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- Some Aspects Of Sengupta's Transformation

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Some Aspect Of Sengupta's Transformation

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1 Introduction

In this work,lorentz transformation and sengupta's transformation are discussed.In section 2 ,lorentz transformation its non relativistic reductions are discussed.we also discuss about electric limit and how four vectors are transforming under electric limit and invariance of continuity equation and maxwell's equation under this limit.we also study the same things under large spacelike limit $|x^0| << |x_i|$ or magnetic limit corresponding to lorentz transformation.

In the third section,we discuss how four vectors transform under Sengupta's Transformation under which we also show that $c^2t^2 - x^2 - y^2 - z^2$ remains invariant like loerntz transformation.We explore the physical meaning of $\vec{v}$ and $\vec{\omega}$ in both transformation .After that we discuss how Electric field and Magnetic field transform under Sengupta's Transformation and consider their electric and magnetic limit,Invariance of continuity equation under both limit .Maxwell's equation and its invariance under both limit.

In both cases we have considered different spatio temporal derivatives for different limits namely electric and magnetic limit . in order to get the relevant results.

2 Gallelian Electromagnetism

When two frame S and S' are moving relative to each other with speed sufficiently comparable to c ,event (x, y, z, t) in frame S is related to the event (x', y', z', t') in frame S' by the Lorentz transformation .If we further consider the two frames are moving with respect to each other along the X-axis with a boost $\vec{v}$ where two frames coincides at time t=0,we get

$$x' = \gamma (x - vt)$$

$$y' = y$$

$$z' = z \tag{1}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

γ is the relativistic factor defined by $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.Using (1) we first show that $c^2t^2 - x^2 - y^2 - z^2$ remains invariant under (1).

$$\begin{aligned}
c^2t'^2 - x'^2 - y'^2 - z'^2 &= \gamma^2\left(ct - \frac{v}{c}x\right)^2 - \gamma^2(x - vt)^2 - y^2 - z^2 \\
&= \gamma^2c^2t^2 - 2\gamma^2\frac{v}{c}xct + \gamma^2\left(\frac{v}{c}x\right)^2 - \gamma^2x^2 + 2\gamma^2xvt - \gamma^2v^2t^2 - y^2 - z^2 \\
&= c^2t^2 - x^2 - y^2 - z^2
\end{aligned}$$

It implies that the space time interval is invariant nder lorentz transformation . (1) tells us that the only components that are transforming under lorentz transformation is the x component which is along the direction of boost $\vec{v}$ and the time .the y and z component are uneffecteted under the Lorentz transformation.We can generalized (1) when boost is along any arbitrary direction.In this case ,we decomposed the spatial component along two direction ,namely,,one parralel to boost and another one perpendicular to the direction of boost.These are $\vec{x}_{||}$ and $\vec{x}_{\perp}$ respectively.The zeroth component of four vector transforms as

$$x'^0 = \gamma\left(x^0 - \vec{\beta}\cdot\vec{x}\right) \tag{2}$$

The $\vec{x}_{\perp}$ component remains uneffected under the lorentz transformation but $\vec{x}_{||}$ transforms as

$$\vec{x'}_{||} = \gamma\left(\vec{x}_{||} - \vec{\beta}x^0\right) \tag{3}$$

Therefore, the transformed spatial component in primed frame can be written as

$$\begin{aligned}
\vec{x'} &= \vec{x'}_{\perp} + \vec{x'}_{||} \\
&= \vec{x}_{\perp} + \gamma\left(\vec{x}_{||} - \vec{\beta}x^0\right) \\
&= \vec{x} - \vec{x}_{||} + \gamma\left(\vec{x}_{||} - \vec{\beta}x^0\right) \\
&= \vec{x} + (\gamma - 1)\vec{x}_{||} - \gamma\vec{\beta}x^0
\end{aligned} \tag{4}$$

we can see that $\vec{x}_{||}$ can be written in terms of $\vec{x}$ and $\vec{\beta}$ as

$$\vec{x}_{||} = \frac{\vec{\beta}\left(\vec{\beta}\cdot\vec{x}\right)}{\beta^2} \tag{5}$$

Therefore, (4) becomes

$$\vec{x'} = \vec{x} + (\gamma - 1)\frac{\vec{\beta}\left(\vec{\beta}\cdot\vec{x}\right)}{\beta^2} - \gamma\vec{\beta}x^0 \tag{6}$$

Now,the Lorentz transformation of a four vector$(u^0, \vec{u})$ where the four components have the same unit ,is given by (6)

$$u^{'0} = \gamma \left(u^0 - \frac{\vec{v} \cdot \vec{u}}{c} \right) \tag{7}$$

and

$$\vec{u}' = \vec{u} - \gamma \frac{\vec{v} u^0}{c} + (\gamma - 1) \frac{\vec{v} \, (\vec{v} \cdot \vec{u})}{v^2} \tag{8}$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ with a relative velocity $\mathbf{v}$ [boost in arbitrary direction] and $\vec{\beta} = \frac{\vec{v}}{c}$ in the limit $\vec{v} << c$ or $\gamma \to 1$, (7) becomes

$$u^{'0} = u^0 - \frac{\vec{v} \cdot \vec{u}}{c} \tag{9}$$

and (8) becomes

$$\vec{u}^{'} = \vec{u} - \frac{\vec{v} u^0}{c} \tag{10}$$

now when $|u^0| >> |\vec{u}|$,(9) becomes

$$u^{'0} = u^0 \tag{11}$$

as $|\vec{v}| << c$ and $|u^0| >> |\vec{u}|$, we drop the term $\frac{\vec{v} \cdot \vec{u}}{c}$. $(|\frac{\vec{v} \cdot \vec{u}}{c}| << |u^0|)$ (10) becomes

$$\vec{u}^{'} = \vec{u} - \frac{\vec{v} u^0}{c} \tag{12}$$

it implies that (9) remains same in the large timelike limit,i.e,in $|u^0| >> |\vec{u}|$ limit. here we cannot neglect $\frac{\vec{v} u^0}{c}$ term .in spite of being $|\vec{v}| << c$, $|u^0|$ is much greater than $|\vec{u}|$ $(|u^0| >> |\vec{u}|$) .this is the reason why we keep the term $\frac{\vec{v} u^0}{c}$ in (9).

now in the limit $|u^0| << |\vec{u}|$

$$u^{'0} = u^0 - \frac{\vec{v} \cdot \vec{u}}{c} \tag{13}$$

as we can not neglect the $\frac{\vec{v} \cdot \vec{u}}{c}$ term in this case.the other equation (10) reads as

$$\vec{u}^{'} = \vec{u} \tag{14}$$

here we have dropped the term $\frac{\vec{v} u^0}{c}$ as $|\vec{v}| << c$ and $|u^0| << |\vec{u}|$. now assuming $u^0 = ct$ and $\mathbf{u} = \mathbf{x}$, we get

$$c\Delta t' = c\Delta t \tag{15}$$

$$\Delta \vec{r}' = \Delta \vec{r} - \vec{v} \Delta t \tag{16}$$

it only holds if $|\Delta \vec{r}| << c|\Delta t|$ .this coincides with the famous gallalian transformation where time remains universal and only spatial coordinates are transformed.We can also derive this by considering the limit $c \to \infty$ in (9) and (10) but it would be naive as it ignores the cases of large timelike limit and large spacelike limit. let us calculate spatio- temporal derivatives corresponding to gallelian transformation

$$\begin{aligned}\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial t'} \\ &= \frac{\partial}{\partial t} + \frac{\partial x^j}{\partial t'}\frac{\partial}{\partial x^j}\end{aligned} \tag{17}$$

as $\frac{\partial t}{\partial t'} = 1$ and $\frac{\partial x^j}{\partial t'} = v^j$ where v^j is the jth component of the velocity vector. therefore (17) can be written as

$$= \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \tag{18}$$

$$\frac{\partial}{c\partial t'} = \frac{\partial}{c\partial t} + \frac{\vec{v} \cdot \nabla}{c} \tag{19}$$

let $x^0 = ct$,therefore (19) can be written as

$$\frac{\partial}{\partial x'^0} = \frac{\partial}{\partial x^0} + \frac{\vec{v} \cdot \nabla}{c} \tag{20}$$

now as we know $x^0 = -x_0$,(20) becomes

$$-\frac{\partial}{\partial x'_0} = -\frac{\partial}{\partial x_0} + \frac{\vec{v} \cdot \nabla}{c} \tag{21}$$

$$\frac{\partial}{\partial x'_0} = \frac{\partial}{\partial x_0} - \frac{\mathbf{v} \cdot \nabla}{c} \tag{22}$$

let's calculate the ∇'

$$\frac{\partial}{\partial x'^j} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x'^j} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial x'^j} \tag{23}$$

now $\frac{\partial t}{\partial x'^j} = 0$ as t does not depend on x'^j and $\frac{\partial x^j}{\partial x'^j} = 1$ (from 15 and 16) hence ,(23) becomes

$$\frac{\partial}{\partial x'^j} = \frac{\partial}{\partial t}(0) + \frac{\partial}{\partial x^j}(1) \tag{24}$$

$$\frac{\partial}{\partial x'^j} = \frac{\partial}{\partial x^j} \tag{25}$$

$$\nabla' = \nabla \tag{26}$$

and finally from the lorentz transformation of electric and magnetic field ,we get

$$\vec{E}' = \gamma\left(\vec{E} + \vec{v} \times \vec{B}\right) + (1-\gamma)\frac{\vec{v}\left(\vec{v} \cdot \vec{E}\right)}{v^2} \tag{27}$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right) + (1-\gamma)\frac{\vec{v}\left(\vec{v}\cdot\vec{B}\right)}{v^2} \tag{28}$$

The field Tensor $F^{\mu\nu}$ reads $\partial^\mu A^\nu - \partial^\nu A^\mu$ where A^μ are the components of the four potential.Now ,under Lorentz transformation the field tensor transforms like

$$F'^{\lambda\rho} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}F^{\mu\nu} \tag{29}$$

using (2) and (6), we get

$$F'^{0\rho} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^\rho}{\partial x^\rho}F^{0\rho} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^\rho}{\partial x^\nu}F^{0\nu} + \frac{\partial x'^0}{\partial x^\nu}\frac{\partial x'^\rho}{\partial x^0}F^{\nu 0} + \frac{\partial x'^0}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\rho}F^{\mu\rho} + \frac{\partial x'^0}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}F^{\mu\nu} \tag{30}$$

We write (29) as the form (30) because the transformed field tensor component that corresponds to the electric field will depend on all 16 field tensor component and four of them are automatically zero as $F^{\nu\nu} = \partial^\nu A^\nu - \partial^\nu A^\nu = 0$ and among the remaining 12 we have to know just six component which is actually consistent with the the three component of electric field and three components of magnetic field as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -\partial^\nu A^\nu + \partial^\mu A^\nu = -F^{\nu\mu}$ and finally μ and ν runs from 0,1,2,3.Now using (2) and (6) ,we get

$$F'^{0\rho} = \gamma F^{0\rho} + \gamma(\gamma-1)\frac{\beta^\rho\beta_\nu}{\beta^2}F^{0\nu} - \gamma^2\beta^\rho\beta_\nu F^{0\nu} + \gamma\beta_\mu F^{\rho\mu} + \gamma(\gamma-1)\frac{\beta^\rho\beta_\nu\beta_\mu}{\beta^2}F^{\nu\mu} \tag{31}$$

Now let us calculate $F'^{\lambda\rho}$

$$F'^{\lambda\rho} = \frac{\partial x'^\lambda}{\partial x^0}\frac{\partial x'^\rho}{\partial x^\rho}F^{0\rho} + \frac{\partial x'^\lambda}{\partial x^\lambda}\frac{\partial x'^\rho}{\partial x^\nu}F^{\lambda\nu} + \frac{\partial x'^\lambda}{\partial x^\lambda}\frac{\partial x'^\rho}{\partial x^0}F^{\lambda 0} + \frac{\partial x'^\lambda}{\partial x^\lambda}\frac{\partial x'^\rho}{\partial x^\rho}F^{\lambda\rho} + \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\rho}F^{\mu\rho} + \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}F^{\mu\nu} \tag{32}$$

$$F'^{\lambda\rho} = -\gamma\beta^\lambda F^{0\rho} + \gamma\beta^\rho F^{0\lambda} + (\gamma-1)\frac{\beta^\rho\beta_\nu}{\beta^2}F^{\lambda\nu} + F^{\lambda\rho} + (\gamma-1)\frac{\beta^\lambda\beta_\mu}{\beta^2}F^{\mu\rho} + (\gamma-1)^2\frac{\beta^\lambda\beta^\rho\beta_\mu\beta_\nu}{\beta^4}F^{\mu\nu} \tag{33}$$

We can proceed in a similar way in the case of sengupta's transformation and where we use $\tilde{\gamma}$ and $\tilde{\vec{\beta}}$ instead of γ and $\vec{\beta}$ .Now we can easily calculate the components of field tensors according to the definition of it and we will finally get (27) and (28) .Now when $|\vec{v}| << |c|$ and $\gamma \to 1$,eq 27 and 28 reduce to

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B} \tag{34}$$

$$\vec{B}' = \vec{B} - \frac{\vec{v}\times\vec{E}}{c^2} \tag{35}$$

2.1 Electric Limit

Let us investigate how current 4 vector $(c\rho, \mathbf{j})$ changes under lorentz transformation.the transformed zeroth component and $\vec{j}$ are given by eq 7 and eq 8

$$c\rho' = \gamma\left(c\rho - \frac{\vec{v}\cdot\vec{j}}{c}\right) \tag{36}$$

and

$$\vec{j}' = \vec{j} - \gamma \frac{\vec{v} c\rho}{c} + (\gamma - 1) \frac{\vec{v}\left(\vec{v} \cdot \vec{j}\right)}{v^2} \tag{37}$$

now in the case $c|\rho| >> |\vec{j}|$,which implies that $E >> cB$.This is the electric limit (subscript e) which corresponds to the transformation law(given by eq. and eq.) for $\left(c\rho, \vec{j}\right)$

$$\rho_e^{'} = \rho_e \tag{38}$$

$$\vec{j}_e^{'} = \vec{j}_e - \mathbf{v}\rho_e \tag{39}$$

and the continuity equation reads

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j}_e = 0 \tag{40}$$

now let's check wheather continuity equation is valid or not in under the transformed current four vector.

$$\frac{\partial \rho_e^{'}}{\partial t^{'}} + \nabla^{'} \cdot \vec{j}_e^{'} = \left(\frac{\partial}{\partial t} \vec{v} \cdot \nabla\right) \rho_e + \nabla \cdot \left(\vec{j}_e - \mathbf{v}\rho_e\right) \tag{41}$$

$$= \frac{\partial \rho_e}{\partial t} + \vec{v} \cdot \nabla \rho_e + \nabla \cdot \vec{j}_e - \nabla \cdot (\vec{v}\rho_e) \tag{42}$$

now $\nabla \cdot (\vec{v}\rho_e) = \rho_e \nabla \cdot \vec{v} Vecv \cdot \nabla \rho_e$ using the above identity in equation 33 ,we get

$$= \frac{\partial \rho_e}{\partial t} + \vec{v} \cdot \nabla \rho_e + \nabla \cdot \vec{j}_e - \rho_e \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \rho_e \tag{43}$$

now when $\mathbf{v}$ is uniform ,we get $\nabla \cdot \vec{v} = 0$

$$= \frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j}_e \tag{44}$$

$$= 0 \tag{45}$$

therefore , continuiuty equation holds. taking into account that $|\vec{E}_e| >> c|\vec{B}_e|$ we can derive the following transformation law for the electromagnetic field from equation 34 and 35.

$$\vec{E}_e^{'} = \vec{E}_e \tag{46}$$

$$\vec{B}_e^{'} = \vec{B}_e - \frac{\vec{v} \times \vec{E}_e}{c^2} \tag{47}$$

the last two equations shows that the motion of the electric field induces a magnetic field (generally any time variation) , on the other hand the motion of the magnetic field does not induce an electric field.therefore the faraday's law of the induction is no longer true in this limit.there can be no faraday term. there fore curl $\vec{E}_e$ becomes zero.and maxwell's equation which is originally

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{48}$$

$$\nabla \cdot \vec{B} = 0 \tag{49}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{50}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{51}$$

becomes

$$\nabla \cdot \vec{E}_e = \frac{\rho_e}{\epsilon_0} \tag{52}$$

$$\nabla \cdot \vec{B}_e = 0 \tag{53}$$

$$\nabla \times \vec{E}_e = 0 \tag{54}$$

$$\nabla \times \vec{B}_e = \mu_0 \vec{j}_e + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e}{\partial t} \tag{55}$$

let's check wheather equations (52) to (56) are invariant under gallelian transformation or not

$$\nabla^{'} \cdot \vec{E}_e^{'} = \frac{\rho_e^{'}}{\epsilon_0} \tag{56}$$

it implies that $\nabla \cdot \vec{E}_e = \frac{\rho_e}{\epsilon_0}$ we get it using equation

$$\begin{aligned} &\nabla^{'} \cdot \vec{B}_e^{'} = 0 \\ \implies &\nabla^{'} \cdot \vec{B}_e^{'} = \nabla \cdot \left(\vec{B}_e - \frac{\vec{v} \times \vec{E}_e}{c^2} \right) \\ \implies &\nabla^{'} \cdot \vec{B}_e^{'} = 0 - \nabla \cdot \left(\frac{\vec{v} \times \vec{E}_e}{c^2} \right) \end{aligned} \tag{57}$$

now $\nabla \cdot \left(\vec{v} \times \vec{E}_e \right) = \vec{E}_e \cdot (\nabla \times \vec{v}) - \vec{v} \cdot \left(\nabla \times \vec{E}_e \right)$ as $\nabla \times \vec{v} = 0$ and $\nabla \times \vec{E}_e = 0$ therefore,$\nabla^{'} \cdot \vec{B}_e^{'} = 0$

$$\begin{aligned} &\nabla^{'} \times \vec{E}_e^{'} = 0 \\ \implies &\nabla^{'} \times \vec{E}_e^{'} = \nabla \times \vec{E}_e \end{aligned} \tag{58}$$

or,$\nabla^{'} \times \vec{E}_e^{'} = 0$ from (54)

$$\nabla^{'} \times \vec{B}_e^{'} = \mu_0 \vec{j}_e^{'} + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e^{'}}{\partial t} \tag{59}$$

So,maxwell's equations are invariant under gallelian transformations, physically the theory based on these equations will describe situations where isolated electric charges move with low velocities .Now electric field and magnetic field can be derived from scalar and vector potential

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{60}$$

$$\vec{B} = \nabla \times \vec{A} \tag{61}$$

and as four potential is given by $\left(\frac{\phi}{c}, \vec{A}\right)$ and in case of electric limit $|\phi| >> c|\vec{A}|$,it gives us

$$\vec{E}_e = -\nabla\phi_e \tag{62}$$

$$\vec{B}_e = \nabla \times \vec{A}_e \tag{63}$$

and $\left(\frac{\phi_e}{c}, \vec{A}_e\right)$ transforms as

$$\phi_e^{'} = \phi_e \tag{64}$$

$$\vec{A}_e^{'} = \vec{A}_e - \epsilon_0\mu_0\vec{v}\phi_e \tag{65}$$

the lorentz force is given by

$$\vec{F} = \int d^3\vec{r}[\rho\left(\vec{r}\right)\vec{E}\left(\vec{r}\right) + \vec{j}\left(\vec{r}\right) \times \vec{B}\left(\vec{r}\right)] \tag{66}$$

under electric limit (57) reduces to

$$\vec{F}_e = \int d^3\vec{r}\rho_e\left(\vec{r}\right)\vec{E}_e\left(\vec{r}\right) \tag{67}$$

thus in electric limit ,the magnetic field does exist but it has no effect .

