

Notes on Complex Analysis

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1 Analytic Functions

1.1 The Complex Number System

Definition 1.1. Consider the following maps:

- Addition

$$\begin{aligned}\mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\end{aligned}$$

- Scalar Multiplication

$$\begin{aligned}\mathbb{R} \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ \alpha(x, y) &= (\alpha x, \alpha y)\end{aligned}$$

- Multiplication

$$\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

These operations, together with the set \mathbb{R}^2 , form a vector space over \mathbb{R} , which we call the complex number system, denoted by \mathbb{C} . We identify a real number x with the pair $(x, 0)$, and $(0, 1)$ will be denoted by i . With this we recover the conventional notation, for

$$(x, y) = (x, 0) + (0, 1)(y, 0) = x + iy.$$

We call the x -axis and y -axis by *real* and *imaginary* axis, respectively. Given $z = a + bi \in \mathbb{C}$, we call a the *real part* of z , denoted by $\Re(z)$, and b the *imaginary part* of z , denoted by $\Im(z)$. Finally, z is said to be a *pure imaginary number* if $\Re(z) = a = 0$.

1.1.1 Algebraic Properties

Proposition 1.0.1. Let z be a non-zero complex number, then there exists $z' \in \mathbb{C}$, such that

$$z \cdot z' = 1,$$

called the *inverse* of z .

Proof. Let $z = a + bi$ and $z' = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}$. $z \neq 0$ implies that $a^2 + b^2 \neq 0$. Furthermore,

$$\begin{aligned}z \cdot z' &= (a + bi) \left(\frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2} \right) \\ &= \left(\frac{a^2 + abi - abi + b^2}{a^2 + b^2} \right) \\ &= 1.\end{aligned}$$

□

The inverse of a complex number z is unique, and represented by z^{-1} ; the symbol z/w means zw^{-1} .

Theorem 1.1. \mathbb{C} , together with the previously defined addition and multiplication, is a field.

1.1.2 Roots of Quadratic Equations

Proposition 1.1.1. Let $z \in \mathbb{C}$. Then there exists a complex number $w \in \mathbb{C}$ such that $w^2 = z$.

[To-Finish]

1.2 Properties of Complex Numbers

1.2.1 Polar Representation

The *modulus* of a complex number $z = a + bi$ is its norm, i.e., $\|a + bi\| = \|(a, b)\| = \sqrt{a^2 + b^2}$, conventionally written as $|z|$. Let θ be the angle that z makes with the positive real axis, where $0 \leq \theta < 2\pi$, and $r = |z|$. Then z may be rewritten as

$$a + bi = r \cos \theta + (r \sin \theta)i = r(\cos \theta + i \sin \theta).$$

This way of writing z is called the *polar coordinate representation*. The angle θ is called the *argument* of z and is denoted $\theta = \arg z$. [To-Do: Insert Figures] The interval $[0, 2\pi[$ is an arbitrary choice, any other interval of length 2π could be specified and the resulting representation would be unique, granted that the relevant complex number is not equal to zero. [True? What if it's a closed interval?] Alternatively, $\arg z$ may be defined as the set of values $\{\theta + 2n\pi : n \in \mathbb{Z}\}$. Specifying a particular suitable interval for the angle is known as choosing a *branch of the argument*.

1.2.2 Multiplication of Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \theta_1 \cdot \cos \theta_2 - \sin \theta_1 \cdot \sin \theta_2)] + i [\cos \theta_1 \cdot \sin \theta_2 + \cos \theta_2 \cdot \sin \theta_1] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \end{aligned}$$

Which proves the following proposition.

Proposition 1.1.2. For any complex numbers z_1, z_2 ,

$$|z_1 z_2| = |z_1| |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$$