CMT-3

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1. Mostre que a verossimilhança $f_{\tilde{\boldsymbol{X}}}(\boldsymbol{y} \mid \theta)$ pode ser escrita na forma

$$f_{\tilde{\boldsymbol{X}}}(\boldsymbol{y} \mid \theta) = g_{\tilde{\boldsymbol{X}}}(\boldsymbol{y}|\boldsymbol{\beta}, \sigma^2) h_{\tilde{\boldsymbol{X}}}(\boldsymbol{y}|\sigma^2).$$

$$f(y_i \mid \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\tilde{x}_i\beta - y_i)^2\right)$$

$$f_x(Y \mid \beta, \sigma^2) = \prod^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\tilde{x}_i\beta - y_i)^2\right)$$

$$L(\beta, \sigma^2 \mid Y) = \frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(\tilde{x}_i\beta - y_i)^2\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}\left((\tilde{x}_i\beta)^2 - 2x_i\beta y_i + y_i^2\right)\right)$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left(-\frac{1}{2\sigma^2}\left(\sum y_i^2\right)\right)}_{h_{\tilde{x}}(y|\sigma^2)} \cdot \underbrace{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}\left((x_i\beta)^2 - 2x_i\beta y_i\right)\right)}_{g_{\tilde{x}}(y|\beta,\sigma^2)}$$

2. Utilize o resultado anterior para deduzir que a priori conjugada para este caso é da forma

$$\pi_{B,S}(\boldsymbol{\beta}, \sigma^2) = \pi_{B|S}(\boldsymbol{\beta} \mid \sigma^2)\, \pi_S(\sigma^2).$$

Em particular, mostre que

$$\pi_{B,S}(\boldsymbol{\beta}, \sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{a+(P+1)/2+1} \times \exp\left(-\frac{1}{\sigma^2}\left\{b + \frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)^T \boldsymbol{V}_\beta^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)\right\}\right),$$

onde $\boldsymbol{\mu}_\beta) \in \mathbb{R}^{P+1}$, $\boldsymbol{V}_\beta$ é uma matriz positiva definida e $a, b \in \mathbb{R}_+$.

Dica: Que escolhas para $\pi_{B|S}$ e π_S eu preciso fazer?

priori conjugada

pela 1.,

$$\pi(\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^a \exp\left(\left(\frac{1}{\sigma^2}\right) b\right) \rightarrow \text{gamma-inversa }(a, b)$$

$$\pi(\beta \mid \sigma^2) \propto \exp\left(-\frac{1}{2\sigma^2}(\text{forma quadrática de } \beta)\right) \rightarrow \text{normal multivariada }(\mu, \sigma^2 V_\beta)$$

$$\pi(\sigma^2) = \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1} \exp\left(-\frac{b}{\sigma^2}\right)$$

$$\pi(\beta \mid \sigma^2) = \frac{1}{(2\pi)^{\frac{P+1}{2}} \det(\sigma^2 V_\beta)^{-1/2}} \exp\left(-\frac{1}{2}(\beta - \mu)^T(\sigma^2 V_\beta)^{-1}(\beta - \mu)\right)$$

$$= (2\pi)^{\frac{P+1}{2}} (\sigma^2)^{-\frac{P+1}{2}} \det(V_\beta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\beta - \mu)^T(V_\beta)^{-1}(\beta - \mu)\right)$$

3. A priori anterior chama-se **normal inversa gama** (NIG) e tem quatro parâmetros: $\boldsymbol{m}$, $\boldsymbol{V}$, a e b. Mostre que a posteriori de θ também é NIG e exiba seus hiperparâmetros.

