Advanced Calculus

Contents

1 Vector Spaces

1.1 Fundamental Notions

1.1.1 Summary

Not much is covered here. Vector spaces and subspaces are defined.

1.1.2 First set of exercises

In what follows, A1, A2, A3, A4, S1, S2, S3, S4 refer to the axioms presented in the book.

Problem 1. Prove S3 for $\mathbb{R}^3$ using the explicit display form $\{x_1, x_2, x_3\}$ for ordered triples.

Solution: With $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $x \in \mathbb{R}$.

$$\begin{aligned}
x((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= x(x_1 + y_1, x_2 + y_2, x_3 + y_3)8 \\
&= (x(x_1 + y_1), x(x_2 + y_2), x(x_3 + y_3)) \\
&= (xx_1 + xy_1, xx_2 + xy_2, xx_3 + xy_3) \\
&= (xx_1, xx_2, xx_3) + (xy_1, xy_2, xy_3) \\
&= x(x_1, x_2, x_3) + x(y_1, y_2, y_3).
\end{aligned}$$

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Problem 2. Show that given α, the β postualted in A4 is unique.

Solution: Let $\alpha, \beta, \beta' \in V$ a vector space over $\mathbb{R}$, such that $\alpha + \beta = 0$ and $\alpha + \beta' = 0$. By transitivity

$$\begin{aligned}
\alpha + \beta &= \alpha + \beta' \\
(\beta + \alpha) + \beta &= (\beta + \alpha) + \beta' \\
0 + \beta &= 0 + \beta' \\
\beta &= \beta'
\end{aligned}$$

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Problem 3. Prove similarly that $0\alpha = 0$, $x0 = 0$, and $(-1)\alpha = -\alpha$.

Solution: For the first equality,

$$\begin{aligned}
0\alpha &= (0 + 0)\alpha \\
&= 0\alpha + 0\alpha,
\end{aligned}$$

by S2. Subtracting 0α from both sides yields the desired identity. Proving $x0 = 0$ is identical, except S3 is used instead of S2. As for the last equation:

$$\begin{aligned} \alpha + (-1)\alpha &= (1-1)\alpha \\ &= 0\alpha \\ &= 0. \end{aligned}$$

We already determined that $-\alpha$ is unique, so it follows that $(-1)\alpha = -\alpha$. ∎

Problem 4. Prove that if $x\alpha = 0$, then either $x = 0$ or $\alpha = 0$.

Solution: Assume $a \neq 0$ and $x \neq 0$. Since $\alpha \in \mathbb{R}$ has an inverse α^{-1}.

$$\begin{aligned} x\alpha = 0 &= 0 \\ x\alpha\alpha^{-1} &= 0\alpha^{-1} \\ x &= 0. \end{aligned}$$

Contradiction. ∎

Problem 5. Prove S1 for a function space $\mathbb{R}^4$. Prove S3.

Solution: Let x, y be elements of $\mathbb{R}$, f and g real functions on A, and a an element of A.
Since $f(a)$ is a real number, $(xy)f(a) = x(yf(a))$ is just a consequence of associativity in $\mathbb{R}$. As for S3, note also that $g(a) \in \mathbb{R}$, thus

$$\begin{aligned} x(f+g)(a) &= x(f(a) + g(a)) \\ &= xf(a) + xg(a). \end{aligned}$$

Because no conditions were imposed on a, both equalities are valid for all elements of A. ∎

The following theorem is not mentioned (so far) in the book but it is quite useful for checking whether or not a certain set is a subspace.

Theorem 1.1. *A subset U of V is a subspace of V if and only if U satisfies the following conditions:*

1. $0 \in U$,
2. $\alpha, \beta \in U$ *implies* $\alpha + \beta \in U$,
3. $x \in \mathbb{R}$ *and* $\alpha \in U$ *implies* $x\alpha \in U$.

Problem 6. Given that α is any vector in a vector space V, show that the set $A = \{x\alpha \mid x \in \mathbb{R}\}$ of all scalar multiples of α is a subspace of V.

Solution: We can see that $0 \in A$ because $0\alpha = 0$. Now take β and γ elements of A,

$$\begin{aligned}\beta + \gamma &= x\alpha + y\alpha \\ &= (x+y)\alpha,\end{aligned}$$

which is clearly an element of A. Finally, taking $y \in \mathbb{R}$, $y(x\alpha) = (yx)\alpha \in A$. By theorem 1.1, A is a subspace of V. ■

Problem 7. Given that α and β are any two vectors in V, show that the set of all vectors $x\alpha + y\beta$, where x and y are any real numbers, is a subspace of V.