2.2 Magnetic limit

now let's study the case where $c|\rho| << |\vec{j}|$ and $|\vec{E}| << |c\vec{B}|$ in order to obtain the magnetic limit we use the transformation relation 7 and 8 and thus we get

$$c\rho_m^{'} = c\rho_m - \frac{\vec{v} \cdot \vec{j}_m}{c} \tag{68}$$

it implies that

$$\rho_m^{'} = \rho_m - \frac{\vec{v} \cdot \vec{j}_m}{c^2} \tag{69}$$

and $\vec{j}$ becomes

$$\vec{j}_m^{'} = \vec{j}_m \tag{70}$$

now taking into account that $|\vec{E}_m| << c|\vec{B}_m|$ we obtain the field transformation law

$$\vec{E}_m^{'} = \vec{E}_m + \vec{v} \times \vec{B}_m \tag{71}$$

$$\vec{B}_m^{'} = \vec{B}_m \tag{72}$$

four potential transforms as

$$\frac{\phi_m^{'}}{c} = \frac{\phi_m}{c} - \frac{\vec{v}\cdot\vec{A}_m}{c} \tag{73}$$

it implies that

$$\phi_m^{'} = \phi_m - \vec{v}\cdot\vec{A}_m \tag{74}$$

$$\vec{A}_m^{'} = \vec{A}_m \tag{75}$$

according to the transformation law 7 and 8 and putting $u^0 = ct$ and $\vec{u} = \vec{x}$ we get

$$ct^{'} = ct - \frac{\vec{v}\cdot\vec{x}}{c} \tag{76}$$

$$t^{'} = t - \frac{\vec{v}\cdot\vec{x}}{c^2} \tag{77}$$

$$\vec{x}^{'} = \vec{x} \tag{78}$$

we now calculate the spatio temporal derivative corresponding to transformation law 77 and 78

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial t'} \tag{79}$$

(77) can be written as

$$t = t^{'} + \frac{\vec{v}\cdot\vec{x}^{'}}{c^2} \tag{80}$$

using (78).

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \tag{81}$$

as $\vec{x}$ does not depend on t and $\frac{\partial t}{\partial t'} = 1$ and ∇ transforms as

$$\frac{\partial}{\partial x'^j} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x'^j} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial x'^j} \tag{82}$$

$$\frac{\partial}{\partial x'^j} = \frac{\partial}{\partial t}\frac{v_j}{c^2} + \frac{\partial}{\partial x^j} \tag{83}$$

$$\nabla^{'} = \frac{\partial}{\partial t}\frac{\vec{v}}{c^2} + \nabla \tag{84}$$

Maxwell's equations (48-51) become

$$\nabla\cdot\vec{E}_m = \frac{\rho_m}{\epsilon_0} \tag{85}$$

$$\nabla\cdot\vec{B}_m = 0 \tag{86}$$

$$\nabla\times\vec{E}_m = -\frac{\partial\vec{B}_m}{\partial t} \tag{87}$$

$$\nabla\times\vec{B}_m = \mu_0\vec{j}_m \tag{88}$$

From equation 71 we obtain that the motion of a magnetic field induces an electric field .This is the reason behind the appearance of faraday term in the maxwell's equation but equation 88 just gives us the ampere's law.The electric and magnetic field can also be expressed in terms of four potential.Electric field will depend upon both ϕ_m, $\mathbf{A}_m$ and as transformed electric field depends upon both electric and magnetic field in unprimed reference frame.But the magnetic field only depends on $\vec{A}_m$ .The following equations

$$\vec{E}_m = -\nabla\phi_m - \frac{\partial \vec{A}_m}{\partial t} \tag{89}$$

$$\vec{B}_m = \nabla \times \vec{A}_m \tag{90}$$

$$\vec{E}_m^{'} = -\nabla'\phi_m^{'} - \frac{\partial \vec{A}_m^{''}}{\partial t'} \tag{91}$$

$$\vec{B}_m^{'} = \nabla^{'} \times \vec{A}_m^{''} \tag{92}$$

Let's explicitly calculate the electric and magnetic field in primed frame under magnetic limit.

$$\vec{E}_m^{'} = -(\frac{\partial}{\partial t}\frac{\vec{v}}{c^2} + \nabla)(\phi_m - \vec{v}\cdot\vec{A}_m) - \frac{\partial A_m}{\partial t} \tag{93}$$

$$= -\frac{\vec{v}}{c^2}\frac{\partial \phi_m}{\partial t} - \nabla\phi_m + \frac{\vec{v}}{c^2}\frac{\partial\left(\vec{v}\cdot\vec{A}_m\right)}{\partial t} + \nabla\left(\vec{v}\cdot\vec{A}_m\right) - \frac{\partial A_m}{\partial t}$$

$$= -\frac{\vec{v}}{c^2}\frac{\partial \phi_m}{\partial t} + \vec{E}_m + \frac{\vec{v}}{c^2}\frac{\partial\left(\vec{v}\cdot\vec{A}_m\right)}{\partial t} + \left(\vec{v}\cdot\nabla\right)\vec{A}_m + \vec{v}\times\left(\nabla\times\vec{A}_m\right)$$

Similarly,for magnetic field we get

$$= -\frac{\vec{v}}{c^2}\frac{\partial \phi_m}{\partial t} + \mathbf{E}_m + \frac{\vec{v}}{c^2}\frac{\partial\left(\vec{v}\cdot\vec{A}_m\right)}{\partial t} + \left(\vec{v}\cdot\nabla\right)\vec{A}_m + \vec{v}\times\mathbf{B}_m$$

now using (71) ,we conclude that

$$-\frac{\vec{v}}{c^2}\frac{\partial \phi_m}{\partial t} + \frac{\vec{v}}{c^2}\frac{\partial\left(\vec{v}\cdot\vec{A}_m\right)}{\partial t} + \left(\vec{v}\cdot\nabla\right)VecfA_m = 0 \tag{94}$$

$$\vec{B}_m^{'} = \nabla' \times \vec{A}_m^{''}$$

$$\vec{B}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \times \vec{A}_m$$

$$\vec{B}_m^{'} = \nabla \times \vec{A}_m + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t}$$

$$\vec{B}_m^{'} = \vec{B}_m + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t}$$

As (72) tells us that magnetic field remains invariant under lorentz transformation when magnetic limit is considered,we get

$$\vec{B}_m^{'} = \vec{B}_m^{'} + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t}$$

$$\frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t} = 0$$

We are interested to check wheather divergence of the magnetic field or the flux of magnetic field vanishes under large spacelike transformation

$$\nabla' \cdot \vec{B}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \cdot \left(\vec{B}_m\right) \tag{95}$$

$$\nabla' \cdot \vec{B}_m^{'} = \nabla \cdot \vec{B}_m + \frac{\vec{v}}{c^2} \cdot \frac{\partial \nabla \times \vec{A}_m}{\partial t} \tag{96}$$

As we know $\nabla \cdot \left(\vec{v} \times \vec{A}_m\right) = \vec{A}_m \cdot (\nabla \times \vec{v}) - \vec{v} \cdot \left(\nabla \times \vec{A}_m\right)$ but as $\vec{v}$ is uniform therefore $\nabla \times \vec{v} = 0$ and $\nabla \left(\vec{v} \times \vec{A}_m\right) = -\vec{v} \cdot \left(\nabla \times \vec{A}_m\right)$ and time derivative will be zero.

$$\nabla' \cdot \vec{B}_m = \nabla \cdot \vec{B}_m = 0 \tag{97}$$

$$\nabla' \cdot \vec{E}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \cdot \left(\vec{E}_m + \mathbf{v} \times \vec{B}_m\right)$$

$$= \nabla \cdot \vec{E}_m + \frac{\vec{v}}{c^2} \cdot \frac{\partial \vec{E}_m}{\partial t} + \nabla \cdot \left(\vec{v} \times \vec{B}_m\right) + \frac{\vec{v}}{c^2} \cdot \frac{\partial \left(\vec{v} \times \vec{B}_m\right)}{\partial t}$$

here we are neglecting $\frac{\vec{v}}{c^2} \cdot \frac{\partial \vec{E}_m}{\partial t}$ term as it has c^2 in its denominator.

$$= \frac{\rho_m}{\epsilon_0} - \frac{\epsilon_0 \mu_0 \vec{v} \cdot \vec{j}}{\epsilon_0} \tag{98}$$

$$= \frac{\rho_m^{'}}{\epsilon_0} \tag{99}$$

Let's verify the invarience of faraday's law .

$$\nabla' \times \vec{E}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \times \left(-\nabla' \phi_m^{'} - \frac{\partial \vec{A}_m^{'}}{\partial t'}\right) \tag{100}$$

$$\nabla' \times \vec{E}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \times \left(-\nabla\left(\phi_m - \vec{v} \cdot \vec{A}_m\right) - \frac{\vec{v}}{c^2}\frac{\partial \left(\phi_m - \vec{v} \cdot \vec{A}_m\right)}{\partial t} - \frac{\partial \vec{A}_m}{\partial t}\right)$$

$$= -\left(\nabla \times \frac{\vec{v}}{c^2}\frac{\partial \left(\phi_m - \vec{v} \cdot \vec{A}_m\right)}{\partial t} + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t} \times \nabla\left(\phi_m - \vec{v} \cdot \vec{A}_m\right) + \nabla \times \frac{\partial \vec{A}_m}{\partial t} + \frac{\partial \left(\frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t}\right)}{\partial t}\right)$$

Now,first two term cancels as $\nabla \times \frac{\vec{v}}{c^2}\frac{\partial\left(\phi_m - \vec{v}\cdot\vec{A}_m\right)}{\partial t} = \nabla\frac{\partial\left(\phi_m - \vec{v}\cdot\vec{A}_m\right)}{\partial t} \times \frac{\vec{v}}{c^2} = -\frac{\vec{v}}{c^2} \times \nabla\frac{\partial\left(\phi_m - \mathbf{v}\cdot\vec{A}_m\right)}{\partial t}$.we also have $\frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}_m}{\partial t} = 0$ and $\nabla \times \vec{A}_m = \mathbf{B}_m$

Therefore,

$$\nabla' \times \vec{E}_m^{'} = -\frac{\partial \vec{B}_m}{\partial t} = -\frac{\partial \vec{B}_m^{'}}{\partial t'} \tag{101}$$

In case of Ampere"s Law in primed frame,(79) reads

$$\nabla' \times \vec{B}_m^{'} = \left(\nabla + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \times \vec{B}_m \tag{102}$$

$$= \nabla \times \vec{B}_m + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{B}_{m]}}{\partial t}$$

$$= \mu_0 \vec{j}_m + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{B}_m}{\partial t}$$

here we are neglecting $\frac{\vec{v}}{c^2} \times \frac{\partial \vec{B}_m}{\partial t}$ term as it has c^2 in its denominator. Hence,

$$\nabla' \times \vec{B}_m^{'} = \mu_0 \vec{j}_m^{'} \tag{103}$$

Finally, the continuity equation

$$\frac{\partial \rho_m^{'}}{\partial t'} + \nabla' \cdot \vec{j}_m^{'}$$

$$= \frac{\partial\left(\rho_m - \epsilon_0\mu_0\vec{v}\cdot\vec{j}_m\right)}{\partial t} + \left(\nabla + \epsilon_0\mu_0\vec{v}\cdot\frac{\partial}{\partial t}\right)\cdot\vec{j}_m$$

$$= \frac{\partial \rho_m}{\partial t} + \nabla \cdot \vec{j}_m = 0$$

So,the continuity equation holds. the lorentz force is given by

$$\vec{F} = \int d^3\vec{r}[\rho\left(\vec{r}\right)\vec{E}\left(\vec{r}\right) + \vec{j}\left(\vec{r}\right) \times \vec{B}\left(\vec{r}\right)]$$

under magnetic limit the above relation reduces to

$$\vec{F}_m = \int d^3\vec{j}\left(\vec{r}\right) \times \vec{B}\left(\vec{r}\right)]$$

Thus in magnetic limit electric field does exist but have no effect.

3 Carrollian Electromagnetism

Under Sengupta's transformation, the four vector transforms like

$$x^{'} = \tilde{\gamma}\left(x - \tilde{\beta}x^{0}\right); y^{'} = y; z^{'} = z \tag{104}$$

$$x^{'0} = \tilde{\gamma}\left(x^{0} - \tilde{\beta}x\right) \tag{105}$$

where we have considered boost velocity ω along the x-axis.the Sengupta factors are defined as $\tilde{\gamma} = \frac{1}{\sqrt{1-\tilde{\beta}^{2}}}$ and $\tilde{\beta} = \frac{c}{\omega}$.Using (104) and (105) we first show that $c^{2}t^{2} - x^{2} - y^{2} - z^{2}$ remains invariant under (104)and (105).

$$\begin{aligned}\left(x^{\prime 0}\right)^{2} - x^{\prime 2} - y^{\prime 2} - z^{\prime 2} &= \tilde{\gamma}^{2}\left(x^{0} - \tilde{\beta}x\right)^{2} - \tilde{\gamma}^{2}\left(x - \tilde{\beta}x^{0}\right)^{2} - y^{2} - z^{2} \\ &= \tilde{\gamma}^{2}\left(x^{0}\right)^{2} - 2\tilde{\beta}\tilde{\gamma}^{2}x^{0}x + \tilde{\gamma}^{2}\left(\tilde{\beta}x\right)^{2} - \tilde{\gamma}^{2}x^{2} + 2\tilde{\gamma}^{2}x\tilde{\beta}x^{0} - \tilde{\gamma}^{2}\tilde{\beta}^{2}\left(x^{0}\right)^{2} - y^{2} - z^{2} \\ &= \left(x^{0}\right)^{2} - x^{2} - y^{2} - z^{2}\end{aligned}$$

We can generalize the above treatment for a boost along any arbitrary direction.Noting that the position vector $\vec{x}$ can be written in terms of two components -one is parallel to the boost by denoting $\vec{x}_{||}$ and the another is perpendicular to it .we denote the perpendicular component by $\vec{x}_{\perp}$.Therefore, $\vec{x} = \vec{x}_{||} + \vec{x}_{\perp}$.The parallel component must be given by $\frac{\vec{\tilde{\beta}}\left(\vec{\tilde{\beta}}\cdot\vec{x}\right)}{\tilde{\beta}^{2}}$.$\tilde{\beta} = \frac{c\vec{\omega}}{\omega^{2}}$.Now we get the transformation law for parallel and perpendicular components

$$\vec{x'}_{||} = \tilde{\gamma}\left(\vec{x}_{||} - \vec{\tilde{\beta}}x^{0}\right); \vec{x'}_{\perp} = \vec{x}^{\perp}. \tag{106}$$

Now , we can find the transformation of the position vector in the primed frame.

$$\begin{aligned}\vec{x'} &= \vec{x'}_{||} + \vec{x'}_{\perp} \\ &= \tilde{\gamma}\left(\vec{x}_{||} - \vec{\tilde{\beta}}x^{0}\right) + \vec{x}_{\perp} \\ &= \tilde{\gamma}\left(\vec{x}_{||} - \vec{\tilde{\beta}}x^{0}\right) + \vec{x} - \vec{x}_{||} \\ &= \vec{x} + \left(\tilde{\gamma} - 1\right)\vec{x}_{||} - \tilde{\gamma}\vec{\tilde{\beta}}x^{0}\end{aligned} \tag{107}$$

and the temporal coordinate changes as

$$x^{\prime 0} = \tilde{\gamma}\left(x^{0} - \vec{\tilde{\beta}}\cdot\vec{x}\right) \tag{108}$$

Though both $\vec{v}$ and $\vec{\omega}$ have dimension of velocity but they have different physical interpretations.As γ and $\tilde{\gamma}$ are dimensionless quantities , the denominators in both $\gamma = \frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}$ and $\tilde{\gamma} = \frac{1}{\sqrt{1-\frac{c^{2}}{\omega^{2}}}}$ will be dimensionless .If they are not , γ and $\tilde{\gamma}$ won't be dimensionless quantities.From dimensional analysis,we can conclude that denominators of both γ and $\tilde{\gamma}$ must be dimensionless which implies that $\frac{v}{c}$ and $\frac{c}{\omega}$ must be dimensionless quantities.Therefore ,dimensions of $\vec{v}$ and $\vec{\omega}$ are same .In fact ,their dimensions will be same as c which has the dimension of velocity.In order to understand their physical meaning,we find,

$$\begin{aligned} x'^0 &= \tilde{\gamma}\left(x^0 - \tilde{\vec{\beta}} \cdot \vec{x}\right) \\ &= \tilde{\gamma}\left(x^0 - \tilde{\beta}_i x^i\right) \end{aligned} \tag{109}$$

Now putting $x^0 = ct$ and $\tilde{\beta}_i = \frac{c\omega_i}{\omega^2}$ in (105), we get upon simplifying (105)

$$t' = \tilde{\gamma}\left(t - \frac{\tilde{\beta}_i}{c} x^i\right) \tag{110}$$

Now,setting $\Delta t' = 0$,we get from (106)

$$0 = \tilde{\gamma}\left(\Delta t - \frac{\tilde{\beta}_i}{c} \Delta x^i\right) \tag{111}$$

therefore from (106)

$$\Delta t = \tilde{\beta}_i \Delta x^i \tag{112}$$

now as $\tilde{\beta}_i = \frac{c\omega_i}{\omega^i \omega_i}$,(112) implies that $\omega_i = \frac{\Delta x^i}{\Delta t}$ .Finally,taking $\Delta t \to 0$ we get $\omega_i = \frac{dx^i}{dt}$. Similar result can be obtained in case of Lorentz Transformation.Recalling (6),we can write (6) in an alternate form ,namely

$$x'^i = x^i - \gamma v^i t + (\gamma - 1)\frac{v^i v_j}{v^2} x^j \tag{113}$$

Now ,let us compute the derivative of (113) with respect to x^i.We get

$$\frac{dx'^i}{dx^i} = 1 - \gamma v^i \frac{dt}{dx^i} + (\gamma - 1)\frac{v^i v_j}{v^2}\frac{dx^j}{dx^i} \tag{114}$$

Now setting left hand side of the equation (114) to zero ,we get

$$0 = 1 - \gamma v^i \frac{dt}{dx^i} + (\gamma - 1)\frac{v^i v_j}{v^2}\delta_i^j \tag{115}$$

