Aluna: Lara

Álgebra Linear Lista 3

① Decomposição LU da matriz $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

1ª linha de U

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix}}^{L} \cdot \overbrace{\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}}^{U}$$

$a_{11} = 1 = 1 \cdot U_{11} + 0 \cdot 0 + 0 \cdot 0 = U_{11} = 1$

$a_{12} = 3 = 1 \cdot U_{12} + 0 \cdot U_{22} + 0 \cdot 0 = U_{12} = 3$

$a_{13} = 0 = 1 \cdot U_{13} + 0 \cdot U_{23} + 0 \cdot U_{33} = U_{13} = 0$

1ª coluna de L

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$a_{21} = 2 = (L_{21}) \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = L_{21} = 2$

$a_{31} = 2 = (L_{31}) \, 1 + (L_{32}) \cdot 0 + 1 \cdot 0 = L_{31} = 2$

2ª linha de U

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & L_{32} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$a_{22} = 4 = 2 \cdot 3 + 1 \cdot U_{22} + 0 \cdot 0 \Rightarrow U_{22} = 4 - 6 = -2$

$a_{23} = 0 = 2 \cdot 0 + 1 \cdot U_{23} + 0 \cdot U_{33} \Rightarrow U_{23} = 0$

2° coluna de L

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & L_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$a_{32} = 0 = 2 \cdot 3 + L_{32}(-2) + 1 \cdot 0 \Rightarrow L_{32} = 6/2 = 3$$

3° linha de U

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$a_{33} = 1 = 2 \cdot 0 + 3 \cdot 0 + 1\, U_{33} \Rightarrow U_{33} = 1$$

$$\underbrace{\begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

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② Decomposição da matriz simétrica

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$\begin{matrix} L_2 - L_1 \\ L_3 - L_1 \\ L_4 - L_1 \end{matrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \quad \begin{matrix} L_3 - L_2 \\ L_4 - L_2 \\ \longrightarrow \end{matrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$\begin{matrix} L_4 - L_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U$$

$$E_{21}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{31}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{41}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{42}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad E_{43}(1) \hookrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$A = \underbrace{E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{41}^{-1} \cdot E_{32}^{-1} \cdot E_{42}^{-1} \cdot E_{43}^{-1}}_{L} \cdot U$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{41}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{41}^{-1} \cdot E_{32}^{-1} \cdot E_{42}^{-1} \cdot E_{43}^{-1}$$

$$\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-b \end{pmatrix}$$

$$A = L \cdot U$$

Condição para ter 4 pivôs:

$b \neq a \quad a \neq 0$

$c \neq b \qquad a, b, c, d \neq 0$

$d \neq b$

(3)

a) $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3}$ $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

permutação do set $\{1,2,3\}$: $1 \leftrightarrow 2$ e $2 \leftrightarrow 3$ da matriz I

b) $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $S^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $S^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

permutação do set $\{1,2,3,4\}$:
$2 \leftrightarrow 3$
$2 \leftrightarrow 4$

$S^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$S^4 = S$, pois $S \neq I$
$S^4 = S^3 \cdot S$ e $S^3 = I$
então $S^4 \neq I$

(4)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

a) Se $A = A^T$, temos que escolher as 4 entradas da diagonal e também as 6 entradas inferiores à diagonal principal, pois aí saberemos todas as 6 entradas acima da diagonal principal. Então, $6 + 4 = 10$ entradas.

b) Se $A^T = -A$, as entradas diagonais são zeros, então, basta escolher 6 entradas inferiores ou superiores à entrada diagonal. Então, são 6 entradas.

(5) $A = L \cdot U$

Provar que

$\underline{A = L}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \xrightarrow[L_3 - bL_1]{L_2 - aL_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$\xrightarrow{L_3 - cL_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = E_{32}(c) \cdot E_{31}(b) \cdot E_{21}(a)$$
$$L = E_{32}^{-1} \cdot E_{31}^{-1} \cdot E_{21}^{-1}$$

$$E_{32}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \qquad E_{32}^{-1}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$E_{31}(b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \qquad E_{31}^{-1}(b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

$$E_{21}(a) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}^{-1}(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} = L$$

Se $A = LU$ e $A = L$ então $U = I$

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(6)

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

a) $\begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \xrightarrow[L_3 - 3L_1]{L_2 - 2L_1} \begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 5-3c & 1 \end{bmatrix} \xrightarrow{4-2c \neq 0}$

