The Angular Momentum of the Solar Wind

‘*The Angular Momentum of the Solar Wind*’ Weber 1966

A steady-state model of the solar-wind flow in the equatorial plane including the effects of pressure gradients, gravitation, and magnetic forces is developed and solved for both the radial and azimuthal motions

Ideal MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta c}{4\pi} \nabla^2 \mathbf{B}$$

Assumptions:

1.MHD equations with infinite conductivity, zero viscosity and a scalar pressure.

2.The Sun is assumed to have a general magnetic field that depends on latitude.

3. The local irregularities in the field, the polarity reversals and wind velocity fluctuations of the sector structure, and the waves superimposed on the smooth field in interplanetary space are ignored.

4. Steady-state, complete axial symmetry in which in the equatorial plane of the Sun the field is combed out by the solar wind and has no component normal to this plane, no $\phi-\text{dependence}$.

5. In the steady-state solar wind the velocities and magnetic fields as well as their derivatives are continuous, smooth functions of position, i.e., no shocks exist anywhere.

Solar Wind Equations

Solar wind velocity:

$$\boldsymbol{v} = u\boldsymbol{e_r} + v_\phi \boldsymbol{e_\phi}$$

Magnetic field:

$$\boldsymbol{B} = B_r\boldsymbol{e_r} + B_\phi \boldsymbol{e_\phi}$$

Conservation of mass:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\rho u) + \frac{1}{rsin\theta}\frac{\partial}{\partial\phi}\left(v_\phi sin\theta\right) = 0$$

since the solar wind is axisymmetric and $\phi -$ independent:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\rho u)=0$$

$$r^2\rho u = \text{constant}$$

Since the solar wind is perfect conductor (conductivity $\rightarrow \infty, \boldsymbol{E} = 0$):

$$\boldsymbol{E} = -\frac{\boldsymbol{v} \times \boldsymbol{B}}{c}$$

$$c(\nabla \times \boldsymbol{E})_{\phi} = -\boldsymbol{v}(\nabla \cdot \boldsymbol{B}) + \boldsymbol{B}(\nabla \cdot \boldsymbol{v}) = \boldsymbol{B}(\nabla \cdot \boldsymbol{v})$$

$$\frac{1}{\mathrm{r}}\frac{d}{dr}\left[r\left(uB_{\phi} - v_{\phi}B_{r}\right)\right] = 0$$

In a perfectly conducting fluid, v is parallel to $\boldsymbol{B}$ in a frame that rotates with the Sun:

$$r\left(uB_{\phi} - v_{\phi}B_{r}\right) = constant = -\Omega r^{2}B_{r}$$

Since

$$\nabla \cdot \boldsymbol{B} = 0$$

we have

$$\frac{d}{dr}(r^2 B_r) = 0$$

$$r^2 B_r = constant = r_0^2 B_0$$

$\phi - symmetry \rightarrow$ the momentum and magnetic-force term are the only terms that enter the steady ϕ-equation of motion:

$$\rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{\boldsymbol{j} \times \boldsymbol{B}}{c}$$

$$\rho \frac{u}{r} \frac{d}{dr}\left(r v_\phi\right) = \frac{1}{c}(\boldsymbol{j} \times \boldsymbol{B})_\phi = \frac{1}{4\pi}[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]_\phi = \frac{B_r}{4\pi r} \frac{d}{dr}(r B_r)$$

Since:

$$r^2 B_r = constant$$
$$r^2 \rho u = \text{constant}$$

we have:

$$\frac{B_r}{4\pi\rho u} = \frac{B_r r^2}{4\pi\rho u r^2} = constant$$

Thus

$$\frac{d}{dr}\left(rv_{\phi}\right) = \frac{B_r}{4\pi\rho u}\frac{d}{dr}(rB_r)$$

then we can integrate the azimuthal equation of motion and get the equation of momentum:

$$rv_{\phi} - \frac{B_r}{4\pi\rho u} rB_{\phi} = constant = L$$

Ordinary angular momentum per unit mass:

$$r v_\phi$$

Torque associated with the magnetic stresses:

$$-\frac{B_r}{4\pi\rho u} r B_\phi$$

Total angular momentum carried away from the Sun per unit mass loss:

$$L = r v_\phi - \frac{B_r}{4\pi\rho u} r B_\phi$$

Introduce *radial Alfvenic Mach number*:

