Notes on Topology

January 14, 2020

Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Topological Spaces

1.1 Topology

Definition 1.1. Let E be a set. The set $\tau \subseteq \mathcal{P}(E)$ is said to be a *topology on E* if it satisfies the following axioms:

1. $\emptyset, E \in \tau$,
2. $O_1, O_2 \in \tau \Rightarrow (O_1 \cap O_1 \in \tau)$,
3. $(O_j \in \tau : j \in J) \Rightarrow (\cup_{j \in J} O_j) \in \tau$.

A *topological space* is an ordered pair (E, τ) where E is a set and τ a topology on E. An element of $X \in \mathcal{P}(E)$ is said to be an *open set* if $X \in \tau$, and said to be a *closed set* if $E \setminus X \in \tau$.

Definition 1.2. A topological space (E, τ) is said to *Hausdorff* if

$$\forall a, b \in E \text{ such that } a \neq b, \text{ there exists } O_a, O_b \in \tau \text{ and } O_a \cap O_b \neq \emptyset$$

Definition 1.3. Let τ_1, τ_2 be two topologies on a set E. The topology τ_1 is said to be *finer* than τ_2 if $\tau_2 \subseteq \tau_1$. If in addition $\tau_1 \neq \tau_2$, then τ_1 is said to be *strictly finer* than τ_2.

Two topologies are said to be *comparable* if one is finer than the other.

1.2 Interior, exterior, boundary and closure

Definition 1.4. Let (E, τ) be a topological space, a an element of E, and $X \in \mathcal{P}(E)$. Then:

$$\begin{aligned}
a \text{ is an interior point of } X &\quad\Leftrightarrow\quad & \exists O \in \tau, a \in O \text{ and } O \subseteq X, \\
a \text{ is an exterior point of } X &\quad\Leftrightarrow\quad & \exists O \in \tau, a \in O \text{ and } O \subseteq E \setminus X, \\
a \text{ is a boundary point of } X &\quad\Leftrightarrow\quad & \forall O \in \tau, a \in O, O \cap X \neq \emptyset \text{ and } O \cap (E \setminus X)
\end{aligned}$$

Notation 1. Note that the following sets depends on the fixed topology.

- $\operatorname{int} X$ is the set of all interior points of X,

- $\operatorname{ext} X$ is the set of all exterior points of X,
- ∂X is the set of all boundary points of X,
- The set $\overline{X} = \operatorname{int} X \cup \partial X$ is called the *closure* of X.

Remark 1. The elements of $\overline{X}$ are called *adherent* point of X. It follows that

$$a \text{ is an adherent point of } X \Leftrightarrow \forall O \in \tau, a \in O \Rightarrow (O \cap X) \neq \emptyset.$$

Definition 1.5. Let $(E, \tau$ be a topological space, let X be a subset of E and $a \in E$. The element a is said to be an *accumulation* point of X if:

$$\forall O \in \tau, a \in O \Rightarrow \exists b, a \neq b.$$

The set of such points is called the *derivative* of X, denoted X'. It follows that $X \cup X' = \overline{X}$.

Proposition 1.1. *Let (E, τ) be a topological space and $X \in \mathcal{P}(E)$. Then*

1. $a \in \overline{X} \Leftrightarrow \forall O \in \tau, a \in O \Rightarrow (O \cap X) \neq \emptyset$,
2. $\operatorname{int} X \subseteq X \subseteq \overline{X}$,
3. $X \in \tau \Leftrightarrow X = \int X$,
4. $(E \setminus X) \in \tau \Leftrightarrow \overline{X} = X$,
5. $\overline{X} = X \cup \partial X$,
6. $X \subseteq Y \Rightarrow \operatorname{int} X \subseteq \operatorname{int} Y$ *and* $\overline{X} \subseteq \overline{Y}$,
7. $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ *and* $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$,
8. $\overline{\overline{X}} = X$.

Definition 1.6. Let (E, τ) be a topological space, and $X, Y \subseteq E$. The set X is said to be *dense* on Y if $Y \subseteq \overline{X}$.

Definition 1.7. Let (E, τ) be a topological space; it is said to be *separable* if there exists a subset X of E that is both dense and countable.

1.3 Neighborhoods

Definition 1.8. Let (E, τ) be a topological space and $a \in E$. A subset X of E is said to be a *neighborhood* of a if there exists $O \in \tau$, such that $a \in O$ and $O \subseteq X$.

The set X is said to be a neighborhood of $A \subseteq E$, if for every element $a \in A$, X is a neighborhood of a. The set of all neighborhoods of a point a is denoted by $\mathcal{V}_a$, and $\mathcal{V}_A$ denotes the set of all neighborhoods of an arbitrary set A.