$\xrightarrow{L_3 - \left(\frac{5-3c}{4-2c}\right) L_2} \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 - \frac{5-3c}{4-2c} \end{bmatrix} = U$

a) Se $c = 2$, o segundo pivô é igual a zero
Para evitar isso $4 - 2c \neq 0$ e $c \neq 2$

$$L = E_{21}(2)^{-1} \cdot E_{31}(3)^{-1} \cdot E_{32}\left(\frac{5-3c}{4-2c}\right)^{-1}$$

$$E_{21}(2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}(2)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad E_{31}(3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E_{32}\left(\frac{5-3c}{4-2c}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-5+3c}{4-2c} & 1 \end{bmatrix} \qquad E_{32}\left(\frac{5-3c}{4-2c}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5-3c}{4-2c} & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5-3c}{4-2c} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{5-3c}{4-2c} & 1 \end{bmatrix}$$

$$\begin{array}{ccccc} L & & U & & \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{5-3c}{4-2c} & 1 \end{bmatrix} & \cdot & \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\left(\frac{5-3c}{4-2c}\right) \end{bmatrix} & = & \begin{bmatrix} 1 & c & 0 \\ 2 & 2c+1 & 1 \\ 3 & \frac{6c^2-9c-5}{2c-4} & 1 \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

$U = L\,A$ apenas é valido se $\boxed{C = 1.5}$

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b) $1 - \left(\frac{5-3c}{4-2c}\right) = 0$

$4 - 2c - (5 - 3c) = 0$

$-5c - 1 = 0$

$c = -1/5$

Se $c = \frac{-1}{5}$ o terceiro pivô é zero. Para evitar isso $c \neq \frac{1}{5}$

D S T Q Q S S

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$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad B = \begin{bmatrix} d & f \\ f & g \end{bmatrix}$$

$A^2 - B^2$ e ABA são simétricas.

$$A^2 = \begin{bmatrix} a^2 + c^2 & ac + bc \\ ac + bc & b^2 + c^2 \end{bmatrix} \qquad B^2 = \begin{bmatrix} d^2 + f^2 & df + fg \\ df + fg & f^2 + g^2 \end{bmatrix}$$

$$A^2 - B^2 = \begin{bmatrix} a^2 + c^2 - d^2 - f^2 & ac + bc - df - fg \\ ac + bc - df - fg & b^2 + c^2 - f^2 - g^2 \end{bmatrix}$$ É simétrica

$$A + B = \begin{bmatrix} a+d & c+f \\ c+f & b+g \end{bmatrix} \qquad A - B = \begin{bmatrix} a-d & c-f \\ c-f & b-g \end{bmatrix}$$

$$(A+B)(A-B) = \begin{bmatrix} a^2 + c^2 - d^2 - f^2 & ac + bc + cd - af + bf - df - cg - fg \\ ac + bc - cd + af - bf - df + cg - fg & b^2 + c^2 - f^2 - g^2 \end{bmatrix}$$

Não é simétrica

$$BA = \begin{bmatrix} ad + cf & cd + bf \\ af + cg & cf + bg \end{bmatrix} \qquad ABA = \begin{bmatrix} a^2d + 2acf + c^2g & acd + abf + c^2f + bcg \\ acd + abf + c^2f + bcg & c^2d + 2bcf + b^2g \end{bmatrix}$$

→ É simétrica

$$BAB = \begin{bmatrix} ad^2 + bf^2 + 2cdf & cf^2 + adf + cdg + bfg \\ cf^2 + adf + cdg + bfg & af^2 + bg^2 + 2cfg \end{bmatrix}$$

$$ABAB = \begin{bmatrix} \ldots & af^3 + cdf^2 \ldots \\ bf^3 + 2cdf^2 \ldots & \ldots \end{bmatrix}$$

→ Não é simétrica

kajoma

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Supondo que $B = \frac{1}{2}\left(A + A^T\right)$

É simétrica pois:

$$B = B^T \Rightarrow \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left(A^T + A^{TT}\right) = \frac{1}{2}\left(A + A^T\right)$$

Supondo que $C = \frac{1}{2}\left(A - A^T\right)$

É antissimétrica pois:

$$-C = C^T \Rightarrow \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right)$$

$B + C = A$

$$\frac{1}{2}\left(A + \cancel{A^T}\right) + \frac{1}{2}\left(A - \cancel{A^T}\right) = A \;//$$

(9)

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \xrightarrow{L_2 - C\,A^{-1}L_1} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = U$$

$$L = E_{21}^{-1}(CA^{-1})$$

$$L = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} AI & BI \\ C & DI \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{array}{ccccc} A & = & L & & U \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & = & \begin{bmatrix} I & 0 \\ L_{21} A_{11}^{-1} & I \end{bmatrix} & \cdot & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21} \cdot A_{11}^{-1} \cdot A_{12} \end{bmatrix} \end{array}$$