$$M_A^2 = \frac{4\pi\rho u^2}{B_r^2}$$

Solve the azimuthal velocity:

$$r\left(uB_\phi - v_\phi B_r\right) = constant = -\Omega r^2 B_r$$

$$rv_\phi - \frac{B_r}{4\pi\rho u} rB_\phi = constant = L$$

we get

$$v_\phi = \Omega r \frac{M_A^2 L r^{-2}\Omega^{-1} - 1}{M_A^2 - 1}$$

M_A is much smaller near the surface of the Sun,
but at 1 $a.u.$,it is approximately 10.
So there exists a point between Sun and where $M_A = 1$ and let the radial velocity and radius at this point be called u_a, r_a.
It is called the *Alfvenic critical point:*

$$M_A^2 - 1 \rightarrow 0$$

To keep v_ϕ finite, we need:

$$L = \Omega r_a^2$$

Since

$$\frac{B_r}{4\pi\rho u} = \frac{B_r r^2}{4\pi\rho u r^2} = constant$$

we have:

$$M_A^2 = \frac{4\pi\rho u^2}{B_r^2}$$

$$\frac{M_A^2}{u r^2} = constant$$

which may be evaluated at the critical point :

$$M_A^2 = \frac{u r^2}{u_a r_a^2} = \frac{\rho_a}{\rho}$$

The azimuthal velocity reduces to :

$$v_\phi = \frac{\Omega r}{u_a} \frac{u_0 - u}{1 - M_A}$$

The azimuthal magnetic field is given by

$$B_\phi = -B_r \frac{\Omega r}{u_a} \frac{r_a^2 - r^2}{r_a^2 (1 - M_A^2)}$$

Note that L is determined only from the conditions at the Alfvenic critical point:

$$L = \Omega r_a^2$$

For $r \gg r_a$:

$$u \approx constant$$

$$M_A \propto r, v_\phi \propto \frac{1}{r}, B_\phi \propto \frac{1}{r}$$

For $r \ll r_a$, where $u \ll u_a, M_A^2 \ll 1$:

$$B_\phi = -B_r \frac{\Omega r}{u_a}[1 - \frac{r^2}{r_a^2}\left(1 - \frac{u}{u_a}\right) + \cdots]$$

whereas

$$v_\phi = \Omega r[1 - \frac{u}{u_0}\left(1 - \frac{r^2}{r_a^2}\right) + \cdots]$$

$$L = rv_\phi - \frac{B_r}{4\pi\rho u} rB_\phi$$

Near the surface of the Sun where r is small, according to the equation of angular momentum above, most of the momentum loss is due to the torque exerted by the magnetic fields. As r increases, the azimuthal fluid velocity v_ϕ increases and the magnetic stress decreases until at large distances the relative contributions to the angular momentum loss are $[1 - (u_a/u_\infty)]$ and u_∞, respectively.

Take the radial momentum equation:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \rho \boldsymbol{g}$$

$$\rho u \frac{du}{dr} = -\frac{d}{dr} p - \rho \frac{G M_{\odot}}{r^2} + \frac{1}{c} (\boldsymbol{j} \times \boldsymbol{B})_r + \rho \frac{v_\phi^2}{r^2}$$

In a fully ionized gas of pure hydrogen the effective particle mass is only half the hydrogen m, the equation of state is :

$$p = \frac{2kT}{m} \rho$$

Assume the temperature is equal for ions and electrons.

The polytrope law yields:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$
$$\frac{p}{p_a} = \left(\frac{\rho}{\rho_a}\right)^{\gamma}$$

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} = p_a \left(\frac{\rho}{\rho_a}\right)^{\gamma}$$

The magnetic force:

$$\frac{1}{c}(j \times B)_r = -\frac{1}{4\pi r} B_\phi \frac{d}{dr}(rB_r)$$

Substitute the magnetic force into the equation of momentum:

$$\frac{d}{dr}\left\{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1} - \frac{GM_\odot}{r}\right\} = \frac{v_\phi^2}{r} - \frac{1}{8\pi\rho r^2}\frac{d}{dr}\left(rB_\phi\right)^2$$

with the R.H.S. equals to zero, we'll have *the Parker's equation of motion for the solar wind:*

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1} - \frac{GM_\odot}{r} = constant$$