$$p(\beta, \sigma^2 \mid y) = \frac{p(y \mid \beta, \sigma^2)\, L(\beta, \sigma^2)}{p(y)} \quad p(y) \rightarrow \text{marginal}$$

$$p(y) = \int p(\beta, \sigma^2)\, p(y \mid \beta, \sigma^2)\, d\beta\, d\sigma^2$$

$$p(\beta, \sigma^2 \mid y) \propto p(y \mid \beta, \sigma^2)\, p(\beta, \sigma^2)$$

$$\left(\frac{1}{\sigma^2}\right)^{n/2}$$

$$L(\beta,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(x_i\beta - y)^T(x_i\beta - y)\right) \cdot \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1}\exp\left(-\frac{b}{\sigma^2}\right)$$

$$p(\beta,\sigma^2 \mid y) \propto (2\pi)^{\frac{p+1}{2}} (\sigma^2)^{-\left(\frac{p+1}{2}\right)} \det(V_\beta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\beta-\mu)^T(V_\beta)^{-1}(\beta-\mu)\right)\frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1}\exp\left(-\frac{b}{\sigma^2}\right)$$

expoente:

$$\left\{-\frac{1}{2\sigma^2}(\beta-\mu)^T(V_\beta)^{-1}(\beta-\mu) - \frac{b}{\sigma^2} - \frac{1}{2\sigma^2}\left((x_i\beta-y)^T(x_i-y)\right)\right\}$$

$$= \left\{-\frac{1}{\sigma^2}\left(b + \frac{1}{2}\left(\underbrace{(\beta-\mu)^T(V_\beta)^{-1}(\beta-\mu_\beta) + (x_i\beta-y)^T(x_i\beta-y)}_{\text{forma quadrática de } \beta}\right)\right)\right\}$$

$$(\beta-\mu)^T(V_\beta)^{-1}(\beta-\mu_\beta) = \beta^T V_\beta^{-1}\beta - \beta^T V_\beta^{-1}\mu_\beta - \mu_\beta^T V_\beta^{-1}\beta + \mu_\beta^T \cancel{V_\beta^{-1}}\mu_\beta$$

$$(x_i\beta-y)^T(x_i\beta-y) = \beta^T X^T X\beta - \beta^T X^T y - y^T X\beta + \cancel{y^T y}$$

não dependem de β

somando:

$$\beta^T V_\beta^{-1}\beta - 2\beta^T V_\beta^{-1}\mu_\beta - 2y^T X\beta + \beta^T X^T X\beta$$

$$= \beta^T(X^TX + V_\beta^{-1})\beta - 2\beta^T(V_\beta^{-1}\mu_\beta + X^T y)$$

Temos a dica:

- **Completando o "quadrado" em múltiplas dimensões**: tome $\boldsymbol{A}$ matriz simétrica positiva definida $d \times d$ e $\boldsymbol{\alpha}, \boldsymbol{u} \in \mathbb{R}^d$. Vale que:

$$\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} - 2\boldsymbol{\alpha}^T\boldsymbol{u} = (\boldsymbol{u} - \boldsymbol{A}^{-1}\boldsymbol{\alpha})^T\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{A}^{-1}\boldsymbol{\alpha}) - \boldsymbol{\alpha}^T\boldsymbol{A}\boldsymbol{\alpha}. \quad (1)$$

  Dica: Expanda o produto e procure por cancelamentos de termos da forma $\boldsymbol{a}^T\boldsymbol{M}^{-1}\boldsymbol{a}$.

$$\alpha^T\mu = \mu^T\alpha$$

$$\underbrace{\begin{array}{cccc} \mu_\beta & V_B, & a, & b \\ \downarrow & \downarrow & \downarrow & \downarrow \\ m, & V & a^* & b^* \end{array}}_{\text{hiperparâmetros}}$$

então: $A = (X^TX + V_\beta^{-1})$ $\quad \mu = \beta \quad \alpha = V_\beta^{-1}\mu_\beta + X^T y$

$$\beta^T(X^TX + V_\beta^{-1})\beta - 2\beta^T(V_\beta^{-1}\mu_\beta + X^T y) = \mu^T A\mu - 2\alpha^T\mu$$

