– LECTURE 0 –

A SICKNESS UNTO DEATH

OR, AN ENQUIRY INTO THE THEORY OF MEASURE AS IT CONCERNS THOSE PROCESSES OF A STOCHASTIC KIND

UNDERGROUND RESEARCH DIVISION

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Nihil – $\emptyset$

"It is clear that I can only deliver to you, to each of you, what you are already on the verge of absorbing."

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1. The Doctrine of Chances

The general problem of measure is our starting point. Given $A \subseteq \Omega$ we want to be able to assign a quantity $m(A)$ to this set, which can be said, in some sense, to be its "measure". Take, for example, $X = \mathbb{R}$ and $A = [a, b]$, for which we can set $m(A) = b - a$. Some questions immediately arise: what properties must m obey? and what is its proper domain, i.e. which sets can be said to be *measurable*? The answer to this last question requires the introduction of the elementary structure of measure theory– the σ-algebra.

Definition 1. A **σ-algebra over a set** Ω is a set $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ satisfying

1. $\emptyset \in \mathcal{F}$,
2. $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$,
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then

$$\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}.$$

Definition 2. Let $\mathcal{F}$ be a σ-algebra over a set Ω. A **probability measure** $\mathbb{P}$ is a function $\mathbb{P} : \mathcal{P}(\mathcal{F}) \to [0, 1]$ such that:

1. $\mathbb{P}(\Omega) = 1$,
2. for disjoint $A_1, A_2, ... \in \mathcal{F}$:

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

Definition 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A \in \mathcal{F}$.
If $\mathbb{P}(A) = 1$ we say that A occurs **almost surely** or **almost always**.
If $\mathbb{P}(A) = 0$ we say that A occurs **almost never**.

Definition 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **random variable** X is function $X : \Omega \to \mathbb{R}$ such that

$$\{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F},$$

for every Borel subset B or $\mathbb{R}$.

We will also denote the set $\{\omega \in \Omega \mid X(\omega) \in B\}$ simply by $\{X \in B\}$.

Definition 5. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The **distribution measure of** X is the probability measure μ_X that assigns to each Borel subset B of $\mathbb{R}$ the **mass**

$$\mu_X(B) = \mathbb{P}(\{X \in B\}).$$

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1.1. **Expected Value.**

In data's dark, a guardian sleeps
Expected Value, veiled, mystery deep
Probability's weights, measures guide
Assigning worth, uncertainty's tide
Chancing fate, in shadows cast
Converging sums, destiny forecast
Lo, a shadow of horror is risen
In Eternity! Unknown, unprolific?
Self-closd, all-repelling; what Demon
Hath form'd this abominable void
This soul-shudd'ring vacuum?
Some said
"It is Urizen". But unknown, abstracted
Brooding secret, the dark power hid.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a random variable X. Then

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$$

for the finite case,

$$\mathbb{E}X = \sum_{k=1}^{\infty} X(\omega_k)\mathbb{P}(\omega_k),$$

for the contable case.

The uncontable case is problematic. Consider

$$X^+(\omega) = \max\{X(\omega), 0\} \text{ and } X^-(\omega) = \max\{-X(\omega), 0\}$$

$$\int_\Omega X^+(\omega)d\mathbb{P}(\omega) \text{ and } \int_\Omega X^-(\omega)d\mathbb{P}(\omega)$$

and then

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) = \int_\Omega X^+(\omega)d\mathbb{P}(\omega) - \int_\Omega X^-(\omega)d\mathbb{P}(\omega)$$

Theorem 6. *Let X and Y be a random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then:*

(1) if X takes a finite amount of values $x_1, x_2, ..., x_n$, then

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) = \sum_{k=1}^n x_k \mathbb{P}(\{X = x_k\})$$

(2) $X^+(\omega) = \max\{X(\omega), 0\} < \infty,\ X^-(\omega) = \max\{-X(\omega), 0\} < \infty$

$$\textit{if and only if} \qquad \int_\Omega |X(\omega)|d\mathbb{P}(\omega) < \infty$$

(3) If $X \leq Y$ almost surely,

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) \leq \int_\Omega Y(\omega)d\mathbb{P}(\omega) \qquad \textit{(if both integrals exist)}$$

and the equality is realized if $X = Y$ almost surely

(4) if $\alpha, \beta \in \mathbb{R}$

$$\int_\Omega (\alpha X(\omega) + \beta Y(\omega))d\mathbb{P}(\omega) = \alpha \int_\Omega X(\omega)d\mathbb{P}(\omega) + \beta \int_\Omega Y(\omega)d\mathbb{P}(\omega)$$

The **indicator function** will often be useful:

$$\mathbb{I}_A(\omega) = 1,\ \text{if } \omega \in A \text{ else } \mathbb{I}_A(\omega) = 0,$$

for any set A.

