# Lecture 5. Regression models
# Applied Statistics in R

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# Agenda

1. Linear regression
2. Quantile regression
3. Ridge regression

# Linear regression

We have a sample:

```
y<-c(132,143,153,162,154,168,137,149,159,128,166)
x1<-c(52,59,67,73,64,74,54,61,65,46,72)
x2<-c(173,184,194,211,196,220,188,188,207,167,217)
```

We want to have the model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2.$$

We use functions `lm` and `summary`:

```
res<-lm(y~x1+x2)
summary(res)
```

## Results:

Call:

lm(formula = y ~ x1 + x2)

Residuals:

| Min | 1Q | Median | 3Q | Max |
|---|---|---|---|---|
| -3.4640 | -1.1949 | -0.4078 | 1.8511 | 2.6981 |

Coefficients:

| | Estimate | Std. Error | t value | $Pr(>\|t\|)$ | |
|---|---|---|---|---|---|
| (Intercept) | 30.9941 | 11.9438 | 2.595 | 0.03186 | * |
| x1 | 0.8614 | 0.2482 | 3.470 | 0.00844 | ** |
| x2 | 0.3349 | 0.1307 | 2.563 | 0.03351 | * |

---

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.318 on 8 degrees of freedom

Multiple R-squared: 0.9768, Adjusted R-squared: 0.9711

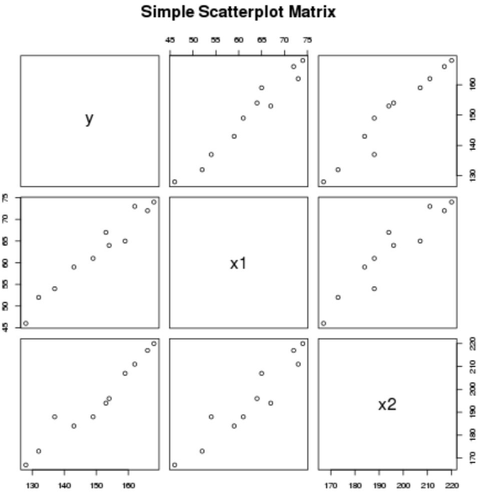
F-statistic: 168.8 on 2 and 8 DF, p-value: 2.874e-07

- We obtain quantiles of levels 0.25, 0.5 and 0.75, estimators $\beta_0$, $\beta_1$ and $\beta_2$, their standard errors $\hat{\beta}_0/t_{\beta_0}$, $\hat{\beta}_1/t_{\beta_1}$, $\hat{\beta}_2/t_{\beta_2}$, values of statistics $t_{\beta_0}$, $t_{\beta_1}$, $t_{\beta_2}$ for hypothesis $H_0 : \beta_i = 0$, $i = 0, 1, 2$.
- In column $Pr(> |t|)$ we obtain corresponding $p$-values. If $p - value > \alpha$ (by default, $\alpha = 0.05$), then $H_0 : \beta_i = 0$ is accepted and coefficient is not significant. Otherwise, null hypothesis is rejected and coefficient is accepted to be significant. In example, all coefficients are significant. Value `Residual standard error` is statistics $S$.
- `Multiple R-squared` is the value of $R^2$. Statistics $F$ for hypothesis $H_0 : \beta_1 = \ldots = \beta_k = 0$, is 168.8. And $p$-value is 2.874e-07, which is less than 0.05. Therefore, the null hypothesis is rejected and we may state that the linear regression model is significant in general.

Use function `pairs`:

```
pairs(y~x1+x2, main="Simple Scatterplot Matrix")
```

# Graphs


Simple Scatterplot Matrix
y
x1
x2
45 50 55 60 65 70 75
130 140 150 160
170 180 190 200 210 220

# Quantile regression

# Quantile regression

The model in a matrix form:

$$Y = X\beta + \varepsilon,$$

where $Y = (y_1, \ldots, y_n)^\intercal$, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^\intercal$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\intercal$,

$$X = \left( \begin{array}{ccccc} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ \ldots & \ldots & \ldots & \ldots & \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{array} \right).$$

# Quantile regression

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$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ \ldots & \ldots & \ldots & \ldots & \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{pmatrix}.$$

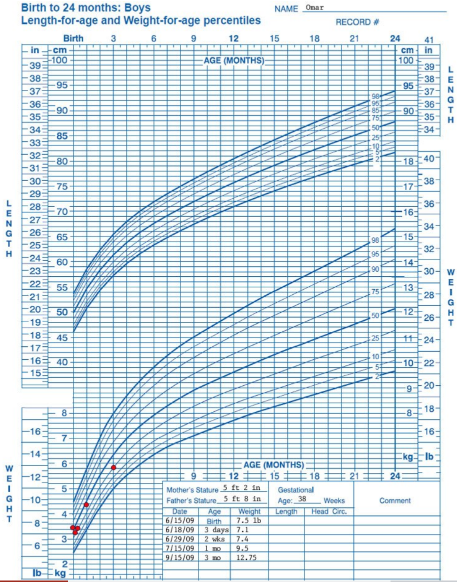
The problem:

$$\min_{\beta(\tau)} \left[ \sum_{i\colon y_i \geq (X\beta)_i} \tau \left| y_i - (X\beta)_i \right| + \sum_{i\colon y_i < (X\beta)_i} (1-\tau) \left| y_i - (X\beta)_i \right| \right], \quad \tau \in (0,1).$$

LAD estimator: $\hat{\beta}(\tau)$.