Proposition 1.2.

$$X \in \tau \Leftrightarrow X \textit{ is a neighborhood of } x \in X,$$
$$V_1, V_2 \in \mathcal{V}_a \Rightarrow (V_1 \cap V_2) \in \mathcal{V}_a,$$
$$V \in \mathcal{V}_a \textit{ and } V \subseteq W \Rightarrow W \in \mathcal{V}_a,$$
$$V \in \mathcal{V}_a \Rightarrow \mathrm{int}(V) \in \mathcal{V}_a.$$

Definition 1.9. Let $\mathcal{W}$ be a class of neighborhoods of a given point a (or a given set X) on the topological space (E, τ). We call $\mathcal{W}$ a *fundamental neighborhood system* of a (or X) if for every neighborhood V of a (or X), there exists $W \in \mathcal{W}$ such that $W \subseteq V$.

Definition 1.10. A topological space (E, τ) is said to satisfy the *first axiom of countability* if every $a \in E$, has a countable fundamental neighborhood system.

Definition 1.11. Let $(J, \leq)$ be a well-ordered set. A class of neighborhoods $(V_j)_{j \in J}$ of a, indexed on J, is said to be a *nested* fundamental system of neighborhoods of a, if $V_k \subseteq V_j$ whenever $j \leq k$.

Proposition 1.3. *Every topological space that satisfies the first axiom of countability has a nested fundamental neighborhood system.*

Proof. Let (E, τ) be a topological space, and let $(W_n)_{n \in \mathbb{N}}$ be a fundamental neighborhood system of $a \in E$. For each $n \in \mathbb{N}$, define $V_n = \cap_{k=0}^{n} W_k$. Then $(V_n)_{n \in \mathbb{N}}$ is a fundamental neighborhood system of a, and $V_k \subseteq V_i$ whenever $i \leq k$. Therefore $(V_n)_{n \in \mathbb{N}}$ is a nested fundamental neighborhood system of a. □

Theorem 1.4. *Let (E, τ) be a topological space; (E, τ) is separated if and only if, for every $a \in E$, the intersectiion of closed neighborhood of a is equal to $\{a\}$.*

Proof. Exercise! □

Corollary 1.4.1. *For every Hausdorff (separated) topological space E and $a \in E$, the singleton $\{a\}$ is a closed set.*

Definition 1.12. In a topological space E, a point $a \in E$ is said to be *isolated* if $\{a\}$ is open, or equivalently, if $\{a\}$ is a neighborhood of a.

If E is a topological space, the class of all open sets of E, such that $a \in E$ is a fundamental neighborhood system of a. Therefore every point has a fundamental system of open neighborhoods. It is not true that every point has a fundamental system of closed neighborhoods. Hence the following definition.

Definition 1.13. A topological space E is said to be *regular* if it is Hausdorff and if every point $a \in E$ has a fundamental system of closed neighborhoods.

Definition 1.14. The class $\mathcal{B} \subseteq \mathcal{P}(E)$ is said to be a basis of a topological space (E, τ) (or a basis of τ) if, $\mathcal{B} \subseteq \tau$ and every open set of E is the intersection of elements of $\mathcal{B}$.

Remark 2. A basis for a topology τ determines τ in its entirety, since every open is the intersection of elements from the basis. If a topological space has a countable basis, then it is said to satisfy the *second axiom of countability*.

Theorem 1.5. *Let (E, τ) be a topological space and $\mathcal{B} \in \tau$. The class $\mathcal{B}$ is a basis of E if and only if, for every $a \in E$, the class $\{X \in \mathcal{B} : a \in X\}$ is a fundamental neighborhood system of a.*

Proof. Exercise! □

Theorem 1.6. *Let $\mathcal{B} \subseteq \mathcal{P}(E)$. Then there exists a unique topology τ in E, for which $\mathcal{B}$ is a basis of the topological space (E, τ) if and only if, $\mathcal{B}$ satisfies the following two conditions:*

1. $$\left(\bigcup_{A \in \mathcal{B}} A\right) = E.$$

2. *For all $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists $C \in \mathcal{B}$, such that $x \in C$ and $C \subseteq (A \cap B)$.*

Proof. □

2 Continuity and Limits

2.1 Continuity

Definition 2.1. Let (E_1, τ_1) and (E_2, τ_2) be topological spaces; let $f : (E_1, \tau_1) \to (E_2, \tau_2)$ be a function and a a point of E_1. The function f is *continuous* on a if:

$$\forall O_2 \in \tau_2, f(a) \in O_2 : \exists O_1 \in \tau_1, a \in O_1 \Rightarrow f(x) \in O_2.$$

This may be equivalently stated as:

$$\forall V \in \mathcal{V}_{f(a)}, \exists U \in \mathcal{V}_a : f(U) \subseteq V.$$

We'll prove that these two definitions are indeed equivalent.

