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EXPLAINAIBILITY OF POSSIBILISTIC AND FUZZY RULE-BASED SYSTEMS

PhD Thesis Defense

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Ismaïl Baaj

LIP6 cea CentraleSupélec

Objectives

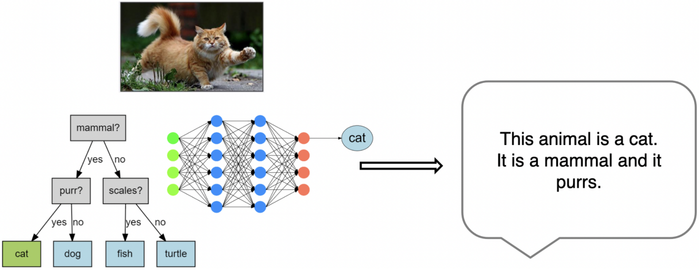
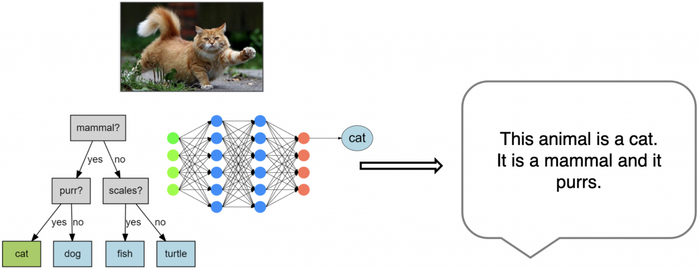


Figure 1: Explainable Artificial Intelligence (XAI)



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In this thesis, we focus on two objectives:

i) the establishment of meeting points between *Knowledge Representation and Reasoning* (KRR) and *Machine Learning* (ML)

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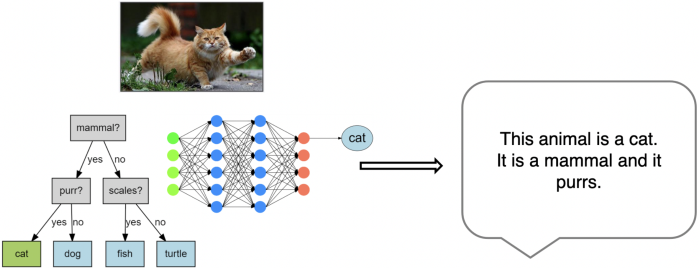


Figure 1: Explainable Artificial Intelligence (XAI)

In this thesis, we focus on two objectives:

i) the establishment of meeting points between *Knowledge Representation and Reasoning* (KRR) and *Machine Learning* (ML)

ii) the elaboration of a processing chain for XAI, in order to generate AI explanations in natural language and to evaluate them:

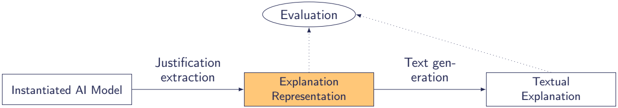
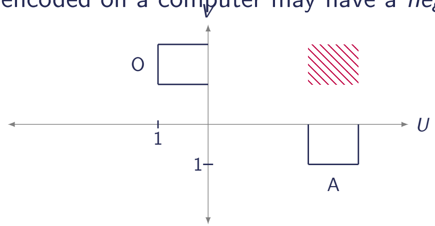
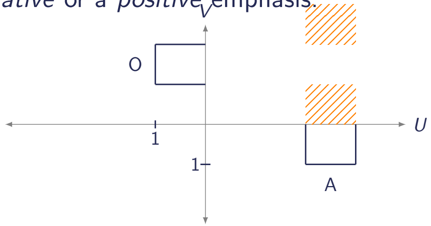


Figure 2: Proposed XAI processing chain (Baaj et al. 2019)

As (Dubois, Prade and Ughetto 2003) developed the idea that information encoded on a computer may have a *negative* or a *positive* emphasis:



(a) Examples of the rule $A \times O$



(b) Counterexamples of the rule $\overline{A} \cup O$

Figure 3: Complete representation of a rule “If X is A then Z is O”, $A \subseteq U, O \subseteq V$