If we want to integrate a ramdom variable X over a subset A of Ω, we may use the **indicator function** to define

$$\int_A X(\omega)d\mathbb{P}(\omega) = \int_\Omega \mathbb{I}_A X(\omega)d\mathbb{P}(\omega)$$

Definition 7. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The **expected value** of X is

$$\mathbb{E}X = \int_\Omega X(\omega)d\mathbb{P}(\omega)$$

Care must be taken when the integrals diverge.

Theorem 8. *Let X and Y be a random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

(1) if X takes a finite amount of values $x_1, x_2, ..., x_n$, then

$$\mathbb{E}X = \sum_{k=1}^{n} x_k \mathbb{P}(\{X = x_k\})$$

(2) $X^+(\omega) = \max\{X(\omega), 0\} < \infty, \; X^-(\omega) = \max\{-X(\omega), 0\} < \infty$

if and only if $\quad \mathbb{E}|X| < \infty$

(3) If $X \leq Y$ almost surely,

$$\mathbb{E}X \leq \mathbb{E}Y d\mathbb{P}(\omega) \qquad \textit{(if both integrals exist)}$$

and the equality is realized if $X = Y$ almost surely

(4) if $\alpha, \beta \in \mathbb{R}$

$$\mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}X + \beta \mathbb{E}Y$$

(5) (Jensen's inequality) If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex and $\mathbb{E}|X| < \infty$, then

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X)$$

Times on times he divided, & measur'd
Space by space in his ninefold darkness
Unseen, unknown: changes appeard
In his desolate mountains rifted furious
By the black winds of perturbation
In realms of measure, sets unfold
Borel's shadow, secrets told
Sigma fields converge to bind
Lebesgue's dream, precision aligned
Cantor's dust, infinite unfold
Measurable paths, stories untold
Open closed, boundaries blur
Topology's dance, precision furor

Definition 9. The σ-algebra of **Borel subsets of** $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated from the closed real intervals $[a, b]$.
Informally, this is no more than collection of subsets of $\mathbb{R}$ that can be contructed from closed intervals using a countable amount of intersections, unions and complements "Most" real subsets are Borel subsets. It takes a substancial amount of effort to find contra-examples.

Definition 10. The Lebesgue measure on $\mathbb{R}$ which we denote by $\mathcal{L}$, maps each $B \in \mathcal{B}(\mathbb{R})$ to $[0, \infty]$ (that is, a non-negative real of ∞) such that

(1) $\mathcal{L}([a,b]) = b - a$, when $a \leq b$
(2) if $B_1, B_2, B_3, ...$ is a sequence of disjoint sets in $\mathcal{B}(\mathbb{R})$, then

$$\mathcal{L}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty}$$

Theorem 11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded real function, and let $a < b$, be reals.*

(1) The Riemann integral of $\int_a^b f(x)dx$ exists if and only if the subset of $[a,b]$ where f is not continuous has Lebesgue measure zero
(2) If the Riemann integral of $\int_a^b f(x)dx$ exists then f is Borel measurable (in particular, the Lebesgue integral $\int_a^b f(x)d\mathcal{L}(x)x$ exists) and the integrals are the same.

Definition 12. Given a property, if the set of reals that fail to have is a set with Lebesgue measure zero, we say that the property **holds almost aeverywhere**.

As the Riemann and Lebesgue integrals agree (when the Riemann integral exists), we will use simply the term *integral* to refer to either, and abuse the notation when no confuse arises. $\int_a^b f(x)dx$ will denote the Lebesgue integral (instead of $\int_{[a,b]} f(x)\mathcal{L}x$) We will also write $\int_A f(x)dx$, when A is not an interval.

1.2. **Convergence of integrals.**

For he strove in battles dire
In unseen conflictions with shapes
Bred from his forsaken wilderness.
Of beast, bird, fish, serpent & element
Combustion, blast, vapour and cloud
In infinite realms, areas unfold
Riemann's wings, summation told
Limits converge, precision's sway
Integration's dance, the mathematic way
Strong Law's grasp, a siren's call
Independent trials, destined to enthrall

Definition 13. Let $X_1, X_2, ...$ and be a sequence of random variables and X be a random variable, all defined on the same probability space

$(\Omega, \mathcal{F}, \mathbb{P})$. We say that $X_1, X_2, ...$ **converges to X almost surely** and write

$$\lim_{n\to\infty} X_n = X \qquad \text{almost surely}$$

if

$$\mathbb{P}(\{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X\}) = 1$$

Equivalently

$$\mathbb{P}(\{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) \neq X\}) = 0$$

A famous and important example is the **Strong Law of Large Numbers**.

