Notes of Linear Algebra

August 17, 2019

1 Vector Spaces

1.1 Fundamental Notions

The following theorem characterizes all linear transformations that have $\mathbb{R}^n$ as domain space.

Theorem 1.1. *If $\beta_1, \dots, \beta_n$ is any fixed list of vectors in a vector space W, then its "linear combination mapping" $\mathbf{x} \mapsto \sum_1^n x_i\beta_i$ is a linear transformation T from $\mathbb{R}^n$ to W, and $T(\delta^j) = \beta_j$ for $j = 1, \dots, n$. Conversely, if T is any linear mapping from $\mathbb{R}^n$ to W, and if we set $\beta_j = T(\delta^j)$ for $j = 1, \dots, n$, then T is the linear transformation mapping $\mathbf{x} \mapsto \sum_1^n x_i\beta_i$.*

Proof. We will only prove the converse statement. Let $T : \mathbb{R}^n \to W$ be a linear map and let $\beta_j = T(\delta^j)$. For any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have $T(\mathbf{x}) = T(\sum_1^n x_i\delta^i) = \sum_1^n x_iT(\delta^i) = \sum_1^n x_i\beta_i$. □

If $\alpha = (\alpha_1, \dots, \alpha_n) \in W^n$ we will denote its corresponding linear combination mapping by $L_\alpha : \mathbb{R}^n \to W$. If $T : \mathbb{R}^n \to W$ is a linear map, we call the n-tuple $(T(\delta^1), \dots, T(\delta^n))$ the *skeleton* of T. With this terminology in mind the previous theorem can be restated as follows.

Theorem 1.2. *For each n-tuple α in W^n, the map $L_\alpha : \mathbb{R}^n \to W$ is linear and its skeleton is α. Conversely, if T is any linear map from $\mathbb{R}^n$ to W, then $T = L_\beta$ where β is the skeleton of T.*

Or again:

Theorem 1.3. *The map $\alpha \mapsto L_\alpha$ is a bijection from W^n to the set of linear maps from $\mathbb{R}^n$ to , and $T \mapsto \mathsf{skeleton}(T)$ is its inverse.*

Definition 1.1. A *linear functional* is a linear transformation from a vector space V to the scalar field $\mathbb{R}$.

One special class of linear functionals are the so-called *coordinate functionals*. The i-th coordinate functional is a linear functional π_i from $\mathbb{R}^I$ to $\mathbb{R}$ and $i \in I$, defined by $\pi_i(f) := f(i)$.

Theorem 1.4. *Every linear mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ determines the $m \times n$ matrix $\boldsymbol{t} = \{t_{ij}\}$ having the skeleton of T as its columns, and the expression of the equation $\boldsymbol{y} = T(\boldsymbol{x})$ in linear combination form is equivalent to the m scalar equations*

$$y_i = \sum_{j=1}^{n} t_{ij} x_j$$

Conversely, each $m \times n$ matrix $\boldsymbol{t}$ determines the linear combination mapping having the columns of $\boldsymbol{t}$ as its skeleton, and the mapping $\boldsymbol{t} \mapsto T$ is therefore a bijection from the set of all $m \times n$ matrices to the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The skeleton of this linear transformation is $(T(\delta^1), \ldots, T(\delta^n))$, each of elements is an element of $\mathbb{R}^m$, now consider the following map:

$$\begin{bmatrix} T(\delta^1) \\ \vdots \\ T(\delta^n) \end{bmatrix} \mapsto \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & & \vdots \\ b_{1m} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix}$$

Where $\beta_i = T(\delta^i) = (b_{i1}, \ldots, b_{im})$ with $i = 1, \ldots, n$. The columns determine the skeleton of T and thus determine T itself, this process applies to any matrix and hence the map is invertible. (TO-DO: finish) $\square$

Theorem 1.5. *If $T : V \to W$ is linear, then the T-image of the linear span of any subset $A \subset V$ is the linear span of the T-image of A, i.e., $T[L(A)] = L(T[A])$. In particular, if A is a subspace, then so is $T[A]$. Furthermore, if Y is a subspace of W, then $T^{-1}[Y]$ is a subspace of V.*

Definition 1.2. Given $T : V \to W$ linear, the subspace $T^{-1}(0) = \{\alpha \in V : T(\alpha) = 0\}$ is called the *null space*, or *kernel*, of T, and is designated $\ker(T)$. The range of T is the subspace $T[V]$ of W, which will be designated by $\operatorname{Im}(T)$

Lemma 1.6. *A linear mapping T is injective if and only if its null space is $\{0\}$.*

Definition 1.3. Two vector spaces V and W are said to be isomorphic if there exists a bijective map $T : V \to W$ (also called an isomorphism).

Definition 1.4. Let $T : V \to V$ be a linear map and α a vector in V. If $T(\alpha) = x\alpha$ for some x in $\mathbb{R}$, then α is called an *eigenvector* and x is the corresponding *eigenvalue*.

1.2 Vector Spaces and Geometry