Separable

eg: $|0\rangle \otimes |1\rangle$

$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$

$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

Entanglement

eg: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

cannot be written as

$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

* How to create Entanglement

1] Apply H to q_1

2] Apply CNOT(q_1, q_0)

control target

Asja applies Hadamard operator to her qubit.

The quantum state of Asja's qubit is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Then, Asja and Balvis combine their qubits. Their quantum state is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Asja and Balvis apply CNOT operator on two qubits.

The new quantum state is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

At this moment, Asja's and Balvis' qubits are correlated to each other.

If we measure both qubits, we can observe either state $|00\rangle$ or state $|11\rangle$.

Suppose that Asja observes her qubit secretly.

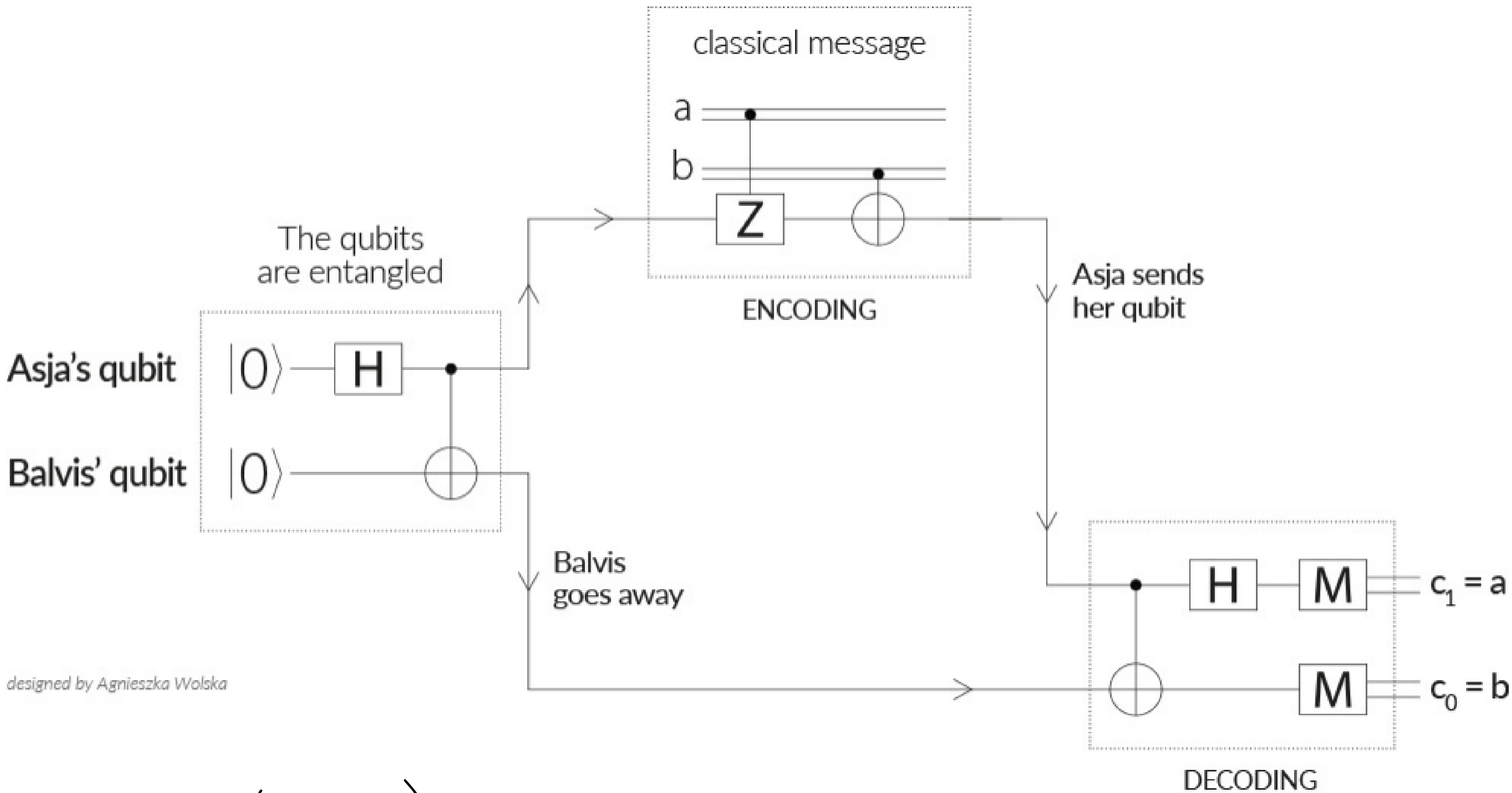
- When Asja sees the result $|0\rangle$, then Balvis' qubit also collapses to state $|0\rangle$. Balvis cannot observe state $|1\rangle$.
- When Asja sees the result $|1\rangle$, then Balvis' qubit also collapses to state $|1\rangle$. Balvis cannot observe state $|0\rangle$.

