Ch 3: Lagrange and Barycentric Interpolation

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One practical method for polynomial interpolation (and can be improved via Barycentric Lagrange interpolation)

Setup

⚠ we have $n+1$ nodes since starting at 0

Given $n+1$ distinct nodes $\{x_0, x_1, \dots, x_n\}$ with values $\{y_0, y_1, \dots, y_n\}$,

we want to find the unique n-degree polynomial that interpolates these values. ← we proved previously

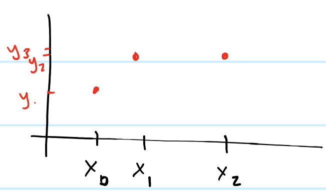
GOAL: Avoid using the Vandermonde matrix

(Note: we could find an interpolating polynomial of degree $> n$, but then ① it's not unique, ② more likely to overfit)

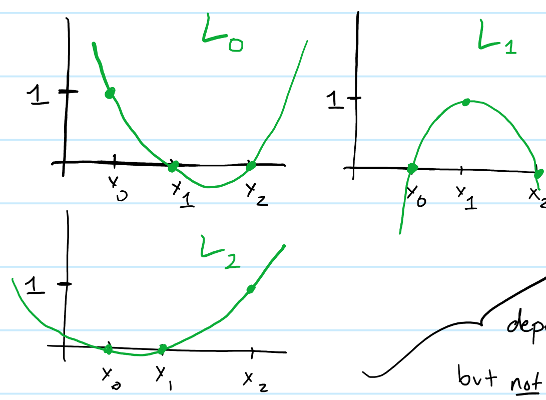
One solution: Lagrange polynomials

it's not "the" Lagrange polynomials, since they will change based on the nodes ($\neq$ Lagrange, Laguerre, Legendre, L'Hôpital)

Idea: (for illustration, let $n=2$, so looking for a quadratic polynomial to interpolate 3 points)



need not be equispaced



depends on $\{x_i\}$ but not on $\{y_i\}$

Now, putting it all together is easy:

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

similarly for L_1, L_2

Quadratic

$L_0(x_0) = 1$

$L_0(x_1) = 0$

$L_0(x_2) = 0$

$$p(x) = y_0 \cdot L_0(x) + y_1 \cdot L_1(x) + y_2 \cdot L_2(x)$$

So $p(x_0) = y_0 \cdot \underbrace{L_0(x_0)}_{=1} + y_1 \underbrace{L_1(x_0)}_{=0} + y_2 \underbrace{L_2(x_0)}_{=0} = y_0$ ✓

and similarly for x_1, x_2

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General Form: (Sometimes we write $L_{n,k}$ to make it clear the dependence on n)

$$(\star) \quad L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdot\ldots\cdot(x-x_{k-1})(x-x_{k+1})\cdot\ldots\cdot(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdot\ldots\cdot(x_k-x_{k-1})(x_k-x_{k+1})\cdot\ldots\cdot(x_k-x_n)}$$

Lagrange Polynomial

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} \qquad \left(\text{so } L_{n,i}(x_k) = \begin{cases} 1 & k=i \\ 0 & k\neq i \end{cases} \right)$$

and the interpolating polynomial is

$$p(x) = \sum_{k=0}^{n} y_k \cdot L_{n,k}(x)$$

Lagrange form of the interpolating polynomial

Comments:

(+) The slowest part of the algorithm depends only on the nodes $\{x_i\}$ not the function values $\{y_i\}$. Sometimes we need to repeatedly interpolate different sets of $\{y_i\}$ but with the same nodes, so we can amortize the slow part.

(−) Evaluating $p(x)$ takes $O(n^2)$ flops) we often want to evaluate at many points

(−) Adding a new data point (x_{n+1}, y_{n+1}) requires you to start from scratch

ex: we keep adding points until the error in approximation is small

(−) Computation can be unstable

Fix is the

Barycentric version of the Lagrange formula (cf. Berrut and Trefethen, SIAM Review 2004)

p. 3: Barycentric formulation

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Barycentric version of the Lagrange formula (cf. Berrut and Trefethen, SIAM Review 2004)

Define $\ell(x) = (x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)$ (no missing terms)

So $L_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} = \frac{\prod_{i\neq k}(x-x_i)}{\prod_{i\neq k}(x_k-x_i)} = \frac{\ell(x)/(x-x_k)}{\prod_{i\neq k}(x_k-x_i)} = \ell(x)\cdot\frac{w_k}{(x-x_k)}$.

$\prod_{i\neq k}(x_k-x_i) = w_k^{-1}$

$$\boxed{\text{So } p(x) = \sum_{k=0}^{n} y_k L_{n,k}(x) = \ell(x)\cdot\sum_{k=0}^{n}\frac{w_k}{(x-x_k)}\,y_k}$$

"First Form of the Barycentric Formula"

Precompute w_k in $O(n^2)$ flops* (each w_k takes $O(n)$ flops, and do this for $O(n)$ values of k)

*See code on github

Amortize: Spread out the cost over a long period of time (like buying a season pass... it makes sense if you use it a lot)

but now we can evaluate $p(x)$ at many different x, each one costing only $O(n)$. So we've amortized the $O(n^2)$ cost

evaluate $\underbrace{\ell(x)}_{O(n)}$ once, then $O(n)$ multiplies and sums

Furthermore, we have fast updates:

to add (x_{n+1}, y_{n+1}), update old w_k $(k=0,\ldots,n)$ by $\div$ by $(x_k - x_{n+1})$ and compute new w_{n+1}

which is $O(n)$. Nice.