Solution: Setting $x = y = 0$ shows that the additive identity is in the set. Let $\gamma = x\alpha + y\beta$ and $\delta = x'\alpha + y'\beta$.

$$\begin{aligned}\gamma + \delta &= (x\alpha + y\beta) + x'\alpha + y'\beta \\ &= (x + x')\alpha + (y + y')\beta.\end{aligned}$$

Finally, if $z \in \mathbb{R}$ then $z(x\alpha + y\beta) = (zx)\alpha + (zy)\beta$. ■

Problem 8. Show that the set of triples $\mathbf{x}$ in $\mathbb{R}^3$ such that $x_1 - x_2 + 2x_3 = 0$ is a subspace M. If N is the similar subspace $\{\mathbf{x} \mid x_1 + x_2 + x_3 = 0\}$, find a nonzero vector $\mathbf{a}$ in $M \cap N$. Show that $M \cap N$ is the set $\{x\mathbf{a} \mid x \in \mathbb{R}\}$ of all scalar multiples of $\mathbf{a}$.

Solution: The intersection $M \cap N$ is the set of all triples $\mathbf{x} = (x_1, x_2, x_3)$ that satisfy the system:

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

The nonzero triple $\mathbf{a} = (3, -1, -2)$ satisfies this system.
Let $A = \{x\mathbf{a} \mid x \in \mathbb{R}\}$. Clearly, if $x \in \mathbb{R}$, then $x\mathbf{a} \in M \cap N$, that is, $A \subset M \cap N$.
Now let $\alpha = (a_1, a_2, a_3)$ be an element of $M \cap N$, then

$$\begin{aligned}a_1 - a_2 + 2a_3 &= a_1 + a_2 + a_3 \\ -2a_2 + a_3 &= 0 \\ a_3 &= 2a_2,\end{aligned}$$

and now substituting a_3 by $2a_2$,

$$\begin{aligned}a_1 + a_2 + 2a_2 &= 0 \\ a_1 &= -3a_2.\end{aligned}$$

So $\alpha = (a_1, a_2, a_3) = (a_1, -\frac{1}{3}a_1, -\frac{2}{3}a_1) = a_1(1, -\frac{1}{3}, -\frac{2}{3}) = a_1 3\mathbf{a}$

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Problem 9. Let A be the open interval $(0,1)$, and let V be $\mathbb{R}^A$. Given a point x in $(0,1)$, let V_x be the set of functions in V that have a derivative at x. Show that V_x is a subspace of V.

Solution: The constant function $I(x) = 0$ has a derivative at x and it is the identity of $\mathbb{R}^A$. Given $f(x), g(x)$ functions in V_x, we know that $(f+g)'(x) = f'(x) + g'(x) \Rightarrow f + g \in V_x$. Also, given $y \in \mathbb{R}$, $(yf(x))' = yf'(x) \Rightarrow yf(x) \in V_x$. ■

Problem 10. For any subsets A and B of a vectors space V we define the set sum $A+B$ by $A+B = \{\alpha + \beta \mid \alpha \in A \text{ and } \beta \in B\}$. Show that $(A+B)+C = A+(B+C)$.

Solution:

$$\begin{aligned}(A+B)+C &= \{(\alpha+\beta)+\gamma \mid \alpha \in A, \beta \in B, \gamma \in C\} \\ &= \{\alpha+(\beta+\gamma) \mid \alpha \in A, \beta \in B, \gamma \in C\} \\ &= A+(B+C).\end{aligned}$$

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Problem 11. Let $A \subset V$ and $X \subset \mathbb{R}$, we similarly define, $XA = \{x\alpha \mid x \in X \text{ and } \alpha \in A\}$. Show that a nonvoid set A is a subspace if and only if $A+A = A$ and $\mathbb{R}A = A$.

