Adjacency matrix and Incidence matrix

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1 Adjacency matrix

It is very glad share two types of matrixs in Linear Algebra and numerical analysis, which is the Adjacency and Laplacian matrix.

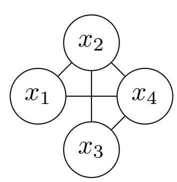
2 definition

In graph theory and computer science, an adjacency matrix is a **square matrix** used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.

3 lemma* and application

In the special case of a finite simple graph, the adjacency matrix is a (0,1)-matrix with zeros on its diagonal. If the graph is undirected (i.e. all of its edges are bidirectional), the adjacency matrix is symmetric. The relationship between a graph and the eigenvalues and eigenvectors of its adjacency matrix is studied in **spectral graph theory**.

4 Representing Graphs using Matrix

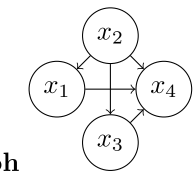


Adjacency Matrix of undirected graph

$$\text{A=}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

4.1 remarks

rows and columns represent vertices
rowsum = degree of the vertex that the row represents



Adjacency Matrix of directed graph

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5 Applications of graph theory

An example of a graph is the route map that most airlines (or railways) produce. A copy of the northern route map for Cape Air from May 2001 is Figure 2. This map is available at the web page:*www.flycapeair.com*. Here the vertices are the cities to which Cape Air flies, and two vertices are connected if a direct flight flies between them.

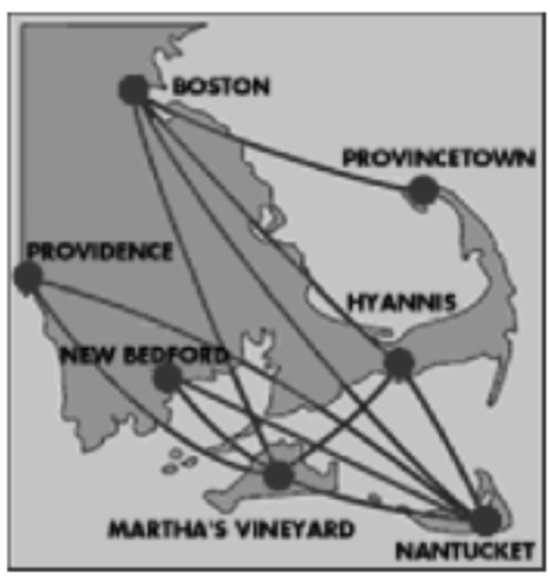


Figure 2: Northern Route Map for Cape Air -- May 2001

In the route map, Provincetown and Hyannis are connected by a two-edge sequence, meaning that a passenger would have to stop in Boston while flying between those cities on Cape Air. It might be important to know if it is possible to get from a vertex to another vertex. It is impossible to go from vertex D to any other vertex, but a passenger on Cape Air can get from any city in their network to any other city given enough flights.

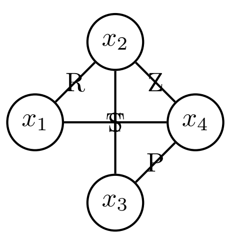
If the vertices in the Cape Air graph respectively correspond to Boston, Hyannis, Martha's Vineyard, Nantucket, New Bedford, Providence, and Provincetown, then the adjacency matrix for Cape Air is

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6 Incidence Matrix

The incidence matrix A of an undirected graph has a row for each vertex and a column for each edge of the graph. The element $A_{[[i,j]]}$ of A is 1 if the ith vertex is a vertex of the jth edge and 0 otherwise.
The incidence matrix A of a directed graph has a row for each vertex and a column for each edge of the graph. The element $A_{[[i,j]}$ of A is 1 if the ith vertex is an initial vertex of the jth edge, 1 if the ith vertex is a terminal vertex, and 0 otherwise.

6.1 Example



$$\mathbf{S} = \begin{array}{c} \begin{array}{ccccc} \mathrm{R} & \mathrm{T} & \mathrm{P} & \mathrm{S} & \mathrm{Z} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \end{array}$$

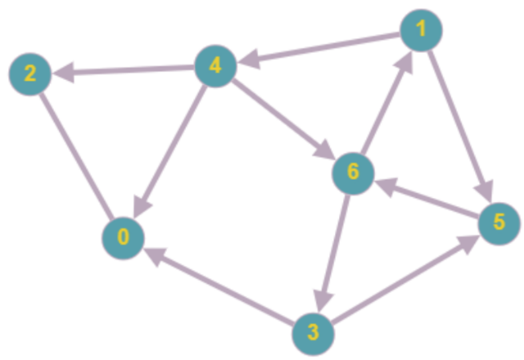
6.2 remark*

rows represent vertices. Columns represent edges.

row sum = degree; column sum = 2

7 Applications of Adjacency List

Here is the graph of Unweighted, directed graph



First we need to find the adjacency matrix of the graph.As you can see, the graph now contains directions. This will alter the matrix since nodes are connected only in a specific direction. For example, in row 1, there now contains only 2 rather than 3 since node 1 is still connected to node 4 and 5, however there is no longer a connection to 6 as there is a different edge direction. We can see that this is mirrored throughout the new matrix, with connections only being present when the direction is correct.

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The use of adjacency matrices in the representation of graphs, an alternative method would be the implementation of an adjacency list. An adjacency list is similar to an adjacency matrix in the fact that it is a way of representing a graph, however it uses linked lists to store the connections between nodes. Each element in the array is a linked list containing all connected vertices.

Adjacency List			
0:	2		
1:	4	5	6
2:	0		
3:	0	5	
4:	0	2	6
5:	6		
6:	3		

8 Time/Space Complexities

8.1 Adjacency Matrix:

Space complexity: O(N * N)
Time complexity for checking if there is an edge between 2 nodes: O(1)
Time complexity for finding all edges from a particular node: O(N)

8.2 Adjacency List:

Space complexity: O(N+M)
Time complexity for checking if there is an edge between 2 nodes: O(degree of node)
Time complexity for finding all edges from a particular node: O(degree of node)