Experimental results have confirmed that this happens even if there is a physical distance between Asja's and Balvis' qubits.

It seems correlated quantum particles can "affect each other" instantly, even if they are in the different part of the universe.

If two qubits are correlated in this way, then we say that they are **entangled**.

* Superdense Encoding

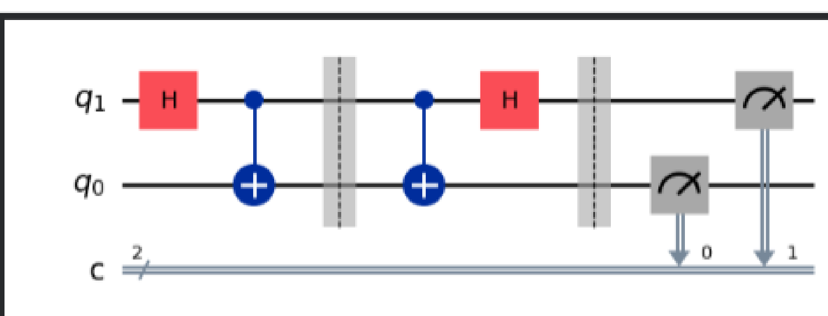


$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

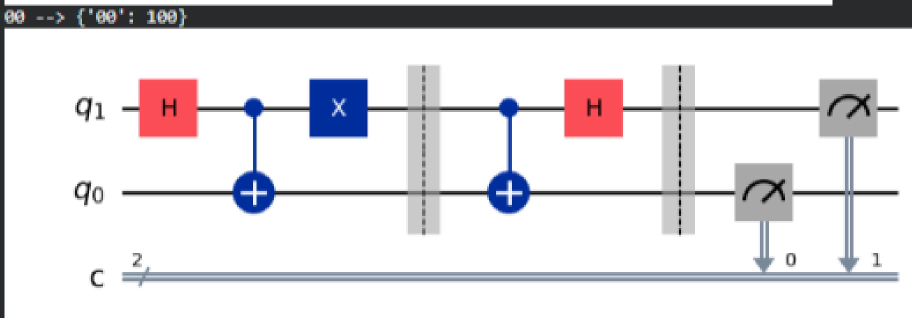
$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Initially	ENTANGLEMENT		ENCODING by Alice		DECODING		measurement
	Alice H	CNOT (Alice, Bob)	a b Alice	Z ⊕ Alice	CNOT (Alice, Bob)	Alice H	a b Alice Bob
$\|00\rangle$	$\|+0\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$	0 0	$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle+\|10\rangle)$	$\|00\rangle$	00
			0 1	$\frac{1}{\sqrt{2}}(\|01\rangle+\|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle+\|11\rangle)$	$\|01\rangle$	01
			1 0	$\frac{1}{\sqrt{2}}(\|00\rangle-\|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle-\|10\rangle)$	$\|10\rangle$	10
			1 1	$\frac{1}{\sqrt{2}}(-\|01\rangle+\|10\rangle)$	$-\frac{1}{\sqrt{2}}(\|01\rangle-\|11\rangle)$	$-\|11\rangle$	11