$$= (\mu - A^{-1}\alpha)^T A(\mu - A^{-1}\alpha) - \alpha^T A^{-1}\alpha$$

$$(\beta - m)^T A(\beta - m) -$$

$$\downarrow$$

$$V = A^{-1}$$
$$V^{-1} = A$$

$$\alpha^T A^{-1}\alpha = \alpha^T A^{-1} A A^{-1}\alpha = (A^{-T}\alpha)^T A (A^{-1}\alpha)$$

$$= (A^{-1}\alpha)^T A (A^{-1}\alpha)$$

- A simétrico: $A^T = A$

$$= m^T V^{-1} m$$

$$m = A^{-1}\alpha = (X^TX + V_\beta^{-1})^{-1}(V_\beta^{-1}\mu_\beta + X^T y)$$

$$V = (X^TX + V_\beta^{-1})^{-1} = A^{-1}$$

voltando pro expoente:

$$\left\{-\frac{1}{\sigma^2}\left(b + \frac{1}{2}\left((\beta - m)^T V^{-1}(\beta - m)\right)\right)\right\}$$

$$b^* = b + \frac{1}{2}\left(\mu_\beta^T V_\beta^{-1}\mu_\beta + y^T y - m^T V^{-1} m\right)$$

outros termos:

$$\left(\frac{1}{\sigma^2}\right)^{a+1} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{m}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{p+1}{2}} \;.= \left(\frac{1}{\sigma^2}\right)^{\underbrace{a+\frac{m}{2}}_{a^*}+\frac{p+1}{2}+1}$$

$$a^* = a + \frac{m}{2}$$

4. **Distribuições marginais**

Dica: Antes de começar os cálculos para essa seção, vale considerar a seguinte representação do nosso modelo:

$$\boldsymbol{y} = \tilde{\boldsymbol{X}}^T\boldsymbol{\beta} + \boldsymbol{\epsilon}_1, \text{ com } \boldsymbol{\epsilon}_1 \sim \mathrm{MVN}_n(\mathbf{0}_n, \boldsymbol{\Sigma}_1),$$
$$\boldsymbol{\beta} = \boldsymbol{\mu}_\beta + \boldsymbol{\epsilon}_2, \text{ com } \boldsymbol{\epsilon}_2 \sim \mathrm{MVN}_{P+1}(\mathbf{0}_{P+1}, \boldsymbol{\Sigma}_2),$$

onde $\boldsymbol{\epsilon}_1$ e $\boldsymbol{\epsilon}_2$ são erros independentes.

(a) Determine $\boldsymbol{\Sigma}_1$ e $\boldsymbol{\Sigma}_2$;

(b) Compute a verossimilhança marginal com respeito a σ^2:

$$\tilde{f}_{\tilde{\boldsymbol{X}}}(\boldsymbol{y} \mid \sigma^2) := \int_{\mathbb{R}^{P+1}} f_{\tilde{\boldsymbol{X}}}(\boldsymbol{y} \mid \boldsymbol{b}, \sigma^2)\pi_{B|S}(\boldsymbol{b} \mid \sigma^2)\, d\boldsymbol{b}.$$

(c) Usando o item anterior, compute a verossimilhança marginal ou *preditiva a priori*:

$$m_{\tilde{\boldsymbol{X}}}(\boldsymbol{y}) := \int_0^\infty \tilde{f}_{\tilde{\boldsymbol{X}}}(\boldsymbol{y} \mid s)\pi_S(s)\, ds.$$

(d) Mostre $\bar{f}_{\tilde{\boldsymbol{X}}}(\boldsymbol{\beta} \mid \boldsymbol{y})$ e comente sobre como calcular, por exemplo, $\Pr(\beta_1 > a \mid \boldsymbol{y})$, para $a \in \mathbb{R}$.

a) $\Sigma_1 = \sigma^2 I$

$\Sigma_2 = \sigma^2 V_\beta$

b) verossimilhança marginal com respeito a σ^2

$$f^{\tilde{x}}(y, \sigma^2) = \int f^{\tilde{x}}(y \mid b, \sigma^2)\ \pi_{\beta|S}(b \mid \sigma^2)\, db$$

a dist de β será a própria priori

$y = x^T\beta + \epsilon_1$

$\beta = \mu_\beta + \epsilon_2$

$$y = \tilde{x}(\mu_\beta + \epsilon_2) + \epsilon_1 = \tilde{x}\mu_\beta + \tilde{x}\epsilon_2 + \epsilon_1$$

