1) $y'' + x^2 y' + y\sin x = 0 \xrightarrow{x=0} y''_{(0)} = 0$

$\quad \rightarrow y''' + 2x y' + x^2 y'' + y'\sin x + y\cos x = 0 \xrightarrow{x=0} y'''_{(0)} = -y_{(0)}$

$$y^{(4)} + 2y' + 2x y'' + 2x y'' + x^2 y''' + y''\sin x + 2y'\cos x - y\sin x = 0$$

$$\xrightarrow{x=0} y^{(4)}_{(0)} + 2y'_{(0)} + 2y'_{(0)} = 0 \rightarrow y^{(4)}_{(0)} = -4y'_{(0)}$$

$P_{(x)} = 1 \rightarrow$ عادی $x=0 \rightarrow y_g = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}_{(0)}}{n!} x^n$

$$\rightarrow y_g = y_{(0)} + y'_{(0)} x + 0 + \frac{y'''_{(0)}}{3!} x^3 + \frac{y^{(4)}_{(0)}}{4!} x^4 + \dots$$

$$\xrightarrow[y'_{(0)} = a_1]{y_{(0)} = a_0} y_g = a_0 + a_1 x - \frac{a_0}{3!} x^3 - \frac{a_1}{3!} x^4 + \dots$$

۲) $x^4 y'' + \lambda y = 0$

الف) $P(x) = x^4 = 0 \longrightarrow$ منفرد $x=0$

$$\lim_{x\to 0} x\left(\frac{0}{x^4}\right)=0$$

$$\lim_{x\to 0} x^2\left(\frac{\lambda}{x^4}\right)=\infty \longrightarrow$$ $x=0$ منفرد نامنظم

ب) $y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -x^{-2}\dot{y}$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-x^{-2}\dot{y}\right) = 2x^{-3}\dot{y} + x^{-4}\ddot{y}$$

جایگذاری در معادله: $\ddot{y} + 2x\dot{y} + \lambda y = 0 \longrightarrow t\ddot{y} + 2\dot{y} + \lambda t y = 0$

$P(t) = t = 0 \longrightarrow$ منفرد $t=0$

$$\left.\begin{array}{l} \lim_{t\to 0} t\left(\frac{2}{t}\right)=2 \\ \lim_{t\to 0} t^2(\lambda)=0 \end{array}\right\} \longrightarrow$$ منفرد منظم $t=0 \longrightarrow y_g = \sum_{n=0}^{\infty} a_n t^{n+r}$

$$\dot{y} = \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

$$\ddot{y} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}$$

جایگذاری در معادله: $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-1} + 2\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} + \lambda\sum_{n=0}^{\infty} a_n t^{n+r+1} = 0$

$$r(r-1)a_0 t^{r-1} + r(r+1)a_1 t^{r} + 2ra_0 t^{r-1} + 2(r+1)a_1 t^{r}$$

$$+\sum_{n=2}^{\infty}\left[(n+r)(n+r+1)a_n + \lambda a_{n-2}\right]t^{n+r-1} = 0$$

t^{r-1} ضریب $=0 \to r(r+1)a_0=0 \xrightarrow{a_0\neq 0}$ معادله‌ی شاخص: $r(r+1)=0 \to \begin{cases} r_1=0\\ r_2=-1\end{cases}$

t^{r} ضریب $=0 \to (r+1)(r+2)a_1=0 \quad (*)$

t^{n+r-1} ضریب $=0 \to a_n=\dfrac{-\lambda a_{n-2}}{(n+r)(n+r+1)} \quad (**)$

$r_1=0:$

$$\xrightarrow{(*)} 2a_1=0 \to a_1=0 \xrightarrow{(**)} a_{2n-1}=0 \quad (n=1,2,\dots)$$

$$\xrightarrow{(**)} a_{2n}=\frac{-\lambda a_{2(n-1)}}{2n(2n+1)} \quad (n=1,2,\dots)$$

$$a_2=\frac{-\lambda a_0}{3!}$$

$$a_4=\frac{-\lambda a_2}{5\times 4}=\frac{+\lambda a_0}{5!}$$

$$\vdots$$

$$a_{2n}=\frac{(-1)^n a_0}{(2n+1)!}\lambda \quad (n=1,2,\dots)$$

$$y_1=a_0+a_2t^2+a_4t^4+\dots+a_{2n}t^{2n}+\dots=a_0\left(1-\frac{\lambda}{3!}t^2+\frac{\lambda}{5!}t^4+\dots+\frac{(-1)^n\lambda}{(2n+1)!}t^{2n}+\dots\right)$$