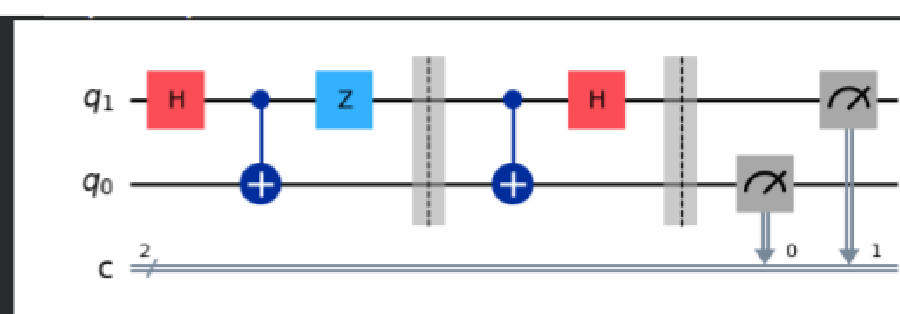
```python
# import all necessary objects and methods for quantum circuits
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit
from qiskit_aer import AerSimulator

all_pairs = ['00','01','10','11']

for pair in all_pairs:

    # create a quantum curcuit with two qubits: Asja's and Balvis' qubits.
    # both are initially set to |0>.
    q = QuantumRegister(2,"q") # quantum register with 2 qubits
    c = ClassicalRegister(2,"c") # classical register with 2 bits
    qc = QuantumCircuit(q,c) # quantum circuit with quantum and classical registers

    # apply h-gate (Hadamard) to the Asja's qubit
    qc.h(q[1])

    # apply cx-gate as CNOT(Asja's-qubit,Balvis'-qubit)
    qc.cx(q[1],q[0])

    # they are separated from each other now

    # if a is 1, then apply z-gate to Asja's qubit
    if pair[0]=='1':
        qc.z(q[1])

    # if b is 1, then apply x-gate (NOT) to Asja's qubit
    if pair[1]=='1':
        qc.x(q[1])

    # Asja sends her qubit to Balvis
    qc.barrier()

    #  apply cx-gate as CNOT(Asja's-qubit,Balvis'-qubit)
    qc.cx(q[1],q[0])

    # apply h-gate (Hadamard) to the Asja's qubit
    qc.h(q[1])

    # measure both qubits
    qc.barrier()
    qc.measure(q,c)

    # draw the circuit in Qiskit's reading order
    display(qc.draw(output='mpl',reverse_bits=True))

    # compare the results with pair (a,b)
    job = AerSimulator().run(qc,shots=100)
    counts = job.result().get_counts(qc)
    print(pair,"-->",counts)
```



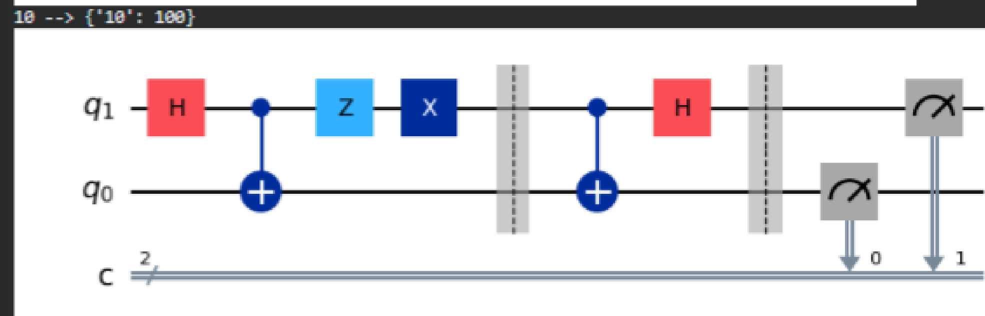
00 --> {'00': 100}



01 --> {'01': 100}



10 --> {'10': 100}



11 --> {'11': 100}

Questions:

① Entangled?

$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

True or False

② Entangled?

$$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \_\_ \otimes \_\_$$

True or False

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$= \frac{1}{2}\left[|0\rangle\left(|0\rangle + |1\rangle\right) + |1\rangle\left(|0\rangle + |1\rangle\right)\right]$$

$$= \frac{1}{2}\left[\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)\right]$$