Linear Algebra; Notes

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Contents

Some conventions:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- A_+ with $A = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$, refers to the respective subset of positive elements,
- A_- is the same as above but for negative elements.

1 Vector Spaces

1.1 Fields

Definition 1.1. A *field* is a set $\mathbb{F}$ together with two binary operations. Addition is a map:

$$+ : \mathbb{F}^2 \to \mathbb{F}; \quad (a, b) \mapsto a + b,$$

such that, if $\alpha, \beta \in \mathbb{F}$, then the following properties are satisfied:

1. Addition is commutative, i.e., $\alpha + \beta = \beta + \alpha$;
2. It is associative, i.e., if $\gamma \in \mathbb{F}$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;
3. There exists an element (called *additive identity*) $z \in \mathbb{F}$, such that $\alpha + z = \alpha$. It will be shown that this element is unique, thus it will always be denoted by 0 and called zero;
4. Every element is invertible, that is, there exists l such that $\alpha + l = 0$. As in the previous property, the additive inverse of an element α in uniquely determined, and thus will be denoted by $-\alpha$.

Multiplication, frequently denoted by $\times$ or $\cdot$, is a map:

$$\cdot : \mathbb{F}^2 \to \mathbb{F}; \quad (a, b) \mapsto a \cdot b,$$

satisfying, for all $\alpha, \beta \in \mathbb{F}$:

1. $\alpha\beta = \beta\alpha$;
2. If $\gamma \in \mathbb{F}$ then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
3. There exists $e \in \mathbb{F}$ (called an *multiplicative identity*) such that $e \neq 0$ and $e\alpha = \alpha$ for every $\alpha \in \mathbb{F}$. This element is unique and denoted by 1;
4. For every $\alpha \neq 0$, there exists $\gamma \in \mathbb{F}$ such that $\alpha\gamma = 1$. The element γ is uniquely determined by α so it will be denoted by α^{-1}.

5. *Multiplication is distributive over addition*, i.e., $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Example 1.1. $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are the most commonly encountered fields.

Example 1.2. The set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ together with the usual addition and multiplication is a field.

Proposition 1.1. *Let $\mathbb{F}$ be a field.*

1. *Both additive and multiplicative identities are unique;*
2. *For all $\alpha \in \mathbb{F}$, $\alpha 0 = 0$;*
3. *For all $\alpha \in \mathbb{F}$, $-\alpha = (-1)\alpha$.*

Proof. **1.):** Let $e, e' \in \mathbb{F}$ both additive identities, then $e = e + e' = e' + e = e'$. The uniqueness of the multiplicative identity follows from an identical argument.

2.) Let $\alpha \in \mathbb{F}$, $\alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$, add the additive inverse of $\alpha 0$ to both sides and we get the desired equality. **3.)** $\alpha + (-1)\alpha = (1 + (-1))\alpha = 0\alpha$ which is equal to 0 by the previous proposition. Thus we get $\alpha + (-1)\alpha = \alpha - \alpha \Leftrightarrow (-1)\alpha = -\alpha$. $\square$

2 Finite Dimensional Vector Spaces

2.1 Span and Linear Independence