Notes on Topology

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Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Topological Spaces

1.1 Topology

Definition 1.1. Let E be a set. The set $\tau \subseteq \mathcal{P}(E)$ is said to be a *topology on E* if it satisfies the following axioms:

1. $\emptyset, E \in \tau$,
2. $O_1, O_2 \in \tau \Rightarrow (O_1 \cap O_1 \in \tau)$,
3. $(O_j \in \tau : j \in J) \Rightarrow (\cup_{j \in J} O_j) \in \tau$.

A *topological space* is an ordered pair (E, τ) where E is a set and τ a topology on E. An element of $X \in \mathcal{P}(E)$ is said to be an *open set* if $X \in \tau$, and said to be a *closed set* if $E \setminus X \in \tau$.

Definition 1.2. A topological space (E, τ) is said to *Hausdorff* if

$$\forall a, b \in E \text{ such that } a \neq b, \text{ there exists } O_a, O_b \in \tau \text{ and } O_a \cap O_b \neq \emptyset$$

Definition 1.3. Let τ_1, τ_2 be two topologies on a set E. The topology τ_1 is said to be *finer* than τ_2 if $\tau_2 \subseteq \tau_1$. If in addition $\tau_1 \neq \tau_2$, then τ_1 is said to be *strictly finer* than τ_2.

Two topologies are said to be *comparable* if one is finer than the other.

1.2 Interior, exterior, boundary and closure

Definition 1.4. Let (E, τ) be a topological space, a an element of E, and $X \in \mathcal{P}(E)$. Then:

$$\begin{aligned}
a \text{ is an interior point of } X \quad &\Leftrightarrow \quad \exists O \in \tau, a \in O \text{ and } 0 \subseteq X, \\
a \text{ is an exterior point of } X \quad &\Leftrightarrow \quad \exists O \in \tau, a \in O \text{ and } 0 \subseteq E \setminus X, \\
a \text{ is a boundary point of } X \quad &\Leftrightarrow \quad \forall O \in \tau, a \in O, O \cap X \neq \emptyset \text{ and } O \cap (E \setminus X)
\end{aligned}$$

Notation 1. Note that the following sets depends on the fixed topology.

- $\operatorname{int} X$ is the set of all interior points of X,

- $\operatorname{ext} X$ is the set of all exterior points of X,
- ∂X is the set of all boundary points of X,
- The set $\overline{X} = \operatorname{int} X \cup \partial X$ is called the *closure* of X.

Proposition 1.1. *Let (E, τ) be a topological space and $X \in \mathcal{P}(E)$. Then*

1. $a \in \overline{X} \Leftrightarrow \forall O \in \tau, a \in O \Rightarrow (O \cap X) \neq \emptyset$,
2. $\operatorname{int} X \subseteq X \subseteq \overline{X}$,
3. $X \in \tau \Leftrightarrow X = \int X$,
4. $(E \setminus X) \in \tau \Leftrightarrow \overline{X} = X$,
5. $\overline{X} = X \cup \partial X$,
6. $X \subseteq Y \Rightarrow \operatorname{int} X \subseteq \operatorname{int} Y$ *and* $\overline{X} \subseteq \overline{Y}$,
7. $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ *and* $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$,
8. $\overline{\overline{X}} = X$.

1.3 Neighborhoods

2 Continuity

3 Metric Spaces