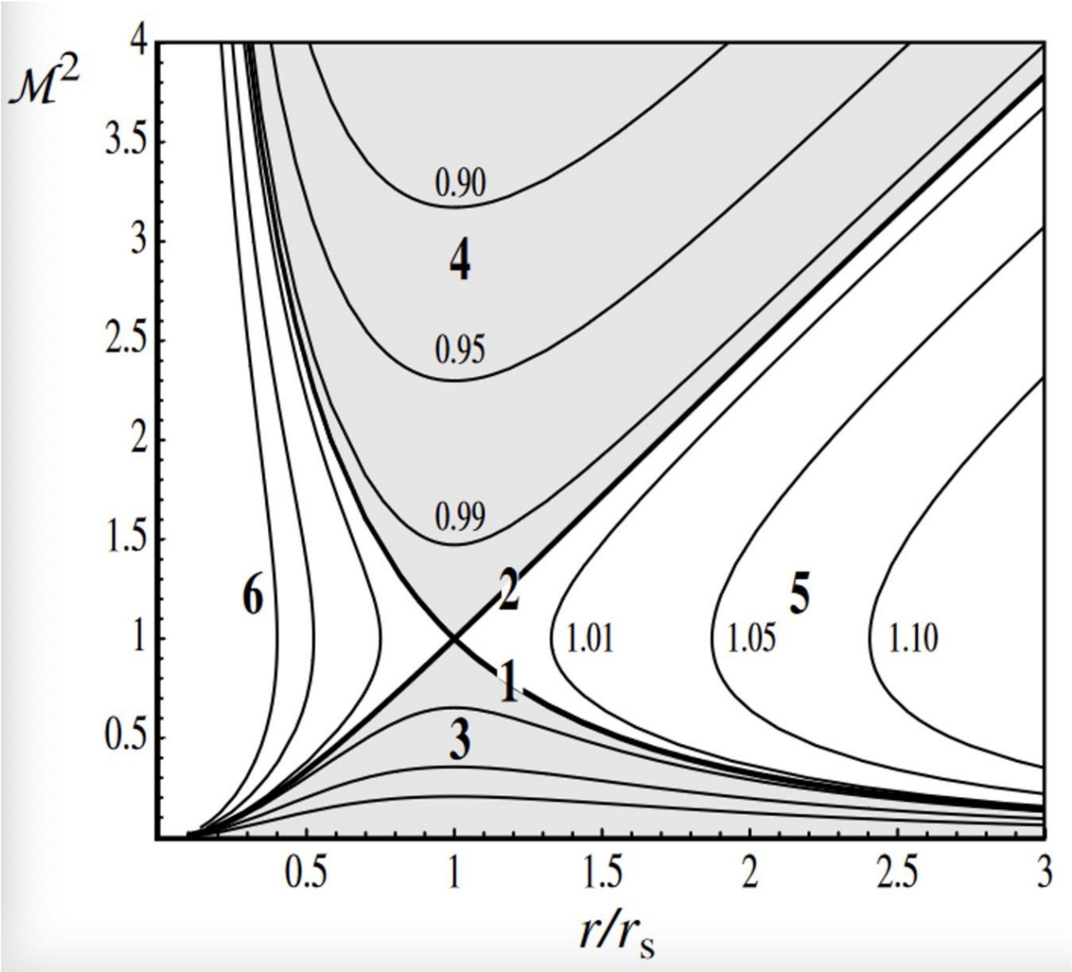
$$v^2(r_s) > c_s^2(r_s) \; everywhere$$
$$\frac{dv^2}{dt} = 0 \Big|_{r=r\_s}$$

