

Notes on Complex Analysis

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1 Analytic Functions

1.1 The Complex Number System

Definition 1.1. Consider the following maps:

- Addition

$$\begin{aligned}\mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\end{aligned}$$

- Scalar Multiplication

$$\begin{aligned}\mathbb{R} \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ \alpha(x, y) &= (\alpha x, \alpha y)\end{aligned}$$

- Multiplication

$$\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

These operations, together with the set \mathbb{R}^2 , form a vector space over \mathbb{R} , which we call the complex number system, denoted by \mathbb{C} . We identify a real number x with the pair $(x, 0)$, and $(0, 1)$ will be denoted by i . With this we recover the conventional notation, for

$$(x, y) = (x, 0) + (0, 1)(y, 0) = x + iy.$$

We call the x -axis and y -axis by *real* and *imaginary* axis, respectively. Given $z = a + bi \in \mathbb{C}$, we call a the *real part* of z , denoted by $\Re(z)$, and b the *imaginary part* of z , denoted by $\Im(z)$. Finally, z is said to be a *pure imaginary number* if $\Re(z) = a = 0$.

1.1.1 Algebraic Properties

Proposition 1.0.1. Let z be a non-zero complex number, then there exists $z' \in \mathbb{C}$, such that

$$z \cdot z' = 1,$$

called the *inverse* of z .

Proof. Let $z = a + bi$ and $z' = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}$. $z \neq 0$ implies that $a^2 + b^2 \neq 0$. Furthermore,

$$\begin{aligned}z \cdot z' &= (a + bi) \left(\frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2} \right) \\ &= \left(\frac{a^2 + abi - abi + b^2}{a^2 + b^2} \right) \\ &= 1.\end{aligned}$$

□

The inverse of a complex number z is unique, and represented by z^{-1} ; the symbol z/w means zw^{-1} .

Theorem 1.1. \mathbb{C} , together with the previously defined addition and multiplication, is a field.

1.1.2 Roots of Quadratic Equations

Proposition 1.1.1. Let $z \in \mathbb{C}$. Then there exists a complex number $w \in \mathbb{C}$ such that $w^2 = z$.

[To-Finish]

1.2 Properties of Complex Numbers

1.2.1 Polar Representation

The *modulus* of a complex number $z = a + bi$ is its norm, i.e., $\|a + bi\| = \|(a, b)\| = \sqrt{a^2 + b^2}$, conventionally written as $|z|$. Let θ be the angle that z makes with the positive real axis, where $0 \leq \theta < 2\pi$, and $r = |z|$. Then z may be rewritten as

$$a + bi = r \cos \theta + (r \sin \theta)i = r(\cos \theta + i \sin \theta).$$

This way of writing z is called the *polar coordinate representation*. The angle θ is called the *argument* of z and is denoted $\theta = \arg z$. [To-Do: Insert Figures] The interval $[0, 2\pi[$ is an arbitrary choice, any other interval $[a, b[$ of length 2π could be specified and the resulting representation would be unique, granted that the relevant complex number is not equal to zero. Alternatively, $\arg z$ may be defined as the set of values $\{\theta + 2n\pi : n \in \mathbb{Z}\}$. Specifying a particular suitable interval for the angle is known as choosing a *branch of the argument*.

1.2.2 Multiplication of Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \theta_1 \cdot \cos \theta_2 - \sin \theta_1 \cdot \sin \theta_2)] + i [\cos \theta_1 \cdot \sin \theta_2 + \cos \theta_2 \cdot \sin \theta_1] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \end{aligned}$$

Which proves the following proposition.

Proposition 1.1.2. For any complex numbers z_1, z_2 ,

$$|z_1 z_2| = |z_1| |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$$

Example 1. Let $z_1 = -1$ and $z_2 = -i$, then $\arg z_1 = \pi$ and $\arg z_2 = 3\pi/2$. Since $z_1 z_2 = i$, then $\arg z_1 z_2 = \pi/2$. Using the previous proposition, we'd get $\arg z_1 + \arg z_2 = \pi + 3\pi/2 = 5\pi/2$, which isn't in the interval $[0, 2\pi[$. Subtracting 2π from this result yields the correct value for $\arg z_1 z_2$.