Quantile regression: $\hat{y}(x,\tau) = \hat{\beta}_0(\tau) + \hat{\beta}_1(\tau)x_1 + \ldots + \hat{\beta}_k(\tau)x_k$.
Median regression: $\hat{y}(x,\frac{1}{2}) = \hat{\beta}_0(\frac{1}{2}) + \hat{\beta}_1(\frac{1}{2})x_1 + \ldots + \hat{\beta}_k(\frac{1}{2})x_k$.

# Using R. . .


Birth to 24 months: Boys
Length-for-age and Weight-for-age percentiles
NAME Omar
RECORD #
AGE (MONTHS)
LENGTH
WEIGHT
Mother's Stature 5 ft 2 in
Father's Stature 5 ft 8 in
Gestational Age: 38 Weeks
Comment
Date | Age | Weight | Length | Head Circ.
6/15/09 | Birth | 7.5 lb
6/18/09 | 3 days | 7.1
6/29/09 | 2 wks | 7.4
7/15/09 | 1 mo | 9.5
9/15/09 | 3 mo | 12.75

# Using R. . .

```
install.packages("quantreg")
```

Example `engel` from `quantreg` demonstrates the function between costs on products and family profit.

```
library(quantreg)
data(engel)
```

We use function `rq`, formula is `foodexp` $\sim$ `income` (`foodexp` is a dependent variable; `income` is an independent variable). Argument `tau` is a parameter $\tau \in (0, 1)$, `data` is a dataset. We also use function `summary`:

```
myqreg <- rq(foodexp ~ income, tau = .5, data = engel)
summary(myqreg)
```

### Results

rq(formula = foodexp $\sim$ income, tau = 0.7, data = engel)

tau: [1] 0.7

Coefficients:

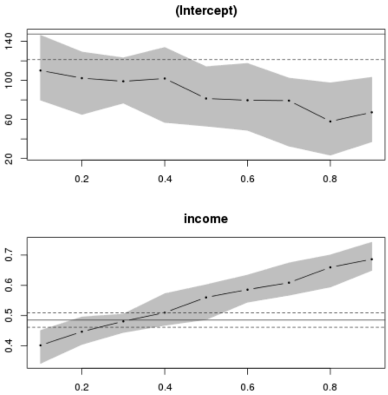
| | coefficients | lower bd | upper bd |
|---|---|---|---|
| (Intercept) | 79.28362 | 32.73534 | 102.35297 |
| income | 0.60885 | 0.56734 | 0.67405 |

We may use function `plot`, for different parameter $\tau$ from 0.1 to 0.9 with step 0.1:

myqreg $<-$ rq(foodexp $\sim$ income, tau $=$ 1:9/10, data $=$ engel)
plot(summary(myqreg))

predict

```
predict(object, newdata, type = "none", interval =
c("none", "confidence"), level = .95, na.action = na.pass,
...)
```




The solid red line is the OLS regression coefficient and the dashed red lines are the confidence intervals around the OLS. Each black dot is the slope coefficient for the quantile indicated on the x-axis. The light gray area around the black dots is the confidence interval around the quantile. The lower quantiles have significant difference below the OLS and the upper quantiles have significant difference above the OLS.

# Ridge regression

# Ridge regression

The model in a matrix form:

$$Y = X\beta + \varepsilon,$$

where $Y = (y_1, \ldots, y_n)^\intercal$, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^\intercal$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\intercal$,

$$X = \left( \begin{array}{ccccc} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ \ldots & \ldots & \ldots & \ldots & \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{array} \right).$$

$|X^\intercal X| = 0$: multicollinearity (two or more predictor variables in a multiple regression model are highly correlated).

# Ridge regression

The problem:

$$\min_{\beta}\left[\sum_{i=1}^{n}(y_i-(X\beta)_i)^2+\lambda\sum_{j=0}^{k}\beta_j^2\right],\quad \lambda>0.$$

Estimator: $\hat{\beta}(\lambda)=(X^TX+\lambda I_{k+1})^{-1}X^TY$.

Ridge regression: $\hat{y}(x,\lambda)=\hat{\beta}_0(\lambda)+\hat{\beta}_1(\lambda)x_1+\ldots+\hat{\beta}_k(\lambda)x_k$.

# Ridge regression

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Estimator: $\hat{\beta}(\lambda)=(X^TX+\lambda I_{k+1})^{-1}X^TY$.

Ridge regression: $\hat{y}(x,\lambda)=\hat{\beta}_0(\lambda)+\hat{\beta}_1(\lambda)x_1+\ldots+\hat{\beta}_k(\lambda)x_k$.

Properties:

1. For any matrix $X$ and any $\lambda>0$, there exists matrix $(X^TX+\lambda I_{k+1})^{-1}$, therefore $\hat{\beta}(\lambda)$ is unique.
2. $\hat{\beta}(\lambda)\to\hat{\beta}$ when $\lambda\to 0$.
3. $\hat{\beta}(\lambda)\to 0$ when $\lambda\to\infty$.

