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Problem 1.

a. $Var(\theta_1) = 1$, $Var(\theta_2) = 2$, $Cov(\theta_1, \theta_2) = \frac{1}{4}$

又 θ_1 , θ_2 be two unbiased estimator of θ

We set $c_1 + c_2 = 1$, then
(from linear combination)

$$E(c_1\hat{\theta}_1 + c_2\hat{\theta}_2) = c_1\theta + c_2\theta = \theta$$

$$Var(c_1\hat{\theta}_1 + c_2\hat{\theta}_2) = c_1^2 Var(\hat{\theta}_1) + c_2^2 Var(\hat{\theta}_2) + 2c_1c_2 Cov(\hat{\theta}_1, \hat{\theta}_2)$$

$$= c_1^2 + 2c_2^2 + \frac{1}{2}c_1c_2$$

$$= c_1^2 + 2(1-c_1)^2 + \frac{1}{2}c_1(1-c_1)$$

$$= c_1^2 + 2 - 4c_1 + 2c_1^2 + \frac{1}{2}c_1 - \frac{1}{2}c_1^2$$

$$= \frac{5}{2}c_1^2 - \frac{7}{2}c_1 + 2$$

$$\frac{\partial}{\partial c_1}\left(\frac{5}{2}c_1^2 - \frac{7}{2}c_1 + 2\right) = 5c_1 - \frac{7}{2} = 0$$

$$\Rightarrow c_1 = 0.7$$

$$c_2 = 1 - 0.7 = 0.3$$

$$Var(c_1\hat{\theta}_1 + c_2\hat{\theta}_2) = \frac{5(49) - 7(7)(10) + 400}{200} = \frac{155}{200} = 0.775$$

The minimum variance estimator of this unbiased class is $0.7\hat{\theta}_1 + 0.3\hat{\theta}_2$

b.

$Var(\theta_3)=1$, $Var(\theta_4)=2$, $Cov(\theta_3,\theta_4)=\frac{3}{4}$

又 θ_3 , θ_4 be two unbiased estimator of θ.

We set $c_3+c_4=1$, then

$$E(c_3\hat{\theta}_3+c_4\hat{\theta}_4) = c_3\theta + c_4\theta = \theta$$

$$\begin{aligned}
Var(c_3\hat{\theta}_3+c_4\hat{\theta}_4) &= c_3^2\,Var(\hat{\theta}_3)+c_4^2\,Var(\hat{\theta}_4)+2\,Cov(\hat{\theta}_3,\hat{\theta}_4)c_3c_4 \\
&= c_3^2+2c_4^2+\frac{3}{2}c_3c_4 \\
&= c_3^2+2(1-c_3)^2+\frac{3}{2}c_3(1-c_3) \\
&= c_3^2+2-4c_3+2c_3^2+\frac{3}{2}c_3-\frac{3}{2}c_3^2 \\
&= \frac{3}{2}c_3^2-\frac{5}{2}c_3+2
\end{aligned}$$

$$\frac{\partial}{\partial c_3}\left(\frac{3}{2}c_3^2-\frac{5}{2}c_3+2\right) = 3c_3-\frac{5}{2}=0$$

$$\Rightarrow \quad c_3=\frac{5}{6}$$

$$\Rightarrow \quad c_4 = 1-\frac{5}{6}=\frac{1}{6}$$

$$\begin{aligned}
Var\left(\frac{5}{6}\hat{\theta}_3+\frac{1}{6}\hat{\theta}_4\right) &= \frac{3}{2}\left(\frac{5}{6}\right)^2-\frac{5}{2}\left(\frac{5}{6}\right)+2 \\
&= \frac{75-150+144}{72} = \frac{69}{72} \\
&= \frac{23}{24}
\end{aligned}$$

The minimum variance estimator of this unbiased class is $\frac{5}{6}\hat{\theta}_3+\frac{1}{6}\hat{\theta}_4$

Problem 2.

a. CDF of the minimum is

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = P(\text{至少一個 } X_1, \cdots, X_n \le x)$$

$$\Rightarrow F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x, \cdots, X_n > x)$$

$\because$ independence

$$= P(X_1 > x) P(X_2 > x) \cdots P(X_n > x)$$

$$= 1 - [P(X_1 > x)]^n$$

$$= 1 - [1 - F(x)]^n$$

$\therefore$ the pdf for the minimum is

$$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x) = \frac{d}{dx}\{1 - [1 - F(x)]^n\}$$

$$= n[1 - F(x)]^{n-1} f(x)$$

Let's find out the distribution of $X_{(1)}$:

$$f_{X_{(1)}}(x) = n e^{-(x-\theta)}\left[1 - \int_{\mu}^{x} e^{-(x-\theta)} dx\right]^{n-1}$$

$$= n e^{-n(x-\theta)} \qquad x \in (\theta, \infty)$$

$\therefore$ Consider $Q = X_{(1)} - \theta$, one can easily prove that $Q \sim e^{(\frac{1}{n})}$ $\Rightarrow Q = X_{(1)} - \theta$ is a pivot quantity.