Now as $\delta_i^j = 0$ for $i \neq j$ and 1 for $i = j$ (115) reduces to the following

$$\begin{aligned} 0 &= 1 - \gamma v^i \frac{dt}{dx^i} + (\gamma - 1)\frac{v^i v_i}{v^2} \\ &= 1 - \gamma v^i \frac{dt}{dx^i} + (\gamma - 1)\frac{v^2}{v^2} \\ &= 1 - \gamma v^i \frac{dt}{dx^i} + \gamma - 1 \\ &= \gamma - \gamma v^i \frac{dt}{dx^i} \end{aligned} \tag{116}$$

(116) implies the following thing

$$
\begin{aligned}
0 &= \gamma\left(1 - v^i \frac{dt}{dx^i}\right) \\
\implies 0 &= 1 - v^i \frac{dt}{dx^i} \\
\implies 1 &= v^i \frac{dt}{dx^i} \\
\implies v^i &= \frac{dx^i}{dt}
\end{aligned} \tag{117}
$$

this is how we prove that $\vec{v}$ is a lorentz boost .Similar for ω^i in sengupta's case. these results imply that ω^i is the rate of a motion of an event in a unprimed frame that occurs at a fixed instant of time in a primed frame. Similarly,v^i gives the rate of motion of an event in a unprimed frame that occurs at a fixed spatial coordinate(or rest) in a primed frame. a four vector $(u^0, \vec{u})$ is transformed according to the sengupta's transformation as

$$u'^0 = \tilde{\gamma}\left(u^0 - \tilde{\vec{\beta}} \cdot \vec{u}\right) \tag{118}$$

$$\vec{u'} = \vec{u} - \tilde{\gamma}\tilde{\vec{\beta}}u^0 + (\gamma - 1)\frac{\tilde{\vec{\beta}}\left(\tilde{\vec{\beta}} \cdot \vec{u}\right)}{\tilde{\beta}^2} \tag{119}$$

The above mentioned sengupta transformation will be physically realizable when $\tilde{\gamma}$ is real.Therefore ,$1 - \frac{c^2}{\omega^2}$ in the denominator must be greater than or equal to zero but denominator can not be zero as $\tilde{\gamma}$ would be undefined in that case.Therefore,$|\omega| > |c|$.In case of Lorentz transformation γ must be real and this gives us $|\vec{v}| < c$. Now,Carroll limit can be achieved in the framework of sengupta's transformation under the following considerations

$$
\begin{aligned}
c &<< \omega; \\
\tilde{\gamma} &\to 1; \\
|\tilde{\vec{\beta}}^i| = \frac{c\omega^i}{\omega^2} &<< 1
\end{aligned} \tag{120}
$$

therefore in the limit given by (120) ,(118) ,(119)reduces to

$$u'^0 = u^0 - \tilde{\vec{\beta}} \cdot \vec{u} \tag{121}$$

$$\vec{u'} = \vec{u} - \tilde{\vec{\beta}}u^0 \tag{122}$$

respectively.Now in the large time like limit (112) and (113) become

$$u'^0 = u^0 \tag{123}$$

$$\vec{u'} = \vec{u} - \tilde{\vec{\beta}}u^0 \tag{124}$$

The reason behind this is the following .In the large time like limit the temporal coordinate of the four vector is much much greater that the spatial components.i.e, $|u^0| >> |\vec{u}|$ or, $|u^0| >> |u^i|$ and from the third relation mentioned in (120), $|\tilde{\vec{\beta}}^i| = \frac{c\omega^i}{\omega^2} << 1$,therefore the term $\tilde{\vec{\beta}} \cdot \vec{u}$ can be dropped as it becomes two small under large timelike limit.In (124) can not drop $\tilde{\vec{\beta}} u^0$ term .Though $\tilde{\vec{\beta}}$ is much much less than 1 ,$|u^0| >> |\vec{u}|$ .therefore the term $\tilde{\vec{\beta}} u^0$ can not be dropped. Under the large spacelike limit, $|u^i| >> |u^0|$ and (121) and (122) reduces to

$$u'^0 = u^0 - \tilde{\vec{\beta}} \cdot \vec{u} \tag{125}$$

$$\vec{u'} = \vec{u} \tag{126}$$

Under large space like limit we can not drop $\tilde{\vec{\beta}} \cdot \vec{u}$ .Though, $\tilde{\vec{\beta}}$ is much much less than 1 ,$|u^i| >> |u^0|$ .Hence,the term $\tilde{\vec{\beta}} \cdot \vec{u}$ can not be dropped from the equation.In (122) the term $\tilde{\vec{\beta}} u^0$ must be dropped as in large spacelike limit both $\tilde{\vec{\beta}}$ and u^0 are very small with respect to u^i. Let's derive how the spatio-temporal derivatives transform under (123) and (124).Setting, $u^0 = ct$ and $\vec{u} = \vec{x}$,we find that

$$t' = t \tag{127}$$

$$\vec{x'} = \vec{x} - \frac{c^2\vec{\omega}}{\omega^2} t \tag{128}$$

therefore ,

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial t'} \\ &= \frac{\partial}{\partial t}(1) + \frac{\partial x^j}{\partial t'}\frac{\partial}{\partial x^j} \end{aligned} \tag{129}$$

As $\frac{\partial t}{\partial t'} = 1$ and $\frac{\partial x^j}{\partial t'} = \frac{c^2\omega^j}{\omega^2}$.Hence,(120) becomes

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{c^2\omega^j}{\omega^2}\frac{\partial}{\partial x^j} \tag{130}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla \tag{131}$$

Let us calculate ∇' in the primed frame.

$$\begin{aligned} \frac{\partial}{\partial x'^j} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial x'^j} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial x'^j} \\ &= \frac{\partial}{\partial t}(0) + \frac{\partial}{\partial x^j}(1) \end{aligned} \tag{132}$$

From (127) and (128) ,we conclude that $\frac{\partial t}{\partial x'^j} = 0$ as t only depends on t' and $\frac{\partial x^j}{\partial x'^j} = 1$ Now,we consider the following transformation which come from (125) and (126)

$$t' = t - \frac{\vec{\omega}}{\omega^2} \cdot \vec{x} \tag{133}$$

$$\vec{x'} = \vec{x} \tag{134}$$

In large spacelike limit , the spatial and temporal derivatives become

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial t'} \\ &= \frac{\partial}{\partial t}(1) + \frac{\partial}{\partial x^j}(0) \end{aligned} \tag{135}$$

From (133) and (134) ,we get that $\frac{\partial t}{\partial t'} = 1$ as t only depends on t' and $\frac{\partial x^j}{\partial t'} = 0$ therefore, under large spsacelike limit the temporal derivative becomes

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \tag{136}$$

Similarly ,we can find how gradient transforms under large space like limit.

$$\begin{aligned} \frac{\partial}{\partial x'^j} &= \frac{\partial}{\partial t}\frac{\partial t}{\partial x'^j} + \frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial x'^j} \\ &= \frac{\partial}{\partial t}\left(\frac{\vec{\omega}}{\omega^2}\right) + \frac{\partial}{\partial x^j}(1) \end{aligned} \tag{137}$$

As (133) can be rewritten as

$$\begin{aligned} t' &= t - \frac{\vec{\omega}}{\omega^2}\cdot\vec{x} \\ \implies t' &= t - \frac{\vec{\omega}}{\omega^2}\cdot\vec{x'} \\ \implies t &= t' + \frac{\vec{\omega}}{\omega^2}\cdot\vec{x'} \end{aligned} \tag{138}$$

here we used the fact $\vec{x} = \vec{x'}$ So (137) becomes

$$\nabla' = \nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \tag{139}$$

(139) tells us that the gradient in primed frame is equal to the gradient in in unprimed frame with an extra term which is temporal derivative operator. in the above calculations ,we see that lorentz transformation and sengupta's transformation are related by $v^i \rightarrow \frac{c^2\omega^i}{\omega^2}$ or equivalently $\omega^i \rightarrow \frac{c^2 v^i}{v^2}$. Now using the above relations we can tell how electric and the magnetic field will look like under the Sengupta's transformation.

$$\vec{E}' = \tilde{\gamma}\left(\vec{E} + c\tilde{\vec{\beta}}\times\vec{B}\right) + (1-\tilde{\gamma})\frac{\tilde{\vec{\beta}}\left(\tilde{\vec{\beta}}\cdot\vec{E}\right)}{\tilde{\vec{\beta}}^2} \tag{140}$$

$$\vec{B}' = \tilde{\gamma}\left(\vec{B} - \frac{\tilde{\vec{\beta}}\times\vec{E}}{c}\right) + (1-\gamma)\frac{\tilde{\vec{\beta}}\left(\tilde{\vec{\beta}}\cdot\vec{B}\right)}{\tilde{\vec{\beta}}^2} \tag{141}$$

Using the limit given in (120) which are

$$
\begin{aligned}
c &<< \omega; \\
\tilde{\gamma} &\rightarrow 1; \\
|\tilde{\beta}^i| = \frac{c\omega^i}{\omega^2} &<< 1
\end{aligned}
$$

Under these limits (140) and (141) reduces to

$$\vec{E}' = \vec{E} + c\tilde{\vec{\beta}} \times \vec{B} \tag{142}$$

$$\vec{B}' = \vec{B} - \frac{\tilde{\vec{\beta}} \times \vec{E}}{c} \tag{143}$$

3.1 Electric Limit

Let us investigate how current 4 vector $(c\rho, \mathbf{j})$ changes under Sengupta's transformation.the transformed ρ' and $\vec{j}$ are given by (118) and (119)

$$c\rho' = \tilde{\gamma}\left(c\rho' - \tilde{\vec{\beta}} \cdot \vec{j}\right) \tag{144}$$

$$\vec{j'} = \vec{j} - \tilde{\gamma}\tilde{\vec{\beta}} c\rho + (\gamma - 1)\frac{\tilde{\vec{\beta}}\left(\tilde{\vec{\beta}} \cdot \vec{j}\right)}{\tilde{\beta}^2} \tag{145}$$

now in the case $c|\rho| >> |\mathbf{j}|$,which implies that $E >> cB$.This is the electric limit (subscript e) which corresponds to the transformation law given by (123) and (124) for $\left(c\rho, \vec{j}\right)$

$$\rho_e^{'} = \rho_e \tag{146}$$

$$\vec{j}_e^{'} = \vec{j}_e - \frac{c^2\vec{\omega}}{\omega^2}\rho_e \tag{147}$$

(146) and (147) imply that under electric limit the motion of the current does not induces a charge density but motion of the current or time varying current induces a volume charge density. and the continuity equation reads

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j}_e = 0 \tag{148}$$

now let's check wheather continuity equation is valid or not in under the transformed current four vector and spatio-temporal gradient

$$
\begin{aligned}
\frac{\partial \rho_e^{'}}{\partial t'} + \nabla' \cdot \vec{j}_e^{'} &= \left(\frac{\partial}{\partial t} + \frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla\right)\rho_e + \nabla \cdot \left(\vec{j}_e - \frac{c^2\vec{\omega}}{\omega^2}\rho_e\right) \\
&= \frac{\partial \rho_e}{\partial t} + \frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla\rho_e + \nabla \cdot \mathbf{j}_e - \nabla \cdot \left(\frac{c^2\vec{\omega}}{\omega^2}\rho_e\right)
\end{aligned} \tag{149}
$$

now $\nabla\cdot\left(\frac{c^2\vec{\omega}}{\omega^2}\rho_e\right)=\rho_e\nabla\cdot\frac{c^2\vec{\omega}}{\omega^2}+\frac{c^2\vec{\omega}}{\omega^2}\cdot\nabla\rho_e$ and $\nabla\cdot\frac{c^2\vec{\omega}}{\omega^2}=c\nabla\cdot\tilde{\vec{\beta}}$.$\tilde{\vec{\beta}}$ being uniform ,$\nabla\cdot\tilde{\vec{\beta}}=0$ as $\nabla\cdot\vec{k}=0$, where $\vec{k}$ is a constant vector. Therefore,

$$\begin{aligned}\frac{\partial\rho_e^{'}}{\partial t^{'}}+\nabla^{'}\cdot\vec{j}_e^{'}&=\frac{\partial\rho_e}{\partial t}+\frac{c^2\vec{\omega}}{\omega^2}\cdot\nabla\rho_e+\nabla\cdot\vec{j}_e-\rho_e\nabla\cdot\frac{c^2\vec{\omega}}{\omega^2}-\frac{c^2\vec{\omega}}{\omega^2}\cdot\nabla\rho_e\\&=\frac{\partial\rho_e}{\partial t}+\nabla\cdot\vec{j}_e-0\\&=0\end{aligned}\tag{150}$$

it implies that continuity equation still holds under electric limit or large timelike limit in segupta's transformation.

Now Electric field and Magnetic field transform as

$$\vec{E}_e^{'}=\vec{E}_e\tag{151}$$

$$\vec{B}_e^{'}=\vec{B}_e-\frac{\tilde{\vec{\beta}}\times\vec{E}_e}{c}\tag{152}$$

(151) and (152) has a similar form to (46) and (47) respectively.these last two equation also tell us that a varying magnetic field or does not induce an electric field under electric limit while the motion of the electric field induces a magnetic field.it implies that faraday term becomes zero in this case also.Maxwell's equations (48-51) become

$$\nabla\cdot\vec{E}_e=\frac{\rho_e}{\epsilon_0}\tag{153}$$

$$\nabla\cdot\vec{B}_e=0\tag{154}$$

$$\nabla\times\vec{E}_e=0\tag{155}$$

$$\nabla\times\vec{B}_e=\mu_0\vec{j}_e+\epsilon_0\mu_0\frac{\partial\vec{E}_e}{\partial t}\tag{156}$$

let's check wheather equations are invariant under transformatiion law (123) and (124) or not

$$\begin{aligned}\nabla^{'}\cdot\vec{E}_e^{'}&=\frac{\rho_e^{'}}{\epsilon_0}\\\implies\nabla\cdot\vec{E}_e&=\frac{\rho_e}{\epsilon_0}\end{aligned}\tag{157}$$

$$\begin{aligned}\nabla^{'}\cdot\vec{B}_e^{'}&=\nabla\cdot\left(\vec{B}_e-\frac{\tilde{\vec{\beta}}\times\mathbf{E}_e}{c}\right)\\\implies\nabla^{'}\cdot\mathbf{B}_e^{'}&=\nabla\cdot\vec{B}_e-\nabla\cdot\frac{\tilde{\vec{\beta}}\times\mathbf{E}_e}{c}\\\implies\nabla^{'}\cdot\mathbf{B}_e^{'}&=0-\nabla\cdot\frac{\tilde{\vec{\beta}}\times\mathbf{E}_e}{c}\end{aligned}\tag{158}$$

now $\nabla \cdot \left(\tilde{\vec{\beta}} \times \vec{E}_e\right) = \mathbf{E}_e \cdot \left(\nabla \times \tilde{\vec{\beta}}\right) - \tilde{\vec{\beta}} \cdot \left(\nabla \times \vec{E}_e\right)$ as $\nabla \times \mathbf{v} = 0$ and $\nabla \times \vec{E}_e$ $=0$

therefore

$$\nabla' \cdot \vec{B}_e' = 0 \tag{159}$$

Now, (155) under the transformation rules (123) and (124)

$$\begin{aligned} \nabla' \times \mathbf{E}_e' &= \nabla \times \mathbf{E}_e \\ &= 0 \end{aligned} \tag{160}$$

$$\begin{aligned} \nabla' \times \vec{B}_e' &= \mu_0 \vec{j}_e' + \epsilon_0 \mu_0 \frac{\partial \vec{E'}_e'}{\partial t} \\ \nabla \times \left(\vec{B}_e - \frac{\vec{\omega} \times \vec{E}_e}{\omega^2}\right) &= \mu_0 \left(\vec{j}_e - \frac{c^2 \vec{\omega}}{\omega^2} \rho_e\right) + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e}{\partial t} + \frac{(\vec{\omega} \cdot \nabla) \vec{E}_e}{\omega^2} \\ \nabla \times \vec{B}_e - \frac{\nabla \times \left(\vec{\omega} \times \vec{E}_e\right)}{\omega^2} &= \mu_0 \left(\vec{j}_e - \frac{c^2 \vec{\omega}}{\omega^2} \rho_e\right) + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e}{\partial t} + \frac{(\vec{\omega} \cdot \nabla) \vec{E}_e}{\omega^2} \\ \nabla \times \vec{B}_e - \frac{\vec{\omega}\left(\nabla \cdot \vec{E}_e\right) + (\vec{\omega} \cdot \nabla) \vec{E}}{\omega^2} &= \mu_0 \left(\vec{j}_e - \frac{c^2 \vec{\omega}}{\omega^2} \rho_e\right) + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e}{\partial t} + \frac{(\vec{\omega} \cdot \nabla) \vec{E}_e}{\omega^2} \\ \nabla \times \vec{B}_e &= \mu_0 \vec{j}_e + \epsilon_0 \mu_0 \frac{\partial \vec{E}_e}{\partial t} \end{aligned}$$

So,maxwell's equations are invariant under transformation laws (123) and (124).Now electric field and magnetic field can be derived from scalar and vector potential

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \tag{161}$$

$$\vec{B} = \nabla \times \vec{A} \tag{162}$$

and as four potential is given by $(\phi, c\mathbf{A})$ and in case of electric limit $|\phi| >> c|\mathbf{A}|$,it gives us

$$\vec{E}_e = -\nabla \phi_e \tag{163}$$

$$\vec{B}_e = \nabla \times \vec{A}_e \tag{164}$$

and $\left(\frac{\phi_e}{c}, \vec{A}_e\right)$ transforms as

$$\phi_e' = \phi_e \tag{165}$$

$$\vec{A}_e' = \vec{A}_e - \frac{\vec{\omega}}{\omega^2} \phi_e \tag{166}$$

the lorentz force is given by

$$\vec{F} = \int d^3 \vec{r} [\rho(\vec{r}) \vec{E}(\vec{r}) + \vec{j}(\vec{r}) \times \vec{B}(\vec{r})] \tag{167}$$

under electric limit (66) reduces to

$$\vec{F}_e = \int d^3\vec{r}\rho_e\left(\vec{r}\right)\vec{E}_e\left(\vec{r}\right) \tag{168}$$

thus in electric limit ,the magnetic field does exist but it has no effect .

