Linear Algebra; Notes

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Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Vector Spaces

1.1 Fields (optional)

Definition 1.1. A *field* is a set $\mathbb{F}$ together with the two following binary operations. Addition is a map:

$$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}; \quad (a, b) \mapsto a + b,$$

such that, if $\alpha, \beta \in \mathbb{F}$, then the following properties are satisfied:

1. Addition is commutative, i.e., $\alpha + \beta = \beta + \alpha$;
2. It is associative, i.e., if $\gamma \in \mathbb{F}$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;
3. There exists an element (called *additive identity*) $z \in \mathbb{F}$, such that $\alpha + z = \alpha$. It will be shown that this element is unique, thus it will always be denoted by 0 and called zero;
4. Every element is invertible, that is, there exists l such that $\alpha + l = 0$. As in the previous property, the additive inverse of an element α in uniquely determined, and thus will be denoted by $-\alpha$.

Multiplication, frequently denoted by $\times$ or $\cdot$, is a map:

$$\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}; \quad (a, b) \mapsto a \cdot b = ab,$$

satisfying, for all $\alpha, \beta \in \mathbb{F}$:

1. $\alpha\beta = \beta\alpha$;
2. If $\gamma \in \mathbb{F}$ then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
3. There exists $e \in \mathbb{F}$ (called an *multiplicative identity*) such that $e \neq 0$ and $e\alpha = \alpha$ for every $\alpha \in \mathbb{F}$. This element is unique and denoted by 1;
4. For every $\alpha \neq 0$, there exists $\gamma \in \mathbb{F}$ such that $\alpha\gamma = 1$. The element γ is uniquely determined by α so it will be denoted by α^{-1}.

5. *Multiplication is distributive over addition*, i.e., $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Example 1.1. $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are the most commonly encountered fields.

Example 1.2. The set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ together with the usual addition and multiplication is a field.

Proposition 1.1. *Let $\mathbb{F}$ be a field.*

1. *Both additive and multiplicative identities are unique;*

2. *For all $\alpha \in \mathbb{F}$, $\alpha 0 = 0$;*

3. *For all $\alpha \in \mathbb{F}$, $-\alpha = (-1)\alpha$.*

Proof. **1.):** Let $e, e' \in \mathbb{F}$ both additive identities, then $e = e + e' = e' + e = e'$. The uniqueness of the multiplicative identity follows from an identical argument.

2.) Let $\alpha \in \mathbb{F}$, $\alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$, add the additive inverse of $\alpha 0$ to both sides and we get the desired equality. **3.)** $\alpha + (-1)\alpha = (1 + (-1))\alpha = 0\alpha$ which is equal to 0 by the previous proposition. Thus we get $\alpha + (-1)\alpha = \alpha - \alpha \Leftrightarrow (-1)\alpha = -\alpha$. $\square$

1.2 Vector Spaces

Definition 1.2. Let V be a set and $\mathbb{F}$ an arbitrary field, provided with with the mappings:

- $(\alpha, \beta) \mapsto \alpha + \beta$; $\quad V \times V \to V$, called *addition*;

- $(x, \alpha) \mapsto x\alpha$; $\quad \mathbb{F} \times V \to V$, called *scalar multiplication*.

V is said to be a *vector space over* $\mathbb{F}$ with respect to these operations if:

1. For all $\alpha, \beta, \gamma \in V$ the equation $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ holds;

2. For all $\alpha, \beta \in$, $\alpha + \beta = \beta + \alpha$;

3. There exists $0 \in V$ such that $\alpha + 0 = \alpha$ for all $\alpha \in V$;

4. For every $\alpha \in V$, there exists $\beta \in V$ such that $\alpha + \beta = 0$;

5. Let $1 \in \mathbb{F}$ be its multiplicative identity, then $1\alpha = \alpha$ for all $\alpha \in V$;

6. For all $x, y \in \mathbb{F}$ and $\alpha, \beta \in V$, $x(\alpha + \beta) = x\alpha + x\beta$ and (x+y) $\alpha = x\alpha + y\alpha$.

If $\mathbb{F} = \mathbb{R}$ then V is said to be a *real vector space*. If $\mathbb{F} = \mathbb{C}$ then V is said to be a *complex vector space*.

The section on fields is optional because we'll be mostly dealing with either real or complex vector spaces, meaning the familiar arithmetic of these sets suffices for most examples. However, at least in these initial chapters, the results that will be proven will not depend on any properties particular to these two specific fields, thus they are equally applicable to the more abstract case.

1.2.1 Subspaces

2 Finite Dimensional Vector Spaces

2.1 Span and Linear Independence