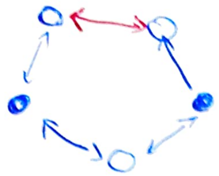
Problem #1. Classical marginal problem.

1.)



In the first case we arrive at a contradiction, therefore no such global distribution exists.

We can find a solution for the case of 4 such bits.

2.) Given P_{12}, P_{23}, P_{34} find $P_{1,2,34}$.

$$P_{12}(a_1, a_2) = \sum_{a_3} P_{123}(a_1, a_2, a_3)$$

$$P_{123}(a_1, a_2, a_3) = \frac{P_{12} \cdot P_{23}}{P_2}$$

$\sum_{a_3}$ gives us the above equation

In the same way we obtain:

$$\boxed{P_{1234} = \frac{P_{123} \cdot P_4}{P_3} = \frac{P_{12} \cdot P_{23} \cdot P_{34}}{P_2 \cdot P_3}}$$

With P_{12}, P_{23}, P_{34} admitted as marginals.

$P_{12} = \begin{pmatrix} 00 & 01 \\ 10 & 11 \end{pmatrix} \rightarrow P_1 = \begin{pmatrix} 00+01 \\ 10+11 \end{pmatrix}, P_2 = (10+00, \ldots)$

We can now calculate $P_{12}, P_{23}, P_{34}, P_2, P_3$. and find P_{1234}.

$\rightarrow$ Calculations with python on github

Problem #2:

The quantum marginal problem.

#1)

$\underbrace{\langle \varphi^+ | \rho_{AB} | \varphi^+ \rangle} = \langle \varphi^+ | \rho_{BC} | \varphi^+ \rangle = 1$

$\underbrace{\langle \varphi^+ | Tr_C \, \rho_{ABC}} | \varphi^+ \rangle = \langle \varphi^+ | Tr_A \rho_{ABC} | \varphi^+ \rangle$

$\rho_{AB} \otimes \rho_C$ because $\langle \varphi^+ | \rho_{AB} | \varphi^+ \rangle = 1$ and $\rho_{AB} \overset{!}{=} |\varphi^+\rangle\langle\varphi^+|$ therefore ρ_C must be separable from AB.

But looking at the right side it is implied that BC are entangled $\rho_{BC} = |\varphi\rangle\langle\varphi|$. Which is a $\lightning$..

#2. Additionally the partial trace over ρ_{ABC} will result the density matrix of a mixed state. $|\varphi^+\rangle$ is a pure state, therefore $\langle \varphi^+ | \rho_{reduced} | \varphi^+ \rangle \overset{!}{<} 1$!

2) $\langle \varphi^+ | \rho_{AB} | \varphi^+ \rangle, \langle \varphi^+ | \rho_{BC} | \varphi^+ \rangle \geq 0.85$

→ linear optimization problem.

Constraints:

- $\rho \geq 0$?
- $Tr \rho = 1$
- $\langle \varphi^+ | \rho | \varphi^+ \rangle < 1$

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Problem #3. Entanglement and Positive Partial Transpose.

1.)
2.) $\boxed{\text{All calculations on Github.}}$! And the results too.

i) ρ_{AB} : $\lambda_0 = \frac{1}{3}$, $\lambda_1 = \frac{2}{3}$, $\underbrace{\lambda_3 = 0}_{\text{2 times (multiplicity)}}$

ii) $\rho_{AB\text{-}pt}$: $\lambda_0 = \frac{1}{3}$, $\lambda_1 = 0.53$, $\lambda_3 = -0.20$

↳ Negative eigenvalue, therefore entangled!

iii) σ_{AB} : $\lambda_0 = \frac{1}{3}$, $\lambda_1 = \frac{2}{3}$, $\underbrace{\lambda_3 = 0}_{2x.}$ $\boxed{\text{Same for the partial transpose of } \sigma_{AB}}$!

↳ All ~~positive~~ non-negative, therefore separable!

iv) ρ_A : $\lambda_0 = \frac{2}{3}$, $\lambda_1 = \frac{1}{3}$

σ_A : Same as for ρ_A.