Note: an equivalent way to write the Barycentric Formula is

($p(x)$ is continuous - use L'Hôpital to see)

$$p(x) = \begin{cases} \dfrac{\sum_{k=0}^{n}\frac{w_k}{x-x_k}\cdot y_k}{\sum_{k=0}^{n}\frac{w_k}{x-x_k}} & \text{if } x\neq x_k \text{ for } k\in\{0,1,\ldots,n\} \\ y_k & \text{if } x = x_k \end{cases}$$

BARYCENTRIC FORMULA ("Second form" or "true form").

USE THIS ONE

you might be worried that if x is close to some node x_k we get subtractive cancellation, but it turns out in this case it's ok since it happens in numerator and denominator

How to derive? If $y_k = 1$, interpolant must be 1 function (unique!) so $1 = \ell(x)\sum\frac{w_k}{x-x_k}$

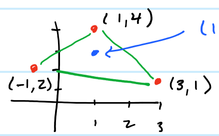
Divide $p(x)$ by 1, cancel the $\ell(x)$

Advantage: we can scale w_k by a constant since it cancels out. Choose a value to prevent overflow and underflow

(Aside: what does Barycenter mean?)

The Barycenter of two bodies (planets, stars...) is their center of mass, so it's a weighted sum

Barycentric coordinates are coordinates with respect to the vertices of a simplex, ex.



(1,3) in Cartesian coordinates

or $(1,3) = 0\cdot(-1,2) + \frac{8}{11}(1,4) + \frac{1}{11}(3,1)$

so $(0, \frac{8}{11}, \frac{1}{11})$ in Barycentric coord.

* *Chebyshev nodes also simplify

p. 4: Special cases and example

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Special Cases * *Chebyshev nodes also simplify, see Berrut + Trefethen

If nodes are $\boxed{\text{equispaced}}$, $\rightarrow x_{k+1} - x_k = h$

then formulas simplify and are faster,

(skip in lecture)

i.e., $\underbrace{(x_k - x_0)\cdot(x_k - x_1)\cdots(x_k - x_{k-1})}_{\text{BEFORE } x_k}\cdot\underbrace{(x_k - x_{k+1})\cdots(x_k - x_n)}_{\text{AFTER } x_k}$

$\quad\;\underbrace{}_{(k)\cdot h}\cdot\underbrace{}_{(k-1)\cdot h}\cdot\ldots\cdot\underbrace{}_{h}\quad\underbrace{}_{-h}\quad\underbrace{}_{-(n-k)\cdot h}$

$$= (-1)^{n-k}\cdot h^n \cdot k!\cdot(n-k)!$$

$$= (-1)^{n-k}\, h^n \left(\binom{n}{k}\Big/ n!\right)^{-1}$$

Please do not memorize! Not that important

Example

Take $x_0 = 1$, $y_0 = 1$ (eg, $y = 1/x$)

$x_1 = 2$, $y_1 = 1/2$

$x_2 = 4$, $y_2 = 1/4$. Find quadratic interpolant p and evaluate $p(3)$

Solution: ① One way, compute $L_0(x) = (x-x_1)(x-x_2)\cdot\overbrace{\left(\frac{1}{(x_0-x_1)(x_0-x_2)}\right)}^{\omega_0}$

$$= (x-2)(x-4)\left(\frac{1}{(1-2)(1-4)}\right)$$

$$\frac{1}{(-1)(-3)} = \frac{1}{3} = \omega_0$$

$$L_1(x) = (x-x_0)(x-x_2)\cdot\left(\frac{1}{(x_1-x_0)(x_1-x_2)}\right)$$

$$\omega_1 = \frac{1}{(2-1)(2-4)} = \frac{-1}{2}$$

$$L_2(x) = (x-x_0)(x-x_1)\left(\frac{1}{(x_2-x_0)(x_2-x_1)}\right)$$

$$\omega_2 = \frac{1}{(4-1)(4-2)} = \frac{1}{6}$$

then

$$p(x) = 1\cdot L_0(x) + \tfrac{1}{2}\cdot L_1(x) + \tfrac{1}{4} L_2(x)$$

and plug $x=3$ in (I won't do this now since I'll do it the 2nd way)

② Second way, Barycentric formula. Uses $\omega_0 = 1/3$, $\omega_1 = -1/2$, $\omega_2 = 1/6$ as before

$$p(x) = \frac{\sum_{k=0}^{n} \frac{\omega_k}{x - x_k}\cdot y_k}{\sum_{k=0}^{n}\frac{\omega_k}{x - x_k}} = \frac{\frac{1}{3}\cdot\frac{1}{(3-1)}\cdot 1 + \frac{-1}{2}\cdot\frac{1}{(3-2)}\cdot\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{(3-4)}\cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{(3-1)} + \frac{-1}{2}\cdot\frac{1}{(3-2)} + \frac{1}{6}\cdot\frac{1}{(3-4)}}$$