The presence of the R.H.S. is due to the inclusion of the magnetic force and the azimuthal velocity.

$$\frac{d}{dr}\left\{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1} - \frac{GM_\odot}{r}\right\} = \frac{v_\phi^2}{r} - \frac{1}{8\pi\rho r^2}\frac{d}{dr}\left(rB_\phi\right)^2$$

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Now express ρ, v_ϕ, B_ϕ in terms of u, r using

$$v_\phi = \frac{\Omega r}{u_a} \frac{u_0 - u}{1 - M_A}$$

$$B_\phi = -B_r \, \frac{\Omega r}{u_a} \frac{r_a^2 - r^2}{r_a^2(1 - M_A^2)}$$

$$M_A^2 = \frac{u r^2}{u_a r_a^2} = \frac{\rho_a}{\rho}$$

we have:

$$\frac{du}{dr} = \frac{u}{r}\left\{\left(\frac{2\gamma p_a}{\rho_a M_A^{2(\gamma-1)}} - \frac{GM_\odot}{r}\right)(M_A^2-1)^3 + \Omega^2 r^2\left(\frac{u}{u_a}-1\right)\left[(M_A^2+1)\frac{u}{u_a} - 3M_A^2+1\right]\right\} \times \left[\left(u^2 - \frac{\gamma p_a}{\rho_a M_A^{2(\gamma-1)}}\right)(M_A^2-1)^3 - \Omega^2 r^2\left(\frac{r_a^2}{r^2}-1\right)^2\right]^{-1}$$

Then this equation is in only two variables u, r .

And the Alfvenic critical point $r = r_a$ is also a critical point for radial equation.

Integrate the radial equation of motion and give the total energy flux per steradian which is a constant for our solution:

$$F = \rho u r^2 \left\{ \frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_a}{\rho_a} M_A^{2(\gamma-1)} - \frac{GM_\odot}{r} + \frac{\Omega^2 r^2}{2r^2} \left[1 + \frac{2(M_A^2 - 1)(r^2 - r_a^2)^2}{r_a^4 (M_A^2 - 1)^2} \right] \right\} = constant$$

Kinetic energy associated with the radial velocity: $\frac{u^2}{2}$

the sum of the enthalpy and the energy transported by thermal conduction, magnetic heating etc.: $\frac{\gamma}{\gamma-1} \frac{p_a}{\rho_a} M_A^{2(\gamma-1)}$

gravitational energy: $-\frac{GM_\odot}{r}$

sum of the magnetic and rotational energies: $\frac{\Omega^2 r^2}{2r^2} \left[1 + \frac{2(M_A^2 - 1)(r^2 - r_a^2)^2}{r_a^4 (M_A^2 - 1)^2} \right]$

Using:

$$v_\phi = \frac{\Omega r}{u_a}\frac{u_0 - u}{1 - M_A}$$

$$B_\phi = -B_r\ \frac{\Omega r}{u_a}\frac{r_a^2 - r^2}{r_a^2(1 - M_A^2)}$$

We can write the fourth term as :

$$F_{rot+mag} = \rho u r^2 (\frac{v_\phi^2}{2} - \frac{B_\phi B_r}{4\pi\rho}\frac{\Omega r}{u})$$

Kinetic energy associated with the azimuthal velocity: $\frac{v_\phi^2}{2}$

the energy transported out by the magnetic field, the Poynting energy flux: $-\frac{B_\phi B_r}{4\pi\rho}\frac{\Omega r}{u}$

Topology of the Solution

In the neighborhood of the origin, the asymptotic forms of the radial velocity are:

$$u = a_0 r^{\frac{3-2\gamma}{\gamma-1}}(1 + a_1 r - a_2 r^3 + \cdots)$$

$$u = b_0 r^{-\frac{1}{2}}\left(1 + b_1 r - b_2 r^{(5-3\gamma)/2} + \cdots\right)$$

The model as well as the Parker's model shows that there is no solution in which $u \to 0$ for small r when $\gamma > \frac{3}{2}$.

As $r \to 0$, the density increases as $r^{-\frac{1}{\gamma-1}}$ for the first expansion of u, and $r^{-\frac{3}{2}}$ for the second.

The coefficients are determined by the initial condition and the parameters entering function of the total energy flux.

