Leveraged Accumulator Analysis

Course: 553.649, Advanced Equity Derivatives
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Abstract

The Leveraged Stock Accumulator is a structured financial derivative designed for investors seeking to acquire shares of a stock at a discounted price while leveraging potential market opportunities. This product obligates the investor to purchase a fixed number of shares daily at a pre-determined strike price, typically set at a discount to the stock's initial closing price. The contract incorporates dynamic features to manage risk and optimize performance:

1. Leverage Mechanism: If the stock price falls below the strike price, the investor purchases twice the fixed number of shares daily, amplifying their exposure to potential gains or losses.
2. Early Termination (Knock-Out): The contract terminates immediately if the stock price exceeds a pre-defined upper barrier, with all profits or losses realized on that day.

Additionally, this product is extended to include a basket of three underlying stocks. Using market-implied correlations and appropriate weighting of the individual stocks, a basket is constructed to serve as the underlying asset. The accumulator's mechanics are applied to the basket price instead of a single stock price. Monte Carlo simulations are employed to evaluate the performance of the basket-based accumulator, capturing the stochastic dynamics and correlation effects of the constituent stocks.

The Leveraged Stock Accumulator, with its single-stock and basket extensions, offers cost savings and the potential for enhanced returns in bullish markets while exposing the investor to significant risks in bearish conditions. By integrating predefined risk controls such as the knock-out feature, the product balances the opportunity for gain with the management of downside risks. This derivative is ideal for investors with a bullish outlook on the underlying assets and a willingness to accept leveraged exposure in exchange for potentially discounted share accumulation.

Contents

1 Executive Summary (Sales Pitch)

The Leveraged Stock Accumulator is an innovative structured financial product that empowers investors to acquire shares of a stock at a discounted price while strategically leveraging market movements. Designed for sophisticated investors with a bullish outlook, this product offers an efficient and cost-effective way to build a stock position while managing risks through predefined mechanisms.

1.1 Why This Product is Attractive

1. Discounted Entry Price:
   - The Leveraged Stock Accumulator allows investors to purchase shares daily at a strike price discounted from the stock's initial market price. This feature reduces the effective cost of acquiring shares, enhancing return potential when the market aligns with the investor's bullish outlook.
2. Controlled Leverage:
   - The product dynamically adjusts share purchases based on market performance. When the stock price drops below the strike price, investors purchase twice the daily fixed number of shares at the discounted strike price, allowing them to accumulate more shares at favorable prices.
3. Early Termination Feature:
   - The inclusion of an upper barrier ensures that investors lock in profits and cease further obligations if the stock performs exceptionally well. This automatic profit-taking mechanism provides a balance between return optimization and risk mitigation.
4. Simplicity and Daily Position Building:
   - Investors benefit from a straightforward and predictable structure. By accumulating a fixed number of shares daily, they steadily build their position in a disciplined manner without timing the market.

1.2 Why This Product is Reasonable

1. Market Dynamics Alignment:
   - This product aligns well with investor sentiment during moderate to bullish market conditions. By offering a lower entry price, it addresses concerns about potential short-term price fluctuations while positioning investors for long-term gains.
2. Predefined Risk Controls:
   - The knock-out feature and the structured nature of the product limit downside exposure while ensuring upside participation. These features provide a clear framework for risk and reward, making the investment decision more transparent.

3. Customizable Design:
   - The product can be tailored to suit specific investor needs, such as adjusting the strike price, upper barrier, or leverage factor, providing flexibility to align with individual risk tolerance and return expectations.
4. Cost-Effective Leverage:
   - Unlike traditional margin trading, which may require significant collateral and carry interest costs, the Leveraged Stock Accumulator builds leverage into the product design. This makes it an efficient tool for investors to amplify returns without additional financing costs.
5. Diversified Applicability:
   - The Leveraged Stock Accumulator is suitable for both institutional investors looking to enhance portfolio performance and individual investors seeking cost-effective stock accumulation strategies.

The Leveraged Stock Accumulator is a compelling choice for investors who want to take advantage of market opportunities with discipline and precision. Its discounted pricing, dynamic leverage mechanism, and risk controls provide an attractive balance of potential returns and mitigated risks, making it an excellent tool for portfolio growth in favorable market conditions. With its intuitive design and clear benefits, this product is well-suited for investors aiming to strategically enhance their exposure to high-quality stocks.

2 Problem Statement

The Leveraged Stock Accumulator is a sophisticated financial product with dynamic features, making its valuation both challenging and crucial for informed decision-making. Accurate pricing is essential to ensure the product's attractiveness to investors while maintaining fair market practices. The goal of this project is to establish a reliable methodology to price the Leveraged Stock Accumulator, addressing the complexities introduced by its unique features such as leverage, knock-out conditions, and daily share accumulation.

To achieve this, we aim to develop and compare two pricing methodologies:

1. Analytical Solution (Closed-Form):
   - Derive a closed-form expression for the product's value, considering the discounted strike price, leverage mechanism, and knock-out feature. Analytical pricing provides insight into the underlying dynamics and computational efficiency for certain market conditions.
2. Numerical Solution (Simulations):
   - Utilize Monte Carlo simulations to price the product, accounting for stochastic price movements of the underlying stock. Numerical methods allow flexibility in modeling the product's complexities, including path-dependent features.

By comparing the results from these two approaches, we can:

- Validate the accuracy and consistency of each method.
- Identify potential discrepancies and their causes.
- Ensure convergence between the solutions to establish a robust and fair pricing framework.

Ultimately, this study aims to determine a fair value for the Leveraged Stock Accumulator, providing transparency and confidence for both issuers and investors. The findings will also shed light on the sensitivity of the product's value to market parameters, contributing to more informed product design and risk management.

3 Methodology

3.1 Product Description

A typical accumulator contract obligates an investor to purchase a specific quantity of stock on pre-specified observation days during the term of the contract at a strike price K, which is generally set at a discount to the initial spot price S_0 of the stock. This discounted price, K, remains constant throughout the contract's duration. The observation days are represented as t_1, t_2, ..., t_n, where $t_i \leq N_t$, and N represents the total length of the contract. If the closing price S_i on the i^{th} observation day is greater than or equal to K, the purchase quantity is fixed at Q. However, if S_i is less than K, the purchase quantity doubles to $2Q$. Additionally, the contract includes a knock-out feature that automatically terminates the agreement if the stock's closing price exceeds a specified barrier B during the contract's term.

Accumulator contracts are typically structured as zero-cost products, meaning neither party pays an upfront premium. Settlement terms are another critical feature of these contracts and can be categorized as either immediate or delayed. Immediate settlement involves the delivery of the fixed quantity (Q or $2Q$) on the same observation day (t_i). However, this type of settlement is relatively uncommon in practice. More frequently, settlement occurs periodically, either weekly or monthly. For example, under a weekly settlement arrangement, all shares purchased within the same week are cleared on the final trading day of that week. In general, there can be m settlement days throughout the life of the contract, which may differ from the observation days. This discussion focuses exclusively on immediate settlement.

Another important aspect of accumulator contracts is the knock-out barrier B, which can be implemented either discretely or continuously. In the case of a discrete barrier, the knock-out condition is only evaluated at the end of observation days, whereas a continuous barrier is monitored throughout the trading days within the contract's term. For the purposes of this discussion, we focus on a discrete barrier.

