Notes on Complex Analysis

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Contents

1 Analytic Functions

1.1 The Complex Number System

Definition 1.1. Consider the following maps:

- **Addition**

$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

- **Scalar Multiplication**

$$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$
$$\alpha(x, y) = (\alpha x, \alpha y)$$

- **Multiplication**

$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

These operations, together with the set $\mathbb{R}^2$, form a vector space over $\mathbb{R}$, which we call the complex number system, denoted by $\mathbb{C}$. We identify a real number x with the pair $(x, 0)$, and $(0, 1)$ will be denoted by i. With this we recover the conventional notation, for

$$(x, y) = (x, 0) + (0, 1)(y, 0) = x + iy.$$

We call the x-axis and y-axis by *real* and *imaginary* axis, respectively. Given $z = a + bi \in \mathbb{C}$, we call a the *real part* of z, denoted by $\Re(z)$, and b the *imaginary part* of z, denoted by $\Im(z)$. Finally, z is said to be a *pure imaginary number* if $\Re(z) = a = 0$.

1.1.1 Algebraic Properties

Proposition 1.0.1. *Let z be a non-zero complex number, then there exists $z' \in \mathbb{C}$, such that*

$$z \cdot z' = 1,$$

called the inverse of z.

Proof. Let $z = a + bi$ and $z' = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$. $z \neq 0$ implies that $a^2 + b^2 \neq 0$. Furthermore,

$$\begin{aligned} z \cdot z' &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}\right) \\ &= \left(\frac{a^2 + abi - abi + b^2}{a^2 + b^2}\right) \\ &= 1. \end{aligned}$$

□

The inverse of a complex number z is unique, and represented by z^{-1}; the symbol z/w means zw^{-1}.

Theorem 1.1. *$\mathbb{C}$, together with the previously defined addition and multiplication, is a field.*

1.1.2 Roots of Quadratic Equations

Proposition 1.1.1. *Let $z \in \mathbb{C}$. Then there exists a complex number $w \in \mathbb{C}$ such that $w^2 = z$.*

[To-Finish]

1.2 Properties of Complex Numbers

1.2.1 Polar Representation

The *modulus* of a complex number $z = a + bi$ is its norm, i.e., $\|a + bi\| = \big\|(a, b)\big\| = \sqrt{a^2 + b^2}$, conventionally written as $|z|$. Let θ be the angle that z makes with the positive real axis, where $0 \leq \theta < 2\pi$, and $r = |z|$ Then z may be rewritten as

$$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

This way of writing z is called the *polar coordinate representation.* The angle θ is called the *argument* of z and is denoted $\theta = \arg z$. [To-Do: Insert Figures] The inverval $[0, 2\pi[$ is an arbitrary choice, any other interval $[a, b[$ of length 2π could be specified and the resulting representation would be unique, granted that the relevant complex number is not equal to zero. Alternatively, $\arg z$ may be defined as the set of values $\{\theta + 2n\pi : n \in \mathbb{Z}\}$. Specifying a particular suitable interval for the angle is known as choosing a *branch of the argument.*

1.2.2 Multiplication of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1z_2 &= r_1r_2[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2)] + i[\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Which proves the following proposition.

Proposition 1.1.2. *For any complex numbers z_1, z_2,*

$$|z_1z_2| = |z_1||z_2| \text{ and } \arg(z_1z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$$

Example 1. Let $z_1 = -1$ and $z_2 = -i$, then $\arg z_1 = \pi$ and $\arg z_2 = 3\pi/2$. Since $z_1z_2 = i$, then $\arg z_1z_2 = \pi/2$. Using the previous proposition, we'd get $\arg z_1 + \arg z_2 = \pi + 3\pi/2 = 5\pi/2$, which isn't in the interval $[0, 2\pi[$. Subtracting 2π from this result yields the correct value for $\arg z_1z_2$.