If the solutions are required to run through the critical points, there constants depend on the properties there.

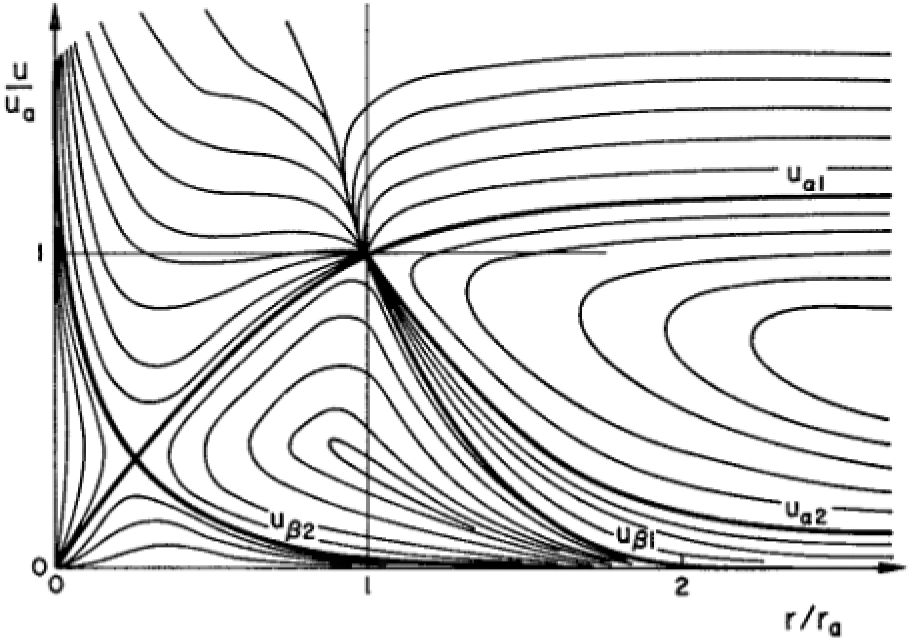
The asymptotic behaviors at large distances are given by

$$u_\alpha = a_0\left[1 + a_1\frac{1}{r^{2(\gamma-1)}} + a_2\frac{1}{r} + a_3\frac{1}{r^2} + \cdots\right]$$

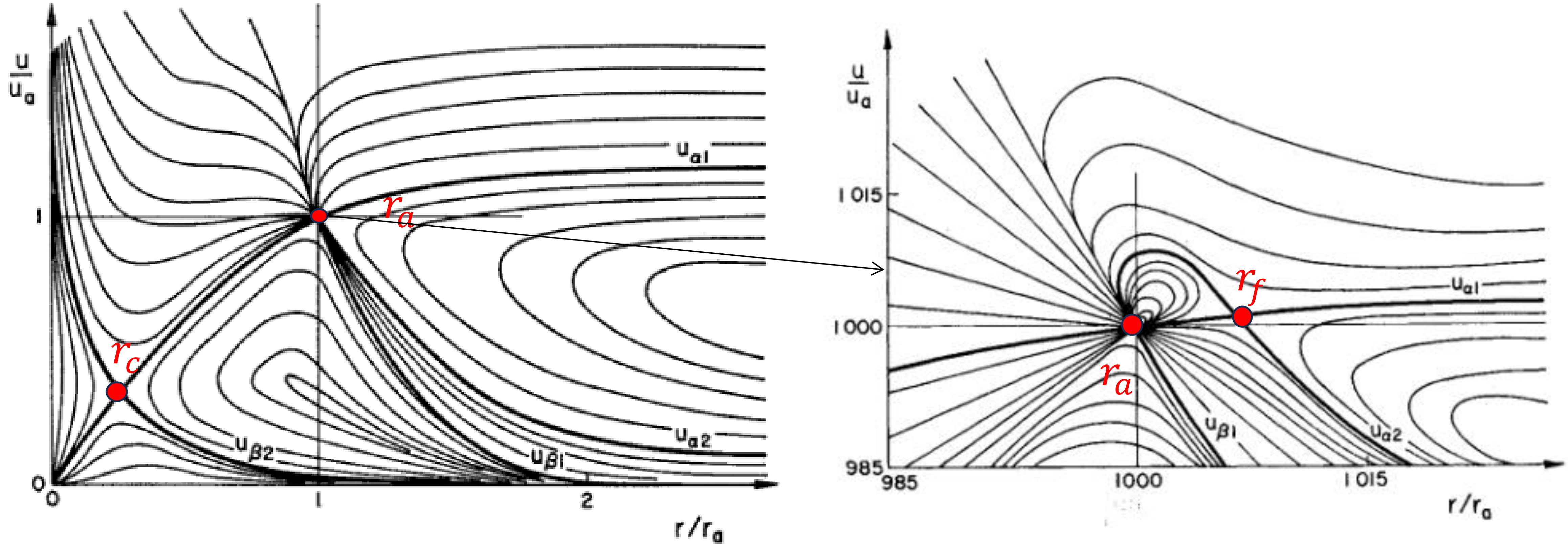
$$u_\beta = \frac{\beta_0}{r^2}\left[1 - \frac{1}{r^2}\left(\beta_1 - \frac{\beta_2}{r}\right) + \cdots\right]$$