Solution: Let $A \neq \emptyset$.
($\Rightarrow$) Suppose A is a subspace, clearly $A+A = A$ by closure of addition on vector spaces. Likewise, $\mathbb{R}A = A$ by closure under scalar multiplication.
($\Leftarrow$) Suppose $A+A = A$ and $\mathbb{R}A = A$. Since $A \neq \emptyset$ then $0 \in A$. Let α, β be elements of A, $\alpha + \beta \in A + A = A$ by hypothesis. Finally, let x be a real number and α and element of A, then $x\alpha \in \mathbb{R}A = A$. By 1.1, A is a subspace of V. ■

Problem 12. Let V be $\mathbb{R}^2$, and let M be the line through the origin with slope k. Let $\mathbf{x}$ be any nonzero vector in M. Show that M is the subspace $\mathbb{R}\mathbf{x} = \{t\mathbf{x} \mid t \in \mathbb{R}\}$.

Solution: M is the set $\{(x, kx) \in \mathbb{R}^2\} \subset \mathbb{R}^2$. Let $\mathbf{x} = (x_1, kx_1) \neq 0$, if $t \in \mathbb{R}$ then $t\mathbf{x} = (tx_1, tkx_1) \in M$, hence $\mathbb{R}\mathbf{x} \subset M$. Conversely, if $(y, ky) \in M$ and $(y, ky) = y(1, k) = y\frac{1}{x_1}(x_1, kx_1) = yx_1^{-1}\mathbf{x} \in \mathbb{R}\mathbf{x} \Rightarrow M \subset \mathbb{R}\mathbf{x}$. Thus $M = \mathbb{R}\mathbf{x}$. ■

Problem 13. Show that any other line L with the same slope k is of the form $M + \mathbf{a}$ for some $\mathbf{a}$.

Solution: Let L be a line with slope k

$$\begin{aligned} L &= \{(x, kx+b) \mid x \in \mathbb{R}\} \\ &= \{(x, kx) + (0, b) \mid x \in \mathbb{R}\} \\ &= \{(x, kx) \mid x \in \mathbb{R}\} + (0, b) \\ &= M + (0, b). \end{aligned}$$

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Problem 14. Let M be a subspace of a vector space V, and let α and β be any two vectors in V. Given $A = \alpha + M$ and $B = \beta + M$, show that either $A = B$ or $A \cap B = \emptyset$. Show also that $A + B = (\alpha + \beta) + M$.

Problem 15. State more carefully and prove what is meant by "a subspace of a subspace is a subspace".

Solution: Let V be a vector space and A a subspace of it. If B is a subspace of A then B is also a subspace of V.
To prove this, note that $B \subset A \subset V \Rightarrow B \subset V$, by hypothesis, all of the propositions of theorem 1.1 are true and thus B is a subspace of V. ∎

Problem 16. Prove that the intersection of two subspaces of a vector space is always itself a subspace.

Solution: Let V be a vector space and A, B subspaces of V. $0 \in A$ and $0 \in B \Rightarrow 0 \in A \cap B$. If $\alpha, \beta \in A \cap B$ then $\alpha, \beta \in A$ and $\alpha, \beta \in B$, since both are subspaces, $\alpha + \beta \in A$ and $\alpha + \beta \in B \Rightarrow \alpha + \beta \in A \cap B$. If $x \in \mathbb{R}$ and $\alpha \in A \cap B$.

$$\begin{aligned} \alpha \in A \cap B &\Rightarrow \alpha \in A \wedge \alpha \in B \\ &\Rightarrow x\alpha \in A \wedge x\alpha \in B \\ &\Rightarrow x\alpha \in A \cap B. \end{aligned}$$

∎

Problem 17. Prove more generally that the intersection $W = \cap_{i \in I} W_i$ of any family $\{W_i \mid i \in I\}$ of subspaces of V is a subspace of V.

Solution: By definiton of subspace, 0 must be an element of every W_i and is consequently an element of W. To check closure of addition consider α, β elements of W,

$$\begin{aligned} \alpha, \beta \in W &\Rightarrow \alpha, \beta \in W_i && \forall i \in I \\ &\Rightarrow \alpha + \beta \in W_i && \forall i \in I \\ &\Rightarrow \alpha + \beta \in W. \end{aligned}$$

Closure under scalar multiplication follows by similar reasoning. ■

Problem 18. Let V again be $\mathbb{R}^{(0,1)}$, and let W be the set of all functions f in V such that $f'(x)$ exists for every x in $(0,1)$. Show that W is the intersection of the collection of subspaces of the form V_x that were considered in problem 9.

Solution: ■