

# Advanced Calculus

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# 1 Vector Spaces

## 1.1 Fundamental Notions

### 1.1.1 First set of exercises

In what follows, A1, A2, A3, A4, S1, S2, S3, S4 refer to the axioms presented in the book.

**Problem 1.** Prove S3 for  $\mathbb{R}^3$  using the explicit display form  $\{x_1, x_2, x_3\}$  for ordered triples.

**Solution:** With  $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$  and  $x \in \mathbb{R}$ .

$$\begin{aligned} x((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= x(x_1 + y_1, x_2 + y_2, x_3 + y_3)8 \\ &= (x(x_1 + y_1), x(x_2 + y_2), x(x_3 + y_3)) \\ &= (xx_1 + xy_1, xx_2 + xy_2, xx_3 + xy_3) \\ &= (xx_1, xx_2, xx_3) + (xy_1, xy_2, xy_3) \\ &= x(x_1, x_2, x_3) + x(y_1, y_2, y_3). \end{aligned}$$

■

**Problem 2.** Show that given  $\alpha$ , the  $\beta$  postulated in A4 is unique.

**Solution:** Let  $\alpha, \beta, \beta' \in V$  a vector space over  $\mathbb{R}$ , such that  $\alpha + \beta = 0$  and  $\alpha + \beta' = 0$ . By transitivity

$$\begin{aligned} \alpha + \beta &= \alpha + \beta' \\ (\beta + \alpha) + \beta &= (\beta + \alpha) + \beta' \\ 0 + \beta &= 0 + \beta' \\ \beta &= \beta' \end{aligned}$$

■

**Problem 3.** Prove similarly that  $0\alpha = 0$ ,  $x0 = 0$ , and  $(-1)\alpha = -\alpha$ .

**Solution:** For the first equality,

$$\begin{aligned} 0\alpha &= (0 + 0)\alpha \\ &= 0\alpha + 0\alpha, \end{aligned}$$

by S2. Subtracting  $0\alpha$  from both sides yields the desired identity. Proving  $x0 = 0$  is identical, except S3 is used instead of S2. As for the last equation:

$$\begin{aligned} \alpha + (-1)\alpha &= (1 - 1)\alpha \\ &= 0\alpha \\ &= 0. \end{aligned}$$

We already determined that  $-\alpha$  is unique, so it follows that  $(-1)\alpha = -\alpha$ . ■

**Problem 4.** Prove that if  $x\alpha = 0$ , then either  $x = 0$  or  $\alpha = 0$ .

**Solution:** Assume  $a \neq 0$  and  $x \neq 0$ . Since  $\alpha \in \mathbb{R}$  has an inverse  $\alpha^{-1}$ .

$$\begin{aligned} x\alpha &= 0 = 0 \\ x\alpha\alpha^{-1} &= 0\alpha^{-1} \\ x &= 0. \end{aligned}$$

Contradiction. ■

**Problem 5.** Prove S1 for a function space  $\mathbb{R}^4$ . Prove S3.

**Solution:** Let  $x, y$  be elements of  $\mathbb{R}$ ,  $f$  and  $g$  real functions on  $A$ , and  $a$  an element of  $A$ .

Since  $f(a)$  is a real number,  $(xy)f(a) = x(yf(a))$  is just a consequence of associativity in  $\mathbb{R}$ . As for S3, note also that  $g(a) \in \mathbb{R}$ , thus

$$\begin{aligned} x(f+g)(a) &= x(f(a) + g(a)) \\ &= xf(a) + xg(a). \end{aligned}$$

Because no conditions were imposed on  $a$ , both equalities are valid for all elements of  $A$ . ■

The following theorem is not mentioned (so far) in the book but it is quite useful for checking whether or not a certain set is a subspace.

**Theorem 1.1.** *A subset  $U$  of  $V$  is a subspace of  $V$  if and only if  $U$  satisfies the following conditions:*

1.  $0 \in U$ ,
2.  $\alpha, \beta \in U$  implies  $\alpha + \beta \in U$ ,
3.  $x \in \mathbb{R}$  and  $\alpha \in U$  implies  $x\alpha \in U$ .

**Problem 6.** Given that  $\alpha$  is any vector in a vector space  $V$ , show that the set  $A = \{x\alpha \mid x \in \mathbb{R}\}$  of all scalar multiples of  $\alpha$  is a subspace of  $V$ .

**Solution:** We can see that  $0 \in A$  because  $0\alpha = 0$ . Now take  $\beta$  and  $\gamma$  elements of  $A$ ,

$$\begin{aligned} \beta + \gamma &= x\alpha + y\alpha \\ &= (x+y)\alpha, \end{aligned}$$

which is clearly an element of  $A$ . Finally, taking  $y \in \mathbb{R}$ ,  $y(x\alpha) = (yx)\alpha \in A$ . By theorem 1.1,  $A$  is a subspace of  $V$ . ■

**Problem 7.** Given that  $\alpha$  and  $\beta$  are any two vectors in  $V$ , show that the set of all vectors  $x\alpha + y\beta$ , where  $x$  and  $y$  are any real numbers, is a subspace of  $V$ .

**Solution:** Setting  $x = y = 0$  shows that the additive identity is in the set. Let  $\gamma = x\alpha + y\beta$  and  $\delta = x'\alpha + y'\beta$ .

$$\begin{aligned}\gamma + \delta &= (x\alpha + y\beta) + x'\alpha + y'\beta \\ &= (x + x')\alpha + (y + y')\beta.\end{aligned}$$

Finally, if  $z \in \mathbb{R}$  then  $z(x\alpha + y\beta) = (zx)\alpha + (zy)\beta$ . ■

**Problem 8.** Show that the set of triples  $\mathbf{x}$  in  $\mathbb{R}^3$  such that  $x_1 - x_2 + 2x_3 = 0$  is a subspace  $M$ . If  $N$  is the similar subspace  $\{\mathbf{x} \mid x_1 + x_2 + x_3 = 0\}$ , find a nonzero vector  $\mathbf{a}$  in  $M \cap N$ . Show that  $M \cap N$  is the set  $\{x\mathbf{a} \mid x \in \mathbb{R}\}$  of all scalar multiples of  $\mathbf{a}$ .