3.2 Magnetic limit

now let's study the case where $c|\rho| << |\mathbf{j}|$ and $|\mathbf{E}| << |c\mathbf{B}|$ in order to obtain the magnetic limit we use the transformation relation 7 and 8 and thus we get

$$c\rho_m^{'} = c\rho_m - \frac{c\vec{\omega}\cdot\mathbf{j}_m}{\omega^2} \tag{169}$$

it implies that

$$\rho_m^{'} = \rho_m - \frac{\vec{\omega}\cdot\mathbf{j}_m}{\omega^2} \tag{170}$$

and $\mathbf{j}$ becomes

$$\vec{j}_m^{'} = \vec{j}_m \tag{171}$$

now taking into account that $|\vec{E}_m| << c|\vec{B}_m|$ we obtain the field transformation law

$$\vec{E}_m^{'} = \vec{E}_m + \frac{c\vec{\omega}\times\mathbf{B}_m}{\omega^2} \tag{172}$$

$$\vec{B}_m^{'} = \vec{B}_m \tag{173}$$

four potential transforms as

$$\phi_m^{'} = \phi_m - \frac{c^2\vec{\omega}\cdot\mathbf{A}_m}{\omega^2} \tag{174}$$

$$\vec{A}_m^{'} = \vec{A}_m \tag{175}$$

Maxwell's equations (48-51) become

$$\nabla\cdot\vec{E}_m = \frac{\rho_m}{\epsilon_0} \tag{176}$$

$$\nabla\cdot\vec{B}_m = 0 \tag{177}$$

$$\nabla\times\vec{E}_m = -\frac{\partial\vec{B}_m}{\partial t} \tag{178}$$

$$\nabla\times\vec{B}_m = \mu_0\vec{j}_m \tag{179}$$

From (172) we obtain that the motion of a magnetic field induces an electric field .This is the reason behind the appearance of faraday term in the maxwell's equation but equation (179) just gives us the ampere's law.The electric and magnetic field can also be expressed in terms of four potential.Electric field will depend upon both ϕ_m $\vec{A}_m$ and as transformed electric field depends upon both electric and magnetic field in unprimed reference frame.But the magnetic field only depends on $\vec{A}_m$ .The following equations

$$\vec{E}_m = -\nabla\phi_m - \frac{\partial \vec{A}_m}{\partial t} \tag{180}$$

$$\vec{B}_m = \nabla \times \vec{A}_m \tag{181}$$

$$\vec{E}_m^{'} = -\nabla'\phi_m^{'} - \frac{\partial \vec{A}_m^{''}}{\partial t'} \tag{182}$$

$$\vec{B}_m^{'} = \nabla^{'} \times \vec{A}_m^{''} \tag{183}$$

Let's explicitly calculate the electric and magnetic field in primed frame under magnetic limit.

$$\vec{E}_m^{'} = -(\frac{\partial}{\partial t}\frac{\vec{\omega}}{\omega^2} + \nabla)(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m) - \frac{\partial A_m}{\partial t} \tag{184}$$

$$\vec{E}_m^{'} = -\frac{\vec{\omega}}{\omega^2}\frac{\partial \phi_m}{\partial t} - \nabla\phi_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial\left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m\right)}{\partial t} + \nabla\left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m\right) - \frac{\partial A_m}{\partial t} \tag{185}$$

$$\vec{E}_m^{'} = -\frac{\vec{\omega}}{\omega^2}\frac{\partial \phi_m}{\partial t} + \vec{E}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial\left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m\right)}{\partial t} + \left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla\right)\mathbf{A}_m + \frac{c^2\vec{\omega}}{\omega^2} \times (\nabla \times \mathbf{A}_m) \tag{186}$$

From, (182) we write

$$= -\frac{\vec{\omega}}{\omega^2}\frac{\partial \phi_m}{\partial t} + \vec{E}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial\left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \vec{A}_m\right)}{\partial t} + \left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla\right)\frac{\vec{\omega}}{\omega^2}\vec{A}_m + \frac{c^2\vec{\omega}}{\omega^2} \times \vec{B}_m \tag{187}$$

now using (173) ,we conclude that

$$-\frac{\vec{\omega}}{\omega^2}\frac{\partial \phi_m}{\partial t} + \frac{\vec{\omega}}{\omega^2}\frac{\partial\left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m\right)}{\partial t} + \left(\frac{c^2\vec{\omega}}{\omega^2} \cdot \nabla\right)\mathbf{A}_m = 0 \tag{188}$$

Now (175) tells us

$$\begin{aligned}
\vec{B}_m^{'} &= \nabla' \times \vec{A}_m^{''} \\
&= \left(\nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t}\right) \times \vec{A}_m \\
&= \nabla \times \vec{A}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \times \vec{A}_m \\
&= \vec{B}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \times \vec{A}_m \\
&= \vec{B'}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \times \vec{A}_m
\end{aligned} \tag{189}$$

As (175) tells us that magnetic field remains invariant under sengupta's transformation when magnetic limit is considered,we get

$$\begin{aligned}\vec{B}'_m &= \vec{B'}_m + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \times \vec{A}_m \\ \implies \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t} \times \vec{A}_m &= 0\end{aligned} \tag{190}$$

We are interested to check wheather divergence of the magnetic field or the flux of magnetic field vanishes under large spacelike transformation

$$\nabla' \cdot \vec{B}'_m = \left(\nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t}\right) \cdot \left(\vec{B}_m\right) \tag{191}$$

$$\nabla' \cdot \vec{B}'_m = \nabla \cdot \vec{B}_m + \frac{\vec{\omega}}{\omega^2} \cdot \frac{\partial \nabla \times \vec{A}_m}{\partial t} \tag{192}$$

As we know $\nabla \cdot \left(\frac{c^2\vec{\omega}}{\omega^2} \times \mathbf{A}_m\right) = \mathbf{A}_m \cdot \left(\nabla \times \frac{c^2\vec{\omega}}{\omega^2}\right) - \frac{c^2\vec{\omega}}{\omega^2} \cdot (\nabla \times \mathbf{A}_m)$ but as $\vec{\omega}$ is uniform therefore $\nabla \times \frac{c^2\vec{\omega}}{\omega^2} = 0$ and $\nabla \left(\frac{c^2\vec{\omega}}{\omega^2} \times \mathbf{A}_m\right) = -\frac{c^2\vec{\omega}}{\omega^2} \cdot (\nabla \times \mathbf{A}_m)$ and time derivative will be zero.

$$\nabla' \cdot \vec{B}_m = \nabla \cdot \vec{B}_m = 0 \tag{193}$$

Let's check wheather the flux of electric field also gives us relevant result or not

$$\begin{aligned}\nabla' \cdot \vec{E}'_m &= \left(\nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t}\right) \cdot \left(\vec{E}_m + \frac{c^2\vec{\omega}}{\omega^2} \times \vec{B}_m\right) \\ &= \nabla \cdot \vec{E}_m + \frac{\vec{\omega}}{\omega^2} \cdot \frac{\partial}{\partial t}\vec{E}_m + \nabla \cdot \left(\frac{c^2\vec{\omega}}{\omega^2} \times \vec{B}_m\right) + \frac{\vec{\omega}}{\omega^2} \cdot \frac{\partial \frac{c^2\vec{\omega}}{\omega^2} \times \vec{B}_m}{\partial t} \\ &= \frac{\rho_m}{\epsilon_0} - \frac{\mu_0 \vec{\omega} \cdot \vec{j}}{\epsilon_0 \mu_0 \omega^2}\end{aligned} \tag{194}$$

here we are neglecting $\frac{\vec{\omega}}{\omega^2} \cdot \frac{\partial \mathbf{E}_m}{\partial t}$ term as it has $|\vec{\omega}|^2$ in its denominator.

$$\nabla' \cdot \vec{E}'_m = \frac{\rho'_m}{\epsilon_0} \tag{195}$$

Let's verify the invarience of faraday's law .

$$\nabla' \times \mathbf{E}'_m = \left(\nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t}\right) \times \left(-\nabla' \phi'_m - \frac{\partial \vec{A}''_m}{\partial t'}\right) \tag{196}$$

$$\nabla' \times \mathbf{E}'_m = \left(\nabla + \frac{\vec{\omega}}{\omega^2}\frac{\partial}{\partial t}\right) \times \left(-\nabla\left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \vec{A}_m\right) - \frac{\vec{\omega}}{\omega^2}\frac{\partial\left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \vec{A}_m\right)}{\partial t} - \frac{\partial \vec{A}_m}{\partial t}\right) \tag{197}$$

after simplifying,we get

$$\nabla' \times \mathbf{E}_m^{'} = -\left(\nabla \times \frac{\vec{\omega}}{\omega^2} \frac{\partial \left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m \right)}{\partial t} + \frac{\vec{\omega}}{\omega^2} \frac{\partial}{\partial t} \times \nabla \left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m \right) + \nabla \times \frac{\partial \mathbf{A}_m}{\partial t} + \frac{\partial \left(\frac{\vec{\omega}}{\omega^2} \times \frac{\partial \mathbf{A}_m}{\partial t} \right)}{\partial t} \right) \tag{198}$$

Now,first two term cancels as

$$\begin{aligned} \nabla \times \frac{\vec{\omega}}{\omega^2} \frac{\partial \left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m \right)}{\partial t} &= \nabla \frac{\partial \left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m \right)}{\partial t} \times \frac{\vec{\omega}}{\omega^2} \\ &= -\frac{\vec{\omega}}{\omega^2} \times \nabla \frac{\partial \left(\phi_m - \frac{c^2\vec{\omega}}{\omega^2} \cdot \mathbf{A}_m \right)}{\partial t} \end{aligned} \tag{199}$$

.we also have $\frac{\vec{\omega}}{\omega^2} \times \frac{\partial \mathbf{A}_m}{\partial t} = 0$ and $\nabla \times \mathbf{A}_m = \mathbf{B}_m$ Therefore,

$$\nabla' \times \mathbf{E}_m^{'} = -\frac{\partial \mathbf{B}_m}{\partial t} = -\frac{\partial \vec{B}_m^{'}}{\partial t'} \tag{200}$$

In case of Ampere"s Law in primed frame

$$\begin{aligned} \nabla' \times \vec{B}_m^{'} &= \left(\nabla + \frac{\vec{\omega}}{\omega^2} \frac{\partial}{\partial t} \right) \times \vec{B}_m \\ &= \nabla \times \vec{B}_m + \frac{\vec{\omega}}{\omega^2} \frac{\partial}{\partial t} \times \mathbf{B}_m \\ &= \mu_0 \vec{j}_m + \frac{\vec{\omega}}{\omega^2} \frac{\partial}{\partial t} \times \mathbf{B}_m \end{aligned} \tag{201}$$

Here ,we are neglecting $\frac{\vec{\omega}}{\omega^2} \frac{\partial}{\partial t} \times \mathbf{B}_m$ term as it has ω^2 in its denominator and $|\vec{\omega}| >> c$ .Hence,

$$\nabla' \times \vec{B}_m^{'} = \mu_0 \vec{j'}_m \tag{202}$$

Finally ,the continuity equation in primed frame tells us

$$\begin{aligned} \frac{\partial \rho_m'}{\partial t'} + \nabla' \cdot \vec{j'}_m =& \frac{\partial \rho_m - \frac{\vec{\omega}}{\omega^2} \cdot \vec{j}_m}{\partial t} + \left(nabla + \frac{\vec{\omega}}{\omega^2} \right) \cdot \vec{j'}_m \\ =& \frac{\partial \rho_m}{\partial t} + \nabla \cdot \vec{j}_m \\ =& 0 \end{aligned} \tag{203}$$

So,the continuity equation also holds in the primed frame.

4 Appendix

In this section,we shall use some mapping to get the above results namely, maxwell's law in different limits.

The maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{204}$$

$$\nabla \cdot \vec{B} = 0 \tag{205}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{206}$$

now in gallelian limits .it consists of two different limits namely electric and magnetic limit under electric limit ,we consider the following transformation $\vec{E} \rightarrow c\vec{E'}_e$and $\vec{B} \rightarrow \vec{B'}_e$ accordingly four current also changes. Now,(204) transforms under above mentioned mapping as

$$\nabla \cdot \vec{E}_e = \frac{\rho_e}{\epsilon_0} \tag{207}$$

So,it remains invariant.(205) also remains invariant under this above mentioned mapping. (206) implies electrostatics.here it is

$$\begin{aligned} \nabla \times c\vec{E}_e &= -\frac{\partial \vec{B}_e}{\partial t} \\ \nabla \times \vec{E}_e &= -\frac{\partial \vec{B}_e}{c\partial t} \end{aligned} \tag{208}$$

Let's set $c \rightarrow \infty$ that means c is very large and in this case $|\vec{B}_e|$ is much much smaller than $|\vec{E}_e|$ Therefore $\nabla \times \vec{E}_e = 0$

$$\nabla \times \vec{B}_e = \mu_0 \vec{j}_e + \frac{\partial \vec{E}_e}{c\partial t} \tag{209}$$

here the last term won't cancel as both $|\vec{E}_e|$ and c are very large. hence this approach gives us the exact sdame result as our previous approach in the section 2.Now,under the magnetic limit in lorentz transformation ,we consider two such mapping - $\vec{E} \rightarrow \frac{\vec{E'}_m}{c}$ and $\vec{B} \rightarrow \vec{B'}_m$ the equations related to the flux of electric field and magnetic field remains invariant. fourth equation gives

$$\nabla \times \vec{B}_m = \mu_0 \vec{j}_e + \frac{\partial \vec{E}_e}{c\partial t} \tag{210}$$

Now as $|\vec{E}_m| << c|\vec{B}_m|$ .therefore the last equation reduces to the ampere's one. faraday's law under this limit reduces to

$$\nabla \times \frac{\vec{E}_m}{c} = -\frac{\partial \vec{B}_m}{\partial t} \tag{211}$$

under $c \rightarrow \infty$ it becomes $\frac{\partial \vec{B}_m}{\partial t} = 0$ eqn.(34) and (35) tell us

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B} \tag{212}$$

$$\vec{B}' = \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \tag{213}$$

under electric limit ,the above equations become

$$\vec{E}_e' = \vec{E}_e \tag{214}$$

$$\vec{B}_e' = \vec{B}_e - \frac{\vec{v} \times \vec{E}_e}{c} \tag{215}$$

The last term in the second equation won't be zero as both $|\vec{E}_e|$ and c are large. Under magnetic limit these equation will transform like

$$\vec{E}_m' = \vec{E}_m + c\vec{v} \times \vec{B}_m \tag{216}$$

$$\vec{B}_m' = \vec{B}_m \tag{217}$$

Now,using the transformation law $\vec{E} \to \vec{E'}_e$ and $\vec{B} \to c\vec{B'}_e$, we get

$$\nabla \cdot \vec{E}_e = \frac{\rho_e}{\epsilon_0} \tag{218}$$

$$\begin{aligned} \nabla \cdot c\vec{B}_e &= 0 \\ \nabla \cdot \vec{B}_e &= 0 \end{aligned} \tag{219}$$

and the faraday's law become

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{E}_e &= -\frac{\partial c\vec{B}_e}{\partial t} \end{aligned} \tag{220}$$

Now $c \to 0$ gives

$$\nabla \times \vec{E}_e = 0 \tag{221}$$

implying electrostatics .the last maxwell's equation gives

$$\nabla \times c\vec{B}_e = \mu_0 c\vec{j}_e + \epsilon_0\mu_0 \frac{\partial \vec{E}_e}{\partial t} \tag{222}$$

taking the limit $c \to 0$ we get $\frac{\partial \vec{E}_e}{\partial t} = 0$ we can also get this without using any mapping inthat case we have to just ignore the $\vec{j}_e$ and $\vec{B}_e$ terms in(160) as they are very small compared to the electric field. In magnetic limit also,we can use the transformation law $\vec{E} \to c\vec{E}_m$ and $\vec{B} \to \vec{B}_m$ therefore, maxwell's equations under this limit and the corresponding mapping become

$$\nabla \cdot \vec{E}_m = \frac{\rho_m}{\epsilon_0} \tag{223}$$

$$\nabla \cdot \vec{B}_m = 0 \tag{224}$$

$$\nabla \times c\vec{E}_m = -\frac{\partial \vec{B}_m}{\partial t} \tag{225}$$

this tells us that $\frac{\partial \vec{B}_m}{\partial t} = 0$

$$\nabla \times \vec{B}_e = \mu_0 \vec{j}_e + \frac{\partial \vec{E}_e}{c\partial t} \tag{226}$$

according to (146) and (147)

$$\vec{E}' = \vec{E} + c\tilde{\vec{\beta}} \times \vec{B} \tag{227}$$

$$\vec{B}' = \vec{B} - \frac{\tilde{\vec{\beta}} \times \vec{E}}{c} \tag{228}$$

Now using the mapping $\vec{E} \to \vec{E}_e$ and $\vec{B} \to c\vec{B}_e$ in the above relation we get

$$\vec{E'}_e = \vec{E}_e + c^2\tilde{\vec{\beta}} \times \vec{B}_e \tag{229}$$

$$\vec{B'}_e = \vec{B}_e - \frac{\tilde{\vec{\beta}} \times \vec{E}_e}{c^2} \tag{230}$$

Now, in $c \to 0$ limit we can not drop the second term in (211) as it is formed by multiplying a large quantity to a small quantity so it would be relevant. therefore,

$$\vec{E}'_e = \vec{E}_e \tag{231}$$

$$\vec{B'}_e = \vec{B}_e - \frac{\tilde{\vec{\omega}} \times \vec{E}_e}{\omega^2} \tag{232}$$

Under Magnetic limit we use the transformation (216) and (217) to get

$$c\vec{E'}_m = c\vec{E}_e + c\tilde{\vec{\beta}} \times \vec{B}_m \tag{233}$$

$$\vec{B'}_e = \vec{B}_e - \tilde{\vec{\beta}} \times \vec{E}_e \tag{234}$$

Now,as $|\tilde{\vec{\beta}}| << 1$ and $|\vec{E}_e| << |\vec{B}_e|$ we dropped the second term .eqn.(222) becomes

$$\vec{E'}_m = \vec{E}_e + \tilde{\vec{\beta}} \times \vec{B}_m \tag{235}$$

4.1 Galilean group

The equations that describe Galilean transformations are

$$t' = t, \quad \mathbf{x}' = \mathbf{x} + \mathbf{v}t$$

Including rotations and space-time translations we can form the Galilean group G. The modified equations now become,

$$t' = t + s$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{a}$$

In matrix form, they can be written as

$$\begin{pmatrix} \mathbf{x}' \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{v} & \mathbf{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix}$$

Now we can write the matrices from it of various generators. Let's do the computation for the generator of rotation in the Y-Z plane.. The transformation matrix for this rotation can be written as

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

For $\theta = 0$ we can get the infinitesimal transformation and could write this in a 5×5 matrix form. We can also do this for the other 2 generators of rotation in a plane. They are given below

$$M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, M_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, M_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The subscripts of M denote the plane in which the rotation is occurring. We can similarly write the other generators in matrix form. The matrices corresponding to time translation (P_0) and space translation (P_i) are simple and easy to state. They are given below

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the gallelian boost are given by

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The indices μ, ν of the generators written in table 1 run from 1 to 5, however indices i and j run from 1 to 3.

Type of generators	Mathematical form
Space translation generator	$(P_i)_{\mu\nu} = \delta_{i\mu}\delta_{\nu 5}$
Time translation generator	$(P_0)_{\mu\nu} = \delta_{4\mu}\delta_{\nu 5}$
Space rotation generator	$\epsilon_{ijk}J_k = (M_{ij})_{\mu\nu} = -\delta_{i\mu}\delta_{j\nu} + \delta_{j\mu}\delta_{i\nu}$
Galilean boost generator	$(B_i)_{\mu\nu} = \delta_{i\mu}\delta_{4\nu}$

Table 1: Galilean generators

The complete algebra of the gallelian group is given by

$[\mathrm{P}_0, P_i]$=0
$[P_i, P_j] = 0$
$[B_i, B_j] = 0$
$[M_ij, P_0] = 0$
$[B_i, P_0] = P_i$
$[M_ij, M_kl] = \delta_{ik}L_{jl} - \delta_{il}L_{jk} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}$
$[M_ij, P_k] = \delta_{ik}P_j - \delta_{jk}P_i$
$[M_ij, B_k] = \delta_{ik}B_j - \delta_{jk}B_i$
$[P_i, B_j] = 0$

4.2 Carroll group

first we write the transformation equation in carroll limit.the relevant equations are

$$t' = t - u_i x^i x' = x$$

We add translations to it.