3.) If $|\psi\rangle_{AB}$ is entangled $\Leftrightarrow$ $Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$ is mixed.

~~ρ of mixed: $\rho^2 \neq \rho$~~

But σ_{AB} is separable, but σ_A is mixed again

So above criterion doesn't seem very useful

4.) Github $\longrightarrow$ $1 \geq \lambda > 1/3$, ~~$\lambda \in [0,1]$~~ entangled $\lambda \in [0,1]$?

$0 \leq \lambda \leq 1/3$ separable

5.) No. PPT is necessary and sufficient for bipartite $\otimes$ bipartite $(d=2)$.

But only necessary and not sufficient for $d=3$.

Exception: $\underset{d=2}{\text{qubit}} \otimes \underset{d=3}{\text{qutrit}}$.

Problem #3

b) $E_D(\rho) = \sup \{ r \geq 0 \mid \lim_{n\to\infty} \inf_{\Lambda \in LOCC} \| \Lambda(\rho^{\otimes n}) - |\Phi^+\rangle\langle\Phi^+|^{\otimes \lfloor rn \rfloor} \|_1 = 0 \}$

$E_C(\rho) = \inf \{ r \geq 0 \mid \lim_{n\to\infty} \inf_{\Lambda \in LOCC} \| \Lambda(|\Phi^+\rangle\langle\Phi^+|^{\otimes n}) - \rho^{\otimes \lfloor rn \rfloor} \| = 0 \}$

$E_D \leq E_C$

Distillable entanglement.

There has to exist some Λ s.t. our distillation protocol converts n copies of our state into a max. entangl. st.

1. $\| |\psi\rangle\langle\psi| - \Lambda(\rho^{\otimes n}) \| \to 0$

2. $| E_D(\rho) - \frac{1}{n} | \leq \varepsilon/2$ (some conversion rate will match up to some $\varepsilon/2$).

Choosing ent. meas. that is cont.

$1 \to 3$ $| (E(\Lambda(\rho^{\otimes n})) - E(|\psi^-\rangle\langle\psi^-|)) / (1+2n) | \to 0$

n large enough

$| E(\Lambda(\rho^{\otimes n}))/n - E(|\psi^-\rangle\langle\psi^-|)/n | \leq \varepsilon/2$

Λ monotonicity! $E(\Lambda(\rho)) \leq E(\rho)$

$\to E(\rho^{\otimes n})/n \geq E(\Lambda_n(\rho^{\otimes n}))/n \geq E(|\psi^-\rangle\langle\psi^-|)/n - \frac{\varepsilon}{2} = \frac{1}{n} - \frac{\varepsilon}{2} \geq E_D(\rho) - \varepsilon$

Analog for $E_C(\rho)$:

$E(\rho^{\otimes n}) \leq E(\Lambda(|\psi^-\rangle\langle\psi^-|))/n + \frac{\varepsilon}{2} \leq E(|\psi^-\rangle\langle\psi^-|)/n + \frac{\varepsilon}{2} = \frac{1}{n} + \frac{\varepsilon}{2} \leq E_C(\rho) + \varepsilon$

$\Rightarrow E_D(\rho) - \varepsilon \leq E(\rho) = E(\rho^{\otimes n})/n \leq E_C(\rho) + \varepsilon$

Can choose ε to be arbitrarily small $\to n \to \infty$

$\Rightarrow E_D(\rho) \leq E_C(\rho)$

OPRI Problem #4

3) Consider A, B with $(a,b) \in \{(1,1),(1,-1),(-1,1),(-1,-1)\}$

Normalization $p(-1,-1) = 1-(p(1,1)+p(1,-1)+p(-1,-1))$

$\Rightarrow$ Probability simplex: $P = \{p \in \mathbb{R}^4 \mid p_i \geq 0, \sum p_i = 1\}$

$= \{(p_1, p_2, p_3, 1-p_1-p_2-p_3) \in \mathbb{R}^4 \mid p_1, p_2, p_3 \geq 0,$

$1-p_1-p_2-p_3 \geq 0\}$.