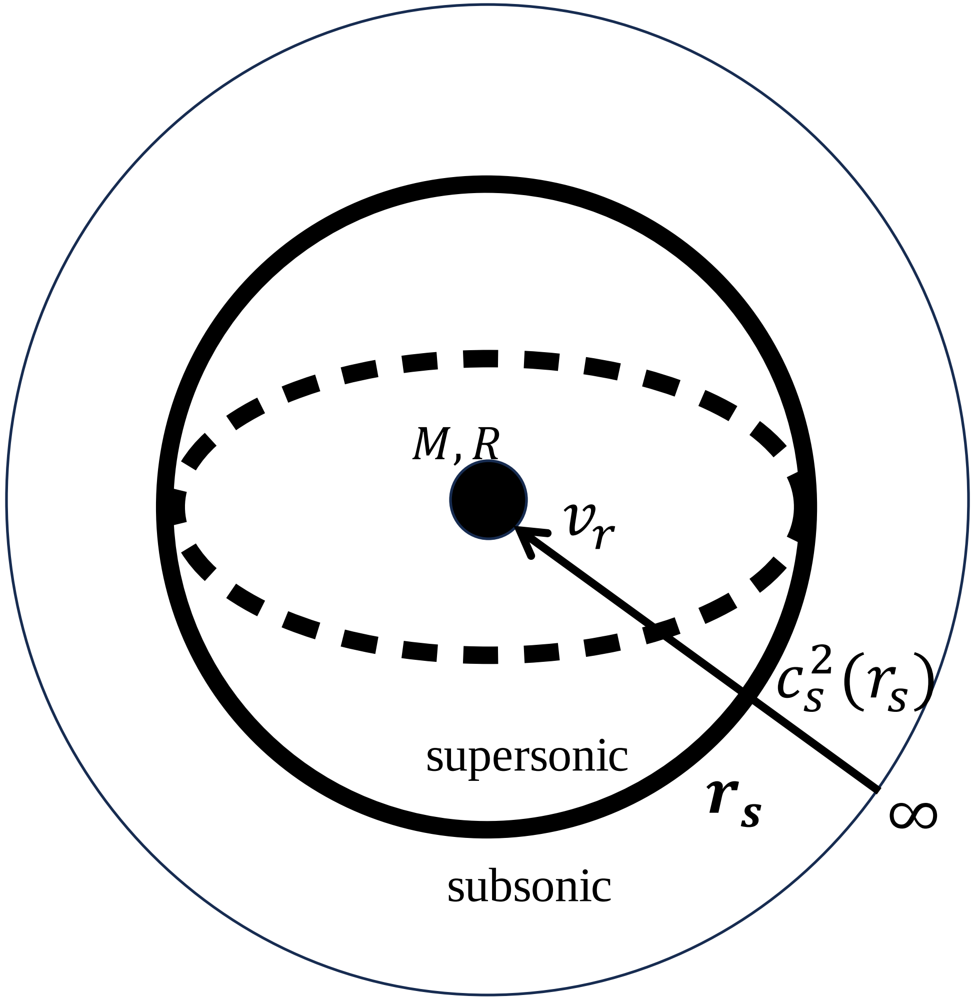
Type 5 :