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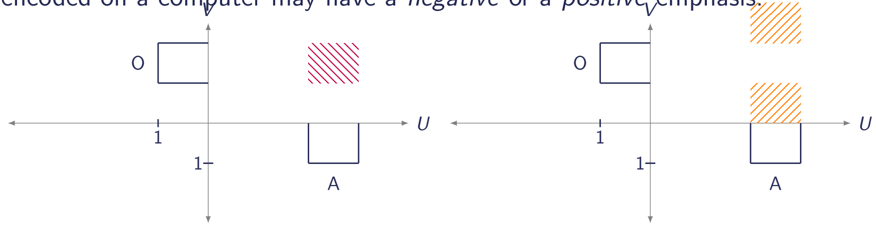

(a) Examples of the rule $A \times O$

(b) Counterexamples of the rule $\overline{A} \cup O$

Figure 3: Complete representation of a rule "If X is A then Z is O", $A \subseteq U, O \subseteq V$

In this thesis, explanatory paradigms are introduced for two AI systems:

- a possibilistic rule-based system, where possibilistic rules encode *negative* information (we focus on this system in this presentation)
- a fuzzy rule-based system composed of possibility rules, which encode *positive* information.

- Possibility Theory is a well-known framework for the representation of incomplete or imprecise information (*What is possible without being certain at all? What is certain to some extent?*)

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- Possibility Theory is a well-known framework for the representation of incomplete or imprecise information (*What is possible without being certain at all? What is certain to some extent?*)

- Dubois and Prade (2020) emphasized the development of possibilistic learning methods that would be *consistent* with if-then rule-based reasoning

- Dubois and Prade (2020) highlighted the approach of Farreny and Prade (1989), who proposed a min-max equation system in order to develop the explanatory capabilities of possibilistic rule based systems

- A set of n if-then possibilistic rules R^1, R^2, $\cdots$, R^n
- R^i: "if p_i then q_i" with an uncertainty propagation matrix:

$$\begin{bmatrix} \pi(q_i|p_i) & \pi(q_i|\neg p_i) \\ \pi(\neg q_i|p_i) & \pi(\neg q_i|\neg p_i) \end{bmatrix} = \begin{bmatrix} 1 & s_i \\ r_i & 1 \end{bmatrix}$$

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- p_i stands for "$a_i(x) \in P_i$" and q_i for "$b(x) \in Q_i$":

  a_i and b: *attributes* applied to an item x

  P_i and Q_i: *subsets* of the respective *attribute domains* (D_{a_i}, D_b)

Background – *Possibilistic rule-based system*

- A set of n if-then possibilistic rules $R^1, R^2, \cdots, R^n$
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Example

"*If a person likes meeting people, then recommended professions are professor or businessman or lawyer or doctor*" (Farreny & Prade 1989)

- attributes: likes-meeting-people and profession
- $P_i = \{yes\} \subset D_{likes-meeting-people} = \{yes, no\}$,
- Q_i = {professor, businessman, laywer, doctor} $\subset D_{profession}$ = {businessman, lawyer, doctor, professor, researcher, architect, engineer, others},
- rule parameters: $s_i = 1, r_i = 0.3$

- Information about $a_i(x)$ represented by a *possibility distribution* $\pi_{a_i(x)} : D_{a_i} \to [0,1]$ *normalized* i.e. $\exists u \in D_{a_i}$, $\pi_{a_i(x)}(u) = 1$
- Evaluation of p_i: "$a_i(x) \in P_i$":

$$\pi(p_i) = \Pi(P_i) = \sup_{u \in P_i} \pi_{a_i(x)}(u) = \lambda_i$$
$$\pi(\neg p_i) = \Pi(\overline{P_i}) = \sup_{u \in \overline{P_i}} \pi_{a_i(x)}(u) = 1 - n(p_i) = \rho_i$$

where $n(p_i) = N(P_i) = \inf_{u \in \overline{P_i}}(1 - \pi_{a_i(x)}(u)) = 1 - \pi(\neg p_i)$

Consequence of the normalization of $\pi_{a_i(x)}$: $\max(\pi(p_i), \pi(\neg p_i)) = 1$

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Example

p_i: "likes-meeting-people(x) $\in$ {yes}", $D_{likes-meeting-people} = \{yes, no\}$
Information: $\pi_{\text{likes-meeting-people}(x)} : \langle yes : 1, no : 0.5\rangle$

$$\pi(p_i) = 1, \quad \pi(\neg p_i) = 0.5, \quad n(p_i) = 0.5$$

- Information about $a_i(x)$ represented by a *possibility distribution* $\pi_{a_i(x)} : D_{a_i} \to [0, 1]$ *normalized* i.e. $\exists u \in D_{a_i}$, $\pi_{a_i(x)}(u) = 1$
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Consequence of the normalization of $\pi_{a_i(x)}$: $\max(\pi(p_i), \pi(\neg p_i)) = 1$