Proof. □

Theorem 2.1. *Let E_1, E_2 be topological spaces and $f : E_1 \to E_2$ a function. The following conditions are equivalent:*

1. *f is continuous;*

2. *If O is an open set of E_2 then its pre-image $f^{-1}(O)$ is an open set of E_1;*

3. *If C is a closed set of E_2, then $f^{-1}(C)$ is closed in E_1.*

Proof. □

Definition 2.2. Let (E_1, τ_1) and (E_2, τ_2) be two topological spaces and $f : (E_1, \tau_1) \to (E_2, \tau_2)$ a function. The function f is said to be *open* (respectively *closed*) if $f(O) \in \tau_2$ (respectively $E_2 \setminus f(O) \in \tau_2$) whenever $O \in \tau_1$ (respectively $E_1 \setminus O \in \tau_1$).

Theorem 2.2. *Let E_1, E_2, E_3 be topological spaces and $f : E_1 \to E_2$, $g : E_2 \to E_3$ two given functions. If f is continuous at $a \in E_1$ and g is continuous at $b = f(a) \in E_2$, then $g \circ f$ is continuous at $a \in E_1$.*

Proof. □

Definition 2.3. If $f : E_1 \to E_2$ is continuous and bijective, then it is called a *homeomorphism*. The topological space E_1 is said to be *homeomorphic* to E_2 if such a function exists.

2.2 Limits

Definition 2.4. Let (E_1, τ_1) and (E_2, τ_2) be topological spaces. Let $f : X \subseteq E_1 \to E_2$ be a function, let $a \in \overline{X}$ and $\lambda \in E_2$.

- We say that $f(x)$ *tends to* λ when x *tends to* a, if:

$$\forall O_2 \in \tau_2, \lambda \in O_2 : \exists O_1 \in \tau_1, a \in O_1 : x \in (O_1 \cap X) \Rightarrow f(x) \in O_2.$$

- If (E_2, τ_2) is a Hausdorff space, then λ is said to be the *limit* of $f(x)$ as x tends to a, and write:

$$\lim_{x \to a} f(x) = \lambda.$$

- Let $D \subseteq X$, suppose $a \in \overline{D}$. We say that $f(x)$ tends to λ when x tends to a by elements of D if f restricted to D tends to λ when x tends to a. In this case we write:

$$\lambda = \lim_{x \to a, x \in D} f(x) \text{ ,which is logically equivalent to:}$$

$$\forall O_2 \in \tau_2, \lambda \in O_2 : O_1 \in \tau_1, a \in O_1 : x \in (O_1 \cap D) \Rightarrow f(x) \in O_2.$$

2.3 Sequences

Definition 2.5. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence defined on a topological space (E, τ). The sequence (x_n) is said to converge to $a \in E$ as n tends to $+\infty$, written as $x_n \to a$ (meaning $a = \lim_{n \to +\infty} x_n$), if:

$$\forall O \in \tau, a \in O : \exists p \in \mathbb{N}, n \geq p \Rightarrow x_n \in O.$$

Theorem 2.3. *Let (E, τ) a topological space, $X \subseteq E$, and $a \in E$.*

3 Metric Spaces

3.1 Metrics

Definition 3.1. A function $d : E \times E \to \mathbb{R}$ is said to be a *metric* on a set E if, for any $x, y, z \in E$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0 \Leftrightarrow x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (E, d) is called a *metric space* if d is a metric on E.

TO-DO: Insert Examples

3.2 The Topology of a Metric Space

Given any metric space (E, d) it is possible to define a topology on it. Consider the sets

$$B_{op}(a; R) \coloneqq \{x \in E : d(x, a) < R\},$$
$$B_{cl}(a; R) \coloneqq \{x \in E : d(x, a) \leq R\}$$

The former is called the *open ball* (the latter, *closed ball*) with center a and radius R. We also define the *sphere* centered at a with radius R to be the set

$$S(a; R) \coloneqq \{x \in E : d(x, a) = R\}.$$

By convention $B(a; R) = B_{op}(a; R)$.

The geometric motivation for these definitions becomes clear once the $\mathbb{R}^2$ case is considered.