فرم جواب دوم: $y_2=cy_1\ln x+\sum_{n=0}^{\infty} b_n x^{n+r_2}=cy_1\ln x+\sum_{n=0}^{\infty} b_n x^{n-1}$

فرم جواب عمومی: $y_g=c_1y_1+c_2y_2=c_1y_1+c_2\left(cy_1\ln x+\sum_{n=0}^{\infty} b_n x^{n-1}\right)$

شعاع همگرایی y_1، y_1 چندجمله‌ای است $\leftarrow$ $R_{y_1}=\infty$

3) $\begin{cases} y'' + 2y' + 2y = \delta(t-\pi) \\ y(0) = y'(0) = 0 \end{cases} \xrightarrow{L} s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 2(sY(s) - \cancel{y(0)}) + 2Y(s) = e^{-\pi s}$

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 2s + 2} = \frac{e^{-\pi s}}{(s+1)^2 + 1} \quad (A)$$

$$L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t} \cdot \sin t \quad (B)$$

$$\xrightarrow{(A),(B)} y_p(t) = U_\pi(t) \cdot e^{-(t-\pi)} \sin(t-\pi) = -U_\pi(t) \cdot e^{\pi - t} \sin t$$

ج)

الف) $L^{-1}\{\sqrt{S-1}-\sqrt{S-۲}\} = L^{-1}\{\sqrt{S-1}\} - L^{-1}\{\sqrt{S-۲}\} = (e^{t} - e^{۲t})L^{-1}\{\sqrt{S}\}$ (A)

$$L\{t^{P}\} = \frac{\Gamma(P+1)}{S^{P+1}} \xrightarrow{P=-\frac{۳}{۲}} L\{t^{-\frac{۳}{۲}}\} = \frac{\Gamma(-\frac{1}{۲})}{S^{-\frac{1}{۲}}} = -۲\sqrt{\pi}\sqrt{S} \rightarrow L^{-1}\{\sqrt{S}\} = \frac{t^{-\frac{۳}{۲}}}{-۲\sqrt{\pi}}$$ (B)

$$\xrightarrow{(A),(B)} L^{-1}\{\sqrt{S-1}-\sqrt{S-۲}\} = \frac{t^{-\frac{۳}{۲}}}{-۲\sqrt{\pi}}(e^{t}-e^{۲t})$$

ب) $L\{f(t)\} = \frac{1}{1-e^{-ST}}\int_0^T e^{-st} f(t)\,dt \xrightarrow{T=۲} L\{f(t)\} = \frac{1}{1-e^{-۲S}}\int_0^۲ e^{-st} f(t)\,dt$

$$\longrightarrow L\{f(t)\} = \frac{1}{1-e^{-۲S}}\int_0^1 e^{-st}\,dt = \frac{-1}{S(1-e^{-۲S})}e^{-st}\Big|_{t=0}^{1} = \frac{-(e^{-S}-1)}{S(1-e^{-۲S})}$$

$$\longrightarrow L\{f(t)\} = \frac{1}{S(1+e^{-S})}$$

ه) $X'_{(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} X_{(t)}$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \xrightarrow{\det(A-\lambda I)=0} (\lambda+1)(\lambda+2-\lambda^2)=0 \longrightarrow \begin{cases} \lambda_1=\lambda_2=-1 \\ \lambda_3=2 \end{cases}$$

$$(A-\lambda_1 I)V_1=\bar{0} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow V_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

به همین ترتیب $V_2=\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ و $V_3=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ مستقل خطی با V_1 به دست می‌آیند.

$$X_g = c_1V_1e^{\lambda_1 t}+c_2V_2e^{\lambda_2 t}+c_3V_3e^{\lambda_3 t}$$

$$\longrightarrow X_g=\begin{bmatrix} c_1e^{-t}+2c_2e^{-t}+c_3e^{2t} \\ -c_1e^{-t}-c_2e^{-t}+c_3e^{2t} \\ -c_2e^{-t}+c_3e^{2t} \end{bmatrix}$$