- In case of a compounded premise $p_i = p_{1,i} \wedge \cdots \wedge p_{k,i}$: $\pi(p_i) = \min_{j=1}^{k} \pi(p_{j,i})$ and $\pi(\neg p_i) = \max_{j=1}^{k} \pi(\neg p_{j,i})$

- Uncertainty propagation: $\begin{bmatrix} \pi(q_i) \\ \pi(\neg q_i) \end{bmatrix} = \begin{bmatrix} 1 & s_i \\ r_i & 1 \end{bmatrix} \Box_{\min}^{\max} \begin{bmatrix} \lambda_i \\ \rho_i \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$

$\Box_{\min}^{\max}$: matricial product with min as product and max as addition

- As $\max(\lambda_i, \rho_i) = 1$ we have:

$$\alpha_i = \max(s_i, \lambda_i) \quad \text{and} \quad \beta_i = \max(r_i, \rho_i)$$

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- Possibility distribution of the attribute b associated to R^i:

$$\pi^{*i}_{b(x)}(u) = \alpha_i \mu_{Q_i}(u) + \beta_i \mu_{\overline{Q_i}}(u) \text{ for any } u \in D_b$$

$\mu_{Q_i}, \mu_{\overline{Q_i}}$: characteristic functions of $Q_i, \overline{Q_i}$

Example

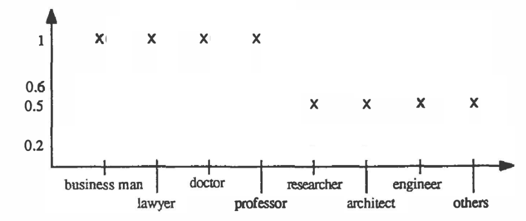

Figure 4: Possibility distribution of *profession* obtained with the rule

- Possibility distribution of b with n rules:

$$\pi^*_{b(x)}(u) = \min(\pi^{*1}_{b(x)}(u), \pi^{*2}_{b(x)}(u), \cdots, \pi^{*n}_{b(x)}(u))$$

Example

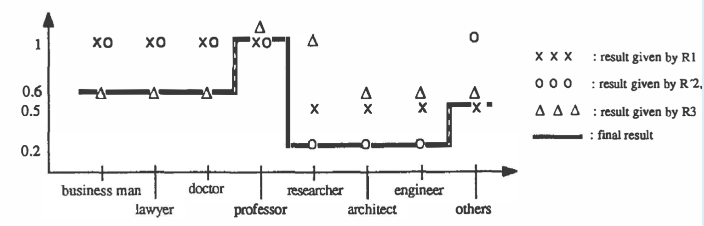

Figure 5: Possibility distribution of *profession* obtained with three rules (Farreny & Prade 1989)

- Two sets of possibilistic rules: $R^1, R^2, \cdots, R^n$ and $R'^1, R'^2, \cdots, R'^m$
- Same attribute b used in both the conclusions of the R^i and the premises of the R'^j.

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- Two sets of possibilistic rules: $R^1, R^2, \cdots, R^n$ and $R'^1, R'^2, \cdots, R'^m$
- Same attribute b used in both the conclusions of the R^i and the premises of the R'^j.
- Each rule R'_j: "if p'_j then q'_j":

uncertainty propagation matrix: $\begin{bmatrix} 1 & s'_j \\ r'_j & 1 \end{bmatrix}$

p'_j stands for "$b(x) \in Q'_j$" and q'_j for "$c(x) \in Q''_j$"

$Q'_j \subseteq D_b$ and $Q''_j \subseteq D_c$

$\lambda'_j = \pi(p'_j)$ and $\rho'_j = \pi(\neg p'_j)$

Cascade of Farreny and Prade (1989)

- First set of possibilistic rules:

  R^1: if a person likes meeting people, then recommended professions are professor or businessman or lawyer or doctor

  R^2: if a person is fond of creation/inventions, then recommended professions are engineer or public researcher or architect

  - R^3: if a person looks for job security and is fond of intellectual speculation, then recommended professions are professor or public researcher

where $D_{profession}$ = {businessman, lawyer, doctor, professor, researcher, architect, engineer, others}, $s_1 = 1$, $r_1 = 0.3$, $s_2 = 0.2$, $r_2 = 0.4$, $s_3 = 1$, $r_3 = 0.3$.