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{a} t' = t - \mathbf{u}\cdot\mathbf{x} + s$$

u is the reciprocal velocity. In the matrix form

$$\begin{pmatrix} \mathbf{x}' \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & 0 & \mathbf{a} \\ -\mathbf{u}^T & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix}$$

The corresponding boosts are

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case the carroll boost are transpose of gallelian boost so $(C_i)_{\mu\nu} = \delta_{i\nu}\delta_{4\mu}$ the corresponding Lie algebra are

$[\mathrm{P}_0, P_i]$=0
$[P_i, P_j] = 0$
$[C_i, C_j] = 0$
$[M_{ij}, P_0] = 0$
$[C_i, P_0] = 0$
$[M_ij, M_kl] = \delta_{ik}L_{jl} - \delta_{il}L_{jk} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}$
$[M_ij, P_k] = \delta_{ik}P_j - \delta_{jk}P_i$
$[M_ij, C_k] = \delta_{ik}C_j - \delta_{jk}C_i$
$[P_i, C_j] = -\delta_{ij}P_0$

In Lorentz group there 6 generators same sengupta's given by $J^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu$ μ runs through 0,1,2,3,ν runs through 0,1,2,3.with tis if we add four generators of translation we get the poincare' group. we can use the method of group contraction from poincare' to galleli group.similarly for carroll.

5 acknowledment

In order to complete this Project My Project guide Prof.Rabin BanerJee helped me .Snbncbs has finiancially supported me for doing this project

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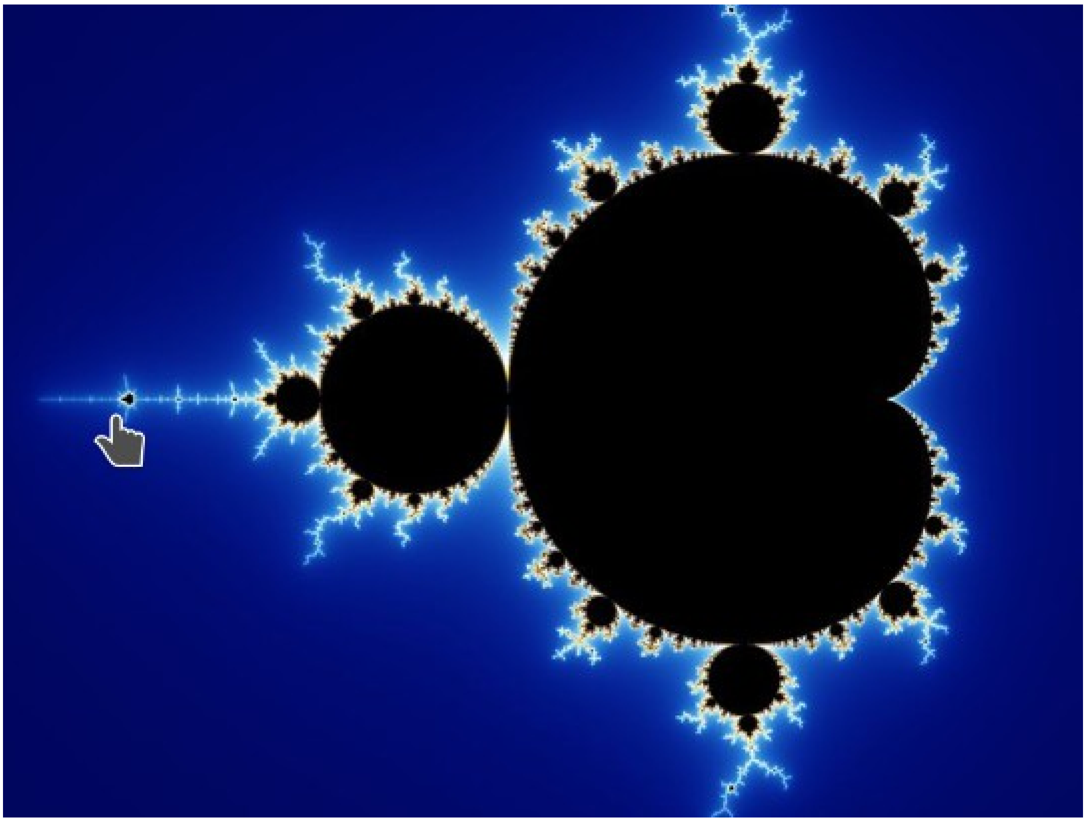
FRACTALS & ITS APPLICATION



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• Name-Indrajit Banerjee

• Semester-3rd

Sub-Project Research(PHY-509)

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Fractals and its application

Indrajit Banerjee

December 2024

1 introduction

The idea of describing natural phenomena by studying statistical scaling laws is not recent. Indeed, many studies were carried out on this topic (Bachelier, 1900; Frish, 1995; Kolmogorov, 1941; Mandelbrot, 1963). However, there has been a recent resurgence of interest in this approach. A great number of physical systems tend to present similar behaviours on different scales of observation. In the 1960s, the mathematician Benoît Mandelbrot used the adjective "fractal" to indicate objects whose complex geometry cannot be characterized by an integral dimension. these fractals are complex geometric shsapes with fine structure at arbitrarily small scale.Ususally they have some degree of self similarity .self similarity means that if we magnify a tiny part of the structure that part will be similar to the whole structure .usually the similarity is approximate or statistical but sometimes it may be exact.

The main attraction of fractal geometry stems from its ability to describe the irregular or fragmented shape of natural features as well as other complex objects that traditional Euclidean geometry fails to analyse. This phenomenon is often expressed by spatial or time-domain statistical scaling laws and is mainly characterized by the power-law behaviour of real-world physical systems. This concept enables a simple, geometrical interpretation and is frequently encountered in a variety of fields, such as geophysics, biology or fluid mechanics. To this end, Mandelbrot introduced the notion of fractal sets (Mandelbrot, 1977), which enables to take into account the degree of regularity of the organizational structure related to the physical system's behaviour.

2 countable and uncountable set

first we shall discuss about set.set is a well defined collection of distinct objects.the objects of the set are called elements.if x is an element of set A it is written as $x \in A$.if x does not belong to A ,we write $x \notin A$.Let A and B be two sets.Now,if every element of A is an element of B ,we say $A \subseteq B$.two sets are equal if and only if they have same elements.i.e, $A \subseteq B$ and $B \subseteq A$.some examples are set of all integers ,set of all natural numbers,set of all real numbers etc. are some infinities larger vthan other?the answer is yes.?Two sets A and

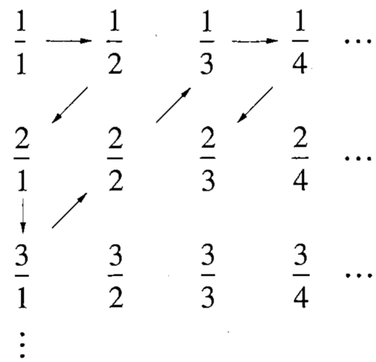

Figure 1:

B are said to have same cardinality.if there is an invertible mapping(bijective mapping) that pairs each element of $x \in A$ with precisely one element of $y \in B$.it is called one-to-one correspondence.set of all positive integers (natural numbers,$\mathbf{N} = \{1, 2, 3, 4, 5, 6, ...\}$)is an example infinite set.$|\mathbf{N}| = \aleph_0$.Now if we can show that a set A can be put into aone to one correspondence with the set of natural numbers ,A is countable otherwise A is uncountable .for set with finite number of elements ,this problem does not arise at all.in that case we can simply count the number of elements in that set .the cardinality of a set is the number of elements in a set. There is also an equivalent definition when an infinite set is countable or not.a set A is countable if it can be written as a list $\{x_1, x_2, x_3, ...\}$ where every element $x \in A$ appears somewhere in the list .therefore it means that given any $x \in A$,there exists some finite n such that $x_n = x$.

Let's study some examples.

example 1

Show that the set of odd natural numbers $\mathbf{O} = \{1, 3, 5, ...\}$ is countable.

Solution: We need to find a one-to-one correspondence between O and N. Such a correspondence is given by the invertible mapping that pairs each natural number n with the odd number $2n - 1$; thus $1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 5, 4 \leftrightarrow 7$, and so on. Hence there are exactly as many odd numbers as natural numbers. as the set $\mathbf{N}$ is countable ,O is also countable

example 2

show that the positive rational numbers are countable.

Solution:to show this ,we have to show that there exists a one to one correspondence between set of all positive rational numbers and the set of natural number .here we have to make a table where the pq th entry is $\frac{p}{q}$.then the rational can be counted according the process shown in the figure.Any given $\frac{p}{q}$ is reached after a finite number of steps, so the rationals are countable.

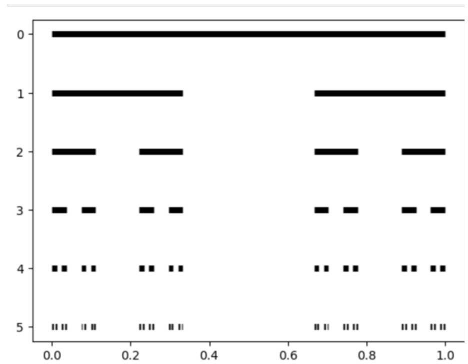

Figure 2: cantor set $S_0, S_1 S_2 S_3 S_4 S_5$

Example 3: Let X denote the set of all real numbers between 0 and 1. Show that X is un- countable. **solution** : The proof is by contradiction. If X were countable, we could list all the real numbers between 0 and 1 as a set $\{x_1, x_2, x_3...\}$. Rewrite these numbers in decimal form:

$x_1 = 0.x_{11}x_{12}x_{13}x_{14}....$

$x_2 = 0.x_{21}x_{22}x_{23}x_{24}....$

$x_3 = 0.x_{31}x_{32}x_{33}x_{34}....$

.

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where x_{ij}, denotes the jth digit of the real number x_i .

To obtain a contradiction, we'll show that there's a number r between 0 and 1 that is not on the list. Hence any list is necessarily incomplete, and so the reals are uncountable. We construct r as follows: its first digit is anything other than x_{11} , the first digit of x_1.second digit is anything other than x_{22} and so on.there fore ,the number r becomes $0.\bar{x_{11}}\bar{x_{22}}\bar{x_{33}}\bar{x_{44}}....$ which is clearly not in the list so the set is uncountable. this argument is called cantor's diagonal argument.

3 cantor set

We start with the closed interval $S_0 = [0, 1]$

and remove its open middle third, i.e., we delete the interval $(\frac{1}{3}, \frac{2}{3})$ and leave the endpoints behind. This produces the pair of closed intervals shown as S_1 . Then we remove the open middle thirds of those two intervals to produce S_2, and so on. The limiting set $C = S_\infty$ is the Cantor set. It is difficult to visualize, but Figure 2 suggests that it consists of an infinite number of infinitesimal pieces, separated by gaps of various sizes. Fractal Properties of the Cantor Set The Cantor set C has several properties that are typical of fractals more generally:

1. Cantor set has measure zero.

2. Cantor set consists uncountably many points

3. C has structure at arbitrarily small scales. If we enlarge part of C repeatedly, we continue to see a complex pattern of points separated by gaps of various sizes. This structure is neverending, like worlds within worlds. In contrast, when we look at a smooth curve or surface under re- peated magnification, the picture becomes more and more featureless.

4. C is self-similar. It contains smaller copies of itself at all scales. For instance, if we take the left part of C (the part contained in the interval $[0, \frac{1}{3}]$) and enlarge it by a factor of three, we get C back again. Simi- larly, the parts of C in each of the four intervals of S_2 are geometrically similar to C, except scaled down by a factor of nine. If you're having trouble seeing the self-similarity, it may help to think about the sets S_n, rather than the mind-boggling set S_n. Focus on the left half of S_2,-it looks just like S_1 , except three times smaller. Similarly, the left half of S_3 is S_2, reduced by a factor of three. In gen- eral, the left half of S_{n+1} , looks like all of S_n , scaled down by three. Now set $n = \infty$. The conclusion is that the left half of S_∞ looks like S_∞ scaled down by three, just as we claimed earlier. The strict self-similarity of the Cantor set is found only in the simplest fractals. More general fractals are only approximately self-similar.

5. The dimension of C is not an integer. As we'll show in the next Section , its dimension is actually $\frac{ln2}{ln3} = 0.63$ .The idea of a noninteger dimen- sion is bewildering at first, but it turns out to be a natural generalization of our intuitive ideas about dimension, and provides a very useful tool for quantifying the structure of fractals.

3.1 cantor set has measure zero

Let us how that the measure of the Cantor set is zero, in the sense that it can be cov- ered by intervals whose total length is arbitrarily small. Solution: Figure 2 shows that each set S_n, completely covers all the sets that come after it in the construction. Hence the Cantor set $C = S_\infty$,is covered by each of the sets S_n,, . So the total length of the Cantor set must be less than the total length of S_n, , for any n . Let L_n, denote the length of S_n,. Then from Figure 2 we see that $L_0 = 1$,$L_1 = \frac{2}{3}$,$L_2 = (\frac{2}{3})(\frac{2}{3}) = (\frac{2}{3})^2$, and in general, $L_n = (\frac{2}{3})^n$,. Since $L_\infty \to 0$, as $n \to 0$, the Cantor set has a total length of zero.

Here's another way to show that the Cantor set has zero total length. In the first stage of construction of the Cantor set, we removed an interval of length $\frac{1}{3}$ from the unit interval [0, 1] . At the next stage we removed $\frac{1}{9}$ from $\left[0, \frac{1}{3}\right]$ and $\frac{1}{9}$ from $\left[\frac{2}{3}, 0\right]$ .we removed total of $\frac{2}{9}$ in step 2.in step 3 we removed total length of $\frac{4}{27}$ and so on. up to step n ,where n tends to infinity.

Let's do the summation

$$
\begin{aligned}
r &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + ... \\
&= \frac{1}{3}(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + ...) \\
&= \frac{1}{3}(\frac{1}{1-\frac{2}{3}}) \\
&= \frac{1}{3}(\frac{1}{\frac{1}{3}}) \\
&= 1
\end{aligned} \tag{1}
$$

so we removed length of amount 1 .there fore the measure of the cantor set is 1-1=0.

Let's prove that any countable subset of the real line has zero measure. The plan for this problem is to upper bound the measure of the rationals with another set that we can shrink to zero measure. First, the set is countable, so we can list them as $x_1, x_2, x_3, x_4, ...$ We can make an interval centered around every rational number r_n with length $\frac{\epsilon}{2^{n-1}}$ where $n \to \infty$. therefore $I_n = (x_n - \frac{\epsilon}{2^n}, x_n + \frac{\epsilon}{2^n})$ This set of intervals is countable, and the union of this countable number of intervals contains $x_1, x_2, x_3, x_4, ...$ and then some. Therefore the measure of the countable set of intervals is an upper bound for the measure of our countable subset.

$$0 \leq \mu(x_1, x_2, x_3, x_4, ...) \leq \mu(\bigcup_{n=1}^{\infty} I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon$$

and this gives the upper bound.We never explicitly chose , so we can scale the size of In and consequently the upper bound for the measure of the numbers to as small as we want, meaning the measure of the this set has to be 0.

as rationals are also countable so we can tell from here that they have zero measure.there fore the set of irrational numbers between 0 and 1 has measure 1-0=1.

3.2 Cantor set C consists of all points $C \in [0, 1]$ that have no 1 ' s in their base-3 expansion.

let's prove that the Cantor set C consists of all points $C \in [0, 1]$ that have no 1 ' s in their base-3 expansion.

The idea of expanding numbers in different bases may be unfamiliar, unless you were one of those children who was taught "New Math" in elementary school. Now you finally get to see why base-3 is useful! First let's remember how to write an arbitrary number $x \in [0, 1]$ in base-3 notation We write $x = 0.a_1a_2a_3a_4a_5.....$ as $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3}$

, where the digits a_n, are 0 , 1, or 2. This expansion has a nice geometric interpretation (Figure 3).

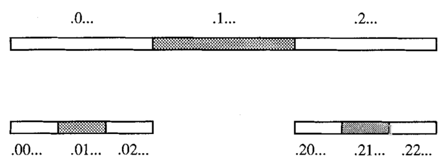

Figure 3:

if we imagine that [O,1] is divided into three equal pieces, then the first digit a_1,tells us whether x is in the left, middle, or right piece. For instance, all numbers with $a_1 = 0$ are in the left piece. (Ordinary base- 10 works the same way, except that we divide [O, 1] into ten pieces instead of three.) The second digit a_2 provides more refined information: it tells us whether x is in the left, middle, or right third of a given piece. For instance, points of the form $x = .02..$. are in the right part of the left third of [O, 1], as shown in Figure 3. Now think about the base-3 expansion of points in the Cantor set C. We deleted the middle third of [O,1] at the first stage of constructing C; this removed all points whose first digit is 1. So those points can't be in C. The points left over (the only ones with a chance of ultimately being in C) must have 0 or 2 as their first digit.Similarly, points whose second digit is 1 were deleted at the next stage in the construction.points having third digit as 1 we remove in the step 3 and so on. By repeating this argument, we see that C consists of all points whose base-3 expansion contains no 1's, as claimed. rn There's still a fussy point to be addressed. What about endpoints like $\frac{1}{3} =, 1000...$? It's in the Cantor set, yet it has a 1 in its base-3 expansion. Does this contradict what we said above? No, because this point can also be written solely in terms of 0's and 2's, as follows: = .1000. . . = .02222. . . . By this trick,each point in the Cantor set can be written such that no 1's appear in its base-3 expansion, as claimed.

3.3 the Cantor set is uncountable

This is just a rewrite of the Cantor diagonal argument , so we'll be brief. Suppose there were a list $c_1, c_2, c_3, ..$ of all points in C . To show that C is uncountable, we produce a point $\bar{c}$ that is in C but not on the list. Let c_{ij} denote the jth digit in the base-3 expansion of c_i . Define $\bar{c} = 0.\bar{c_{11}}\bar{c_{22}}......$, where the overbar means we switch between 0's and 2's: thus if $c_{ii} = 0$,$\bar{c_{ii}} = 2$ and vice versa. Then $\bar{c}$is in C , since it's written solely with 0 ' s and 2's, but it is not on the list, since it differs from c_i, in the ith digit. This contradicts the original assumption that the list is complete. Hence C is uncountable.