$$\frac{dv^2}{dt} \to \infty \; at \; v^2 = c_s^2$$
$$r > r_s \; always$$

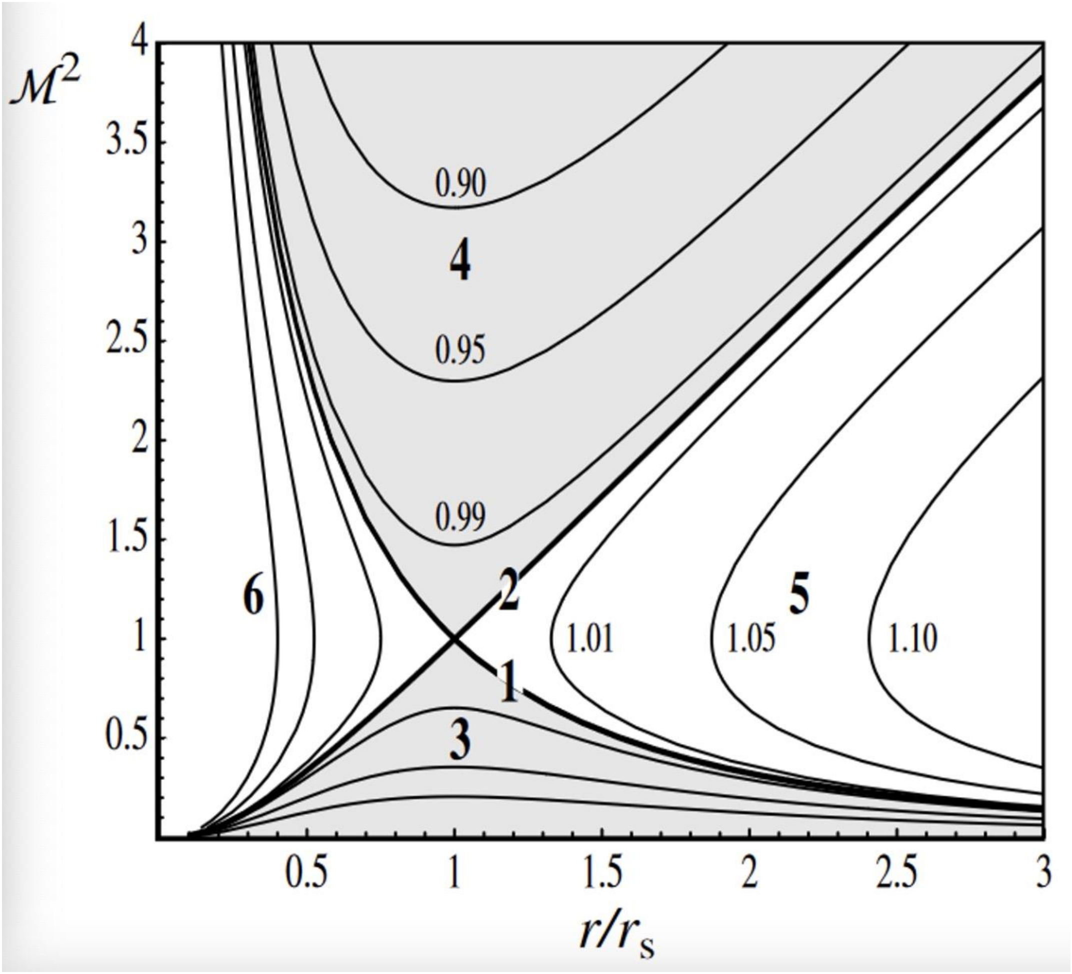
Type 6 :

$$\frac{dv^2}{dt} \to \infty \; at \; v^2 = c_s^2$$
$$r < r_s \; always$$





M, R
v_r
$c_s^2(r_s)$
supersonic
r_s
∞
subsonic



$\mathcal{M}^2$
4
3.5
3
2.5
2
1.5
1
0.5
0.90
4
0.95
0.99
2
1
3
6
1.01
1.05
5
1.10
0.5
1
1.5
2
2.5
3
r/r_s

Now let's integrate it.

From polytropic relation :

$$P = K\rho^{\gamma}$$

$$dP = K\gamma\rho^{\gamma-1}d\rho$$

substitute it in momentum equation:

$$\frac{v^2}{2} + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = const$$

since $K\gamma\rho^{\gamma-1} = \gamma\frac{P}{\rho} = c_s^2$

Bernoulli Integral:

$$\frac{v^2}{2} + \int \frac{dP}{\rho} - \frac{GM}{r} = const$$

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = const$$

since $v \to 0\ as\ r \to \infty$,

$$const = \frac{c_s^2(\infty)}{\gamma - 1}$$

The sonic point condition can relate $c_s(\infty)$ to $c_s(r_s)$:

$$v^2(r_s) = c_s^2(r_s), \frac{GM}{r_s} = 2c_s^2(r_s)$$

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma - 1}$$

we have:

$$c_s(r_S) = c_s(\infty)\left(\frac{2}{5 - 3\gamma}\right)^{1/2}$$

So the accretion rate becomes:

$$\dot{M} = 4\pi r^2 \rho(-v) = 4\pi r^2 \rho(r_s) c_s(r_s)$$

Since $c_s^2 \propto \rho^{\gamma-1}$,

$$\rho(r_s) = \rho(\infty) \left[\frac{c_s(r_s)}{c_s(\infty)}\right]^{\frac{2}{\gamma-1}}$$