The figure below is the family of solutions of (for a given γ, r_a) :

$$\frac{du}{dr} = \frac{u}{r}\left\{\left(\frac{2\gamma p_a}{\rho_a M_A^{2(\gamma-1)}} - \frac{GM_\odot}{r}\right)(M_A^2-1)^3 + \Omega^2 r^2\left(\frac{u}{u_a}-1\right)\left[(M_A^2+1)\frac{u}{u_a} - 3M_A^2+1\right]\right\} \times \left[\left(u^2 - \frac{\gamma p_a}{\rho_a M_A^{2(\gamma-1)}}\right)(M_A^2-1)^3 - \Omega^2 r^2\left(\frac{r_a^2}{r^2}-1\right)^2\right]^{-1}$$



$u_{\alpha 1}$: supersonic, super-Alfvenic wind at infinity.
$u_{\alpha 2}$: remain super-Alfvenic but becomes subsonic after passing the critical points.
For both $u_{\alpha 1}$ and $u_{\alpha 2}$, the pressure tend to zero as r becomes very large.
$u_{\beta 1}$, $u_{\beta 2}$ yield non-zero pressures an infinity.



All these singularities are found at points where flow velocity equals the velocity of a characteristic wave disturbance in the fluid.

r_a : u is equal to the *radial Alfven velocity*.
r_c : u is slightly less than the *pure sound speed*. It is just Parker's critical point displaced slightly due to the magneto-acoustic wave.
r_f : u is nearly equal to the *Alfven velocity* $[(B \cdot B)/4\pi\rho]^{1/2}$ which is slightly larger than the radial Alfven velocity.

Propagation of Disturbances and Stability

If magnetic fields are present,
disturbances may travel both in *Sound waves* and *Alfven waves*.
The direction of the magnetic field establishes a preferred axis and thus introduces anisotropy into the fluid. All possible speeds of the longitudinal wavefronts can be determined from the characteristic condition.
The characteristic condition is:

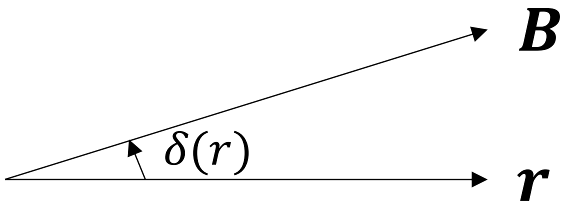
$$c^2[c^4 - (v_{AT}^2 + v_s^2)c^2 + v_s^2 v_{AN}^2] = 0$$

c: the velocity of the disturbance relative to the fluid

$v_s = \left(\frac{2\gamma kT}{m}\right)^{\frac{1}{2}}$: the local sound velocity

$v_{AT} = [(B_r^2 + B_\phi^2)/4\pi\rho]^{\frac{1}{2}}$: the local Alfven velocity

$v_{AN} = \left(\frac{B_r^2}{4\pi\rho}\right)^{\frac{1}{2}}$: the local Alfven velocity along the component of the magnetic field normal to the wavefront of interest.



In the region between the surface of the Sun and the critical radius r_a, the angle $\delta(r)$ between the magnetic field vector and the radius vector ranges from very small to approximately $\frac{1}{7}$ radian, thus:

$$B_\phi \approx -\delta B_r$$

to lowest order in δ we solve the characteristic condition for the characteristic disturbances:

$$c = 0, \pm v_s \left[1 - \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\delta^2\right], \pm v_{AN} [1 - \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\delta^2]$$

It is valid as long as v_{AN} is not close to v_s, which is the case for the Sun. The solution suggests that there are 2 separate wavefronts traveling with 2 different velocities, a “slow” and a “fast” wave.