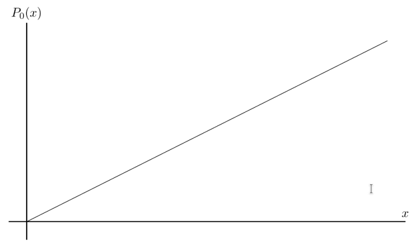

Figure 4: $P_0(x)$

3.4 find an invertible mapping that pairs each point $c \in C$ with precisely one $x \in [0, 1]$

The Cantor set is all ternary numbers in [0,1] that do not contain any 1's in their ternary representation. This satisfies every point in the Cantor set corresponding to a point in [0,1], but not vice versa since any value with a 1 in it's ternary representation is not mapped into the Cantor set. However, we can use binary numbers to uniquely pair every point in [0,1] with a point in the Cantor set, and vice versa. Basically we read the digit in the binary number and pick the left or right third for that subinterval if the digit is 0 or 1 respectively. Then we repeat for the next digit and so on, and the Cantor set point is the left endpoint of the subinterval we end in. For example, the binary number 0.110101 directs us to pick Right Right Left Right Left Right and then the Left endpoint of that subinterval.

3.5 A point in C that is not an endpoint

Endpoints in the cantor set are ternary numbers made entirely of 0's and 2's that either eventually reach endlessly repeating 2's or terminate, which is the same as endlessly repeating 0's. therefore these are all rational numbers in base 3 repesentation.Hence, a number in the Cantor set that is not an endpoint must not terminate and must not eventually become endlessly repeating 2's.there it must be an irrational number. an example of which is 0.020020002....in ternary.

3.6 devil's staircase

Suppose that we pick a point at random from the Can- tor set. What's the probability that this point lies to the left of x , where $0 < x < 1$ is some fixed number? The answer is given by a function P(x) called the devil's staircase. t is easiest to visualize P(x) by building it up in stages. First consider the set S_0 .Since $S_0 = [0, 1]$ has length 1, the probability of a randomly chosen point from being in the interval [0, x] is the length of the interval, so $P_0 = x$

The Devil's staircase is continuous. Clearly, the complement of the Cantor set is continuous since the Devil's staircase is constant there. The tricky part is proving the continuity for points in the Cantor set.

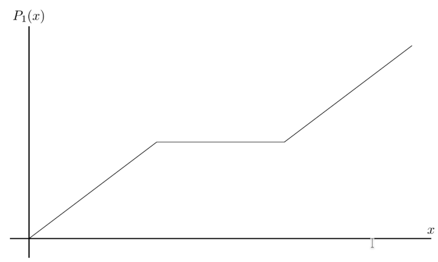

Figure 5: $P_1(x)$

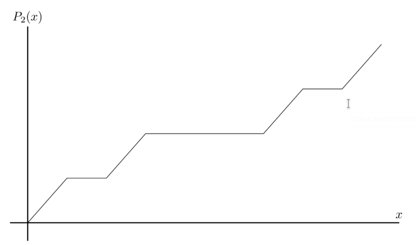

Figure 6: $P_2(x)$

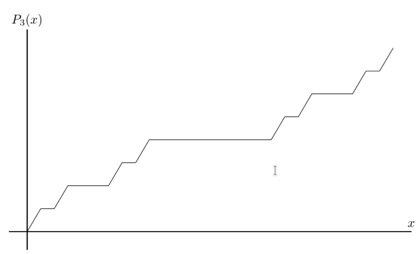

Figure 7: $P_3(x)$

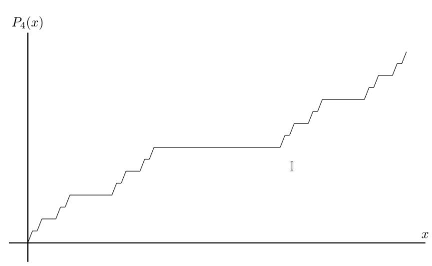

Figure 8: $P_4(x)$

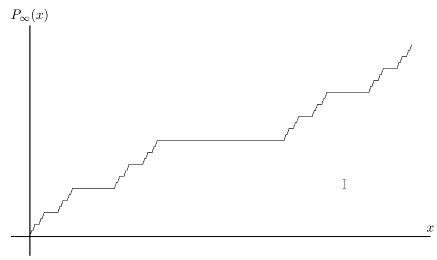

Figure 9: $P_\infty(x)$

The proof uses the fact that for a point (x, y) on the Devil's staircase, where the x-coordinate is in the Cantor set, the y-coordinate is determined by the following process:

- Express the x-coordinate in base-3 (ternary) expansion.
- Change all the 2's in this expansion to 1's.
- Interpret the resulting number as a binary (base-2) number.

This transformation ensures that the Devil's staircase is continuous at points in the Cantor set.

For the ϵ-δ proof, being within 3^{-n} in x is sufficient to be within 2^{-n} in y, since the first n digits in both the binary and ternary expansions are fixed within those distances. Thus, the function behaves continuously at points in the Cantor set.

Derivative of the Devil's Staircase

The derivative of the Devil's staircase is not defined on the Cantor set, as the graph has uncountably many holes that, in total, have zero length. The derivative is defined on the complement of the Cantor set, but these parts of the graph are constant, so the derivative is zero.

Therefore, the graph of the derivative of the Devil's staircase is zero except on the Cantor set, where it is undefined.

4 dimension

What is the "dimension" of a set of points? For familiar geometric objects, the answer is clear-lines and smooth curves are one-dimensional, planes and smooth surfaces are two-dimensional, solids are three-dimensional, and so on. If forced to give a definition, we could say that the dimension is the minimum number of coor- dinates needed to describe every point in the set. For instance, a smooth curve is one-dimensional because .every point on it is determined by one number, the arc length from some fixed reference point on the curve. But when we try to apply this definition to fractals, we quickly run into para- doxes.

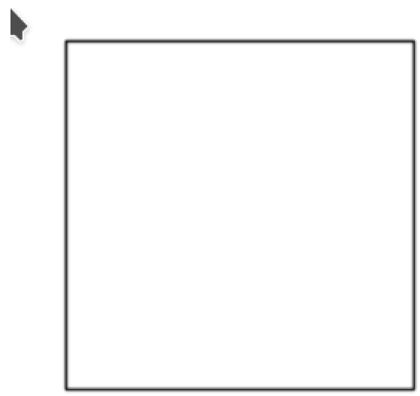

Figure 10:

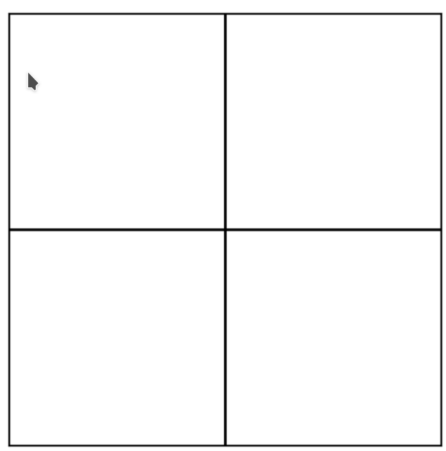

Figure 11: r=2,m=4

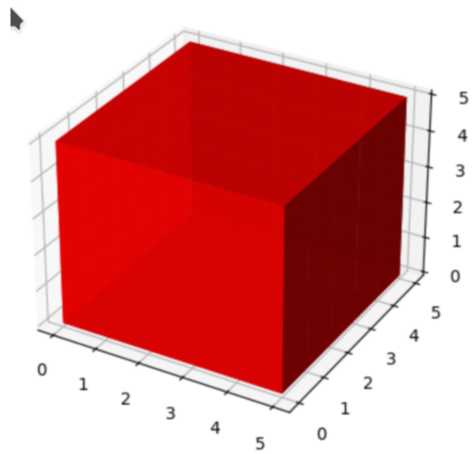

Figure 12: Caption

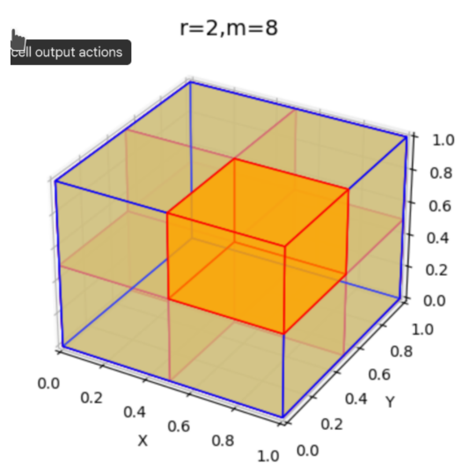

Figure 13: r=3,m=8

4.1 similarity dimension

The simplest fractals are self-similar, i.e., they are made of scaled-down copies of themselves, all the way down to arbitrarily small scales. The dimension of such fractals can be defined by extending an elementary observation about classical self-similar sets like line segments, squares, or cubes. For instance, consider the square region shown in Figure 10 and 11.

If we shrink the square by a factor of 2 in each direction, it takes four of the small squares to equal the whole. Or if we scale the original square down by a factor of 3, then nine small squares are required. In general, if we reduce the linear dimensions of the square region by a factor of r, it takes r^2 of the smaller squares to equal the original. Now suppose we play the same game with a solid cube. The results are differ- ent: if we scale the cube down by a factor of 2, it takes eight of the smaller cubes to make up the original. In general, if the cube is scaled down by r, we need r^3 smaller cubes to make up the larger one. The exponents 2 and 3 are no accident; they reflect the two-dimensionality of the square and the three-dimensionality of the cube. This connection between dimensions and exponents suggests the following definition. Suppose that a self-similar set is composed of m copies of itself scaled down by a factor of r. Then the similarity dimension d is the exponent defined by

$$m = r^d$$

, or equiva- lently,

$$d = \frac{lnm}{lnr}$$

Let's learn this fact through some examples.

4.2 Von Koch Curve

Consider the von Koch curve, defined recursively in Figure We start with a line segment S_0 . To generate S_1 , we delete the middle third of So and replace it with the other two sides of an equilateral triangle. Subsequent stages are generated recursively by the same rule: $S_n$is obtained by replacing the middle third of each line segment in S_{n-1}, by the other two sides of an equilateral triangle. The limiting set $K = S_\infty$, is the von Koch curve.

4.2.1 measure of von koch curve

To see this, observe that if the length of $S_0$is $L_0 = 1$, then the length of S_1 is $L_1 = \frac{4L_0}{3}$
because S_1 contains four segments, each of length $\frac{L_0}{3}$. The length increases by $\frac{4}{3}$ in each stage.at n th stage the length is $L_n = (\frac{4}{3})^n L_0$ as n goes to infinity .L_∞ becomes infinity.
the arc length between any two points on K is infinite, by similar reasoning. Hence points on K aren't determined by their arc length from a partic- ular

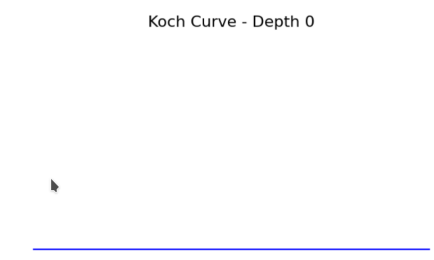

Figure 14: step 0

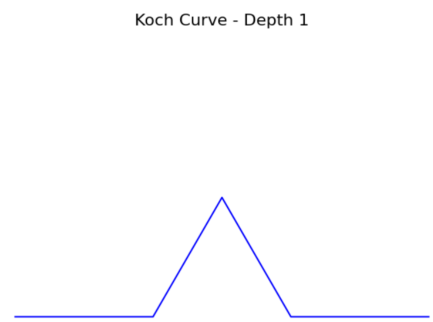

Figure 15: styep 1

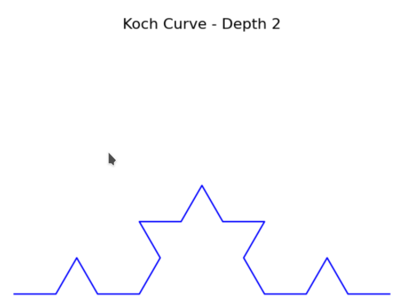

Figure 16: step 2

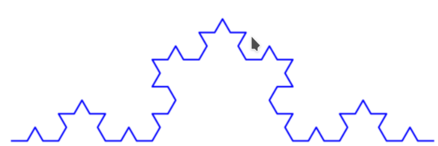

Figure 17: step 3

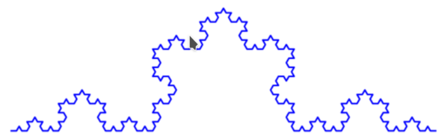

Figure 18: step 4

point, because every point is infinitely far from every other!

This suggests that K is more than one-dimensional. But would we really want to say that K is two-dimensional? It certainly doesn't seem to have any "area." So the dimension should be between 1 and 2, whatever that means.

4.2.2 similarity dimension

The curve is made up of four equal pieces, each of which is similar to the original curve but is scaled down by a factor of 3 in both directions. so m=4 and r=3.therefore,

$$d = \frac{ln4}{ln3}$$

4.3 generalized cantor set

even fifth cantor set

Other self-similar fractals can be generated by changing the recursive procedure. For instance, to obtain a new kind of Cantor set, divide an interval into five equal pieces, delete the second and fourth subintervals, and then repeat this process indefinitely We call the limiting set the even-fifths Cantor set, since the even fifths are re- moved at each stage.

Let the original interval be denotedS_0 , and letS_n, denote the nth I stage of the construction. If we scale S_n, down by a factor of five, we get one third of the set S_{n-1} . Now setting $n = \infty$ we see that the even-fifths Cantor set S, is made of three copies of itself, shrunken by a factor of 5. Hence m = 3 when r = 5, and so

$$d = \frac{ln3}{ln5}$$

. measure of evenfifth cantor set

Here's another way to show that the Cantor set has zero total length. In the first stage of construction of the evenfifth Cantor set, we removed an total interval of length $\frac{2}{5}$ from the unit interval [0, 1] . At the next stage(step 2) we removed $\frac{6}{25}$ in total.$\frac{2}{5}$ from$\left[0, \frac{1}{5}\right]$ and $\frac{2}{25}$ from $\left[\frac{2}{5}, \frac{3}{5}\right]$ and length of total $\frac{2}{5}$ from the last fifth interval.in step 3 we removed total length of $\frac{18}{125}$ and so on. up to step n ,where n tends to infinity.

Figure 19: step 0

Figure 20: step 1

Figure 21: step 2

Figure 22: step 3

Figure 23: at step 5,S_5 of even fifth cantor set

]
Let's do the summation

$$
\begin{aligned}
r &= \frac{2}{5} + \frac{6}{25} + \frac{18}{125} + \frac{54}{625} + ... \\
&= \frac{2}{5}(1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + ...) \\
&= \frac{2}{5}(\frac{1}{1 - \frac{3}{5}}) \\
&= \frac{2}{5}(\frac{1}{\frac{2}{5}}) \\
&= 1
\end{aligned}
\tag{2}
$$

so we removed length of amount 1 .there fore the measure of the cantor set is 1-1=0. In contrast to even fifth cantor set the middle third cantor set has the similarity dimension of

$$d = \frac{ln2}{ln3}$$

here is how we have got this as in middle three cantor set we remove middle third of the interval and thus we get two copies in the next step . in doing so we also reduce the length by a factor of $\frac{1}{3}$.So

$$m = 2$$

and

$$r = 3$$

there fore,

$$2 = r^d$$

equivalently,

$$d = \frac{ln2}{ln3}$$

Figure 24: step 0 for even seventh

Figure 25: step 1 for even seventh

Generalization of even-fifths Cantor set is The ”even-sevenths Cantor set” is constructed as follows: divide $[0, 1]$ into seven equal pieces; delete pieces 2 , 4, and 6 ; and repeat on sub-intervals. similarity dimension of evenseventh cantor set can be calculated as the following way One iteration of the even-sevenths Cantor set leaves four intervals that will each contain the even-sevenths Cantor set scaled by a factor of seven. Then the similarity dimension is

$$4 = 7^d$$

$$d = \frac{ln4}{ln7}$$

measure of evenseventh cantor set can be found as alaways to be zero.we show it in this way.in the first we removed the length of amount $\frac{3}{7}$.length of the S_1 is $\frac{4}{7}$ in S_2 the length is $\frac{16}{49}$ at n th stage it is $(\frac{4}{7})^n$ .Now at limit $n \to \infty$ it becomes 0. A further generalization of even seventh cantor set can be made which is even n cantor set.

For n odd, one iteration of the even-n’ths Cantor set leaves intervals that will each contain the $\frac{n+1}{2}$ intervals

even-n’ths Cantor set scaled by a factor of n. Then the similarity dimension is as number of copies generated in every step from the previous stage is $\frac{n+1}{2}$

$$\frac{n+1}{2} = n^d$$

equivalently,

$$d = \frac{ln(\frac{n+1}{2})}{ln(n)}$$

consider a generalized Cantor set in which we begin by removing an open interval o f length 0 ¡ a ¡ 1 from the middle of [0,1]. At subsequent stages, we remove an open middle interval from each o f the remaining intervals, and so on.

Figure 26: step 2 for even seventh

Figure 27: step 3 for even seventh

Figure 28: step 0 for emiddle half cantor set

the measure of this set is we remove of length a from the middle of the interval.the remaing length is $1-a$ so the length of the interval after nth iteration is $(1-a)^n$ taking $n \to \infty$ we get 0 as $1-a$ is less than 1 and if we raise this quantity to very high value positive integer it reachs to 0. another we can think about it the total length removed to form the cantor is $\frac{a}{1-(1-a)} = 1$ so the remaining length is 0. similarity dimension : number of copy generated in every iteration is twice of the previous one.and the set has been scaled to $\frac{2}{1-a}$.there fore the similarity dimension is

$$d = \frac{ln2}{ln2 - ln(1-a)}$$

for $a = \frac{1}{2}$ we get the infamous middle halve set. whose measure is also zero and having a similarity dimension of $d = \frac{ln2}{2ln2}$.
Let us take take a subset of $[0,1]$ whose elements can be written without using the digit 8 of the cantor set in decimal representation..the measure of this set will be 0.in first step we remove the interval of length $\frac{1}{10}$ now from the remaing $\frac{1}{10}$ at the right end we remove $\frac{1}{100}$ th of the interval. and a length of amount $\frac{8}{100}$ from the left $\frac{8}{10}$ so in stage 2 we remove $\frac{9}{100}$ length in total.in stage 3 $\frac{81}{1000}$

Figure 29: step 1 for middle half cantor set

Figure 30: step 2 for middle half cantor set

and so on.

$$
\begin{aligned}
r &= \frac{1}{10} + \frac{9}{100} + \frac{81}{1000} + ... \\
&= \frac{1}{10}(1 + \frac{9}{10} + \frac{81}{100} + ...) \\
&= \frac{1}{10}(\frac{1}{1-\frac{9}{10}}) \\
&= \frac{1}{10}(\frac{1}{\frac{1}{10}}) \\
&= 1
\end{aligned} \tag{3}
$$

so we removed length of amount 1 .there fore the measure of the $\frac{8}{10}$ th cantor set is 1-1=0. Now let's calculate the similarity dimension.number copy,

$$m = 9$$

and

$$r = 10$$

the similarity dimension is

$$d = \frac{ln9}{ln10}$$

Figure 31: no 8 cantor set step 0

Figure 32: no 8 cantor set step 1

4.4 topological cantor set

There are so many different Cantor-like sets that mathematicians have abstracted their essence in the following definition.
A closed set S is called a topo- logical Cantor set if it satisfies the following properties: 1. S is "totally disconnected." This means that S contains no connected subsets (other than single points). In this sense, all points in S are separated from each other. For the middle-thirds Cantor set and other sub- 1sets of the real line, this condition simply says that S contains no intervals. 2. On the other hand, S contains no "isolated points." This means that every point in S has a neighbor arbitrarily close by-given any point $a \in S$ and any small distance $\epsilon 0$, there is some other point $b \in S$ within a distance ϵ of p. The paradoxical aspects of Cantor sets arise because the first property says that points in S are spread apart, whereas the second property says they're packed to- gether .the middle-thirds Cantor set satisfies both properties. Notice that the definition says nothing about self-similarity or dimension. These notions are geometric rather than topological; they depend on concepts of distance, volume, and so on, which are too rigid for some purposes. Topological features are more robust than geometric ones. For instance, if we continuously deform a self- similar Cantor set, we can easily destroy its self-similarity but properties 1 and 2 will persist. the cross sections of strange attractors are often topological Cantor sets, although they are not necessarily self-similar.