b. $\because$ We can choose a and b s.t.

$$P(a < Q < b) = 1 - \alpha$$

$\because$ when $b \geq a > 0$, and

$$\int_a^b f_Q(q)\,dq = \int_a^b ne^{-nq}\,dq$$

$$= e^{-an} - e^{-bn}$$

$$= 1 - \alpha$$

$$\Rightarrow P(a < X_{(1)} - \theta < b) = P(X_{(1)} - b < \theta < X_{(1)} - a)$$

$$= 1 - \alpha$$

$\therefore$ $100(1-\alpha)\%$ CI for θ is

$(X_{(1)} - b, X_{(1)} - a)$ where $b \geq a > 0$

and $e^{-an} - e^{-bn} = 1 - \alpha$

If we let $a = 0$, $b = -\frac{\ln(\alpha)}{n}$, then

$$P(a < X_{(1)} - \theta < b) = P\left(0 < X_{(1)} - \theta < -\frac{\ln(\alpha)}{n}\right)$$

$$= P\left(X_{(1)} + \frac{\ln(\alpha)}{n} < \theta < X_{(1)}\right)$$

$$= 1 - \alpha$$

$\therefore$ $100(1-\alpha)\%$ CI for θ is

$\left(X_{(1)} + \frac{\ln(\alpha)}{n}, X_{(1)}\right)$ where $0 < \alpha < 1$

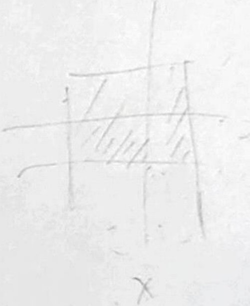
Problem 3.

a. Type I error

$$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$
$$= P(X > 0.92 \mid H_0 : \theta = 1)$$
$$= 0.08$$

b. Type II error

$$P(\text{Type II error}) = P(\text{Not reject } H_0 \mid H_a \text{ is true})$$
$$= P(X \leq 0.92 \mid H_a : \theta = 2)$$
$$= \frac{1}{2} \times 0.92$$
$$= 0.46$$



Problem 4

a. $f(x;\lambda') = P(X=k) = \dfrac{\lambda'^k e^{-\lambda}}{k!}$

$X \sim \text{Poisson}(\lambda')$

Poisson CDF : $e^{-\lambda'} \sum_{i=0}^{\lfloor x \rfloor} \dfrac{\lambda'^i}{i!}$

In this problem,

$\lambda' = n\lambda$ (from origin in Poisson $\lambda = np$)

$\therefore \lambda' = 8(0.5) = 4$

$$\begin{aligned}
\alpha = P(\text{Type I error}) &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\
&= P\left(\sum_{i=1}^{8} X_i \geq 8 \mid H_0 : \lambda = 0.5\right) \\
&= 1 - P\left(\sum_{i=1}^{8} X_i \leq 7 \mid H_0 : \lambda = 0.5\right) \\
&= 1 - P\left(\sum_{i=1}^{8} X_i \leq 7 \mid \lambda' = 4\right) \\
&= 1 - 0.9489 \\
&= 0.0511
\end{aligned}$$

b. $$\begin{aligned}
\beta(\lambda) &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\
&= P\left(\sum_{i=1}^{8} X_i \geq 8 \mid \lambda' = 8\lambda\right) \\
&= 1 - e^{-8\lambda} \sum_{i=0}^{7} \frac{(8\lambda)^i}{i!}
\end{aligned}$$

Problem5

For this problem, I write a program with Python to calculate the answer.

Step0. Calculate some value

$\mu_1 = [4, 5],\ \mu_2 = [8.17, 6.67],\ \mu = [6.08, 5.83]$

Step1. Between-class variance (S_B)

$S_{B1} = \begin{bmatrix} 4.34027778 & 1.73611111 \\ 1.73611111 & 0.69444444 \end{bmatrix},\ S_{B2} = \begin{bmatrix} 4.34027778 & 1.73611111 \\ 1.73611111 & 0.69444444 \end{bmatrix},\ S_B$

Step2. Within-Class Variance (S_W)

$S_{W1} = \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix},\ S_{W2} = \begin{bmatrix} 6.83333333 & 10.33333333 \\ 10.33333333 & 39.33333333 \end{bmatrix},\ S_W = \begin{bmatrix} 10.8333333 \\ 10.3333333 \end{bmatrix}$