While many accumulator contracts use a fixed purchase quantity of Q or $2Q$, more general variants may incorporate a gearing ratio, where the purchase quantity is Q or gQ, with g representing the gearing factor. In practice, Q is typically set as a multiple of the stock's lot size. This paper specifically analyzes a typical accumulator contract with $g = 2$ and $Q = 1$, providing insights into its pricing and risk characteristics. While the formulas derived here apply to this specific structure, they can be easily extended to accommodate general contracts with alternative gearing ratios and quantities.

3.2 Analytical Solution

In this project, we mainly consider an accumulator contract with N-day to expiration and with immediate settlement. The daily payoffs at day t_i, where $i \leq n$ is given by:

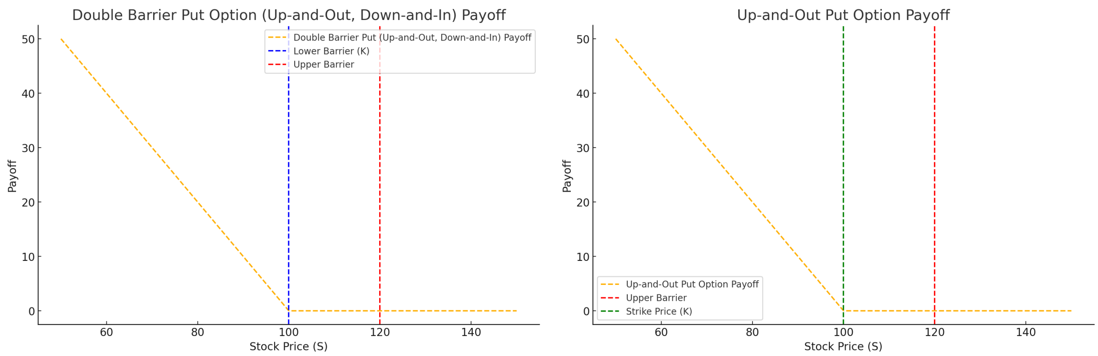
$$\begin{cases} 0 & \text{if } \max_{0\leq\tau\leq t_i} S_\tau \geq B, \\ S_{t_i} - K & \text{if } \max_{0\leq\tau\leq t_i} S_\tau < B,\ S_{t_i} \geq K, \\ 2(S_{t_i} - K) & \text{if } \max_{0\leq\tau\leq t_i} S_\tau < B,\ S_{t_i} < K. \end{cases} \tag{1}$$

This set of payoffs is equivalent to long one up-and-out barrier call option and short two up-and-out barrier put options with expiration at day t_i, and by accumulating all the option payoffs of maturities from t_i to t_n we can replicate the payoff of the accumulator:

$$V = \sum_{i=1}^{n} \{C_{uo}(t_i, S, K, B) - 2P_{uo}(t_i, S, K, B)\} \tag{2}$$

where $C_{uo}(t_i, S, K, B)$ represents the fair value of the up-and-out call option expiring at t_i and $P_{uo}(t_i, S, K, B)$ represents the fair value of the up-and-out put option expiring at t_i.

Note: The leveraged construction of the Leveraged Stock Accumulator, using one up-and-out call option and two up-and-out put options, is mathematically equivalent to a non-leveraged accumulator combined with an option featuring a knock-in mechanism at the strike price (K). This equivalence arises from the property that the payoff of a double-barrier put option matches the payoff of an up-and-out put option when the lower barrier is set to K. This relationship simplifies the decomposition of the leveraged accumulator into more standard option structures, facilitating both valuation and risk analysis[1].



(a) Up-and-Out, Down-and-In Put Payoff (b) Up-and-Out Put Payoff

By using the Reflection Principle we are able to obtain the analytical solution of barrier options, so that the accumulator can be priced analytically[4].

$$\begin{aligned} C_{uo}(t_i, S, K, B) &= [C(t_i, S, K) - C(t_i, S, B) - (B-K)C_d(t_i, S, B)] \\ &\quad - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}}[C(t_i, \frac{B^2}{S}, K) - C(t_i, \frac{B^2}{S}, B) - (B-K)C_d(t_i, \frac{B^2}{S}, B)] \end{aligned} \tag{3}$$

$$P_{uo}(t_i, S, K, B) = P(t_i, S, K) - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}} P(t_i, \frac{B^2}{S}, K) \tag{4}$$

$$
\begin{aligned}
V = \sum_{i=1}^{n} \{ & [C(t_i, S, K) - C(t_i, S, B) - (B-K)C_d(t_i, S, B)] \\
& - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}} [C(t_i, \frac{B^2}{S}, K) - C(t_i, \frac{B^2}{S}, B) - (B-K)C_d(t_i, \frac{B^2}{S}, B)] \\
& - 2[P(t_i, S, K) - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}} P(t_i, \frac{B^2}{S}, K)] \}
\end{aligned} \tag{5}
$$

By simplifying, the final closed form solution should be:

$$
\begin{aligned}
V = \sum_{i=1}^{n} \{ & SN(d_1) - Ke^{-rt_i}N(d_2) - SN(d_1^B) + Ke^{-rt_i}N(d_2^B) \\
& - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}} [\frac{B^2}{S}N(d_1^R) - Ke^{-rt_i}N(d_2^R) - \frac{B^2}{S}N(d_1^{BR}) + Ke^{-rt_i}N(d_2^{BR})] \\
& - 2[Ke^{-rt_i}N(-d_2) - SN(-d_1) - (\frac{S}{B})^{1-\frac{2r}{\sigma^2}} (Ke^{-rt_i}N(-d_2^R) - \frac{B^2}{S}N(-d_1^R))] \}
\end{aligned} \tag{6}
$$

where:

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}}, d_2 = d_1 - \sigma\sqrt{t_i} \\
d_1^B &= \frac{\ln(\frac{S}{B}) + (r + \frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}}, d_2^B = d_1^B - \sigma\sqrt{t_i} \\
d_1^R &= \frac{\ln(\frac{B^2}{SK}) + (r + \frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}}, d_2^R = d_1^R - \sigma\sqrt{t_i} \\
d_1^{BR} &= \frac{\ln(\frac{B}{S}) + (r + \frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}}, d_2^{BR} = d_1^{BR} - \sigma\sqrt{t_i}
\end{aligned} \tag{7}
$$

3.3 Simulation Framework

The pricing of the Leveraged Stock Accumulator relies on a Monte Carlo simulation framework to account for its path-dependent nature and complex payoff structure. To enhance the accuracy and efficiency of the simulation, variance reduction techniques, such as antithetic variates, are employed. This framework consists of the following key steps:

1. Stock Price Simulation

   The first step involves simulating the underlying stock prices using a Geometric Brownian Motion (GBM) model. The GBM assumes that stock prices follow a log-normal distribution, which is widely used in financial modeling due to its ability to capture realistic stock price behaviors, such as positive skewness and the constraint of non-negative prices.

   The dynamics of the stock price S_t are modeled using the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{8}$$

   where:

- μ: Drift term, representing the expected return of the stock.
- σ: Volatility, capturing the uncertainty in the stock price.
- W_t: Wiener process (Brownian motion), representing random shocks to the stock price.

