– LECTURE 0 –

A SICKNESS UNTO DEATH

OR, AN ENQUIRY INTO THE THEORY OF MEASURE AS IT CONCERNS THOSE PROCESSES OF A STOCHASTIC KIND

UNDERGROUND RESEARCH DIVISION

Date: July 1, 2024.

Preface

This preface is temporary. I just want to quickly sketch a road man to guide our study.

A good end goal, for now, would be to prove **Girsanov's Theorem**. It is an important result and will require a lot of the fundamentals:

(1) Probability spaces and Information
  - σ-algebras
    - Filtrations
  - Independence and Conditioning
  - Adapted Stochastic Processes
    - Martigales
    - Markov Processes

(2) Brownian motion
  - Random Walks
  - Martingales and Symmetric Random Walks
  - Distribution of Brownian Motion
  - Filtrations for Brownian Motion
  - Martingales and Brownian Motion

(3) Itô calculus
  - Itô integral (Simple case)
    - Definition, construction and properties
  - Itô integral (General case)
  - Itô-Doeblin formula for Brownian motion
  - Levy's Theorem

(4) Black-Scholes-Merton Equation

(5) Radon-Nikodym
  - Radon-Nikodym Theorem
  - Radon-Nikodym derivatives

(6) Risk-Neutral Measure
  - Girsanov's Theorem
  - Novikov's condition
  - Risk-neutral pricing in the Black-Scholes model

Contents

Nihil – ∅

"It is clear that I can only deliver to you, to each of you, what you are already on the verge of absorbing."

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1. The Doctrine of Chances

The general problem of measure is our starting point. Given $A \subseteq X$ we want to be able to assign a quantity $m(A)$ to this set, which can be said, in some sense, to be its "measure". Take, for example, $X = \mathbb{R}$ and $A = [a, b]$, for which we can set $m(A) = b - a$. Some questions immediately arise: what properties must m obey? and what is its proper domain, i.e. which sets can be said to be *measurable*? The answer to this last question requires the introduction of the elementary structure of measure theory– the σ-algebra.

Definition 1. A σ-algebra over a set X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying

(1) $\emptyset \in \mathcal{F}$,
(2) $A \in \mathcal{F} \implies X \setminus A \in \mathcal{F}$,
(3) If $A_1, A_2, \ldots \in \mathcal{F}$ then

$$\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}.$$