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \left[\frac{2}{5-3\gamma}\right]^{\frac{5-3\gamma}{2(\gamma-1)}}$$

To get $v(r)$, we use $c_s^2 = \gamma \frac{P}{\rho} \propto \rho^{\gamma-1}$ and from :

$$\dot{M} = 4\pi r^2 \rho(-v) = 4\pi r^2 \rho(r_s) c_s(r_s)$$

we have:

$$(-v) = \frac{\dot{M}}{4\pi r^2 \rho(r)} = \frac{\dot{M}}{4\pi r^2 \rho(\infty)} \left[\frac{c_s(\infty)}{c_s(r)}\right]^{\frac{2}{\gamma-1}}$$

Substitute it into the Bernoulli integral an get $c_s(r)$:

$$\frac{\rho^2(r_s) c_s^2(r_s)}{2\rho^2(\infty)} \left[\frac{c_s(\infty)}{c_s(r)}\right]^{\frac{4}{\gamma-1}} + \frac{c_s^2(r)}{\gamma - 1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma - 1}$$

It must be solved numerically.

$$\frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = const$$

At large r :

gravitational pull is weak $\rightarrow \rho(\infty), c_s(\infty), v \cong 0$

As one moves to smaller r , the inflow velocity $(-v)$ increases until it reaches $c_S(\infty)$, the sound speed at infinity. The gravity is the only term can balance this increase.

Since $c_s(r)$ does not greatly exceed $c_s(\infty)$, it must occur at :

$$r \cong r_{acc} = \frac{2GM}{c_S(\infty)^2}$$

at this point $\rho(r), c_s(r)$ begin to increase above ambient values.