**Solution:** The intersection  $M \cap N$  is the set of all triples  $\mathbf{x} = (x_1, x_2, x_3)$  that satisfy the system:

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

The nonzero triple  $\mathbf{a} = (3, -1, -2)$  satisfies this system.

Let  $A = \{x\mathbf{a} \mid x \in \mathbb{R}\}$ . Clearly, if  $x \in \mathbb{R}$ , then  $x\mathbf{a} \in M \cap N$ , that is,  $A \subset M \cap N$ .

Now let  $\alpha = (a_1, a_2, a_3)$  be an element of  $M \cap N$ , then

$$\begin{aligned}a_1 - a_2 + 2a_3 &= a_1 + a_2 + a_3 \\ -2a_2 + a_3 &= 0 \\ a_3 &= 2a_2,\end{aligned}$$

and now substituting  $a_3$  by  $2a_2$ ,

$$\begin{aligned}a_1 + a_2 + 2a_2 &= 0 \\ a_1 &= -3a_2.\end{aligned}$$

So  $\alpha = (a_1, a_2, a_3) = (a_1, -\frac{1}{3}a_1, -\frac{2}{3}a_1) = a_1(1, -\frac{1}{3}, -\frac{2}{3}) = a_1 3\mathbf{a}$  ■

**Problem 9.** Let  $A$  be the open interval  $(0, 1)$ , and let  $V$  be  $\mathbb{R}^A$ . Given a point  $x$  in  $(0, 1)$ , let  $V_x$  be the set of functions in  $V$  that have a derivative at  $x$ . Show that  $V_x$  is a subspace of  $V$ .

**Solution:** The constant function  $I(x) = 0$  has a derivative at  $x$  and it is the identity of  $\mathbb{R}^A$ . Given  $f(x), g(x)$  functions in  $V_x$ , we know that  $(f + g)'(x) = f'(x) + g'(x) \Rightarrow f + g \in V_x$ . Also, given  $y \in \mathbb{R}$ ,  $(yf(x))' = yf'(x) \Rightarrow yf(x) \in V_x$ . ■

**Problem 10.** For any subsets  $A$  and  $B$  of a vectors space  $V$  we define the set sum  $A + B$  by  $A + B = \{\alpha + \beta \mid \alpha \in A \text{ and } \beta \in B\}$ . Show that  $(A + B) + C = A + (B + C)$ .

**Solution:**

$$\begin{aligned}(A + B) + C &= \{(\alpha + \beta) + \gamma \mid \alpha \in A, \beta \in B, \gamma \in C\} \\ &= \{\alpha + (\beta + \gamma) \mid \alpha \in A, \beta \in B, \gamma \in C\} \\ &= A + (B + C).\end{aligned}$$

■

**Problem 11.** Let  $A \subset V$  and  $X \subset \mathbb{R}$ , we similarly define,  $XA = \{x\alpha \mid x \in X \text{ and } \alpha \in A\}$ . Show that a nonvoid set  $A$  is a subspace if and only if  $A + A = A$  and  $\mathbb{R}A = A$ .

**Solution:** Let  $A \neq \emptyset$ .

( $\Rightarrow$ ) Suppose  $A$  is a subspace, clearly  $A + A = A$  by closure of addition on vector spaces. Likewise,  $\mathbb{R}A = A$  by closure under scalar multiplication.

( $\Leftarrow$ ) Suppose  $A + A = A$  and  $\mathbb{R}A = A$ . Since  $A \neq \emptyset$  then  $0 \in A$ . Let  $\alpha, \beta$  be elements of  $A$ ,  $\alpha + \beta \in A + A = A$  by hypothesis. Finally, let  $x$  be a real number and  $\alpha$  an element of  $A$ , then  $x\alpha \in \mathbb{R}A = A$ . By 1.1,  $A$  is a subspace of  $V$ . ■

**Problem 12.** Let  $V$  be  $\mathbb{R}^2$ , and let  $M$  be the line through the origin with slope  $k$ . Let  $\mathbf{x}$  be any nonzero vector in  $M$ . Show that  $M$  is the subspace  $\mathbb{R}\mathbf{x} = \{t\mathbf{x} \mid t \in \mathbb{R}\}$ .

**Solution:**  $M$  is the set  $\{(x, kx) \in \mathbb{R}^2\} \subset \mathbb{R}^2$ . Let  $\mathbf{x} = (x_1, kx_1) \neq 0$ , if  $t \in \mathbb{R}$  then  $t\mathbf{x} = (tx_1, tkx_1) \in M$ , hence  $\mathbb{R}\mathbf{x} \subset M$ . Conversely, if  $(y, ky) \in M$  and  $(y, ky) = y(1, k) = y \frac{1}{x_1}(x_1, kx_1) = yx_1^{-1}\mathbf{x} \in \mathbb{R}\mathbf{x} \Rightarrow M \subset \mathbb{R}\mathbf{x}$ . Thus  $M = \mathbb{R}\mathbf{x}$ . ■

**Problem 13.** Show that any other line  $L$  with the same slope  $k$  is of the form  $M + \mathbf{a}$  for some  $\mathbf{a}$ .

**Solution:**

■