To simulate stock prices, this equation is discretized using the Euler-Maruyama method:

$$S_{t+\Delta t} = S_t \exp\left((\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z\right) \tag{9}$$

where:

- Δt: Time step size.
- Z: A standard normal random variable $Z \sim \mathcal{N}(0, 1)$

2. Variance Reduction with Antithetic Variates

   - To reduce the variance of the simulation and improve convergence:
     (a) For each random variable Z used to simulate the stock price path, generate its antithetic counterpart -Z.
     (b) Simulate stock prices for both the original random variables and their antithetic counterparts.
     (c) Calculate the payoffs for both paths and average the results. This approach reduces the variance because the positive and negative deviations in the stock price paths tend to offset each other.

   By incorporating antithetic variates, the efficiency of the simulation is improved without increasing the computational cost significantly.

3. Payoff Calculation

   Once the stock price paths are generated, the payoff of the accumulator contract is calculated for each simulated path:

   (a) Observation and Quantity Determination:
   - On each observation day t_i, determine the stock price S_i relative to the strike price K:
     - If $S_i \geq K$, the investor purchases Q shares at the strike price.
     - If $S_i < K$, the investor purchases $2Q$ shares at the strike price, doubling their daily obligation.
   - Incorporate the knock-out feature:
     - If $S_i \geq B$ (the knock-out barrier), the contract terminates immediately. No further shares are purchased, and the aggregation of payoffs ends on this day.

   (b) Aggregation of Payoffs:
   - For each simulated path, aggregate the daily payoffs as:

   $$\text{Payoff for Day } t_i = Q \times (S_i - K) \text{ or } 2Q \times (S_i - K), \tag{10}$$

   depending on whether the stock price is above or below the strike price.

- The aggregation process stops if the barrier B is hit. Otherwise, the process continues until the contract matures.

(c) Total Payoff:

- The total payoff for a given path is the sum of daily payoffs up to the knock-out day or the final observation day if no knock-out occurs.

4. Discounting to Present Value

   Before calculating the aggregated payoff for each simulated path, discount each individual payoff to the present value to account for the time value of money:

$$\text{Present Value of } i^{\text{th}} \text{ Payoff} = \text{Total Payoff} \times e^{-rt_i} \tag{11}$$

   where:

   - r: Risk-free rate.
   - t_i: Time to i^{th} day in years.

5. Aggregating Results

   To estimate the fair price of the accumulator contract:

   (a) Perform the simulation process across a large number of paths (N), typically in the range of 10,000 to 100,000 paths, to ensure accuracy and reduce statistical error.

   (b) Incorporate antithetic variates to halve the variance of the estimates.

   (c) Calculate the mean of the discounted payoffs:

$$\text{Simulated Accumulator Price} = \frac{1}{N}\sum_{i=1}^{N} \text{Present Value of Payoff for Path } i \tag{12}$$

By incorporating variance reduction techniques such as antithetic variates, this simulation framework captures the path-dependent features of the Leveraged Stock Accumulator, reduces computational noise, and ensures accurate and efficient pricing of this complex derivative structure.

3.4 Convergence Analysis

To ensure the reliability of the Monte Carlo simulation results for pricing the Leveraged Stock Accumulator, we conducted a convergence analysis focusing on the following methods:

1. Simulation Stability:

   - The simulated price was calculated for varying numbers of paths (N) to observe the stabilization of the mean price as N increased.
   - Standard deviations (SD) and 95% confidence intervals (CI) were computed for each simulation, reflecting the precision of the results.

2. Comparison with Analytical Results:
   - The analytical price of the Leveraged Stock Accumulator served as a benchmark.
   - We compared the analytical price against the simulated mean price to assess the alignment of the results.
   - The 95% confidence interval of the simulated price was checked to ensure it consistently included the analytical price.
3. Variance Reduction:
   - Antithetic variates were used to reduce the variance of simulated results, improving the precision of estimates without increasing computational effort.

This approach confirmed the robustness of the simulation framework and its consistency with theoretical pricing results.

4 Results

4.1 Calibration

To estimate parameters or calibrate the model, market option prices are used as inputs to the Black-Scholes or similar pricing models. The calibration process involves finding implied volatility by minimizing the difference between observed market prices and model-predicted prices. We used one-year ATM call option prices for calibration purpose. For correlations, historical price data of multiple assets is analyzed. Considering the maturity of one year for our product, we took 5-years historical data in estimating correlation matrix. As for interest rate, we used one-year US treasury yield(4.3%) as our constant interest rate in this paper.

Implied Volatilities

- AMZN IV = 33.85%
- AAPL IV = 24.71%
- META IV = 36.83%

Correlation

Ticker	**AAPL**	**AMZN**	**META**
AAPL	1.000000	0.619072	0.591930
AMZN	0.619072	1.000000	0.613559
META	0.591930	0.613559	1.000000

Table 1: Correlation Matrix of Stocks

Based on Fisher Transformation, we can obtain 95% confidence interval of correlation:

Ticker	AAPL	AMZN	META
AAPL	1.000000	0.583761	0.554796
AMZN	0.583761	1.000000	0.577871
META	0.554796	0.577871	1.000000

Table 2: Lower Bound of the 95% Confidence Interval

Ticker	AAPL	AMZN	META
AAPL	1.000000	0.652044	0.626710
AMZN	0.652044	1.000000	0.646905
META	0.626710	0.646905	1.000000

Table 3: Upper Bound of the 95% Confidence Interval

4.2 Simulation Results

Parameters

Based on market quotes as of Dec 18,2024, we have the following parameters:

$$S_0 = 220.52 \quad \text{(Initial stock price)}$$
$$T = \frac{252}{252} = 1 \quad \text{(Time to maturity in years)}$$
$$r = 0.043 \quad \text{(Risk-free rate)}$$
$$\sigma = 33.85\% \quad \text{(Volatility)}$$
$$\text{Steps} = 252 \quad \text{(Total number of observations)}$$
$$\text{Shares} = 1 \quad \text{(Daily shares to buy)}$$
$$\text{N} = 100,000 \quad \text{(Number of simulations)}$$

We manually set the barrier B as 120% of the initial price S_0, and chose K to make the initial value of the product close to 0.

$$K = S_0 \times 0.8623 = 220.52 \times 0.8623 \quad \text{(Strike price)}$$
$$B = S_0 \times 1.2 = 220.52 \times 1.2 \quad \text{(Knock-out barrier)}$$

Results

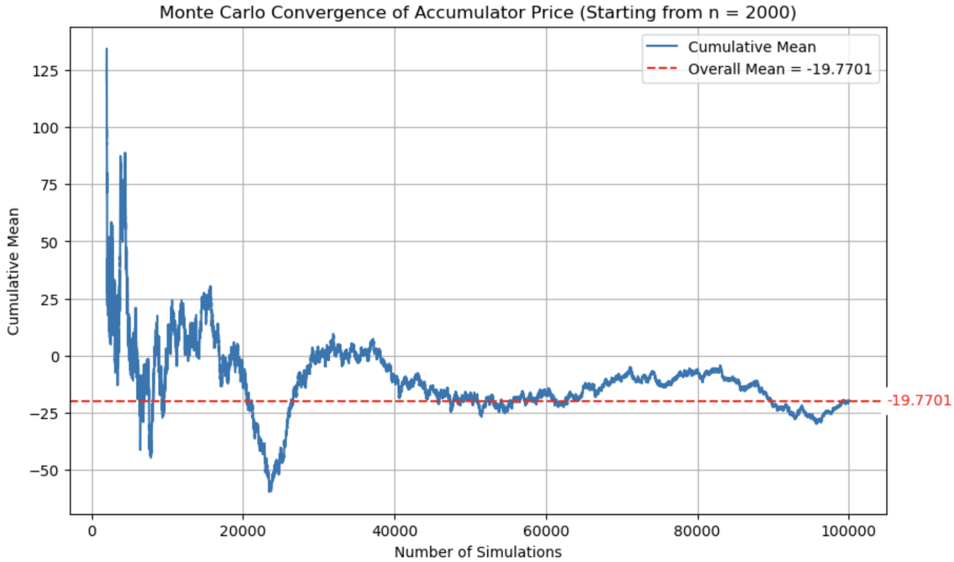


Figure 2: Simulated Results

- Mean: -19.77
- SD: 6677.2715
- CI: (-61.16, 21.62)