1.2.3 De Moivre's Formula

Proposition 1.1.3. *If $z = r(\cos \theta + i \sin \theta)$ and n is a positive integer, then*

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Proof. Use induction. □

As an application, consider the equation $z^n = w$, with $w \in \mathbb{C}$. Suppose that $w = r(\cos \theta + i \sin \theta)$ and $z = \rho(\cos \psi + i \sin \psi)$. By De Moivre's formula, $z^n = \rho^n(\cos n\psi + i \sin n\psi)$, which implies that $\rho^n = r = |w|$ and $n\psi = \theta + 2k\pi$, where k is some integer. Thus

$$z = \sqrt[n]{r} \left[\cos \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right].$$

Each value of $k = 0, 1, \dots, n-1$, gives different values of z . Any other value for k repeats one of the values of z corresponding to $k = 0, 1, \dots, n-1$. Thus there are exactly n n -th roots of a nonzero complex number.

Corollary 1.1.1. *Let $w \in \mathbb{C} \setminus \{0\}$, with polar representation $w = r(\cos \theta + i \sin \theta)$. The n th roots of w are given by the n complex numbers*

$$z_k = \sqrt[n]{r} \left[\cos \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right], \quad k = 0, 1, \dots, n-1.$$

1.2.4 Complex Conjugation

The transformation $a + bi \mapsto a - bi$ is called *complex conjugation*, and $a - bi$ is the *conjugate* of $a + bi$. The conjugate of $z \in \mathbb{C}$ is denoted by \bar{z} .

Proposition 1.1.4. *Let $z, w \in \mathbb{C}$, then*

1. $\overline{z + w} = \bar{z} + \bar{w}$,
2. $\overline{z\bar{w}} = \bar{z}w$,
3. If $w \neq 0$, then $\overline{z/w} = \bar{z}/\bar{w}$,
4. $z\bar{z} = |z|^2$,
5. $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$,
6. $\Re z = (z + \bar{z})/2$ and $\Im z = (z - \bar{z})/2i$,
7. $\overline{\bar{z}} = z$.

Proposition 1.1.5. *Let $z, w, z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$, then*

1. $|zw| = |z||w|$,
2. $-|z| \leq \Re z \leq |z|$ and $-|z| \leq \Im z \leq |z|$,
3. $|\bar{z}| = |z|$,
4. $|z + w| \leq |z| + |w|$,
5. $|z - w| \geq ||z| - |w||$,
6. $|z_1 w_1 + \dots + z_n w_n| \leq \sqrt{|z_1|^2 + \dots + |z_n|^2} \sqrt{|w_1|^2 + \dots + |w_n|^2}$.

1.3 Elementary Functions

1.3.1 The Exponential Function

Definition 1.2. Let $z = x + iy \in \mathbb{C}$, we define

$$\exp(z) = e^z = e^x(\cos y + i \sin y).$$

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be *periodic* if there exists $w \in \mathbb{C}$ such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. The following proposition establishes some of the properties of the exponential function.

Proposition 1.1.6. Let $z, w \in \mathbb{C}$,

1. $e^{z+w} = e^z e^w$,
2. $e^z \neq 0$,
3. If $x \in \mathbb{R}$, then $e^x > 1$ when $x > 0$ and $0 < e^x < 1$ when $x < 0$,
4. $|e^{x+yi}| = e^x$,
5. e^z is periodic; each period for e^z has the form $2k\pi i$ for some integer k ,
6. $e^z = 1 \Leftrightarrow z = 2k\pi i$ for some integer k .

1.3.2 Trigonometric Functions

Definition 1.3. Let z be a complex number.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Proposition 1.1.7. Let $z, w \in \mathbb{C}$

1. $\sin^2 z + \cos^2 z = 1$,
2. $\sin(z + w) = \sin z \cos w + \cos z \sin w$,
3. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

1.3.3 Logarithm Function

Since the complex exponential function is not bijective, we cannot naively define its inverse, the logarithm.

Proposition 1.1.8. *Consider the set $A_{y_0} = \{x + iy : x \in \mathbb{R} \text{ and } y_0 \leq y < y_0 + 2\pi\}$. Then the exponential function with its domain restricted to this set, $e^z : A_{y_0} \rightarrow \mathbb{C} \setminus \{0\}$, is injective.*

Definition 1.4. The function $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, with $y_0 \leq \Im \log z < y_0 + 2\pi$, is defined by

$$\log z = \log|z| + i \arg z,$$

where $\Im \log z = \arg z \in [y_0, y_0 + 2\pi[$, and $\log|z|$ is the usual logarithm where $|z|$ is a real number.

The logarithm function is only well-defined when we fix a branch, that is an interval of length 2π , in which $\arg z$ takes its values.