4) Let $p = (p_1, p_2, p_3)$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$

$\tilde{p}_1 := p(A=1) = p(A=1, B=1) + p(A=1, B=-1) = p_1 + p_2$

$\tilde{p}_2 := p(B=1) = p(A=1, B=1) + p(A=-1, B=1) = p_1 + p_3$

$\tilde{p}_3 := p(A=1, B=1) = p_1$

$\Leftrightarrow \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

$\rightarrow$ linear and invertible

Inequalities

i) $p_1 \geq 0 \Leftrightarrow \tilde{p}_3 \geq 0 \Leftrightarrow p(A=B=1) \geq 0$

ii) $p_2 \geq 0 \Leftrightarrow \tilde{p}_1 \geq \tilde{p}_3 \Leftrightarrow p(A=1) \geq p(A=B=1)$

iii) $p_3 \geq 0 \Leftrightarrow \tilde{p}_2 \geq \tilde{p}_3 \Leftrightarrow p(B=1) \geq p(A=B=1)$

iv) $p_1+p_2+p_3 \leq 1 \Leftrightarrow \tilde{p}_1 + \tilde{p}_2 - \tilde{p}_3 \leq 1 \Leftrightarrow p(\overset{=1}{A}) + p(B=1) - p(A=B=1) \leq 1$

5) Let $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) = (\langle A \rangle, \langle B \rangle, \langle AB \rangle)$

$\hat{p}_1 = \langle A \rangle = p(A=1) - p(A=-1) = 2\tilde{p}_1 - 1$

$\hat{p}_2 = \langle B \rangle = 2\tilde{p}_2 - 1$

$\hat{p}_3 = \langle AB \rangle = \ldots = p_1 - p_2 - p_3 + (1-p_1-p_2-p_3) = 1-2(\tilde{p}_1+\tilde{p}_2)+4\tilde{p}_3$

$\Rightarrow \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 4 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

OPQI #4

6) Simplex inequalities for 3 variables

Let
$$p = \begin{pmatrix} p(1,1,1) \\ p(1,1,-1) \\ p(1,-1,1) \\ \vdots \\ p(-1,-1,-1) \end{pmatrix} \begin{matrix} = p_1 \\ = p_2 \\ \vdots \\ \vdots \\ = p_7 \end{matrix} \longrightarrow \begin{matrix} \tilde{p}_1 = p(A=1) \\ \tilde{p}_2 = p(B=1) \\ \tilde{p}_3 = p(C=1) \\ \tilde{p}_4 = p(A=B=1) \\ \vdots \\ \tilde{p}_7 = p(A=B=C=1) \end{matrix}$$
$= p_8 = 1 - \sum p_i$

Mapping: $\tilde{p}_1 = p_1 + p_2 + p_3 + p_4$ $\quad \tilde{p}_4 = p_1 + p_2$ $\quad \tilde{p}_7 = p_1$

$\tilde{p}_2 = p_1 + p_2 + p_5 + p_6$ $\quad \tilde{p}_5 = p_1 + p_5$

$\tilde{p}_3 = p_1 + p_3 + p_5 + p_7$ $\quad \tilde{p}_6 = p_1 + p_3$

Inequalities

i) $p_1 = \tilde{p}_7 \geq 0 \Leftrightarrow p(A=B=C=1) \geq 0$

ii) $p_2 = \tilde{p}_4 - \tilde{p}_7 \geq 0 \Leftrightarrow p(A=B=1) \geq p(A=B=C=1)$

iii) $p_3 = \tilde{p}_6 - \tilde{p}_7 \geq 0 \Leftrightarrow p(A=C=1) \geq p(A=B=C=1)$

iv) $p_5 = \tilde{p}_5 - \tilde{p}_7 \geq 0 \Leftrightarrow p(B=C=1) \geq p(A=B=C=1)$