Antithetic variates method was used to control the simulation variance:

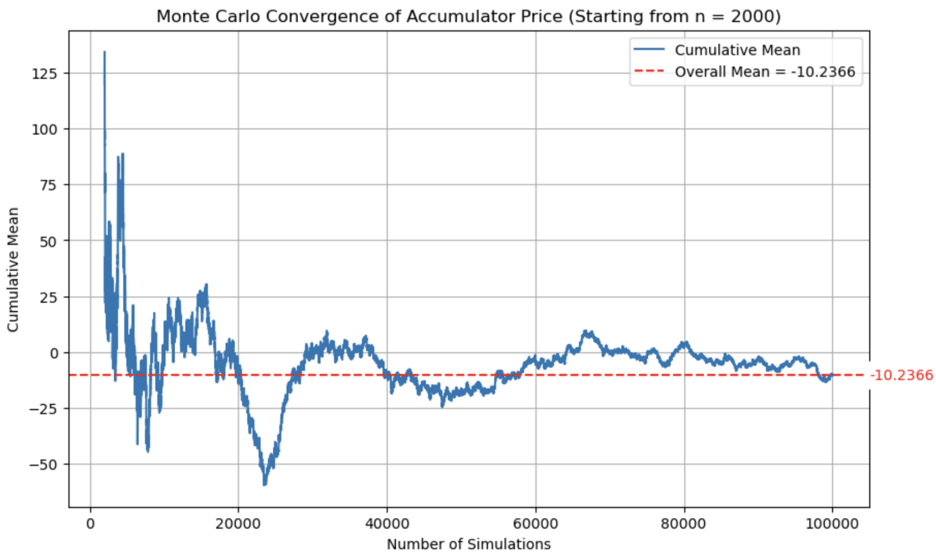


Figure 3: Antithetic Control

- Mean: -10.24
- SD: 6670.4346
- CI: (-51.58, 31.11)

Variate control shows no significant improvement.

4.3 Historical Backtest

Product performance is back tested using moving window of historical data from Jan 01,2010 to Dec 18,2024. Due to the magnitude of this period, stock prices changed dramatically, for the purpose of comparison, we added a weight to the product ($100 \div S_0$).

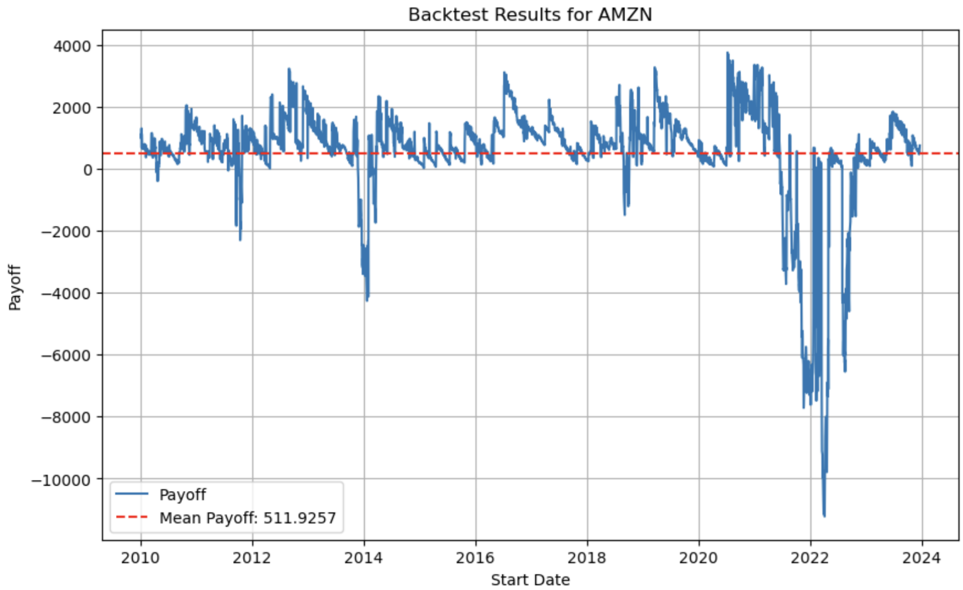


Figure 4: Back Test

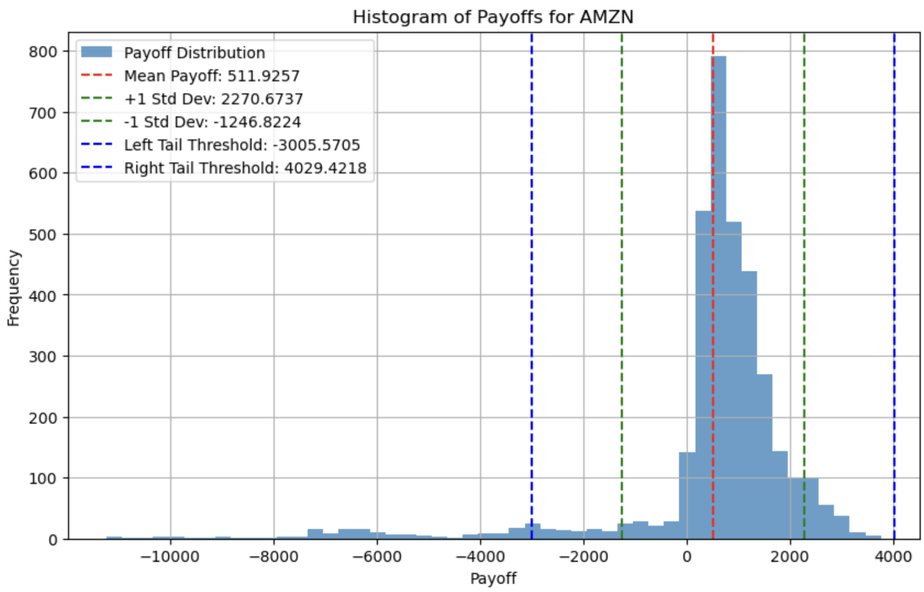


Figure 5: Distribution

- Mean: 511.9257
- SD: 1758.7481

The tail on the left implies this product is highly risky in extreme case, investor can suffer during periods when the price is going down (2021-2022).

