Linear Algebra; Notes

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Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Vector Spaces

1.1 Fields

Definition 1.1. A *field* is a set $\mathbb{F}$ together with two binary operations. Addition is a map:

$$+ : \mathbb{F}^2 \to \mathbb{F}, \ (a, b) \mapsto a + b,$$

such that, if $\alpha, \beta \in \mathbb{F}$, then the following properties are satisfied:

1. Addition is commutative, i.e., $\alpha + \beta = \beta + \alpha$;
2. It is associative, i.e., if $\gamma \in \mathbb{F}$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;
3. There exists an element (identity) $z \in \mathbb{F}$, such that $\alpha + z = \alpha$. It will be shown that this element is unique, thus it will always be denoted by 0 and called zero;
4. Every element is invertible, that is, there exists l such that $\alpha + l = 0$. As in the previous property, the additive inverse of an element α in uniquely determined, and thus will be denoted by $-\alpha$.

Multiplication, frequently denoted by $\times$ or $\cdot$, is a map:

$$\cdot : \mathbb{F}^2 \to \mathbb{F}, \ (a, b) \mapsto a \cdot b;$$

satisfying, for all $\alpha, \beta \in \mathbb{F}$:

1. $\alpha\beta = \beta\alpha$;
2. If $\gamma \in \mathbb{F}$ then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
3. There exists $e \in \mathbb{F}$ such that $e\alpha = \alpha$ for every $\alpha \in \mathbb{F}$. This element is unique and denoted by 1;
4. For every $\alpha \neq 0$, there exists $\gamma \in \mathbb{F}$ such that $\alpha\gamma = 1$. The element γ is uniquely determined by α so it will be denoted by α^{-1}.
5. *Multiplication is distributive over addition*, i.e., $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Example 1.1.
- $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are the most commonly encountered fields.
- The set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

Proposition 1.1 (Basic properties of fields)**.** *Let $\mathbb{F}$ be a field.*

1. *Both additive and multiplicative identities are unique;*
2. $0 \neq 1$*;*
3. *For all $\alpha \in \mathbb{F}$, $\alpha 0 = 0$.*

Proof. 1. Let α be an element of $\mathbb{F}$. Let $e, e' \in \mathbb{F}$ both additive identities, then $e = e + e' = e' + e = e'$. The uniqueness of the multiplicative identity follows from an identical argument.

□

2 Finite Dimensional Vector Spaces

2.1 Span and Linear Independence