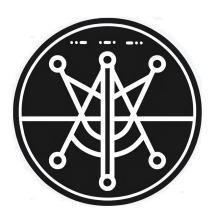
- LECTURE 0 -

A SICKNESS UNTO DEATH

OR, AN ENQUIRY INTO THE THEORY OF MEASURE AS IT CONCERNS THOSE PROCESSES OF A STOCHASTIC KIND

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Preface

This preface is temporary. I just want to quickly sketch a road man to guide our study.

A good end goal, for now, would be to prove **Girsanov's Theorem**. It is an important result and will require a lot of the fundamentals:

- (1) Probability spaces and Information
 - σ -algebras
 - Filtrations
 - Independence and Conditioning
 - Adapted Stochastic Processes
 - Martigales
 - Markov Processes
- (2) Brownian motion
 - Random Walks
 - Martingales and Symmetric Random Walks
 - Distribution of Brownian Motion
 - Filtrations for Brownian Motion
 - Martingales and Brownian Motion
- (3) Itô calculus
 - Itô integral (Simple case)
 - Definition, construction and properties
 - Itô integral (General case)
 - Itô-Doeblin formula for Brownian motion
 - Levy's Theorem
- (4) Black-Scholes-Merton Equation
- (5) Radon-Nikodym
 - Radon-Nikodym Theorem
 - Radon-Nikodym derivatives
- (6) Risk-Neutral Measure
 - Girsanov's Theorem
 - Novikov's condition
 - Risk-neutral pricing in the Black-Scholes model

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Nihil – Ø

"It is clear that I can only deliver to you, to each of you, what you are already on the verge of absorbing."

1. The Doctrine of Chances

The general problem of measure is our starting point. Given $A \subseteq X$ we want to be able to assign a quantity m(A) to this set, which can be said, in some sense, to be its "measure". Take, for example, $X = \mathbb{R}$ and A = [a, b], for which we can set m(A) = b - a. Some questions immediately arise: what properties must m obey? and what is its proper domain, i.e. which sets can be said to be measurable? The answer to this last question requires the introduction of the elementary structure of measure theory—the σ -algebra.

Definition 1. A σ -algebra over a set X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying

- $(1) \emptyset \in \mathcal{F},$
- $(2) A \in \mathcal{F} \implies X \setminus A \in \mathcal{F},$
- (3) If $A_1, A_2, \ldots \in \mathcal{F}$ then

$$\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{F}.$$