$(x=3)$

$$= \frac{\frac{1}{6}\cdot 1 - \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{6}\cdot\frac{1}{4}}{\frac{1}{6} - \frac{1}{2} - \frac{1}{6}} = \frac{\frac{1}{6} - \frac{1}{4} - \frac{1}{24}}{-\frac{1}{2}}$$

$$= \frac{\frac{4-6-1}{24}}{-\frac{1}{2}} = \frac{+3}{24}\cdot 2 = \boxed{\frac{1}{4}}$$

p. 5: Generalized Rolle's Thm

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Thm 1.7: Rolle's Thm (simplified MVT)

If $f \in C[a,b]$ is differentiable on (a,b), then if $f(a)=f(b)$

(special case: $f(a)=0$, $f(b)=0$) there exists $\xi \in (a,b)$ s.t. $f'(\xi)=0$

Thm 1.10: Generalized Rolle's Thm

If $f \in C[a,b]$ is n-times differentiable on (a,b), then if $f(x)=0$ for the $n+1$ points $x_0 < x_1 < \dots < x_n$ (all in $[a,b]$) then $\exists\, \xi \in (x_0, x_n)$ s.t. $f^{(n)}(\xi)=0$

proof (n=2 case only)

$a=x_0,\ b=x_1$

via Rolle's, $\exists\, \xi_1 \in (x_0, x_1)$ st. $f(\xi_1)=0$ since $f(x_0)=f(x_1)$

Similarly, $\exists\, \xi_2 \in (x_1, x_2)$ st. $f(\xi_2)=0$ $\quad a=x_1,\ b=x_2$

Now apply Rolle's to f', using $a=\xi_1$, $b=\xi_2$

So $\exists\, \xi \in (\xi_1, \xi_2)$ st. $\underbrace{(f')'(\xi)}_{\text{i.e. } f''(\xi)}=0$

i.e. $f''(\xi)=0$.

p. 6: Error estimate

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Error

The $\{y_i\}$ can be arbitrary, but sometimes they are generated by a true "underlying" function f. In this case, how close is our interpolant p to f? Should p be a better approximation as we add nodes? (Yes!) Answered by following theorem:

Thm 3.3 (Burden and Faires)

Let $\{x_0, x_1, \ldots, x_n\}$ be distinct points in $[a,b]$ and $f \in C^{n+1}([a,b])$.

Then $\forall x \in [a,b]$, $\exists\ \xi$ (unknown, depends on x) between $x_0, x_1, \ldots, x_n$ (hence $\xi \in (a,b)$) such that if p is the n^{th} degree polynomial interpolant w/ nodes $\{x_i\}_{i=0}^n$ and values $\{y_i = f(x_i)\}_{i=1}^n$, then

$$(*) \quad f(x) = p(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdot\ldots\cdot(x-x_n).$$

(ie., error at $x = x_k$ is 0, and small if x is close to some node)

Similar to Taylor Series except $(x-x_0)\cdot\ldots\cdot(x-x_n)$ instead of $(x-x_0)^{n+1}$.

proof Assume $x \neq x_k$ for any k, else it's trivial

Fix x, define $g(t) = f(t) - p(t) - (f(x)-p(x))\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}$

then note $f \in C^{n+1}([a,b]) \Rightarrow g \in C^{n+1}([a,b])$

and $g(x_k) = 0$ for $k = 0, 1, \ldots, n$

$g(x) = 0$ also.

like MVT

So g has $(n+2)$ distinct zeros in $[a,b]$, so by generalized Rolle's Thm

$\exists\ \xi \in (a,b)$ s.t. $g^{(n+1)}(\xi) = 0$

Compute $g^{(n+1)}(\xi) = f^{(n+1)}(\xi) + \underbrace{p^{(n+1)}(\xi)}_{\text{degree } n \text{ polynomial, so } \frac{d^{n+1}}{dt^{n+1}}(\ ) = 0} - \overbrace{\left(f(x) - p(x)\right)}^{\text{constants}} \frac{d^{n+1}}{dt^{n+1}} \underbrace{\prod \frac{(t-x_i)}{(x-x_i)}}_{\text{degree } n+1}\Bigg|_{\xi}$

so only leading term remains

$$= f^{(n+1)}(\xi) - (f(x)-p(x))\cdot\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$\square$

p. 7: Example

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Test yourself:

Let $x_0 = 2$, $x_1 = 6$

$y_0 = 1/2$, $y_1 = 1/6$. Compute $p(4)$ where p is the linear (degree $n=1$) interpolating polynomial.

Note: your answer should be an integer or fraction (simplify fractions), not a decimal

Answer: $p(x) = \frac{-x}{12} + \frac{2}{3}$

So $p(4) = 1/3$