TO-DO: Insert pictures

From these definitions we may easily deduce that:

$$\begin{aligned} B_{cl}(a; R) = B_{op}(a; R) \cup S(a; R); \\ B_{op}(a; R) \subset B_{cl}(a; R); \\ R < R' \Rightarrow B_{cl}(a; R) \subset B_{op}(a; R). \end{aligned}$$

Definition 3.2. Let (E, d) be a metric space and $X \subset E$. The set X is said to be *bounded* if $\exists a \in E$ and $\exists R \in \mathbb{R}_+$, such that $X \subseteq B_{op}(a; R)$.

Analogously, a function $f : I \to E$ (with I an arbitrary set) is bounded if $f(I)$ is a bounded set. In particular, a sequence $(x_n)_{n \in \mathbb{N}}$ on E is bounded if:

$$\exists a \in E, \exists R \in \mathbb{R}_+ : \forall n \in \mathbb{N} : d(x_n, a) \leq R.$$

Definition 3.3. Let (E, d) be a metric space. Consider the following set:

$$\tau_d := \{O \subset E : \forall a \in O, \exists R \in \mathbb{R}_+ : B(a; R) \subset O\}.$$

This set will be called the *generated topology by the metric* d on E.

Theorem 3.1. *Every metric space is Hausdorff.*

Apply concepts from topology to derive a "definition" of convergence for metric spaces.

Definition 3.4. Let d_1 and d_2 be two metrics on E; they are said to be *equivalent* if $\tau_{d_1} = \tau_{d_2}$.

Proposition 3.2. *Let d_1, d_2 be metrics on E. If there exists $\alpha, \beta \in \mathbb{R}_+$ such that, for every $x, y \in E$:*

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$$

then, d_1 and d_2 are equivalent metrics.

3.3 Continuity

Definition 3.5. Let $(E_1, 1)$ and (E_2, d_2) be metric spaces. A function $f : E_1 \to E_2$ is *continuous at point* $a \in E_1$ if:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E_1 : x \in B(a; \delta) \Rightarrow f(x) \in B(f(a); \epsilon).$$

Two alternative equivalent statements:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E_1 : d_1(x; a) \leq \delta \Rightarrow d_2(f(x); f(a)) \leq \epsilon;$$
$$\forall \epsilon > 0, \exists \delta > 0 : f(B(a; \delta)) \subset B(f(a); \epsilon).$$

The function f is said to be *continuous*, if f is continuous at every point in E_1. If f is continuous and additionally satisfies:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, a \in E_1 : d_1(x; a) \leq \delta \Rightarrow d_2(f(x); f(a)) \leq \epsilon$$

then f is said to be *uniformly continuous.*

Definition 3.6. A function $f : E_1 \to E_2$ is *Lipschitz continuous* if there exists $C \in \mathbb{R}_+$, such that:

$$\forall x, y \in E_1 : d_2(f(x), f(y)) \leq C d_1(x, y).$$

To the real number L_f defined by:

$$L_f = \inf\{C \in \mathbb{R} : \forall x, y \in E_1 : d_2(f(x), f(y)) \leq C d_1(x, y)\},$$

is called a *Lipschitz constant.*

Definition 3.7. A function $f : E_1 \to E_2$ is a *contraction* if it is Lipschitz continuous and $L_f < 1$.

Theorem 3.3. *Let I be a non-degenerate interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function on I and differentiable on (I). Then f is lipschitzian, if and only if, f' is bounded. In this case $L_f = \sup_{y \in \mathrm{int}(I)]} \mid f'(y) \mid$.*

4 Compactness

4.1 Compact Spaces

Definition 4.1. Let A be an arbitrary set.

- A collection $\mathcal{T}$ of subsets of A is said to be a *covering* of A if $\cup_{t\in\mathcal{T}} t = A$;
- A *subcovering* of $\mathcal{T}$ is a covering $\mathcal{T}'$ of A such that $\mathcal{T}' \subset \mathcal{T}$;
- A covering $\mathcal{T}$ is said to be *finite* if $\mathcal{T}$ is a finite collection;
- A covering $\mathcal{T}$ is said to be *open* if its elements are all open sets.

Definition 4.2. A topological space E satisfies the *Heine-Borel-Lebesgue* (H-B-L) property if every open covering of E has finite subcovering.

Definition 4.3. A Hausdorff space E that satisfies the H-L-B is called *compact*.

Theorem 4.1. *Let E be a Hausdorff space; then E is compact, if and only if, for every collection $\{F_j\}_{j\in J}$ of closed subsets of E such that $\cap_{j\in J} F_j = \emptyset$, there exists a finite number of sets $F_1, \ldots, F_n$ such that $\cap_{i=1}^{n} F_i = \emptyset$.*

Definition 4.4. A subset X of a topological space (E, τ) is said to be compact if (X, τ_E) is a compact space, where τ_E is the induced topology.