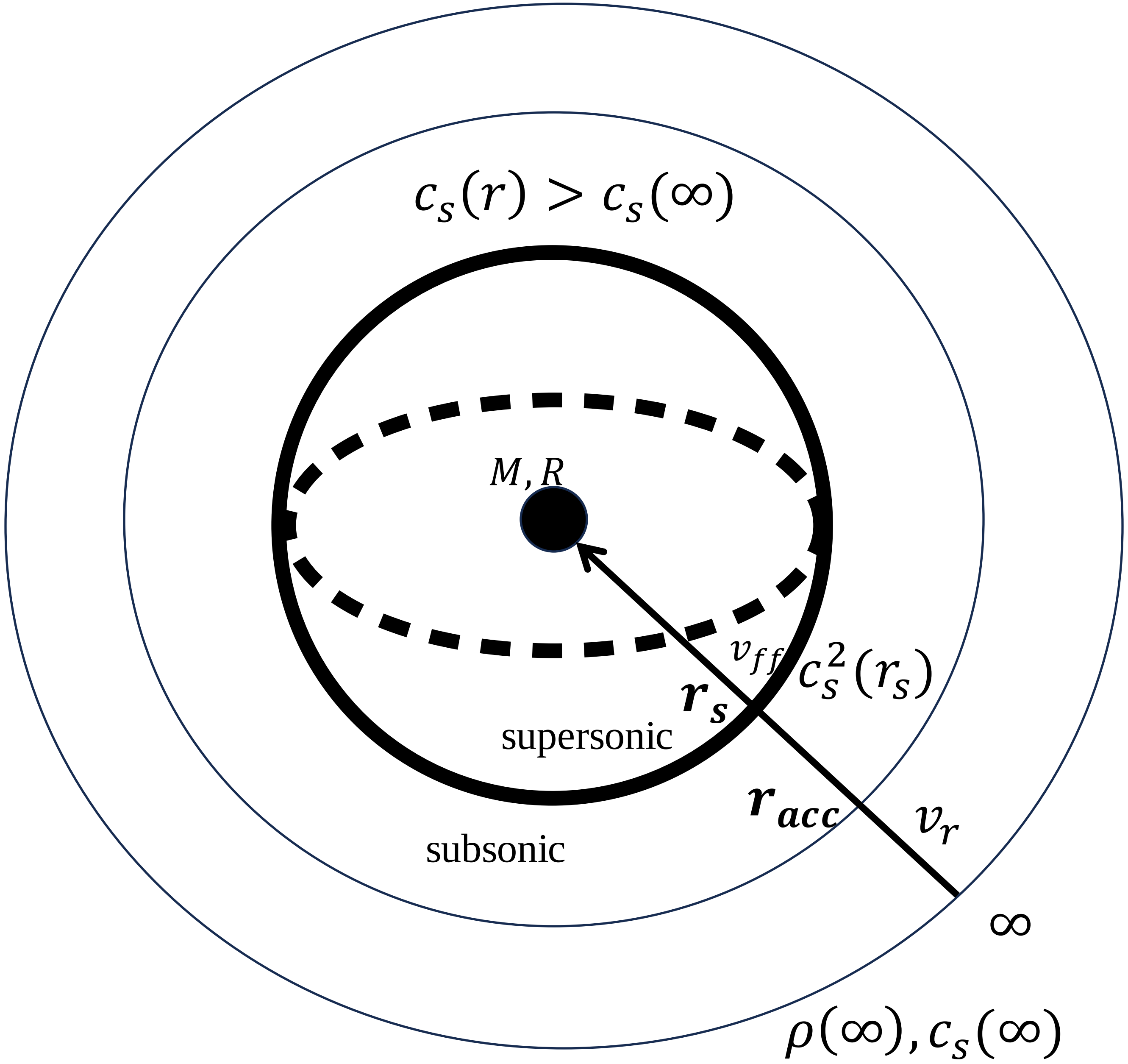
$$r \cong r_{acc} = \frac{2GM}{c_s(\infty)^2}$$

At a radius r the ratio of internal (thermal) energy to gravitational binding energy of a gas element of mass m is :

$$\frac{thermal\ energy}{binding\ energy} \sim \frac{mc_s^2(r)}{2}\frac{r}{GMm} \sim \frac{r}{r_{acc}} \quad for\ \ r \gtrsim r_{acc}$$

since $c_s(r) \sim c_s(\infty)$ for $r > r_{acc}$.

The accretion radius gives the range of influence of the star on the gas cloud.



$$r_s = \frac{GM}{2c_s^2(r_s)}$$

$$r_{acc} = \frac{2GM}{c_s(\infty)^2}$$

$$v^2 \cong \frac{2GM}{r} = v_{ff}^2$$

At the sonic point:

$$v^2 = c_s^2, r = r_s$$

the inflow becomes supersonic, the gas is effectively in free fall, from Bernoulli integral, when $v^2 \gg c_s^2$:

$$v^2 \cong \frac{2GM}{r} = v_{ff}^2$$

Continuity equation gives:

$$\rho \cong \rho(r_s)\left(\frac{r_s}{r}\right)^{\frac{3}{2}} \text{ for } r \lesssim r_s$$

Gas temperature (using the perfect gas law and the polytropic relation):

$$T \cong T(r_S)\left(\frac{r_s}{r}\right)^{\frac{3}{2}(\gamma-1)} \text{ for } r \lesssim r_s$$

steady accretion rate:

$$\dot{M} \sim \pi r_{acc}^2 c_s(\infty)\rho(\infty)$$

The accreting material must eventually join the star with a very small velocity, some way of stopping the highly supersonic accretion flow must be found. → plasma physics