4.4 Model Testing

Parameters

$$
\begin{aligned}
S_0 &= 220.52 && \text{(Initial stock price)}\\
T &= \frac{252}{252} = 1 && \text{(Time to maturity in years)}\\
r &= 0.043 && \text{(Risk-free rate)}\\
\sigma &= 33.85\% && \text{(Volatility)}\\
\text{Steps} &= 252 && \text{(Total number of observations)}\\
\text{Shares} &= 1 && \text{(Daily shares to buy)}\\
\text{N} &= 100,000 && \text{(Number of simulations)}\\
K &= S_0 \times 0.8623 = 220.52 \times 0.8623 && \text{(Strike price)}\\
B &= S_0 \times 1.2 \times e^{0.5826 \times \sigma \times \sqrt{\frac{T}{Steps}}} && \text{(Knock-out barrier)}
\end{aligned}
$$

Due to the discretely monitored nature of back testing, the analytical result calculated using the above parameters has to be adjusted using a correction term $e^{\beta\sigma\sqrt{\frac{T}{Steps}}}$, where $\beta = 0.5826$, to compare with our simulated result[2] [3].

The analytical result using the formulas derived earlier is -2.3930, which lies between the confidence interval of simulation result.

In addition, we took several pairs of strikes and barriers, including some extreme cases. This process checked the simulation results with analytical results to test the simulation correctness, model behavior under extreme cases, and potential monotone behavior with respect to strike and barrier changes.

K_ratio, B_ratio	MC Mean	MC Confidence Interval	Analytical Value
(0.2, 1.2)	24738.636010	(24657.48234311036, 24819.78967689611)	24807.561859
(0.4, 1.2)	17720.070878	(17662.17787906406, 17777.96387784618)	17773.182178
(0.6, 1.2)	10620.386785	(10582.82245563585, 10657.95113896648)	10657.804965
(0.8, 1.2)	2799.764562	(2764.855179174711, 2834.673945350675)	2820.307567
(1.0, 1.2)	-7350.278066	(-7414.796752535322, -7285.759379901364)	-7347.938879
(0.8, 1.1)	828.859475	(801.805060779987, 855.9138897051721)	844.058327
(0.8, 1.2)	2799.764562	(2764.855179174711, 2834.673945350675)	2820.307567
(0.8, 1.3)	4811.013469	(4769.127839701861, 4852.899099067987)	4814.019275
(0.8, 1.4)	6487.836645	(6439.710167755391, 6535.963122783577)	6493.514038
(0.8, 1.5)	7802.030360	(7748.371467820325, 7855.68925294528)	7799.844858

Table 4: Monte Carlo Means, Confidence Intervals, and Analytical Values

The table above indicates that analytical results lie between confidence intervals perfectly in all cases, showing that Monte Carlo simulation is a reliable model in pricing accumulator. With increasing of strikes or decreasing of barriers, the value of accumulator keeps decreasing as expected, showing that the model is doing well in extreme cases.

4.5 Risk Analysis

4.5.1 VaR and CVaR:

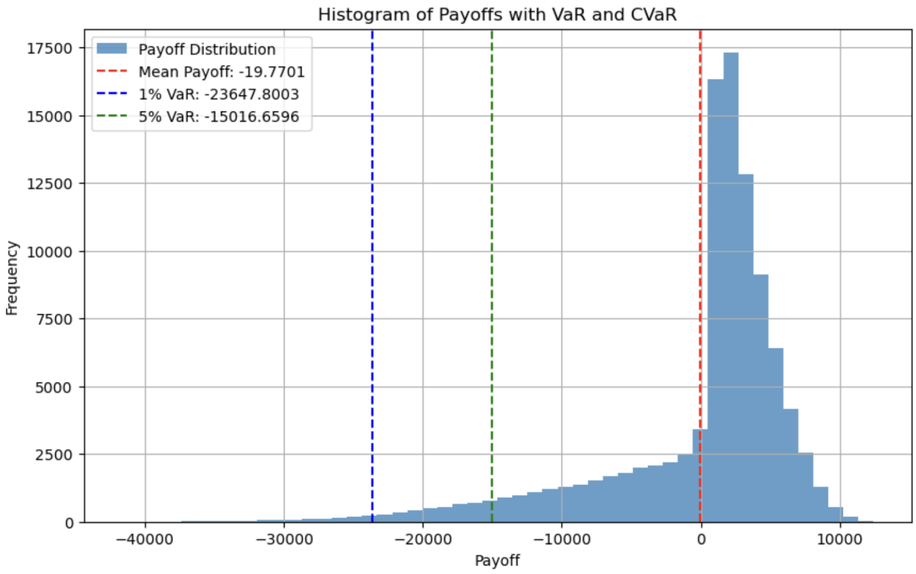


Figure 6: VaR & CVaR

- 1% VaR: -23647.8003
- 5% VaR: -15016.6596
- 1% CVaR: -27570.1082
- 5% CVaR: -20331.8307

4.5.2 Greeks:

- Monte Carlo Simulation

  The Delta and Vega are calculated using finite difference method:

$$\Delta \approx \frac{V(S+\Delta S,\sigma,t)-V(S-\Delta S,\sigma,t)}{2\Delta S} \tag{13}$$

$$\text{Vega} \approx \frac{V(S,\sigma+\Delta\sigma,t)-V(S,\sigma-\Delta\sigma,t)}{2\Delta\sigma} \tag{14}$$

  - Δ: 181.58
  - Vega: -27937.35

- Analytical Method

  Given the closed form solution of this product, the Delta and Vega can be calculated as a combination of Delta and Vega of barrier options.

  - Δ: 107.69
  - Vega: -29357.33

5 Basket Accumulator

5.1 Simulation Framework for Basket Accumulator

To simulate the basket accumulator, we employ a multi-dimensional Geometric Brownian Motion (GBM) to model the correlated dynamics of the underlying assets. This framework incorporates realistic market conditions and accurately reflects the basket accumulator's path-dependent features.

5.1.1 Multi-Dimensional GBM

The price dynamics of n underlying stocks $S_i(t)$ $(i = 1, \ldots, n)$ are described by the following system of stochastic differential equations (SDEs):

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)\, dt + \sigma_i dW_i(t),$$

where:

- r: Risk-free rate.
- q_i: Dividend yield of the i-th stock.
- σ_i: Volatility of the i-th stock.
- $W_i(t)$: Standard one-dimensional Brownian motion.
- ρ_{ij}: Correlation between $W_i(t)$ and $W_j(t)$.

5.1.2 Incorporating Correlation

To account for the correlation structure between the stocks, we use the *Cholesky decomposition* of the correlation matrix C:

$$C = LL^T,$$

where L is the lower triangular matrix obtained from the decomposition.

To simulate correlated Brownian motions, we start with n independent standard normal variables $Z = (Z_1, \ldots, Z_n)^T$ and compute:

$$X = LZ,$$

where $X = (X_1, \ldots, X_n)^T$ represents the correlated standard normal variables. These are then used to simulate the correlated GBM paths for the underlying stocks.

5.1.3 Basket Price Dynamics

The price of the basket at any time t is calculated as the weighted sum of the individual stock prices:

$$B(t) = \sum_{i=1}^{n} w_i S_i(t),$$

where w_i are the weights of the i-th stock, determined based on initial prices to normalize the basket value.

5.1.4 Payoff Calculation

Using the simulated basket prices $B(t)$, the payoff of the basket accumulator is calculated as follows:

1. **Daily Purchase Decision**:
   - If $B(t_i) \geq K$, the investor purchases Q units of the basket at the strike price K.
   - If $B(t_i) < K$, the investor purchases $2Q$ units of the basket at the strike price K.
2. **Knock-Out Condition**: If $B(t_i) \geq B_{\text{knock-out}}$, the contract terminates immediately, and no further shares are purchased.
3. **Aggregation of Payoffs**: The total payoff is the sum of the differences between the market price and the strike price for all accumulated units.