v) $p_4 = \tilde{p}_1 + \tilde{p}_7 - \tilde{p}_4 - \tilde{p}_6 \geq 0 \Leftrightarrow p(A=1) + p(A=B=C=1) \geq p(A=B=1) + p(A=C=1)$

vi) $p_6 = \tilde{p}_2 + \tilde{p}_7 - \tilde{p}_4 - \tilde{p}_5 \geq 0 \Leftrightarrow p(B=1) + p(A=B=C=1) \geq p(A=B=1) + p(B=C=1)$

vii) $p_7 = \tilde{p}_3 + \tilde{p}_7 - \tilde{p}_5 - \tilde{p}_6 \geq 0 \Leftrightarrow p(C=1) + p(A=B=C=1) \geq p(B=C=1) + p(A=C=1)$

viii) $\sum_{i=1}^{7} p_i = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 - \tilde{p}_4 - \tilde{p}_5 - \tilde{p}_6 + \tilde{p}_7 \leq 1 \Leftrightarrow p(A=1) + p(B=1) + p(C=1)$
$- p(A=B=1) - p(B=C=1)$
$- p(A=C=1)$
$+ p(A=B=C=1) \leq 1$

Expectation values:

$\tilde{p}_1 = \langle A \rangle = 2\tilde{p}_1 - 1$, $\tilde{p}_2 = \langle B \rangle = 2\tilde{p}_2 - 1$, $\tilde{p}_3 = \langle C \rangle = 2\tilde{p}_3 - 1$

Bibliography

[1]: Solutions to OPQI, Leander Thiessen, 2022, 1st Edition

OPQT

Problem #6: Causality and Bayesian networks.

1)

$p(A,B) = \begin{pmatrix} 00 \\ 01 \\ 10 \\ 11 \end{pmatrix} \perp \rightarrow \begin{array}{cc} A\,B & A\,B \end{array} \begin{pmatrix} 00 & 01 \\ 10 & 11 \end{pmatrix}$

$G_1 : A \to B$, $G_2 : B \to A$, $G_3 : A \quad B$.

$$\boxed{\begin{aligned} &p(A,B) \\ &= p(A) \cdot p(B|A) \\ &= p(A) \cdot \frac{p(A,B)}{p(A)} \end{aligned}}$$

As long as $p(A) \neq 0$ $\forall$ distribution is compatible.

Same for G_2: compatible with both distributions.

G_3 ?

$p(A,B) \overset{!}{=} p(A) \cdot p(B)$ ← Using this we find that it is incompatible with both distributions.

G_3 : Example calculation:

~~P_4~~

~~$0.083 = 0.2291.$ $A=0, B=0$ ↯~~

~~P_5~~

~~$0.166 = 0.33$ $A=0, B=1$ ↯~~

- P_4 not compatible with G_3
- P_5 is

Calc online jupyter.

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Problem #6. 2) $\boxed{\text{And part 3) is exclusively on Github}}$

G_1' : $A \to B \to C$.

$$p(A,B,C) = p(A)\, p(B|A)\, p(C|B)$$

$$= p(A) \frac{p(A,B)}{p(A)} \frac{p(B,C)}{p(B)}$$

$$\boxed{= \frac{p(A,B)\,p(B,C)}{p(B)}}$$

G_2' : $A \leftarrow B \to C$

$$p(A,B,C) = p(B)\, p(A|B)\, p(C|B)$$

$$= p(B) \frac{p(A,B)}{p(B)} \frac{p(B,C)}{p(B)} = \boxed{\frac{p(A,B)\,p(B,C)}{p(B)}}$$

G_3' : $p(ABC) = p(C) \cdot p(B|C)\, p(A|B)$

$$= p(C) \cdot \frac{p(B,C)}{p(C)} \frac{p(A,B)}{p(B)} = \boxed{\frac{p(A,B)\, p(B,C)}{p(B)}}$$

G_4' : $p(ABC) = \boxed{p(A) \cdot p(B) \cdot p(C)}$

find: $p(A)$, $p(B)$, $p(C)$, $p(A,B)$, $p(B,C)$ → python

G_1', G_2', G_3' compatible with P_4, P_5!
G_4' compatible with NEITHER!