1.2.3 De Moivre's Formula

Proposition 1.1.3. *If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof. Use induction. $\square$

As an application, consider the equation $z^n = w$, with $w \in \mathbb{C}$. Suppose that $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\psi + i\sin\psi)$. By De Moivre's formula, $z^n = \rho^n(\cos n\psi + i\sin n\psi)$, which implies that $\rho^n = r = |w|$ and $n\psi = \theta + 2k\pi$, where k is some integer. Thus

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right].$$

Each value of $k = 0, 1, \ldots, n-1$, gives different values of z. Any other value for k repeats one of the values of z corresponding to $k = 0, 1, \ldots, n-1$. Thus there are exactly n n-th roots of a nonzero complex number.

Corollary 1.1.1. *Let $w \in \mathbb{C} \setminus \{0\}$, with polar representation $w = r(\cos\theta + i\sin\theta)$. The nth roots of w are given by the n complex numbers*

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, 1, \ldots n-1.$$

1.2.4 Complex Conjugation

The transformation $a + bi \mapsto a - bi$ is called *complex conjugation*, and $a - bi$ is the *conjugate* of $a + bi$. The conjugate of $z \in \mathbb{C}$ is denoted by $\overline{z}$.

Proposition 1.1.4. *Let $z, w \in \mathbb{C}$, then*

1. $\overline{z + w} = \overline{z} + \overline{w}$,
2. $\overline{zw} = \overline{zw}$,
3. *If $w \neq 0$, then $\overline{z/w} = \overline{z}/\overline{w}$,*
4. $z\overline{z} = |z|^2$,
5. $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$,
6. $\Re z = (z + \overline{z})/2$ *and* $\Im z = (z - \overline{z})/2i$,
7. $\overline{\overline{z}} = z$.

Proposition 1.1.5. *Let $z, w, z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$, then*

1. $|zw| = |z||w|$,

2. $-|z| \leq \Re z \leq |z|$ *and* $-|z| \leq \Im z \leq |z|$,

3. $|\overline{z}| = |z|$,

4. $|z + w| \leq |z| + |w|$,

5. $|z - w| \geq \big||z| - |w|\big|$,

6. $|z_1 w_1 + \ldots z_n w_n| \leq \sqrt{|z_1|^2 + \ldots |z_n|^2}\sqrt{|w_1|^2 + \ldots + |w_n|^2}$.

1.3 Elementary Functions

1.3.1 The Exponential Function

Definition 1.2. Let $z = x + iy \in \mathbb{C}$, we define

$$\exp(z) = e^z = e^x(\cos y + i \sin y).$$

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *periodic* if there exists $w \in \mathbb{C}$ such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. The following proposition establishes some of the properties of the exponential function.

Proposition 1.1.6. *Let* $z, w \in \mathbb{C}$,

1. $e^{z+w} = e^z e^w$,

2. $e^z \neq 0$,

3. *If* $x \in \mathbb{R}$, *then* $e^x > 1$ *when* $x > 0$ *and* $0 < e^x < 1$ *when* $x < 0$,

4. $\left|e^{x+yi}\right| = e^x$,

5. e^z *is periodic; each period for* e^z *has the form* $2k\pi i$ *for some integer* k,

6. $e^z = 1 \Leftrightarrow z = 2k\pi i$ *for some integer* k.

1.3.2 Trigonometric Functions

Definition 1.3. Let z be a complex number.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Proposition 1.1.7. *Let* $z, w \in \mathbb{C}$

1. $\sin^2 z + \cos^2 z = 1$,

2. $\sin(z + w) = \sin z \cos w + \cos z \sin w$,

3. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

1.3.3 Logarithm Function

Since the complex exponential function is not bijective, we cannot naively define its inverse, the logarithm.

Proposition 1.1.8. *Consider the set* $A_{y_0} = \{x + iy : x \in \mathbb{R}$ *and* $y_0 \leq y < y_0 + 2\pi\}$*. Then the exponential function with its domain restricted to this set,* $e^z : A_{y_0} \to \mathbb{C} \setminus \{0\}$*, is injective.*

Definition 1.4. The function $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, with range $y_0 \leq \operatorname{Im} \log z < y_o + 2\pi$, is defined by

$$\log z = \log|z| + i \arg z,$$

where $\arg z \in [y_0, y_0 + 2\pi[$, and $\log|z|$ is the usual logarithm where $|z|$ is a real number.

The logarithm function is only well-defined when we fix a branch, that is an interval of length 2π, in which $\arg z$ takes its values. (????)