5.1.5 Discounting to Present Value

The total payoff for each path is discounted to its present value to account for the time value of money:

$$\text{PV of Payoff} = \frac{\text{Total Payoff}}{(1+r)^T},$$

where T is the time to maturity or the time to knock-out if triggered.

5.1.6 Monte Carlo Simulations

To estimate the price of the basket accumulator:

1. Perform Monte Carlo simulations with N paths.
2. Compute the mean of the discounted payoffs:

$$\text{Basket Accumulator Price} = \frac{1}{N} \sum_{i=1}^{N} \text{PV of Payoff for Path } i.$$

This simulation framework integrates the correlation between stocks, the leverage mechanism, and the knock-out feature, providing a robust methodology for pricing the basket accumulator under realistic market conditions.

5.2 Basket Accumulator Results

Parameters

Based on market quotes as of Dec 18,2024, we have the following parameters:

$$
\begin{aligned}
S_{0,a} &= 220.52 && \text{(Initial stock a price)} \\
S_{0,b} &= 248.05 && \text{(Initial stock b price)} \\
S_{0,c} &= 597.19 && \text{(Initial stock c price)} \\
w_a &= 1000/S_{0,a} && \text{(Weight of stock a in basket)} \\
w_b &= 1000/S_{0,b} && \text{(Weight of stock b in basket)} \\
w_c &= 1000/S_{0,c} && \text{(Weight of stock c in basket)} \\
S &= S_{0,a} \times w_a + S_{0,b} \times w_b + S_{0,c} \times w_c && \text{(Initial basket price)} \\
T &= \frac{252}{252} = 1 && \text{(Time to maturity in years)} \\
r &= 0.043 && \text{(Risk-free rate)} \\
\sigma_a &= 33.85\% && \text{(Volatility of stock a)} \\
\sigma_b &= 24.71\% && \text{(Volatility of stock b)} \\
\sigma_c &= 36.83\% && \text{(Volatility of stock c)} \\
\rho_{a,b} &= 0.619072 && \text{(Correlation coefficient of stock a and b)} \\
\rho_{a,c} &= 0.591930 && \text{(Correlation coefficient of stock a and c)} \\
\rho_{b,c} &= 0.613559 && \text{(Correlation coefficient of stock b and c)} \\
\text{Steps} &= 252 && \text{(Total number of observations)} \\
\text{Shares} &= 1 && \text{(Daily number of basket to buy)} \\
\text{N} &= 100,000 && \text{(Number of simulations)}
\end{aligned}
$$

We manually set the barrier B as 120% of the initial price S_0, and chose K to make the initial value of the product close to 0.

$$
\begin{aligned}
K &= S \times 0.9018 && \text{(Strike price)} \\
B &= S \times 1.2 && \text{(Knock-out barrier)}
\end{aligned}
$$

Results

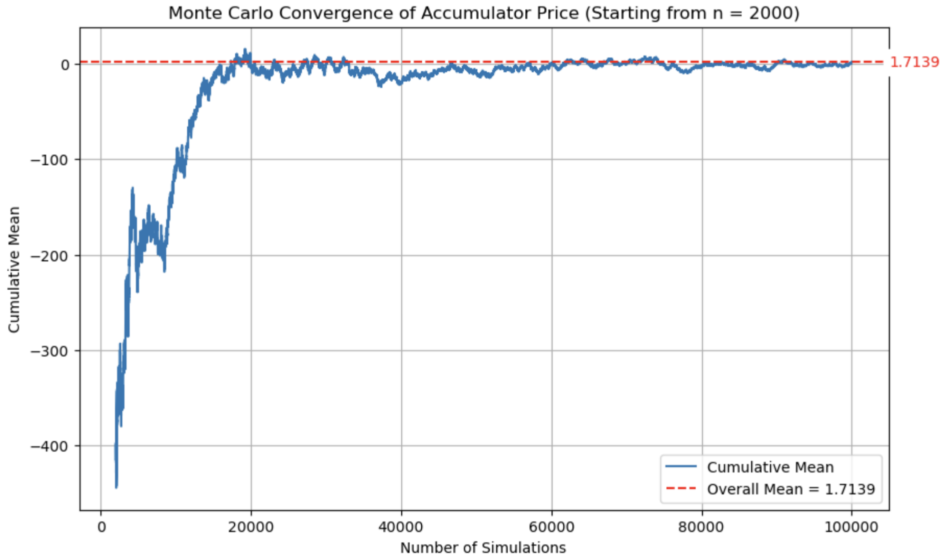


Figure 7: Simulated Results

- Mean: 1.7139
- SD: 8145.3927
- CI: (-48.77, 52.20)

5.3 Basket Accumulator Historical Backtest

Product performance is back tested using moving window of historical data from Jan 01,2010 to Dec 18,2024. Due to the magnitude of this period, stock prices changed dramatically, for the purpose of comparison, we added a weight to the product ($100 \div S$).