5 Von Koch Snowflake

T o construct the famous fractal known as the von Koch snowflake curve, use an equilateral triangle for S_0 . Then do the von Koch procedure as before i.e, removing the middle third of each of the side of the equilateral triangle and replacing it with two sides of an equilateral triangle of length $\frac{1}{3}$.we continue this

Figure 33: no 8 cantor set step 2

Figure 34: no 8 cantor set step 3

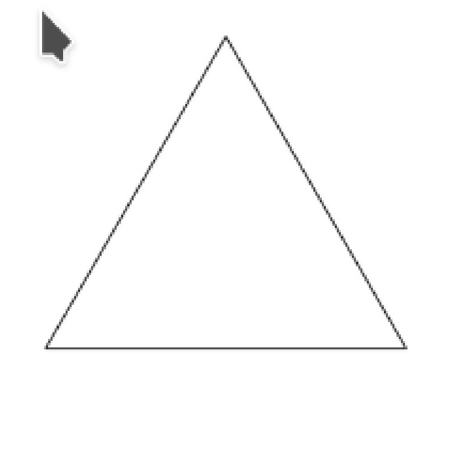

Figure 35: snowflake 0

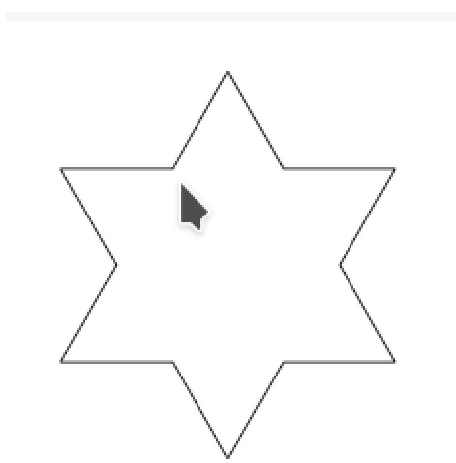

Figure 36: snowflake 1

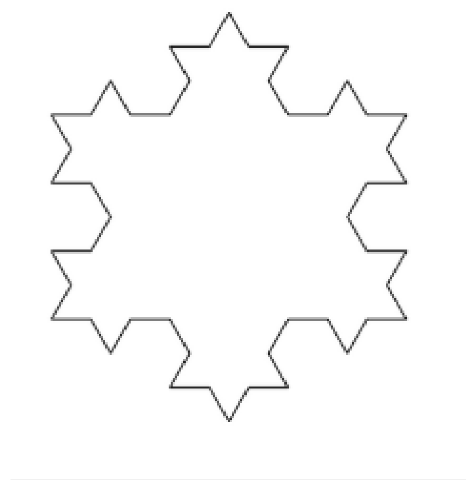

Figure 37: snowflake 2

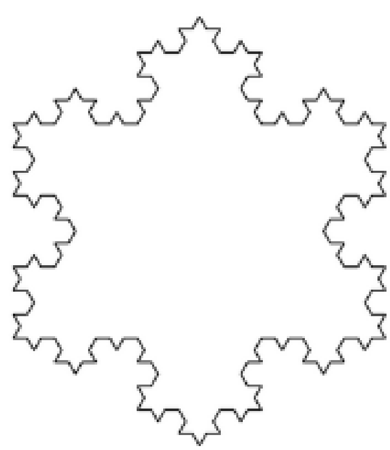

Figure 38: snowflake 3

process and have got the above figures.in the limit $n \to \infty$ it becomes von koch snowflake. let us find the area generated by the snowflake. S_0 has side length one with area $\frac{\sqrt{3}}{4}$ and every iteration adds a new equilateral triangle onto each side that is that is $\frac{1}{3}$the length of the side. side length of s_n is $(\frac{1}{3})^n$,and number of sides is $3(4)^n$ increament area of from S_{n-1} to S_n

$$A_n - A_{n-1} = N_{n-1}\frac{\sqrt{3}}{4}(\frac{1}{3}L_{n-1})^2$$

$$A_n - A_{n-1} = 3(4)^{n-1}\frac{\sqrt{3}}{4}(\frac{1}{3}(\frac{1}{3})^{n-1})^2$$

$$A_n - A_{n-1} = \frac{3\sqrt{3}}{16}\sum_{n=1}^{\infty}(\frac{4}{9})^n$$

If we sum the area of $S_0$and the area added after each iteration we get

$$\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{n=1}^{\infty}(\frac{4}{9})^n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\frac{4}{5} = \frac{\sqrt{3}}{4}(1 + \frac{3}{5}) = \frac{2\sqrt{3}}{5}$$

The Koch snowflake is made of three Koch curves. Each Koch curve requires four copies of itself scaled by a factor of three in order to recreate itself. Therefore the Koch snowflake can be made by taking four complete copies of itself and scaling each one by a factor of three. Then each scaled Koch snowflake is cut into thirds and placed onto the full size Koch snowflake. .there fore number of copies generated(m) is 4 and scaling down by 3(r).Hence, the similarity dimension of the Koch snowflake is

$$d = \frac{ln(4)}{ln(3)}$$

measure of the the perimeter is infinity as

$$\lim_{n\to\infty}(\frac{4}{3})^n = \infty$$

.

6 sierpinski carprt and menger sponge

Consider a square divided it into 9 squares of equal area and delete the central square .here each of the remaing squares has length of $\frac{1}{3}$ to the side of the original square. thus we get S_1 .similarlily we can also divide the remaining 8 squares into 9 small squares and delete the central square.if we reapeat this process infinitely we get sierpinski carpet.there is also a thing called sierpinski triangle where we do the same thing with an equalateral triangle. measure or area of the carpet in the first step we remove area of $\frac{1}{9}$ that of the original one

Figure 39: Carpet 0

.in stage two from each small square we remove area of $\frac{1}{81}$ and $\frac{8}{81}$ in total.thus for sierpinski carpet we remove

$$\begin{aligned} A &= \frac{1}{9}+\frac{8}{81}+\frac{64}{729}+....... \\ &= \frac{1}{9}(1+\frac{8}{9}+\frac{64}{81}+.......) \\ &= \frac{1}{9}(\frac{1}{1-\frac{8}{9}}) \\ &= \frac{1}{9}(\frac{1}{\frac{1}{9}}) \\ &= 1 \end{aligned} \tag{4}$$

so we removed length of amount 1 .there fore the measure of the cantor set is 1-1=0. similarity dimension of sierpinski carpet at each iteration the number of copies ,m=8 and the length is scaled down by $\frac{1}{3}$ therefore$r = 8$ and

$$d = \frac{ln(8)}{ln(3)}$$

menger sponge is the 3-D analog of sierpinski carpet .here we divide the cube into 27 small cube each with length $\frac{1}{3}$ and remove the central cube and centre cube of all 6 faces this gives us remaining $27-2=20$ cube and therefore

$$m = 27$$

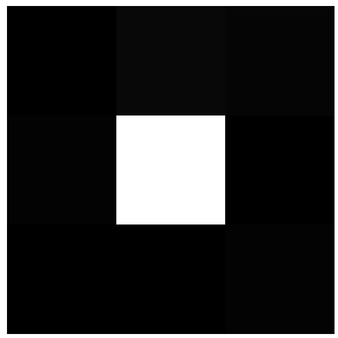

Figure 40: Carpet 1

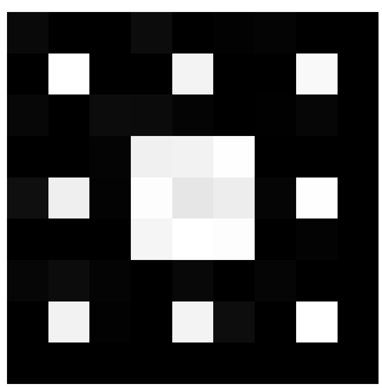

Figure 41: Carpet 2

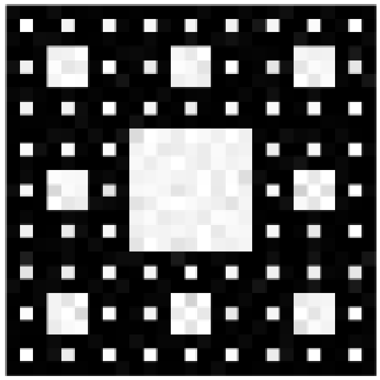

Figure 42: Carpet 3

and

$$r = 3$$

$$d = \frac{ln20}{ln3}$$

is the similarity dimension of the menger sponge.in N dimension we remove only those squares whose corresponding coordinates(at centre) are equal or $N-1$ are equals thus

$$m = 2^{N-1}N + 2^N$$

and

$$r = 3$$

thus similarity dimension is

$$\frac{ln(N2^{N-1} + 2^(N)}{ln3}$$

the volume of menger sponge is zero as at $n \to \infty$ its volume becomes

$$\lim_{n\to\infty} (\frac{20}{27})^N = 0$$

but its surface increases with no bound and blows up at infinity.

7 Box Dimension

To deal with fractals that are not self-similar, we need to generalize our notion of dimension still further. Various definitions have been proposed. All the definitions share the idea of "measurement at a scale ϵ "-roughly speaking, we measure the set in a way that ignores irregulari- ties of size less than ϵ, and then study how the measurements vary as $\epsilon \to 0$.

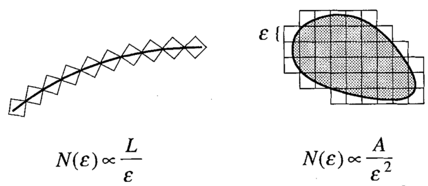

Figure 43:

7.1 Definition of Box Dimension

One kind of measurement involves covering the set with boxes of size ϵ(Fig-43 ure). Figure 43 Let S be a subset of D-dimensional Euclidean space, and let N(ϵ) be the mini- mum number of D-dimensional cubes of side ϵn eeded to cover S. How does N(ϵ) depend on ϵ? To get some intuition, consider the classical sets shown in Figure 43 . For a smooth curve of length L, $N(\epsilon) = \frac{L}{\epsilon}$; for a planar region of area A bounded by a smooth curve, N(E) = $\frac{A}{(\epsilon)^2}$ .The key observation is that the dimen- sion of the set equals the exponent d in thepower law

$$N(E) = \frac{1}{(\epsilon)^d}$$

This power law also holds for most fractal sets S , except that d is no longer an integer. By analogy with the classical case, we interpret d as a dimension, usually called the capacity or box dimension of S. An equivalent definition is

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})}$$

, if the limit exists.

Let us Find the box dimension of the Cantor set. Recall that the Cantor set is covered by each of the sets S,, used in its construction Each $S_n$consists of 2^n intervals of length $\frac{1}{3}$, so if we pick $\epsilon = \frac{1}{(3)^n}$, we need all 2^n of these intervals to cover the Cantor set. $N = 2^n$ when$\epsilon = \frac{1}{(3)^n}$. Since $\epsilon \to 0$ as $n \to \infty$, we find

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \frac{ln2^n}{ln3^n} = \frac{nln2}{nln3} = \frac{ln2}{ln3}$$

.

A fractal that is not self-similar is constructed as follows. A square region is di- vided into nine equal squares, and then one of the small squares is selected at ran- dom and discarded. Then the process is repeated on each of the eight remaining small squares, and so on. Let us find the box dimension of the limiting set. Figure 44 shows the first two stages in a typical realization of this random construction. Figure 44 Pick the unit of length to equal the side of the original square. Then S, is covered (with no wastage) by N = 8 squares of side $\epsilon = \frac{1}{3}$.

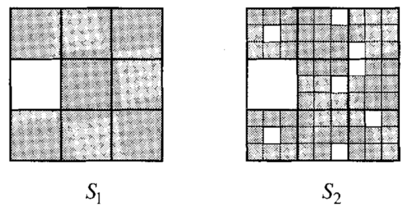

Figure 44:

Similarly, S_2 is covered by $N = 8^2$ squares of side $\epsilon = \frac{1}{9}$ In general,$N = 8^n$ when $\epsilon = \frac{1}{9}$ .hence

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln8^n}{ln3^n} = \lim_{\epsilon \to 0} \frac{nln8}{nln3} = \frac{ln8}{ln3}$$

7.2 box dimension of some fractals

7.2.1 von koch snowflake

Length of a side is along a diagonal of a box of unit length after nth iteration it becomes $\frac{1}{\epsilon} = \sqrt{2}3^n$ and number of sides gives the number of boxes $N(\epsilon) = 3(4)^n$

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln3(4)^n}{ln(3^n)(\sqrt{2})} = \lim_{\epsilon \to 0} \frac{nln4 + ln3}{nln3 + ln\sqrt{2}} = \frac{ln4}{ln3}$$

7.2.2 sierpinski carpet

Length of a side is just the side of the square whose length after nth iteration becomes $\epsilon = (\frac{1}{3})^n$ and number of sides gives the number of boxes $N(\epsilon) = 8^n$

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \frac{ln(8)^n}{ln(3^n)} = \frac{nln8}{nln3} = \frac{ln8}{ln3}$$

7.2.3 menger sponge

Upon each iteration, the box to be descended into is divided into 20 new boxes with the side length reduced by a factor of 3. From this we can calculate

$$\epsilon = \frac{1}{3^n}$$

$$N(\epsilon) = 20^n$$

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln(20)^n}{ln(3^n)} = \lim_{\epsilon \to 0} \frac{nln20}{nln3} = \frac{ln20}{ln3}$$

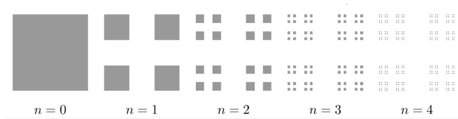

Figure 45: Cantor dust

7.2.4 cantor dust in 2d

cantor dust is the cartesian product of two cantor sets. Upon each iteration, the box to be descended into is divided into 4 new boxes with the side length reduced by a factor of 3. From this we can calculate

$$\epsilon = \frac{1}{3^n}$$

$$N(\epsilon) = 4^n$$

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln(4)^n}{ln(3^n)} = \lim_{\epsilon \to 0} \frac{nln4}{nln3} = \frac{ln4}{ln3}$$

7.2.5 menger hypersponge

it is just an 4 dimensional analog of menger sponge.

$$\epsilon = \frac{1}{3^n}$$

$$N(\epsilon) = 48^n$$

we can calculate N from the previous discussion on menger sponge upon considering $N = 48$.

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln(48)^n}{ln(3^n)} = \lim_{\epsilon \to 0} \frac{nln48}{nln3} = \frac{ln48}{ln3}$$

7.2.6 Tent Map

tent map is defined as

$$x_{n+1} = rx_n$$

when x_n in between of 0 and 0.5 and r is greater than 2 and for x_n and when x_n between 0.5 and 1 it is defined as

$$x_{n+1} = r(1 - x_n)$$

$$0 \leq x_0 \leq 0.5 \implies f(x_0 = rx_0 1$$

and

$$0.5 \leq x_0 \leq 1 \implies f(x_0 = r(1 - x_0)1$$

that means

$$x_0 \in (\frac{1}{r}, \frac{r-1}{r})$$

implies $x_1 1$ for The intervals that escapes after two iterations .for $0 \leq x_1 0.5$ x_0 can come from either first interval or second interval based on that in the interval $0 \leq x_1 0.5$

$$x_2 = r^2 x_0; 0 \leq x_0 0.5$$

$$x_2 = r(1 - r x_0); 0 \leq x_0 0.5$$

and for

$$0.5 \leq x_0 1$$

$$x_2 = r^2(1 - x_0); 0.5 \leq x_0 1$$

$$x_2 = r(1 - r(1 - x_0)); 0.5 \leq x_0 1$$

this two set of equations give us the interval where they escapr.solving, it we get

$$x_0 \in (\frac{1}{r^2}, \frac{r-1}{r^2})$$

another interval is

$$x_0 \in (\frac{r^2 - r + 1}{r^2}, \frac{r^2 - 1}{r^2})$$

thus we get the interval

$$x_0 \in (\frac{1}{r}, \frac{r-1}{r})$$

where $x_2 1$ If we now try to find the set of points that never escape purely by algebra looks like a royal mess. Visible in the graph though is the recursive nature of the iterated function. The problem repeats in small intervals, where points escape from the middle of each interval. We're creating the middle- $\frac{r-2}{r}$ 'ths Cantor set. You can show this analytically by horizontally scaling and shifting in each interval of the iterated graph to transform the domain into [0,1] which gives the f (x) graph. Hence the recursive procedure is exactly the same as the middle $\frac{r-2}{r}$) 'ths Cantor set. Therefore, points in the middle-$\frac{r-2}{r}$ 'ths Cantor set never escape within a finite number of iterations. .Let us find the box dimension of this cantor set.$\epsilon = \frac{1}{r^n}$ as at every step the set is scaled down by $\frac{1}{r}$ and $N = 2^n$.

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln(2)^n}{ln(r^n)} = \lim_{\epsilon \to 0} \frac{nln2}{nlnr} = \frac{ln2}{lnr}$$

This set is called an invariant set.The invariant set is called a strange repeller, for several reasons: it has a fractal structure; it repels all nearby points that are not in the set; and points in the set hop around chaotically under iteration of the tent map.