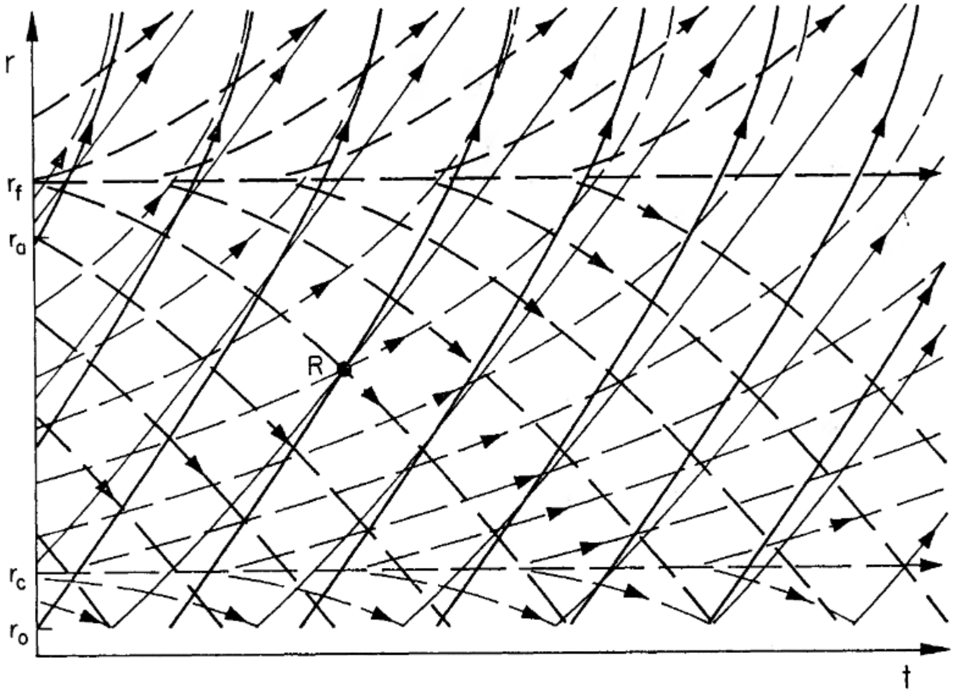
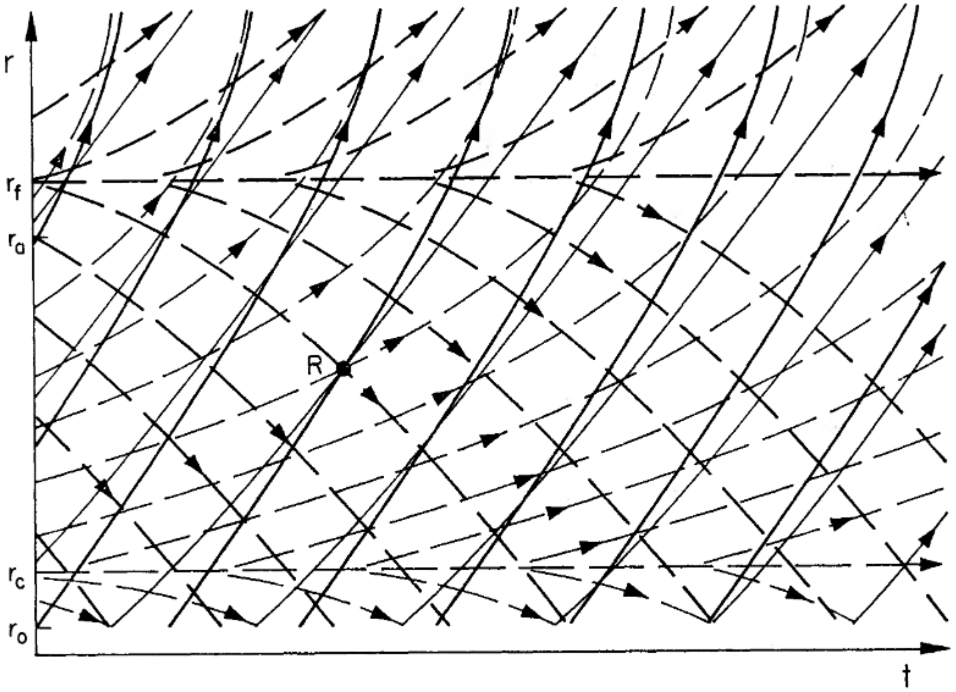


