

# Advanced Calculus

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# 1 Vector Spaces

## 1.1 Fundamental Notions

### 1.1.1 First set of exercises

In what follows, A1, A2, A3, A4, S1, S2, S3, S4 refer to the axioms presented in the book.

**Problem.** Prove S3 for  $\mathbb{R}^3$  using the explicit display form  $\{x_1, x_2, x_3\}$  for ordered triples.

**Solution:** With  $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$  and  $x \in \mathbb{R}$ .

$$\begin{aligned} x((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= x(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x(x_1 + y_1), x(x_2 + y_2), x(x_3 + y_3)) \\ &= (xx_1 + xy_1, xx_2 + xy_2, xx_3 + xy_3) \\ &= (xy_1 + xx_1, xy_2 + xx_2, xy_3 + xx_3) \\ &= (xy_1, xy_2, xy_3) + (xx_1, xx_2, xx_3) \\ &= x(y_1, y_2, y_3) + x(x_1, x_2, x_3). \end{aligned}$$

■

**Problem.** Show that given  $\alpha$ , the  $\beta$  postulated in A4 is unique.

**Solution:** Let  $\alpha, \beta, \beta' \in V$  a vector space over  $\mathbb{R}$ , such that  $\alpha + \beta = 0$  and  $\alpha + \beta' = 0$ . By transitivity

$$\begin{aligned} \alpha + \beta &= \alpha + \beta' \\ (\beta + \alpha) + \beta &= (\beta + \alpha) + \beta' \\ 0 + \beta &= 0 + \beta' \\ \beta &= \beta' \end{aligned}$$

■

**Problem.** Prove similarly that  $0\alpha = 0$ ,  $x0 = 0$ , and  $(-1)\alpha = -\alpha$ .

**Solution:** For the first equality,

$$\begin{aligned} 0\alpha &= (0 + 0)\alpha \\ &= 0\alpha + 0\alpha, \end{aligned}$$

by S2. Subtracting  $0\alpha$  from both sides yields the desired identity. Proving  $x0 = 0$  is identical, except S3 is used instead of S2. As for the last equation:

$$\begin{aligned}\alpha + (-1)\alpha &= (1 - 1)\alpha \\ &= 0\alpha \\ &= 0.\end{aligned}$$

We already determined that  $-\alpha$  is unique, so it follows that  $(-1)\alpha = -\alpha$ . ■

**Problem.** Prove that if  $x\alpha = 0$ , then either  $x = 0$  or  $\alpha = 0$ .

**Solution:** Assume  $a \neq 0$  and  $x \neq 0$ . Since  $\alpha \in \mathbb{R}$  has an inverse  $\alpha^{-1}$ .

$$\begin{aligned}x\alpha &= 0 = 0 \\ x\alpha\alpha^{-1} &= 0\alpha^{-1} \\ x &= 0.\end{aligned}$$

Contradiction. ■

**Problem.** Prove S1 for a function space  $\mathbb{R}^4$ . Prove S3.

**Solution:** Let  $x, y$  be elements of  $\mathbb{R}$ ,  $f$  and  $g$  real functions on  $A$ , and  $a$  an element of  $A$ .

Since  $f(a)$  is a real number,  $(xy)f(a) = x(yf(a))$  is just a consequence of associativity in  $\mathbb{R}$ . As for S3, note also that  $g(a) \in \mathbb{R}$ , thus

$$\begin{aligned}x(f + g)(a) &= x(f(a) + g(a)) \\ &= xf(a) + xg(a).\end{aligned}$$

Because no conditions were imposed on  $a$ , both equalities are valid for all elements of  $A$ . ■

The following theorem is not mentioned (so far) in the book but it is quite useful for checking whether or not a certain set is a subspace.

**Theorem 1.1.** *A subset  $U$  of  $V$  is a subspace of  $V$  if and only if  $U$  satisfies the following conditions:*

1.  $0 \in U$ ,
2.  $\alpha, \beta \in U$  implies  $\alpha + \beta \in U$ ,
3.  $x \in \mathbb{R}$  and  $\alpha \in U$  implies  $x\alpha \in U$ .

**Problem.** Given that  $\alpha$  is any vector in a vector space  $V$ , show that the set  $A = \{x\alpha \mid x \in \mathbb{R}\}$  of all scalar multiples of  $\alpha$  is a subspace of  $V$ .

**Solution:** We can see that  $0 \in A$  because  $0\alpha = 0$ . Now take  $\beta$  and  $\gamma$  elements of  $A$ ,

$$\begin{aligned}\beta + \gamma &= x\alpha + y\alpha \\ &= (x + y)\alpha,\end{aligned}$$

which is clearly an element of  $A$ . Finally, taking  $y \in \mathbb{R}$ ,  $y(x\alpha) = (yx)\alpha \in A$ . By theorem 1.1,  $A$  is a subspace of  $V$ . ■

**Problem.** Given that  $\alpha$  and  $\beta$  are any two vectors in  $V$ , show that the set of all vectors  $x\alpha + y\beta$ , where  $x$  and  $y$  are any real numbers, is a subspace of  $V$ .