Definition 14. Let $f_1, f_2, ...$ and be a sequence of real Borel-measurable functions and f be another such function We say that $f_1, f_2, ...$ **converges to f almost everywhere** and write

$$\lim_{n\to\infty} f_n = f \qquad \text{almost everywhere}$$

if the set

$$\{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \neq f(x)\}$$

has Lebesgue measure zero.

Theorem 15 (Monotone convergence)**.** *Let $X_1, X_2, ...$ be a sequence of random variables convering almost surely to a random variable X. If*

$$0 \leq X_1 \leq X_2 \leq \qquad \textit{almost surely}$$

then

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

Let $f_1, f_2, ...$ be a sequence of Borel-measurable functions on $\mathbb{R}$ converging almost everywhere to a function f. If

$$0 \leq f_1 \leq f_2 \leq \qquad \textit{almost everywhere}$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Theorem 16. *Let X be a nonnegative random variable that takes countably many values $x_1, x_2, ...$ Then*

$$\mathbb{E}X = \sum_{k=1}^{\infty} x_k \mathbb{P}(X = x_k)$$

Theorem 17 (Dominated Convergence)**.** *. Let $X_1, X_2, ...$ be a sequence of random variables convering almost surely to a random variable X. If there is a random variable Y such that $\mathbb{E}Y < \infty$ and $|X_n| \leq Y$ almost surely for every n, then*

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

Let $f_1, f_2, ...$ be a sequence of Borel-measurable functions on $\mathbb{R}$ converging almost everywhere to a function f. If there is a function g such that

$$\int_{-\infty}^{\infty} g(x)dx < \infty$$

and $|f_n| < g$ almost everywhere for every n, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

1.3. **Computation of Expectations.**

Dark revolving in silent activity:
Unseen in tormenting passions;
An activity unknown and horrible;
A self-contemplating shadow,
In enormous labours occupied

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We have defined the expectation of X to be the (Lebesgue) integral:

$$\mathbb{E}X = \int_{\Omega} X(\omega)d\mathbb{P}(\omega),$$

naturally, this approch highlights the linearity of expectation and has a certain intuitive appel. But working on general abstract spaces is quite cancerous. So we will see a few theorems to relate a general Ω to $\mathbb{R}$.

Theorem 18. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let g be a Borel-measurable function on $\mathbb{R}$. Then*

$$\mathbb{E}|g(X)| = \int_{\mathbb{R}} |g(x)|d\mu_X(x)$$

and if $\mathbb{E}|g(X)| < \infty$ then

$$\mathbb{E}g(X) = \int_{\mathbb{R}} g(x)d\mu_X(x)$$

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let g be a real Borel-measurable. Suppose that X has a density f, that is, f is a nonnegative Borel-measurable function such that for every B Borel subset of $\mathbb{R}$

$$\mu_X(B) = \int_B f(x)dx$$

Then

$$\mathbb{E}|g(X)| = \int_{-\infty}^{\infty} |g(x)|f(x)dx$$

and if $\mathbb{E}|g(X)| < \infty$ *then*

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

1.4. **Change of Measure.**

But Eternals beheld his vast forests.
Age on ages he lay, clos'd, unknown,
Brooding shut in the deep; all avoid
The petrific abominable chaos

Theorem 19. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* Z *be an almost surely nonnegative random variable with* $\mathbb{E}Z = 1$ *For* $A \in \mathcal{F}$ *define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega).$$

Then $\tilde{\mathbb{P}}$ *is a probability measure and if* X *is a nonnegative random variable*

$$\tilde{E}X := \int_\Omega X(\omega)d\tilde{\mathbb{P}}(\omega) = E[XZ]$$

If Z *is also surely (strictly positive), then*

$$\mathbb{E}Y = \tilde{E}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.

His cold horrors silent, dark Urizen
Prepar'd; his ten thousands of thunders
Rang'd in gloom'd array stretch out across
The dread world. & the rolling of wheels
As of swelling seas, sound in his clouds
In his hills of stor'd snows, in his mountains
Of hail & ice; voices of terror,
Are heard, like thunders of autumn,
When the cloud blazes over the harvests