Fig. 3 —A sketch showing the characteristics in the r–t plane The characteristics for the "fast" wave are shown in heavy lines, and for the "slow" wave in light lines The solid lines refer to the solution of eq. (32) with the plus sign and the dashed lines refer to the minus sign; r_0 refers to the base of the corona.

The set of equations describing the motion of the disturbance:

$$\frac{dr}{dr} = u \pm c$$

It is along these characteristics that all small-amplitude disturbances travel in the solar wind, except those which travel with the solar wind, i.e., which have $c = 0$.



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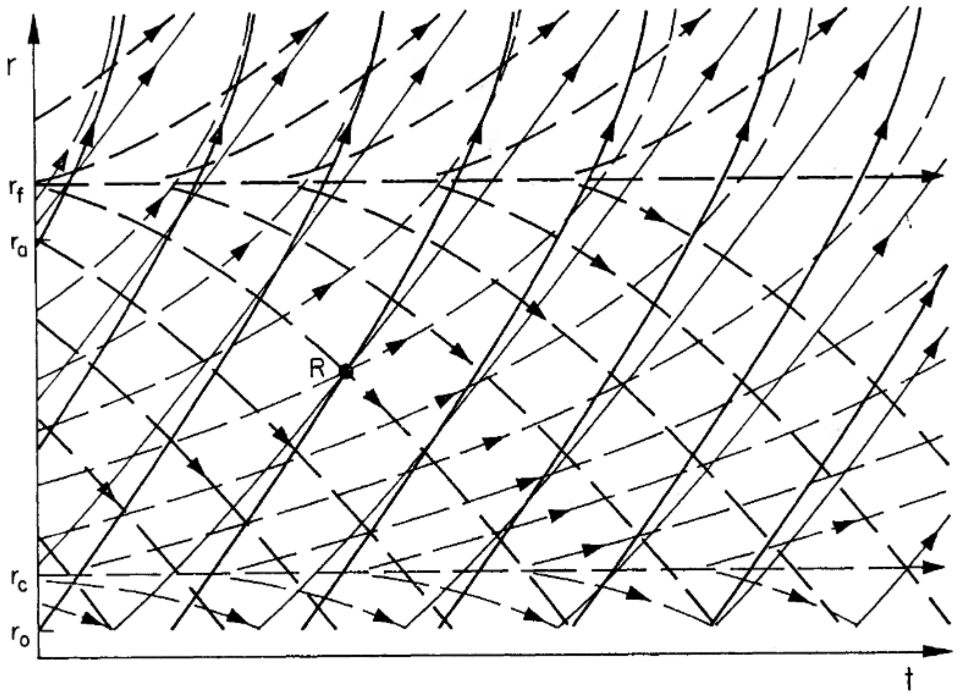
At r_c, u is equal to the velocity of the "slow" wave (the sonic wave).

At r_f, u is equal to the velocity of the "fast" wave.

No small-amplitude hydromagnetic disturbance beyond r_f can be propagated back toward the Sun, i.e., all disturbances originating past rf will be carried out of the region of interest by the wind and thus cannot grow and produce local instabilities.



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All disturbances between r_c and r_f can be carried back toward the Sun by the "fast" wave, but the "slow" wave can carry them only away from the Sun.

At $r < r_c$, disturbances can be carried back by both "fast" and "slow" waves.