$\epsilon_2 \sim MVN(0_{p+1}, \sigma^2 V_\beta)$

$X\epsilon_2 \sim MVN(0_N, \sigma^2 V_\beta X^T)$

$X^T\beta + \epsilon_1 \sim MVN(0_N, \sigma^2 I)$

$$y \sim MVN(X\mu_\beta, \sigma^2(\underbrace{I + XV_\beta X^T}_{W}))$$

A marginal será a densidade.

c) Preditiva a priori

$m_{\tilde{x}}(y) = \int^\infty f \quad (y \mid s)\ \pi(s)\, ds$

c) Preditiva a priori

$$m_{\tilde{x}}(y) = \int_0^\infty f_{\bar{x}}(y|s)\,\pi_s(s)\,ds$$

$$f_{\bar{x}}(y|\sigma^2)\,\pi_s(s) = \frac{1}{(2\pi)^{m/2}(\det W)^{1/2}}\left(\frac{1}{\sigma^2}\right)^{\frac{m}{2}} \cdot \exp\left(\frac{-1}{2\sigma^2}\underbrace{\left(\tilde{x}\mu_\beta - y\right)^T W^{-1}\left(\tilde{x}\mu_\beta - y\right)}_{Q}\right) \cdot \underbrace{\frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1}\exp\left(-\frac{b}{\sigma^2}\right)}_{\pi(s)}$$

$$W = I + \tilde{x}V_\beta\tilde{x}^T$$

Tomando $Q = (X\mu_\beta - y)^T W^{-1}(X\mu_\beta - y)$

$$f_{\bar{x}}(y|\sigma^2)\,\pi_s(s) = \frac{b^a}{(2\pi)^{m/2}(\det W)^{1/2}\,\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{\frac{m}{2}+a+1}\exp\left(-\frac{1}{\sigma^2}\left(b + \frac{Q}{2}\right)\right)$$

$$\alpha = \frac{m}{2} + a \qquad \lambda = b + \frac{Q}{2}$$

$$f_{\bar{x}}(y|\sigma^2)\,\pi_s(s) = \frac{b^a}{(2\pi)^{m/2}(\det W)^{1/2}\,\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{\alpha+1}\exp\left(-\frac{\lambda}{\sigma^2}\right)$$

agora falta integrar com respeito a σ^2

$$m_{\tilde{x}}(y) = \int_0^\infty f_{\bar{x}}(y|s)\,\pi_s(s)\,ds$$

$$= \frac{b^a}{(2\pi)^{m/2}(\det W)^{1/2}\,\Gamma(a)}\int_0^\infty\left(\frac{1}{\sigma^2}\right)^{\alpha+1}\exp\left(-\frac{\lambda}{\sigma^2}\right)d\sigma^2$$

$$= \frac{b^a}{(2\pi)^{m/2}(\det W)^{1/2}\,\Gamma(a)}\cdot\frac{\Gamma(\alpha)}{\lambda^\alpha}\underbrace{\int_0^\infty\frac{\lambda^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\sigma^2}\right)^{\alpha+1}\exp\left(-\frac{\lambda}{\sigma^2}\right)d\sigma^2}_{1}$$

$$= \frac{b^a}{(2\pi)^{m/2}(\det W)^{1/2}\,\Gamma(a)}\cdot\frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$= \frac{b^a}{\sqrt{(2\pi)^m(\det W)}\;\Gamma(a)}\cdot\Gamma\left(\frac{m}{2}+a\right)\left[\frac{1}{b + (\beta - m)^T W^{-1}(\beta - m)}\right]^{a+\frac{m}{2}}$$

$$= \frac{b^a}{\sqrt{(2\pi)^m(\det W)}\;\Gamma(a)}\cdot\Gamma\left(\frac{m}{2}+a\right)\left[b + (\beta - m)^T W^{-1}(\beta - m)\right]^{-a-\frac{m}{2}}$$

d) $f_{\bar{x}}(\beta|y) = \int p(\beta,\sigma^2|y)\,d\sigma^2$

$$\propto \int\left(\frac{1}{\sigma^2}\right)^{a^*+1}\exp\left(-\frac{1}{\sigma^2}\left(b^* + \frac{1}{2}(\beta - m)\,V^{-1}(\beta - m)\right)\right)d\sigma^2$$