Step3. Constructing the Lower Dimensional Space

$S_W^{-1} = \begin{bmatrix} 0.11535464 & -0.02416212 \\ -0.02416212 & 0.02533125 \end{bmatrix},\ S_W^{-1} S_B = \begin{bmatrix} 5.50467654 & 2.20187 \\ -0.73070928 & -0.2922 \end{bmatrix}$

$S_W^{-1} S_B = \lambda W$

$eigenvalue = 5.55111512 * 10^{-17},\quad 5.21239283$

$eigenvector = \begin{bmatrix} 0.99130435 & -0.37139068 \\ -0.13158907 & 0.92847669 \end{bmatrix}$

$\implies optimal\ projection\ vector = \begin{bmatrix} 0.99130435 \\ -0.13158907 \end{bmatrix}$

, and $corresponding\ eigenvalue = 5.21239283$

Code

```
import numpy as np
from numpy import linalg as la
import matplotlib.pyplot as plt
import matplotlib.colors

# Step0. Calculate some value
c1 = np.array([[5,3],[3,5],[3,4],[4,5],[4,7],[5,6]])
c2 = np.array([[9,10],[7,7],[8,5],[8,8],[7,2],[10,8]])
c = np.concatenate((c1,c2), axis=0)

mean_overall = np.mean(c, axis=0)
mean_c1 = np.mean(c1, axis=0)
mean_c2 = np.mean(c2, axis=0)
print("Mean1: " +'\n', mean_c1)
print("Mean2: " +'\n', mean_c2)
print("Mean: " +'\n', mean_overall)
```

```
# Step1. Between-class variance
n_c1 = c1.shape[0]
n_c2 = c2.shape[0]
mean_diff_c1 = (mean_c1 - mean_overall).reshape(2, 1)
SB1 = (mean_diff_c1).dot(mean_diff_c1.T)
mean_diff_c2 = (mean_c2 - mean_overall).reshape(2, 1)
SB2 = (mean_diff_c2).dot(mean_diff_c2.T)
SB = n_c1 * SB1 + n_c2 * SB2
print("SB1: " +'\n', SB1)
print("SB2: " +'\n', SB2)
print("SB: " +'\n', SB)

# Step2. Within-Class Variance
SW_c1 = (c1 - mean_c1).T.dot((c1 - mean_c1))
SW_c2 = (c2 - mean_c2).T.dot((c2 - mean_c2))
SW = SW_c1 + SW_c2
print("SW1: " +'\n', SW_c1)
print("SW2: " +'\n', SW_c2)
print("SW: " +'\n', SW)

#  Step3. Constructing the Lower Dimensional Space
# Determine SW^-1 * SB
SW_=la.inv(SW)
A = la.inv(SW).dot(SB)
print("SW^-1: " +'\n', SW_)
print("SW^-1 * SB: " +'\n', A)
# Get eigenvalues and eigenvectors of SW^-1 * SB
eigenvalues, eigenvectors = la.eig(A)
print("eigenvalues: " +'\n', eigenvalues)
print("eigenvectors: " +'\n', eigenvectors)

n1 = c1.shape[0]
n2 = c2.shape[0]
point_LDA1 = np.dot(c1, eigenvectors.T)
point_LDA2 = np.dot(c2, eigenvectors.T)

# Plot Scatter
plt.figure(figsize=(5, 5))
plt.scatter(c1[:, 0], c1[:, 1], c = 'b')
plt.scatter(c2[:, 0], c2[:, 1], c = 'r')
plt.scatter(point_LDA1[:,0],np.zeros(n1), c='b')
plt.scatter(point_LDA2[:,0],np.zeros(n2), c='r')
plt.show()
```

Output

```
Mean1:
 [4. 5.]
Mean2:
 [8.16666667 6.66666667]
Mean:
 [6.08333333 5.83333333]
SB1:
 [[4.34027778 1.73611111]
 [1.73611111 0.69444444]]
SB2:
 [[4.34027778 1.73611111]
 [1.73611111 0.69444444]]
SB:
 [[52.08333333 20.83333333]
 [20.83333333  8.33333333]]
SW1:
 [[ 4.  0.]
 [ 0. 10.]]
SW2:
 [[ 6.83333333 10.33333333]
 [10.33333333 39.33333333]]
SW:
 [[10.83333333 10.33333333]
 [10.33333333 49.33333333]]
SW^-1:
 [[ 0.11535464 -0.02416212]
 [-0.02416212  0.02533125]]
 SW^-1 * SB:
 [[ 5.50467654  2.20187062]
 [-0.73070928 -0.29228371]]
```

```
eigenvalues:
 [5.21239283e+00 5.55111512e-17]
eigenvectors:
 [[ 0.99130435 -0.37139068]
 [-0.13158907  0.92847669]]
```