Linear Algebra; Notes

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Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Vector Spaces

1.1 Fields (optional)

Definition 1.1. A *field* is a set $\mathbb{F}$ together with the two following binary operations. Addition is a map:

$$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}; \quad (a, b) \mapsto a + b,$$

such that, if $\alpha, \beta \in \mathbb{F}$, then the following properties are satisfied:

1. Addition is commutative, i.e., $\alpha + \beta = \beta + \alpha$;
2. It is associative, i.e., if $\gamma \in \mathbb{F}$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;
3. There exists an element (called *additive identity*) $z \in \mathbb{F}$, such that $\alpha + z = \alpha$. It will be shown that this element is unique, thus it will always be denoted by 0 and called zero;
4. Every element is invertible, that is, there exists l such that $\alpha + l = 0$. As in the previous property, the additive inverse of an element α in uniquely determined, and thus will be denoted by $-\alpha$.

Multiplication, frequently denoted by $\times$ or $\cdot$, is a map:

$$\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}; \quad (a, b) \mapsto a \cdot b = ab,$$

satisfying, for all $\alpha, \beta \in \mathbb{F}$:

1. $\alpha\beta = \beta\alpha$;
2. If $\gamma \in \mathbb{F}$ then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
3. There exists $e \in \mathbb{F}$ (called an *multiplicative identity*) such that $e \neq 0$ and $e\alpha = \alpha$ for every $\alpha \in \mathbb{F}$. This element is unique and denoted by 1;
4. For every $\alpha \neq 0$, there exists $\gamma \in \mathbb{F}$ such that $\alpha\gamma = 1$. The element γ is uniquely determined by α so it will be denoted by α^{-1}.
5. *Multiplication is distributive over addition*, i.e., $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Example 1.1. $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are the most commonly encountered fields.

Example 1.2. The set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ together with the usual addition and multiplication is a field.

Proposition 1.1. *Let $\mathbb{F}$ be a field.*

1. *Both additive and multiplicative identities are unique;*
2. *For all $\alpha \in \mathbb{F}$, $\alpha 0 = 0$;*
3. *For all $\alpha \in \mathbb{F}$, $-\alpha = (-1)\alpha$.*

Proof. **1.):** Let $e, e' \in \mathbb{F}$ both additive identities, then $e = e + e' = e' + e = e'$. The uniqueness of the multiplicative identity follows from an identical argument.

2.) Let $\alpha \in \mathbb{F}$, $\alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$, add the additive inverse of $\alpha 0$ to both sides and we get the desired equality. **3.)** $\alpha + (-1)\alpha = (1 + (-1))\alpha = 0\alpha$ which is equal to 0 by the previous proposition. Thus we get $\alpha + (-1)\alpha = \alpha - \alpha \Leftrightarrow (-1)\alpha = -\alpha$. $\square$

1.2 Vector Spaces

Definition 1.2. Let V be a set and $\mathbb{F}$ an arbitrary field, provided with with the mappings:

- $(\alpha, \beta) \mapsto \alpha + \beta$; $\quad V \times V \to V$, called *addition*;
- $(x, \alpha) \mapsto x\alpha$; $\quad \mathbb{F} \times V \to V$, called *scalar multiplication*.

V is said to be a *vector space over* $\mathbb{F}$ with respect to these operations if:

1. For all $\alpha, \beta, \gamma \in V$ the equation $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ holds;
2. For all $\alpha, \beta \in V$, $\alpha + \beta = \beta + \alpha$;
3. There exists $0 \in V$ such that $\alpha + 0 = \alpha$ for all $\alpha \in V$;
4. For every $\alpha \in V$, there exists $\beta \in V$ such that $\alpha + \beta = 0$;
5. Let $1 \in \mathbb{F}$ be its multiplicative identity, then $1\alpha = \alpha$ for all $\alpha \in V$;
6. For all $x, y \in \mathbb{F}$ and $\alpha, \beta \in V$, $x(\alpha + \beta) = x\alpha + x\beta$ and $(x + y)\alpha = x\alpha + y\alpha$.

If $\mathbb{F} = \mathbb{R}$ then V is said to be a *real vector space*. If $\mathbb{F} = \mathbb{C}$ then V is said to be a *complex vector space*.

Example 1.3. Consider the set $\mathbb{C}$ of complex numbers. This set, together with a scalar multiplication map $(r, z) \in \mathbb{R} \times \mathbb{C} \mapsto rz \in \mathbb{C}$ and the usual complex number addition, form a real vector space.

Notation 1. Let F and A be sets. By F^A we mean the set of functions from A to F. Consider $\mathbb{R}^{\mathbb{R}}$ as an example, this is the set of all real-valued functions of one real variable.

Example 1.4. The set $\mathbb{R}^A$, where A is an arbitrary set, under the natural operations of addition of two functions and multiplication of a function by a real number, is a real vector space. Note that $\mathbb{R}^n$ with $n \in \mathbb{N}$ is a particular example of such sets, since $\mathbb{R}^{\{1,\dots,n\}} = \mathbb{R}^n$. [1]

Proposition 1.2. *Let V be a vector space over $\mathbb{F}$, then:*

1. *The additive identity, denoted 0, is unique;*
2. *Let $\alpha \in V$ then its additive inverse, denoted $-\alpha$, is unique;*
3. $\forall \alpha \in V : 0\alpha = 0$;
4. $\forall x \in \mathbb{F} : x0 = 0$;
5. $\forall \alpha \in V : (-1)\alpha = -\alpha$.

1.2.1 Subspaces

Definition 1.3. Let V be a vector space over $\mathbb{F}$. A subset U of V is said to be a *subspace* of V if it too is a vector space over $\mathbb{F}$.

Proposition 1.3. *Let V be a vector space over $\mathbb{F}$ and $U \subseteq V$. U is a subspace of V if and only if U satisfies the following conditions:*

1. $0 \in U$;
2. $\alpha, \beta \in U \Rightarrow \alpha + \beta \in U$;
3. $x \in \mathbb{F}$ *and* $\alpha \in U \Rightarrow x\alpha \in U$.

Definition 1.4. Let $U_1, \dots, U_m$ be subsets of V. We define its *sum* as

$$U_1 + \cdots + U_m := \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}.$$

2 Finite Dimensional Vector Spaces

2.1 Span and Linear Independence

[1] To be more precise, they are isomorphic. This concept will be explored later.