Advanced Calculus

Contents

1 Vector Spaces

1.1 Fundamental Notions

1.1.1 First set of exercises

In what follows, A1, A2, A3, A4, S1, S2, S3, S4 refer to the axioms presented in the book.

Problem 1. Prove S3 for $\mathbb{R}^3$ using the explicit display form $\{x_1, x_2, x_3\}$ for ordered triples.

Solution: With $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $x \in \mathbb{R}$.

$$\begin{aligned} x((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= x(x_1 + y_1, x_2 + y_2, x_3 + y_3)8 \\ &= (x(x_1 + y_1), x(x_2 + y_2), x(x_3 + y_3)) \\ &= (xx_1 + xy_1, xx_2 + xy_2, xx_3 + xy_3) \\ &= (xx_1, xx_2, xx_3) + (xy_1, xy_2, xy_3) \\ &= x(x_1, x_2, x_3) + x(y_1, y_2, y_3). \end{aligned}$$

■

Problem 2. Show that given α, the β postualted in A4 is unique.

Solution: Let $\alpha, \beta, \beta' \in V$ a vector space over $\mathbb{R}$, such that $\alpha + \beta = 0$ and $\alpha + \beta' = 0$. By transitivity

$$\begin{aligned} \alpha + \beta &= \alpha + \beta' \\ (\beta + \alpha) + \beta &= (\beta + \alpha) + \beta' \\ 0 + \beta &= 0 + \beta' \\ \beta &= \beta' \end{aligned}$$

■

Problem 3. Prove similarly that $0\alpha = 0$, $x0 = 0$, and $(-1)\alpha = -\alpha$.

Solution: For the first equality,

$$\begin{aligned} 0\alpha &= (0 + 0)\alpha \\ &= 0\alpha + 0\alpha, \end{aligned}$$

by S2. Subtracting 0α from both sides yields the desired identity. Proving $x0 = 0$ is identical, except S3 is used instead of S2. As for the last equation:

$$\begin{aligned} \alpha + (-1)\alpha &= (1 - 1)\alpha \\ &= 0\alpha \\ &= 0. \end{aligned}$$

We already determined that $-\alpha$ is unique, so it follows that $(-1)\alpha = -\alpha$. ■

Problem 4. Prove that if $x\alpha = 0$, then either $x = 0$ or $\alpha = 0$.

Solution: Assume $a \neq 0$ and $x \neq 0$. Since $\alpha \in \mathbb{R}$ has an inverse α^{-1}.

$$\begin{aligned} x\alpha = 0 &= 0 \\ x\alpha\alpha^{-1} &= 0\alpha^{-1} \\ x &= 0. \end{aligned}$$

Contradiction. ■

Problem 5. Prove S1 for a function space $\mathbb{R}^4$. Prove S3.

Solution: Let x, y be elements of $\mathbb{R}$, f and g real functions on A, and a an element of A.
Since $f(a)$ is a real number, $(xy)f(a) = x(yf(a))$ is just a consequence of associativity in $\mathbb{R}$. As for S3, note also that $g(a) \in \mathbb{R}$, thus

$$\begin{aligned} x(f+g)(a) &= x(f(a) + g(a)) \\ &= xf(a) + xg(a). \end{aligned}$$

Because no conditions were imposed on a, both equalities are valid for all elements of A. ■

The following theorem is not mentioned (so far) in the book but it is quite useful for checking whether or not a certain set is a subspace.

Theorem 1.1. *A subset U of V is a subspace of V if and only if U satisfies the following conditions:*

1. $0 \in U$,
2. $\alpha, \beta \in U$ *implies* $\alpha + \beta \in U$,
3. $x \in \mathbb{R}$ *and* $\alpha \in U$ *implies* $x\alpha \in U$.

Problem 6. Given that α is any vector in a vector space V, show that the set $A = \{x\alpha \mid x \in \mathbb{R}\}$ of all scalar multiples of α is a subspace of V.

Solution: We can see that $0 \in A$ because $0\alpha = 0$. Now take β and γ elements of A,

$$\begin{aligned} \beta + \gamma &= x\alpha + y\alpha \\ &= (x+y)\alpha, \end{aligned}$$

which is clearly an element of A. Finally, taking $y \in \mathbb{R}$, $y(x\alpha) = (yx)\alpha \in A$. By theorem 1.1, A is a subspace of V. ■

Problem 7. Given that α and β are any two vectors in V, show that the set of all vectors $x\alpha + y\beta$, where x and y are any real numbers, is a subspace of V.

Solution: Setting $x = y = 0$ shows that the additive identity is in the set. Let $\gamma = x\alpha + y\beta$ and $\delta = x'\alpha + y'\beta$.

$$\begin{aligned}\gamma + \delta &= (x\alpha + y\beta) + x'\alpha + y'\beta \\ &= (x + x')\alpha + (y + y')\beta.\end{aligned}$$

Finally, if $z \in \mathbb{R}$ then $z(x\alpha + y\beta) = (zx)\alpha + (zy)\beta$. ∎

Problem 8. Show that the set of triples $\mathbf{x}$ in $\mathbb{R}^3$ such that $x_1 - x_2 + 2x_3 = 0$ is a subspace M. If N is the similar subspace $\{\mathbf{x} \mid x_1 + x_2 + x_3 = 0\}$, find a nonzero vector $\mathbf{a}$ in $M \cap N$. Show that $M \cap N$ is the set $\{x\mathbf{a} \mid x \in \mathbb{R}\}$ of all scalar multiples of $\mathbf{a}$.

Solution: The intersection $M \cap N$ is the set of all triples $\mathbf{x} = (x_1, x_2, x_3)$ that satisfy the system:

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

The nonzero triple $\mathbf{a} = (3, -1, -2)$ satisfies this system.
Let $A = \{x\mathbf{a} \mid x \in \mathbb{R}\}$. Clearly, if $x \in \mathbb{R}$, then $x\mathbf{a} \in M \cap N$, that is, $A \subset M \cap N$.
Now let $\alpha = (a_1, a_2, a_3)$ be an element of $M \cap N$, then

$$\begin{aligned} a_1 - a_2 + 2a_3 &= a_1 + a_2 + a_3 \\ -2a_2 + a_3 &= 0 \\ a_3 &= 2a_2, \end{aligned}$$

and now substituting a_3 by $2a_2$,

$$\begin{aligned} a_1 + a_2 + 2a_2 &= 0 \\ a_1 &= -3a_2. \end{aligned}$$

So $\alpha = (a_1, a_2, a_3) = (a_1, -\frac{1}{3}a_1, -\frac{2}{3}a_1) = a_1(1, -\frac{1}{3}, -\frac{2}{3}) = a_1 3\mathbf{a}$

∎