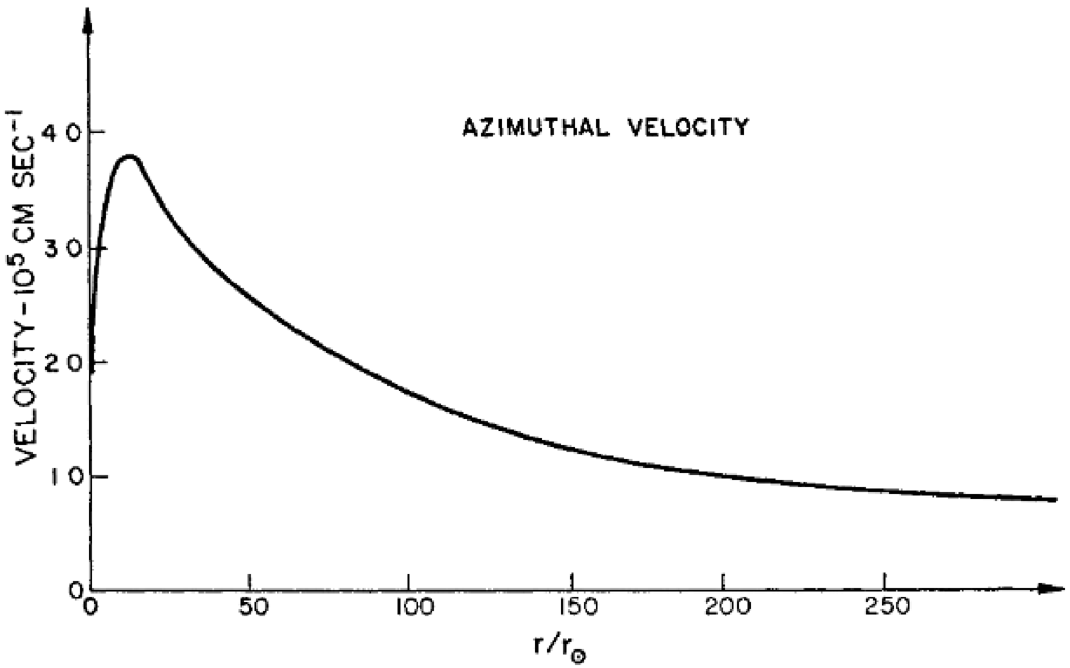

FIG. 4.—Azimuthal velocity of the solar wind

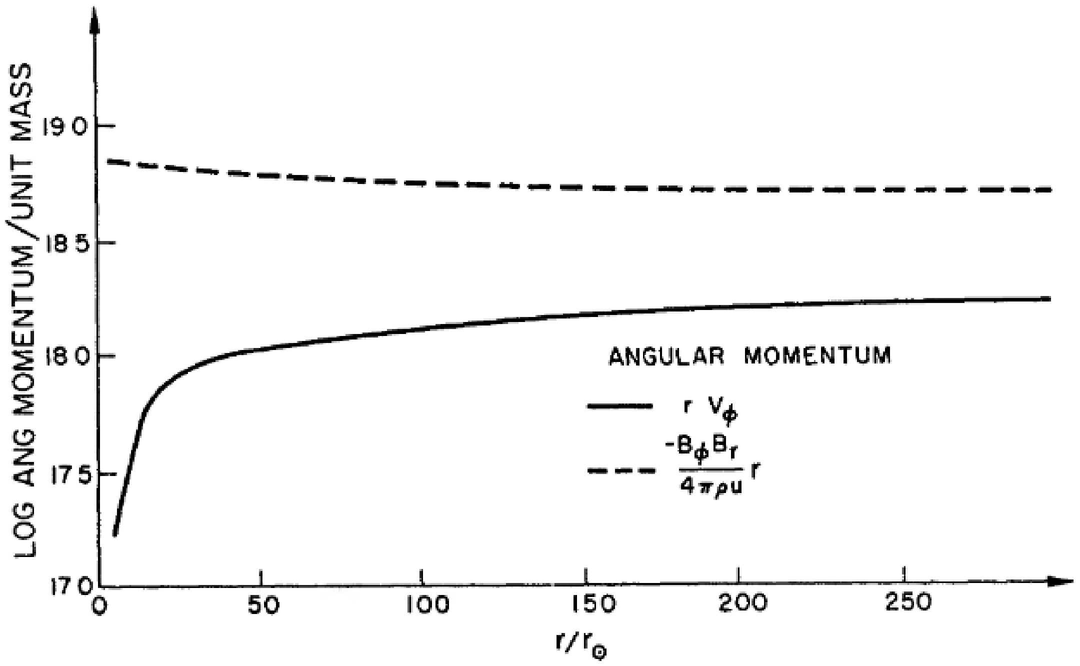

FIG. 5.—Angular momentum and magnetic torque in the solar wind