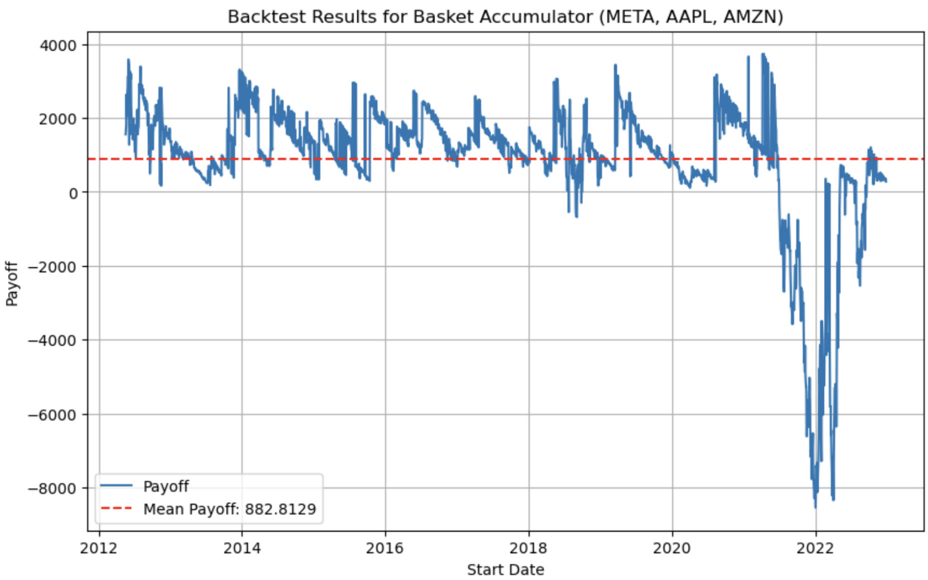


Figure 8: Back Test

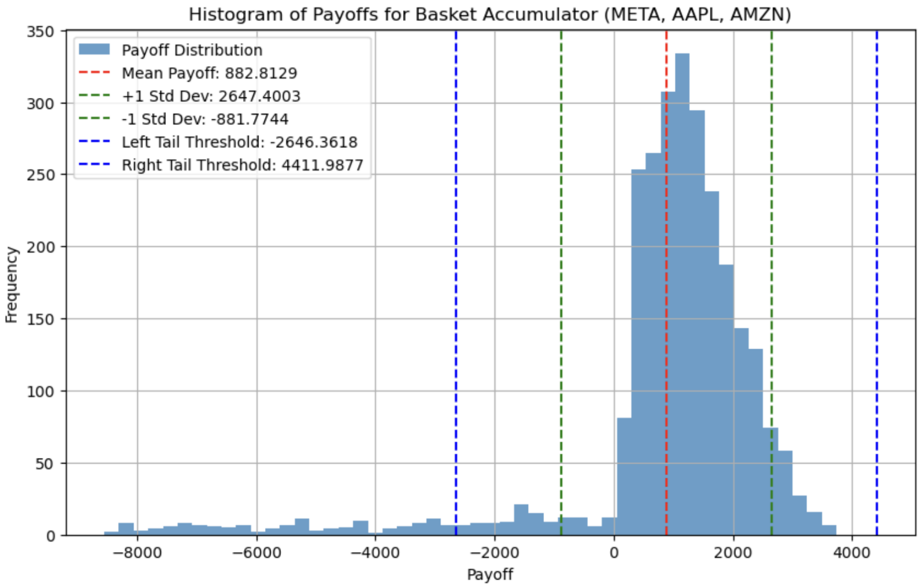


Figure 9: Distribution

- Mean: 882.81
- SD: 1764.59

Comparing with single asset accumulator, basket accumulator has lower standard deviation, and lighter left tail, meaning that basket accumulator has better diversification and relatively less risks.

5.4 Basket Accumulator Risk Analysis

5.4.1 Greeks:

- Monte Carlo Simulation

  The Delta and Vega are calculated using finite difference method:

$$\Delta \approx \frac{V(S+\Delta S,\sigma,t)-V(S-\Delta S,\sigma,t)}{2\Delta S} \tag{15}$$

$$\text{Vega} \approx \frac{V(S,\sigma+\Delta\sigma,t)-V(S,\sigma-\Delta\sigma,t)}{2\Delta\sigma} \tag{16}$$

  - Δ_a: -72.26
  - Δ_b: -57.87
  - Δ_c: -0.41
  - Vega$_a$: -12577.06
  - Vega$_b$: -13116.74
  - Vega$_c$: -12836.35

  where, Δ_a, Δ_b, Δ_c are the partial derivatives of basket value with respect to initial prices of stock a, b, and c, Vega$_a$, Vega$_b$, Vega$_c$ are the partial derivatives of basket value with respect to volatilities of stock a, b, and c.

5.4.2 Correlation Adjustment Analysis:

In this project, we assessed the model risk arising from correlation uncertainty by conducting a correlation adjustment analysis. This method evaluates the sensitivity of the basket accumulator's price to changes in correlation estimates, quantifying the potential impact of correlation risk. The methodology is described as follows:

Initial Correlation Estimation

Using one year of historical observations ($N = 252$), we estimated the pairwise correlations ρ_{12}, ρ_{13}, and ρ_{23} between the three underlying stocks in the basket. These correlations represent the baseline inputs to the pricing model.

Standard Error of Correlation Estimates

For each pair of stocks, we calculated the standard error of the correlation estimates ($\Delta\rho_{12}$, $\Delta\rho_{13}$, $\Delta\rho_{23}$) based on the sample size (N) and the observed correlation levels. These errors quantify the statistical uncertainty in the correlation estimates.

Alpha Adjustment Factor

For each correlation estimate, we computed an adjustment factor (α_i) to scale the correlation by its uncertainty:

$$\alpha_1 = \frac{\Delta\rho_{12}}{1-\rho_{12}}, \quad \alpha_2 = \frac{\Delta\rho_{13}}{1-\rho_{13}}, \quad \alpha_3 = \frac{\Delta\rho_{23}}{1-\rho_{23}}.$$

The average adjustment factor (α) was then calculated as:

$$\alpha = \frac{1}{3}\sum_{i=1}^{3}\alpha_i.$$

Adjusted Correlation Matrix

Using the average adjustment factor (α), we adjusted the original correlations upward to reflect the potential impact of correlation uncertainty:

$$\hat{\rho}_{ij} = \rho_{ij} + \alpha(1-\rho_{ij}),$$

where $\hat{\rho}_{ij}$ denotes the adjusted correlations.

Ticker	**AAPL**	**AMZN**	**META**
AAPL	1.000000	0.619072	0.591930
AMZN	0.619072	1.000000	0.613559
META	0.591930	0.613559	1.000000

Table 5: Correlation matrix (ρ) of stocks AAPL, AMZN, and META.

Ticker	**AAPL**	**AMZN**	**META**
AAPL	1.000000	0.636373	0.610464
AMZN	0.636373	1.000000	0.631111
META	0.610464	0.631111	1.000000

Table 6: Modified correlation matrix ($\hat{\rho}$) of stocks AAPL, AMZN, and META.

Revaluation of the Basket Accumulator

With the adjusted correlations ($\hat{\rho}_{ij}$), we re-evaluated the basket accumulator's price ($\hat{V}$) using the simulation framework. The difference between the revalued price ($\hat{V}$) and the original price (V_0) was calculated as:

$$\Delta V = \hat{V} - V_0 = -91.37$$

Interpretation of Results

The value ΔV represents the model valuation adjustment due to correlation uncertainty, effectively quantifying the correlation risk associated with the basket accumulator. This analysis highlights the sensitivity of the product's price to potential deviations in correlation estimates, providing insights into the robustness of the pricing model.

This correlation adjustment methodology ensures that the model accounts for the impact of statistical uncertainties in correlation estimates, allowing for a more accurate assessment of risk and valuation adjustments for the basket accumulator.

6 Discussion

In this project, we employed a constant volatility assumption for both single-stock and basket pricing models. While this simplification allowed us to streamline computations, it may not fully capture the complexities of market dynamics. Incorporating a stochastic volatility model or a local volatility model could improve the accuracy of pricing by better reflecting the variability in volatility observed in real markets. These models are particularly beneficial for path-dependent derivatives, as they account for changes in volatility over time and across different levels of the underlying asset's price.

The maturity of the accumulator was set to one year in our analysis. While this time frame provides sufficient data to evaluate the product's performance, shorter maturities may be more suitable for risk-averse investors. A shorter maturity would likely reduce exposure to adverse market conditions and result in smaller value-at-risk (VaR) estimates. This adjustment could make the product more appealing to investors with lower risk tolerance, as it would limit the potential downside while preserving its key features.

Given the highly path-dependent nature of the accumulator, Monte Carlo simulations were used to estimate the product's value. However, due to limited computational resources, the number of simulations was not set to a very high level, which may have impacted the convergence of the results. A higher number of simulations could lead to improved convergence between the analytical and simulated results, reducing estimation errors. This highlights the importance of computational resources in achieving reliable pricing for complex derivative structures like the basket accumulator.

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A Appendix

A.1 Single Asset Accumulator Term Sheet

Product Summary

The Leveraged Stock Accumulator is a path-dependent structured financial product designed to allow investors to acquire shares of an underlying stock or a weighted basket of stocks at a discounted price. This product is suitable for investors with a bullish outlook on the underlying asset(s) and a high-risk tolerance.