Calculations on Github.

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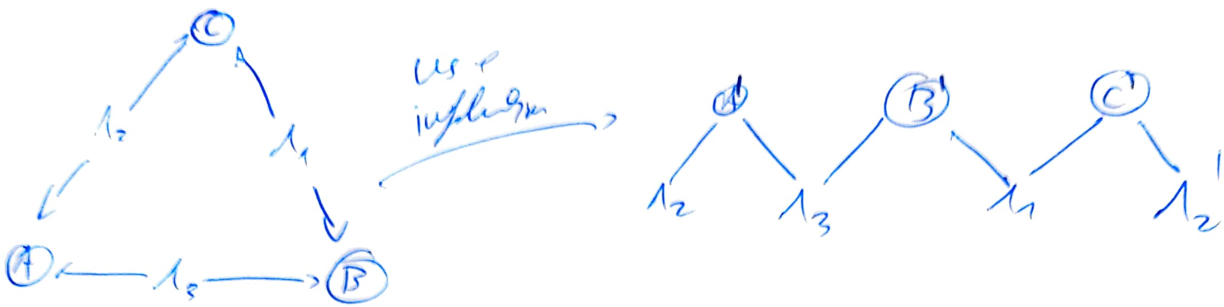
Problem #6. 4.



Prove: $p(0,0,0) = p(1,1,1) = \frac{1}{2}$ are incompatible.

If the above distribution were compatible then so would the marginals

$P_{AC} = P_{BC}$ $\quad P_{AB}(0,0) = P_{AB}(1,1) = 1/2 = P_{BC}(0,0) = P_{BC}(1,1)$

Therefore

$$\underbrace{P_{A'B'}(0,0) = P_{A'B'}(1,1)} = 1/2 = P_{B'C'}(0,0) = P_{B'C'}(1,1)$$

A' has perfect correlation with B'

C' has perfect correlation with B'

This implies A' has perfect correlation with C'!

But they have no common ancestor! ↯

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Problem #6. 5)

$I(X:Y) = H(X) + H(Y) - H(XY)$

$\hookrightarrow I(X:Y) + I(X:Z) \leq I(Y:Z) + H(X)$

$\Leftrightarrow H(X) - H(XY) - H(XZ) + H(YZ) \leq 0$ eq1?

From $I(Z:Y|X) \geq 0$

$\hookrightarrow H(Z|X) - H(Z|XY) \geq 0$

d①: $H(X|Y) = H(XY) - H(Y)$.

d.① $\Leftrightarrow H(XZ) - H(X) - H(XYZ) + H(XY) \geq 0$. (very similar to eq 1)

↳ eq2.

Use. $H(XYZ) \geq H(YZ)$. d②

eq1: $H(XZ) - H(X) - H(YZ) + H(XY) \geq 0$

Using d② and comparing the red terms.

Since eq2 holds and $H(XYZ) \geq H(YZ)$. So must eq 1.

Problem #6. 6.

A', C' are marginally independent. $I(A':C') = 0$.

$$I(A:B) + I(B:C) \leq I(A:C) + H(B)$$

The constraint must hold in the original graph too:
(→ from inflation.

$$I(A':B) + I(B:C) \leq H(B). \quad \text{eq } 1.$$

for $P_{abc} = \begin{cases} 1/2 & \text{correlated} \\ 0 & \text{else} \end{cases} \Rightarrow H(A) = H(B) = H(C) = 1. \quad (2 \cdot \log_2 2 = 1)$

$H(AB) = H(BC) = 1$

eq 1: $(1 + 1 - 1) + (1 + 1 - 1) \leq 1. \Leftrightarrow 2 \leq 1$ ↯.

Once again we see the entropic equations (well inequalities here) not satisfied ~~incompatible~~, therefore P_{abc} is not compatible with the graph.