Figure 46: lopsided fractal step 0

Figure 47: lopsided fractal step 1

7.2.7 lopsided fractal

Divide the closed unit interval [0,1] into four quar- ters. Delete the open second quarter from the left. This produces a set S, . Repeat this construction indefinitely; i.e., generate S_{n+1}, from$S_n$by deleting the second quarter of each of the intervals in S_n box dimension will be for $N = 3^n$ and $\epsilon = \frac{1}{4^n}$

$$d = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(\frac{1}{\epsilon})} = \lim_{\epsilon \to 0} \frac{ln(3)^n}{ln(4^n)} = \lim_{\epsilon \to 0} \frac{nln3}{nln4} = \frac{ln3}{ln4}$$

This also an example of self similar set . However, its self-similar pieces are not the scaled by the same amount, unlike most of the other fractals we've seen.In random fractal we just add another condition .suppose I have coin when head appears we delete the second quarter ,in case of tail we delete the third quarter.the limiting set is called the random fractal. The box dimension is exactly the same as the previous question. Getting heads or tails flips the pattern for that iteration. The distinct interval lengths and the number of intervals of each length will still be there, just not in the same order. Let's we little bit change the condition again we think that tail appears and we Delete the first quarter versus the second quarter does make a difference in that the length of each interval to descend into is $\frac{3}{4}$ versus $\frac{1}{2}$ and $\frac{1}{4}$ respectively. An extreme example would be the coin landing tails forever. Every iteration would take off $\frac{1}{4}$ of the remaining interval. The limiting set $S_\infty = 1$ is a point, which isn't even a fractal, and the box dimension

is 0 since we would need one interval ${\frac{3}{4}}^n$ of length to cover S_n.

7.2.8 fractal cheese

A fractal slice of swiss cheese is constructed as follows: The unit square is divided into p^2 squares, and m^2 squares are chosen at random and discarded. (Here $p > m + 1$, and p, m are positive integers.) The process is re-

Figure 48: lopsided fractal step 3

peated for each remaining square (side = $\frac{1}{p}$) . Assuming that this process is re- peated indefinitely, find the box dimension of the resulting fractal. (Notice that the resulting fractal may or may not be self-similar, depending on which squares are removed at each stage. Nevertheless, we are still able to calculate the box dimen- Upon each iteration we descended into

$$p^2 - m^2$$

new boxes with the side length reduced by a factor of p. From this we can calculate $\frac{1}{p^n} = \epsilon$ and $N = (p^2 - M^2)^n$ there fore ,the box dimension is

$$d = \frac{ln(p^2 - m^2)}{lnp}$$

When computing the box dimension, it is not always easy to find a minimal cover. There's an equivalent way to compute the box dimension that avoids this problem. We cover the set with a square mesh of boxes of side E, count the number of occupied boxes N(E), and then compute d as before. Even with this improvement, the box dimension is rarely used in practice. Its computation requires too much storage space and computer time, compared to other fractals types of fractal dimension (see below). The box dimension also suffers from some mathematical drawbacks. For example, its value is not always what it should be: the set of rational numbers between 0 and 1 can be proven to have a box dimension of 1 (Falconer 1990, p. 44), even though the set has only countably many points. Falconer (1990) discusses other fractal dimensions, the most important of which is the Hausdorffdimensiorz.It is more subtle than the box dimension. The main con- ceptual difference is that the Hausdorff dimension uses coverings by small sets of varying sizes, not just boxes of fixed size E. It has nicer mathematical properties than the box dimension, but unfortunately it is even harder to compute numerically.

8 Pointwise and Correlation Dimensions

Now it's time to return to dynamics. Suppose that we're studying a chaotic system that settles down to a strange attractor in phase space. Given that strange attractors typically have fractal microstructure (as we'll see in Chapter 12), how could we es- timate the fractal dimension?

First we generate a set of many points $\{x_i\}, i = 1, \ldots, n$ on the attractor by letting the system evolve for a long time (after taking care to discard the initial transient, as usual). To get better statistics, we could repeat this procedure for sev- eral different trajectories. In practice, however, almost all trajectories on a strange attractor have the same long-term statistics, so it's sufficient to run one trajectory for an extremely long time. Now that we have many points on the attractor, we could try computing the box dimension, but that approach is impractical, as men- tioned earlier.

Grassberger and Procaccia (1983) proposed a more efficient approach that has become standard. Fix a point x on the attractor A. Let $N_\epsilon(x)$ denote the number of points on A inside a ball of radius ϵ about x (Figure 49).

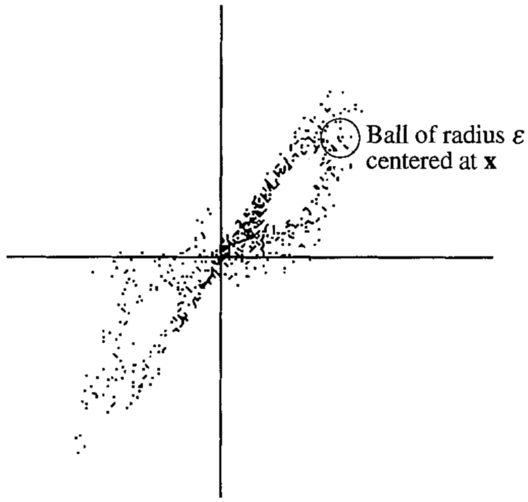

Figure 49: Example of a ball of radius ϵ centered at x on the attractor.

Most of the points in the ball are unrelated to the immediate portion of the trajec- tory through x; instead, they come from later parts that just happen to pass close to x. Thus, $N_\epsilon(x)$ measures how frequently a typical trajectory visits an ϵ-neighbor- hood of x.

Now vary ϵ. As ϵ increases, the number of points in the ball typically grows as a power law:

$$N_\epsilon(x) \sim \epsilon^{-d(x)}$$

where $d(x)$ is called the pointwise dimension at x. The pointwise dimension can de- pend significantly on x; it will be smaller in rarefied regions of the attractor. To get an overall dimension of A, one averages $N_\epsilon(x)$ over many x. The resulting quantity $C(\epsilon)$ is found empirically to scale as:

$$C(\epsilon) \sim \epsilon^{d_C}$$

where d_C is called the correlation dimension.

The correlation dimension takes account of the density of points on the attrac- tor, and thus differs from the box dimension, which weights all occupied boxes equally, no matter how many points they contain. (Mathematically speaking, the correlation dimension involves an invariant measure supported on a fractal, not just the fractal itself.) In general, $d_C \leq d_{box}$, although they are usually very close (Grassberger and Procaccia 1983).

To estimate d_C, one plots $\log C(\epsilon)$ vs. $\log \epsilon$. If the relation $C(\epsilon) = \epsilon^{d_C}$ were valid for all ϵ, we'd find a straight line of slope d_C. In practice, the power law holds only over an intermediate range of ϵ (Figure 50).

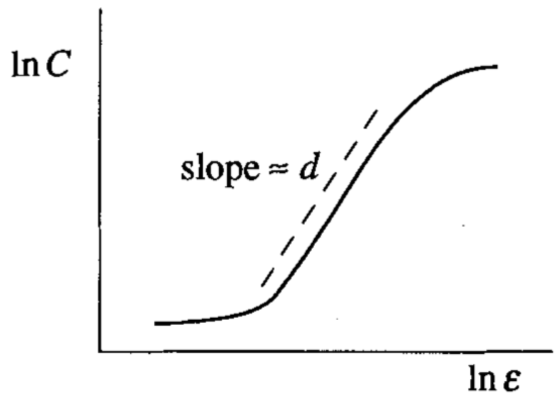

Figure 50: The correlation sum $C(\epsilon)$ versus ϵ for the correlation dimension estimation.

The curve saturates at large ϵ because the ϵ-balls engulf the whole attractor and so $N_\epsilon(x)$ can grow no further. On the other hand, at extremely small ϵ, the only point in each ϵ-ball is x itself. So the power law is expected to hold only in the scaling region where

$$\text{(minimum separation of points on } A) \ll \epsilon \ll \text{(diameter of } A).$$

8.1 Lorenz Attractor

the differential equations that lead to the lorenz attractor are defined by

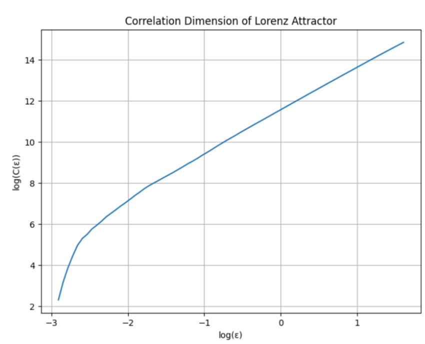

Figure 51: $logCvslog(l/l_0)$

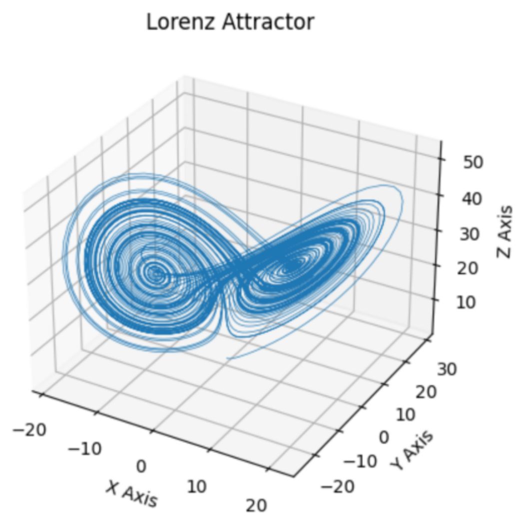

Figure 52: The correlation dimension of the Lorenz attractor, as reported by Grassberger and Procaccia (1983).

$$\dot{x} = -\sigma(x - y)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

We have equilibrium points whenever the following equations are satisfied:

$$\dot{x} = -\sigma(x - y) = 0$$
$$\dot{y} = -xz + rx - y = 0$$
$$\dot{z} = xy - bz = 0$$

Thus, we have the following system of equations:

$$x = y$$
$$x = \pm\sqrt{bz}$$
$$0 = x(-z + r - 1)$$

This leads to the following equilibrium points:

$$x_1^* = (0, 0, 0)$$

$$x_2^* = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \quad \text{for } r \geq 1$$

$$x_3^* = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \quad \text{for } r \geq 1$$

In order to treat stability, we consider the Jacobian matrix for our system:

$$Df(x,y,z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

First, let us consider the stability of the trivial equilibrium point, x_1^*. So we have

$$Df(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-1 & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

The characteristic polynomial is given by

$$p(\lambda)(0,0,0) = \det[\lambda I - Df(0,0,0)] = (\lambda + b)(\lambda^2 + (\sigma+1)\lambda + \sigma(1-r))$$

Thus, the eigenvalues of $Df(0,0,0)$ are

$$\lambda_1 = -b,$$

$$\lambda_2 = -\sigma - 1 - \sqrt{\sigma^2 - 2\sigma + 4\sigma r + 1},$$

and

$$\lambda_3 = -\sigma - 1 + \sqrt{\sigma^2 - 2\sigma + 4\sigma r + 1}.$$

We can see that for $r < 1$, all the eigenvalues are stable. However, for $r > 1$, one of the eigenvalues is positive, so $(0,0,0)$ is unstable. Now, let us consider the stability of the equilibrium point x_2^*, given by (3). So we have

$$Df(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

The characteristic polynomial is given by

$$p(\lambda)(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) = \det[\lambda I - Df(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$$

$$=$$

$$\lambda^3 + (1+b+\sigma)\lambda^2 + b(\sigma+r)\lambda + 2b\sigma(r-1)$$

From [3], we know that if we have a cubic polynomial of the form:

$$p(\lambda) = \lambda^3 - \tau\lambda^2 + \alpha\lambda - \delta \tag{6}$$

Then we know that the roots of $p(\lambda)$ have negative real parts if and only if $\tau < 0$ and $\tau\alpha < \delta < 0$.

So, in our case, we have $\tau = -(1+b+\sigma)$, $\alpha = b(\sigma+r)$, and $\delta = -2b\sigma(r-1)$. For stability, we need

$$-(1+b+\sigma) < 0 \tag{7}$$

and

$$-b(1+b+\sigma)(\sigma+r) < -2b\sigma(r-1) < 0. \tag{8}$$

Clearly, the inequality in (7) is satisfied because σ, b, and r are assumed to be positive. Now, let us consider the inequality in (8). In the same way, we can see that

$$-b(1+b+\sigma)(\sigma+r) < 0 \quad \text{and} \quad -2b\sigma(r-1) < 0,$$

but we need to find the values of σ, b, and r satisfying

$$-b(1+b+\sigma)(\sigma+r) < -2b\sigma(r-1).$$

So, we obtain

$$r < \frac{\sigma(\sigma+3+b)}{\sigma-b-1}.$$

We have that the equilibrium point is stable if r satisfies the inequality in (9). Note that the stability analysis yields the same result for the equilibrium point x_3^*, given by

Estimate the correlation dimension of the Lorenz attractor, for the standard pa- rameter values $r = 28$, $\sigma = 10$, $\beta = \frac{8}{3}$.

Solution: Figure 8.3 shows the results of Grassberger and Procaccia (1983). (Note that in their notation, the radius of the balls is ϵ and the correlation dimen- sion is v.) A line of slope $d_C = 2.05 \pm 0.01$ gives an excellent fit to the data, ex- cept for large ϵ, where the expected saturation occurs.

These results were obtained by numerically integrating the system with a Runge-Kutta method. The time step was 0.25, and 15,000 points were computed. Grassberger and Procaccia also report that the convergence was rapid; the correla- tion dimension could be estimated to within ± 5 percent using only a few thousand points.

8.2 Logistic Map

Consider the logistic map $x_{n+1} = rx_n(1-x_n)$ at the parameter value $r = r_\infty = 3.5699456$, corresponding to the onset of chaos. Show that the attractor is a Cantor-like set, although it is not strictly self-similar. Then compute its correlation dimension numerically.

Solution: We visualize the attractor by building it up recursively. Roughly speaking, the attractor looks like a 2^n-cycle, for $n \gg 1$. Figure schemati- cally shows some typical 2^n-cycles for small values of n.

The dots in the left panel of Figure 53 represent the superstable 2^n-cycles. The right panel shows the corresponding values of x. As $n \to \infty$, the resulting set ap- proaches a topological Cantor set, with points separated by gaps of various sizes. But the set is not strictly self-similar—the gaps scale by different factors depend- ing on their location. In other words, some of the "wishbones" in the orbit diagram are wider than others at the same r.

The correlation dimension of the limiting set has been estimated by Grassberger and Procaccia (1983). They generated a single trajectory of 30,000

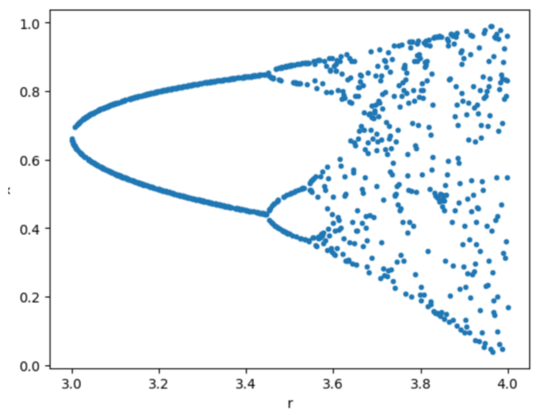

Figure 53: Recursive construction of the logistic map attractor, visualized as a sequence of 2^n-cycles.

points, starting from $x_0 = 0.5$. Their plot of $\log C(\epsilon)$ vs. $\log \epsilon$ is well fit by a straight line of slope $d_C = 0.500 \pm 0.005$ (Figure 11.5.5).

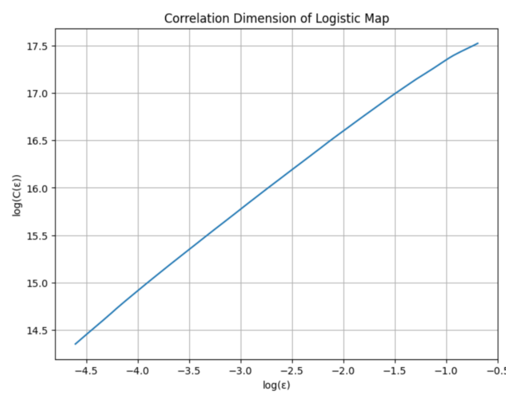

Figure 54: The correlation dimension of the logistic map attractor, as reported by Grassberger and Procaccia (1983).

This is smaller than the box dimension $d_{box} = 0.538$ (Grassberger 1981), as ex- pected.

For very small ϵ, the data in Figure 54 deviate from a straight line. Grassberger and Procaccia (1983) attribute this deviation to residual correlations among the x_n's on their single trajectory. These correlations would be negligible if the map were strongly chaotic, but for a system at the onset of chaos (like this one), the correlations are visible at small scales. To extend the scaling region, one could use a larger number of points or more than one trajectory.

8.3 Multifractals

We conclude by mentioning a recent development, although we cannot go into details. In the logistic attractor , the scaling varies from place to place, unlike in the middle-thirds Cantor set, where there is a uniform scaling by $1/3$ everywhere. Thus we cannot completely characterize the logistic attractor by its dimension, or any other single number—we need some kind of distribution function that tells us how the dimension varies across the attractor. Sets of this type are called multifractals.

The notion of pointwise dimension allows us to quantify the local variations in scaling. Given a multifractal A, let S_a be the subset of A consisting of all points with pointwise dimension a. If a is a typical scaling factor on A, then it will be represented often, so S_a will be a relatively large set; if a is unusual, then S_a will be a small set. To be more quantitative, we note that each S_a is itself a fractal, so it makes sense to measure its "size" by its fractal dimension. Thus, let $f(a)$ denote the dimension of S_a. Then $f(a)$ is called the multifractal spectrum of A or the spectrum of scaling indices (Halsey et al., 1986).

Roughly speaking, you can think of the multifractal as an interwoven set of fractals of different dimensions a, where $f(a)$ measures their relative weights. Since very large and very small a are unlikely, the shape of $f(a)$ typically looks like Figure 55. The maximum value of $f(a)$ turns out to be the box dimension (Halsey et al., 1986).

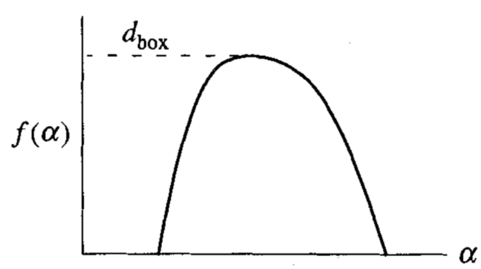

Figure 55: Multifractal spectrum $f(a)$ for a system exhibiting multifractal behavior.

9 conclusion

fractals are complex geometric shapes with fine structure at arbitrarily small scales. Usually they have some degree of self-similarity. In other words, if we magnify a tiny part of a fractal, we will see features reminis- cent of the whole. Sometimes the similarity is exact; more often it is only approx- imate or statistical.

Fractals are of great interest because of their exquisite combination of beauty, complexity, and endless structure. They are reminiscent of natural objects like mountains, clouds, coastlines, blood vessel networks, and even broccoli, in a way that classical shapes like cones and squares can't match. They have also turned out to be useful in scientific applications ranging from computer graphics and image compression to the structural mechanics of cracks and the fluid mechanics of vis- cous fingering.we have studied some crucial examples such as cantor sets their applications and we also found their similarity and box dimensions.we also discussed about countable and uncountable sets using cantor's diagonal argument. The dimensions of the fractal objects that we have studied are mostly non integers.For systems at the onset of chaos, multifractals lead to a more powerful version of the universality theory. The universal quantity is now

a function f (a) of fig 55 , rather than a single number; it therefore offers much more infor- mation, and the possibility of more stringent tests. The theory's predictions have been checked for a variety of experimental systems at the onset of chaos, with striking success.

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