# Using R. . .

## lm.ridge

```
library(MASS)
lm.ridge(formula, data, subset, na.action, lambda = 0,
model = FALSE, x = FALSE, y = FALSE, contrasts = NULL, ...)
```

# Using R. . .

## lm.ridge

```
library(MASS)
lm.ridge(formula, data, subset, na.action, lambda = 0,
model = FALSE, x = FALSE, y = FALSE, contrasts = NULL, ...)
```

## summary

```
summary(object)
```

# Using R. . .

## lm.ridge

```
library(MASS)
lm.ridge(formula, data, subset, na.action, lambda = 0,
model = FALSE, x = FALSE, y = FALSE, contrasts = NULL, ...)
```

## summary

```
summary(object)
```

## plot

```
plot(object)
```

We model datasets: `x1` and `x2`, dependent variable is `y`.

```
x1 <- rnorm(30)
x2 <- rnorm(30,mean=x1,sd=.03)
y <- rnorm(30,mean=1+x1+x2)
```

Let $\lambda = 2$.

```
library(MASS)
lm.ridge(y ~ x1+x2,lambda=2)
lm(y ~ x1+x2)$coef
```

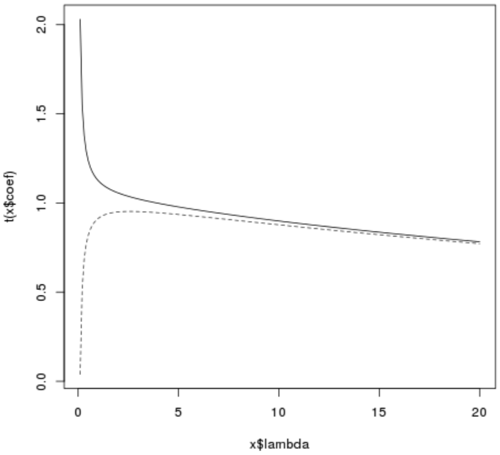
## Results

```
lm.ridge(y ~ x1+x2,lambda=2)
                    x1          x2
 1.0298541   0.9626579   0.8653311
lm(y~ x1+x2)$coef
 (Intercept)          x1           x2
 1.291811     14.763597   -12.865680
```

Construct ridge regression for $\lambda$ from 0.1 to 20 with step 0.1:

```
fit <- lm.ridge(y ~ x1+x2,lambda=seq(0.1,20,by=0.1))
plot(fit)
```

Upper graph is a graph of coef. before `x1`, the lower graph is a graph before `x2`.



## ridge

```
library(ridge)
linRidgeMod <- linearRidge(X1 ~., data=blood)
summary(linRidgeMod)
```

## result

```
Coefficients:
Estimate Scaled estimate Std.  Error t value Pr(>|t|)
(Intercept) 30.9289 NA NA NA NA
X2 0.8505 24.5135 6.2148 3.944 8e-05 ***
X3 0.3387 18.5445 6.2148 2.984 0.00285 **
---
Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1
' ' 1
Ridge parameter:  0.004737444, chosen automatically,
computed using 1 PCs
Deg.  of fr.:  model 1.917, variance 1.84, residual 1.993
```

## predict

```
predicted <- predict(linRidgeMod, blood)
compare <- cbind(actual=blood$X1, predicted)
show(compare)
```

## result

```
 actual predicted
1 132 133.7481
2 143 143.4272
3 153 153.6181
4 162 164.4789
5 154 151.7440
6 168 168.3776
7 137 140.5295
8 149 146.4830
9 159 156.3201
10 128 126.6130
11 166 165.6605
```

## Accuracy

```
mean(apply(compare, 1, min)/apply(compare, 1, max))
[1] 0.9887473
```

## Linear regression model

```
linearRegr <- lm(X1 ~ ., data=blood)
summary(linearRegr)
```

## result

```
Call:
lm(formula = X1 ~ ., data = blood)
Residuals:
Min 1Q Median 3Q Max
-3.4640 -1.1949 -0.4078 1.8511 2.6981
Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) 30.9941 11.9438 2.595 0.03186 *
X2 0.8614 0.2482 3.470 0.00844 **
X3 0.3349 0.1307 2.563 0.03351 *
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
' ' 1
Residual standard error: 2.318 on 8 degrees of freedom
Multiple R-squared: 0.9768, Adjusted R-squared: 0.9711
F-statistic: 168.8 on 2 and 8 DF, p-value: 2.874e-07
```

## result

```
 predicted2 <- predict(linearRegr, blood)
compare2 <- cbind(actual=blood$X1, predicted2)
show(compare2)
actual predicted2
1 132 133.7183
2 143 143.4317
3 153 153.6716
4 162 164.5327
5 154 151.7570
6 168 168.4078
7 137 140.4640
8 149 146.4939
9 159 156.3019
10 128 126.5407
11 166 165.6804
mean(apply(compare2, 1, min)/apply(compare2, 1, max))
[1] 0.9886923
```