Key Features

1. **Purchase Obligation:**
    - The investor is obligated to purchase a fixed number of shares daily (Q) at a pre-determined strike price (K), set at a discount to the initial spot price.
    - If the stock price falls below K, the investor must purchase $2Q$ shares daily.
2. **Leverage Trigger:**
    - If the stock price drops below the strike price, the investor is obligated to purchase 2 shares per day at the strike price until the stock price recovers above the strike price.
3. **Early Termination (Knock-Out Feature):**
    - If the stock price reaches or exceeds the *upper barrier price*, the contract terminates immediately. On the termination day, all profits or losses are realized and settled, and no further share purchases are required.
4. **No Termination at Strike:**
    - If the stock price remains below the strike price for the contract's remaining term, the investor continues to purchase 2 shares per day at the strike price until maturity.
5. **Settlement:**
    - Upon contract termination or at maturity, the investor settles their position based on the aggregate difference between the market price and the strike price for all shares purchased.

Key Parameters

Parameter	Description
Strike Price	Discounted price based on the first day's closing price (e.g., 86.23% of the closing price).
Upper Barrier Price	Pre-determined price (e.g., 120% of the closing price) above which the contract terminates early and profits/losses are realized.
Leverage Factor	If the stock price drops below the strike price, the investor purchases 2 shares instead of 1.
Daily Share Purchase	Fixed number of shares purchased daily, typically 1 share (or 2 shares if the leverage trigger is activated).
Maturity	Length of the contract, typically measured in trading days (e.g., 1 year, 252 trading days).

Illustrative Example

Initial Setup

- **Underlying Stock:** AMZN
- **Starting Date:** Dec 19,2024
- **Strike Price:** \$190.15 (86.23% of the first day's closing price, \$220.52)
- **Upper Barrier Price:** \$264.62 (120% of the first day's closing price)
- **Daily Share Purchase:** 1 share
- **Maturity:** 1 year (252 trading days)
- **Initial Margin:** \$28750.68 (30% of the maximum buying obligation)

Scenario 1

- On Day 50, the stock price reaches \$270 (above the upper barrier).
- The contract terminates, and no further buying obligation is needed.

Scenario 2

- On Day 40, the stock price drops to \$180 (below the strike price).
- From Day 40 onwards, the investor purchases 2 shares per day at \$190.15 until the stock price recovers above \$190.15.
- If the stock price remains below \$190.15 until maturity, the investor continues purchasing 2 shares daily for the rest of the contract period.

Scenario 3

- On Day 60, the stock price is \$200 (above the strike price and below the barrier).
- The investor purchases 1 shares per day at \$190.15 until the stock price either goes above the barrier or drops below the strike.

Risk and Rewards

Upsides

- Opportunity to accumulate shares at a discounted price, providing cost savings.
- Potential for significant gains if the stock price rises before the contract terminates.

Downsides

- Leverage amplifies exposure to downside risk if the stock price drops below the strike price.
- No flexibility to opt-out of purchases until the contract terminates.

Important Notes

Market Conditions

- Performance is subject to market volatility and price movement of the underlying asset.

Obligations

- Investors must meet daily purchase obligations throughout the contract period unless terminated early.

Liquidity

- The product is illiquid; investors cannot exit the contract early except through pre-defined termination conditions.

A.2 Basket Accumulator Term Sheet

Product Summary

The Leveraged Basket Accumulator is a path-dependent structured financial product designed to allow investors to acquire units of a weighted basket of underlying stocks at a discounted price. This product is suitable for investors with a bullish outlook on the basket of assets and a high-risk tolerance.

Key Features

1. **Purchase Obligation:**
   - The investor is obligated to purchase a fixed number of basket units daily (Q) at a pre-determined strike price (K), set at a discount to the initial basket price.
   - If the basket price falls below K, the investor must purchase $2Q$ units daily.
2. **Leverage Trigger:**
   - If the basket price drops below the strike price, the investor is obligated to purchase 2 units per day at the strike price until the basket price recovers above the strike price.
3. **Early Termination (Knock-Out Feature):**
   - If the basket price reaches or exceeds the *upper barrier price*, the contract terminates immediately. On the termination day, all profits or losses are realized and settled, and no further basket purchases are required.
4. **No Termination at Strike:**
   - If the basket price remains below the strike price for the contract's remaining term, the investor continues to purchase 2 basket units daily at the strike price until maturity.
5. **Settlement:**
   - Upon contract termination or at maturity, the investor settles their position based on the aggregate difference between the basket market price and the strike price for all units purchased.

Key Parameters

Parameter	Description
Basket Price	Weighted sum of the initial prices of the underlying stocks, computed as Basket Price $= \sum_{i=1}^{n} w_i \cdot S_i$.
Strike Price (K)	90.18% of the initial basket price.
Upper Barrier Price (B)	120% of the initial basket price.
Leverage Factor	If the basket price drops below the strike price, the investor purchases 2 units instead of 1.
Daily Basket Purchase	Fixed number of basket units purchased daily, typically 1 unit (or 2 units if the leverage trigger is activated).
Maturity	Length of the contract, typically measured in trading days (e.g., 1 year, 252 trading days).

Illustrative Example

Initial Setup

- **Underlying Stocks:** AMZN, AAPL, META
- **Initial Stock Prices:** AMZN: \$220.52, AAPL: \$248.05, META: \$597.19
- **Weights:**

$$w_{\text{AMZN}} = \frac{1000}{220.52}, \quad w_{\text{AAPL}} = \frac{1000}{248.05}, \quad w_{\text{META}} = \frac{1000}{597.19}$$

- **Basket Price:**

$$\text{Basket Price} = w_{\text{AMZN}} \cdot 220.52 + w_{\text{AAPL}} \cdot 248.05 + w_{\text{META}} \cdot 597.19 \approx \$3000.00$$

- **Strike Price:** \$2705.40 (90.18% of \$3000.00)
- **Upper Barrier Price:** \$3600.00 (120% of \$3000.00)
- **Daily Basket Purchase:** 1 unit
- **Maturity:** 1 year (252 trading days)

Scenario 1

- On Day 50, the basket price reaches \$3700 (above the upper barrier).
- The contract terminates, and no further buying obligation is needed.

Scenario 2

- On Day 40, the basket price drops to \$2500 (below the strike price).
- From Day 40 onwards, the investor purchases 2 basket units per day at \$2705.40 until the basket price recovers above \$2705.40.
- If the basket price remains below \$2705.40 until maturity, the investor continues purchasing 2 basket units daily for the rest of the contract period.

Scenario 3

- On Day 60, the basket price is \$3100 (above the strike price and below the barrier).
- The investor purchases 1 basket unit per day at \$2705.40 until the basket price either goes above the barrier or drops below the strike.

Risk and Rewards

Upsides

- Opportunity to accumulate a diversified basket of assets at a discounted price, providing cost savings.
- Potential for significant gains if the basket price rises before the contract terminates.

Downsides

- Leverage amplifies exposure to downside risk if the basket price drops below the strike price.
- No flexibility to opt-out of purchases until the contract terminates.

Important Notes

Market Conditions

- Performance is subject to market volatility and price movements of the underlying assets in the basket.

Obligations

- Investors must meet daily purchase obligations throughout the contract period unless terminated early.

Liquidity

- The product is illiquid; investors cannot exit the contract early except through pre-defined termination conditions.