à posteriori

$$\beta \sim t_{2a^*}\left(m, \left(\frac{b^*}{a^*}V\right)\right)$$

Para avaliar $Pr(\beta_1 > a|y)$ basta integrar

$$Pr(\beta_1 > a|y) = \int_\Omega f_{\bar{x}}(\beta|y)\,d\beta$$

$$\Omega = \mathbb{R}\times(a, +\infty)\times\mathbb{R}\times\ldots\times\mathbb{R}$$

Bônus:

NIG: Normal Inverse Gamma

$$\pi(\beta, \sigma^2) = \pi(\beta|\sigma^2)\,\pi(\sigma^2) = N(\mu_\beta, \sigma^2 \overset{\curvearrowright cI}{V_\beta}) \times IG(a,b) = NIG(\mu_\beta, V_\beta, a, b)$$

$$= \frac{1}{\sqrt{(2\pi)^m |V_\beta|}} \exp\left(-\frac{1}{2}(X-\mu)^T V_\beta^{-1}(X-\mu)\right) \cdot \frac{b^a}{\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+1} \exp\left(-\frac{b}{\sigma^2}\right)$$

$$= \frac{b^a}{(2\pi)^{m/2}|V_\beta|^{1/2}\,\Gamma(a)}\left(\frac{1}{\sigma^2}\right)^{a+\frac{p}{2}+1} \cdot \exp\left(-\frac{1}{\sigma^2}\left\{b+\frac{1}{2}(\beta-\mu_\beta)^T V_\beta^{-1}(\beta-\mu_\beta)\right\}\right)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{a+\frac{p}{2}+1} \cdot \exp\left(-\frac{1}{\sigma^2}\left\{b+\frac{1}{2}(\beta-\mu_\beta)^T V_\beta^{-1}(\beta-\mu_\beta)\right\}\right)$$

Verossimilhança

$$p(y|\beta,\sigma^2) = N(X\beta, \sigma^2 I) = \left(\frac{1}{2\pi\sigma^2}\right)^{m/2} \cdot \exp\left(-\frac{1}{2\sigma^2}(y-X\beta)^T(y-X\beta)\right)$$

Posteriori

$$p(\beta,\sigma^2|y) = \frac{p(y|\beta,\sigma^2)\,\overset{\to \pi(\beta,\sigma^2)}{p(\beta,\sigma^2)}}{p(y) \to \text{marginal}}$$

$$p(y) = \int p(\beta,\sigma^2)\,p(y|\beta,\sigma^2)\,d\beta\, d\sigma^2$$

$$p(\beta,\sigma^2|y) \propto p(y|\beta,\sigma^2)\,p(\beta,\sigma^2)$$

$$\propto \left[\left(\frac{1}{\sigma^2}\right)^{\frac{m}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}\left[\sum(y-\bar{x}\beta)^2 + m(\bar{x}\beta-\beta)^2\right]\right)\right] \cdot \left[\left(\frac{1}{\sigma^2}\right)^{a+\frac{p}{2}+1} \cdot \exp\left(-\frac{1}{2\sigma^2}\left[V_0^{-1}(\beta-\beta_0)^2 + 2b_0\right]\right)\right]$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\underbrace{a+\frac{m}{2}+\frac{p}{2}+1}_{a^*}} \cdot \exp\left(-\frac{1}{\sigma^2}\left[b^* + \frac{1}{2}(\beta-\mu^*)^T V^{*-1}(\beta-\mu^*)\right]\right)$$