Theorem 4.2. *Let E be a Hausdorff space. Let $\{X_1, \ldots, X_n\}$ a collection of compact subsets of E such that: $X = \cup_{i=1}^{n} X_i$. Then, X is compact.*

Theorem 4.3. *Let E be a Hausdorff space and $X \subset E$ compact. Then, X is closed.*

Theorem 4.4. *Let E be a compact topological space and let $X \subset E$ be a closed set. Then, X is compact.*

Theorem 4.5. *Let E be a compact topological space. Then, every $a \in E$ has a fundamental system of compact neighborhoods.*

Corollary 4.5.1. *Every compact space E is regular.*

Theorem 4.6. *Let E_1 be a compact topological space, E_2 a Hausdorff space, and $f : E_1 \to E_2$ a continuous function. Then, $f(E_1)$ is a compact subset of E_2.*

Corollary 4.6.1. *Let E_1 and E_2 be two homeomorphic topological spaces. Then E_1 is compact, if and only if, E_2 is compact.*

Definition 4.5. A topological space E is said to be *sequentially compact* if it is Hausdorff and every sequence $x : \mathbb{N} \to E$ has a convergent subsequence.

Definition 4.6. A topological space is *locally compact* if it is Hausdorff and every point has a compact neighborhood.

4.2 Compactness in Metric Spaces

Theorem 4.7. *Let E be a metrizable topological space. The space E is compact, if and only if, every sequence on E has a convergent subsequence.*

Theorem 4.8. *Let (E, d) be a compact metric space; let (E_0, d_0) any metric space, and let $f : E \to E_0$ be a continuous function. Then f is uniformly continuous.*

Theorem 4.9. *Let $a, b \in \mathbb{R}$ with $a < b$, then the interval $[a, b]$ is compact.*

Corollary 4.9.1. *The space $\mathbb{R}$ is locally compact.*

Theorem 4.10. *Let $(E_1, d_1), \ldots, (E_n, d_n)$ be a list of compact metric spaces, and let $E = E_1 \times \cdots \times E_n$*

5 Connectedness

5.1 Connected Spaces

Definition 5.1. Let E be a topological space. E is *disconnected* if two non-empty disjoint open sets U, V exist such that $E = U \cup V$. The space is said to be *connected* if this isn't the case.

Theorem 5.1. *Let E_1, E_2 be topological spaces and let $f : E_1 \to E_2$ be a function. If f is continuous and E_1 is connected, then $f(E_1)$ is connected.*

Theorem 5.2. *Let E be a topological space, let $J \neq \emptyset$, and let $(X_j)_{j \in J}$ be a collection of connected subsets of E. Then, if $X_i \cap X_j \neq \emptyset$ for every $i, j \in J$, the set $\cup_{j \in J} X_j$ is connected.*

Theorem 5.3. *Let E be a topological space and let X be a subset of E. Then, if X is connected, so is $\overline{X}$.*

Definition 5.2. Let E be a topological space. We define the following equivalence relation on E:

$$xRy \Leftrightarrow \exists S \subset E \land S \text{ connected } \land x, y \in S$$

If $(x, y) \in R$ then x is said to be *connected* to y. The equivalence classes determined by R on E are called the connected components of E.

Theorem 5.4. *Let E be a topological space, a an element of E, and V the connected component to which a belongs. Then*

$$V = \bigcup_{\substack{J \text{ connected subset of } E \\ a \in J}} J,$$

and V is a closed connected subset of E.

Theorem 5.5. *Let X be a subset of $\overline{\mathbb{R}}$. The set X is connected, if and only if X is an interval of $\overline{\mathbb{R}}$*

5.2 Arcwise Connected Spaces

Definition 5.3. Let E be a topological space and let $a, b \in E$. An *arc from a to b* is a continuous map $f : [\alpha, \beta] \subset \mathbb{R} \to E$, such that $\alpha \leq \beta$ and $f(\alpha) = a$, $f(\beta) = b$.

Definition 5.4. A topological space E is *arcwise connected* if, for every $(a, b) \in E \times E$, there exists an arc from a to b.

Theorem 5.6. *Every connected space is also arcwise connected.*

5.3 Locally Connected Spaces

Definition 5.5. A topological space E is said to be *locally connected* if for every point $a \in E$, there exists a fundamental system of connected neighborhoods.

Theorem 5.7. *Every connected component of a locally connected space is both open and closed.*

Definition 5.6. A topological space is said to be *arcwise locally connected* if for every point there exists a fundamental system of arcwise connected neighborhoods.