- Second set of possibilistic rules:

  R'^1: if a person is a professor or a researcher, then her salary is rather low

  R'^2: if a person is an engineer, a lawyer or an architect, her salary is average or high

  R'^3: if a person is a business man or a doctor, then her salary is high

where D_{salary}={low,average,high}, $s'_1 = 1$, $r'_1 = 0.7$, $s'_2 = 0.8$, $r'_2 = 0.2$, $s'_3 = 0.6$ and $r'_3 = 0.4$.

- Farreny and Prade (1989) proposed an equation system denoted $OV = MR \blacksquare IV$ in order to:

  describe the output possibility distribution

  perform a sensitivity analysis

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    describe the output possibility distribution

    perform a sensitivity analysis

- Dubois and Prade (2020) made explicit this equation system for the case of two rules R^1 and R^2:

$$\begin{bmatrix} \Pi(Q_1 \cap Q_2) \\ \Pi(Q_1 \cap \overline{Q_2}) \\ \Pi(\overline{Q_1} \cap Q_2) \\ \Pi(\overline{Q_1} \cap \overline{Q_2}) \end{bmatrix} = \begin{bmatrix} s_1 & 1 & s_2 & 1 \\ s_1 & 1 & 1 & r_2 \\ 1 & r_1 & s_2 & 1 \\ 1 & r_1 & 1 & r_2 \end{bmatrix} \Box^{\min}_{\max} \begin{bmatrix} \lambda_1 \\ \rho_1 \\ \lambda_2 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \min(\alpha_1, \alpha_2) \\ \min(\alpha_1, \beta_2) \\ \min(\beta_1, \alpha_2) \\ \min(\beta_1, \beta_2) \end{bmatrix}$$

$\Box^{\min}_{\max}$: matricial product with max as product and min as addition

- $Q_1 \cap Q_2, Q_1 \cap \overline{Q_2}, \overline{Q_1} \cap Q_2, \overline{Q_1} \cap \overline{Q_2}$ form a *partition* of D_b

- From a possibilistic rule-based system with n rules $R^1, R^2, \cdots, R^n$:

$$O_n = M_n \Box_{\max}^{\min} I_n$$

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- From a possibilistic rule-based system with n rules R^1, R^2, $\cdots$, R^n:

$$O_n = M_n \Box_{\max}^{\min} I_n$$

- To understand the output vector O_n, we introduce:

  $(E_k^{(n)})_{1 \leq k \leq 2^n}$: an explicit partition of D_b constructed with the sets $Q_1, Q_2, \cdots, Q_n$ used in the conclusions of the rules and their complements

  B_n: a matrix constructed inductively w.r.t the number of rules

- For $i = 1, 2, \cdots, n$, the matrices M_i, I_i, B_i are defined according to:

  $s_1, s_2, \cdots, s_i$ and $r_1, r_2, \cdots, r_i$ for M_i

  $\lambda_1, \lambda_2, \cdots, \lambda_i$ and $\rho_1, \rho_2, \cdots, \rho_i$ for I_i

  $\alpha_1, \alpha_2, \cdots, \alpha_i$ and $\beta_1, \beta_2, \cdots, \beta_i$ for B_i

For each $i = 1, 2, \cdots, n$, the partition $(E_k^{(i)})_{1\leq k\leq 2^i}$ is defined by the following two conditions:

$$E_1^{(1)} = Q_1 \text{ and } E_2^{(1)} = \overline{Q_1}$$

$$\text{and for } i > 1\text{: } E_k^{(i)} = \begin{cases} E_k^{(i-1)} \cap Q_i & \text{if } 1 \leq k \leq 2^{i-1} \\ E_{k-2^{i-1}}^{(i-1)} \cap \overline{Q_i} & \text{if } 2^{i-1} < k \leq 2^i \end{cases}$$

Generalized equation system – *Constructions of M_i, I_i, B_i*

- Respective size of M_i, I_i and B_i: $(2^i, 2i)$, $(2i, 1)$ and $(2^i, i)$
- $i = 1$, we take $M_1 = \begin{bmatrix} s_1 & 1 \\ 1 & r_1 \end{bmatrix}$, $I_1 = \begin{bmatrix} \lambda_1 \\ \rho_1 \end{bmatrix}$, $B_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$

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- $i > 1$, we define $M_i = \left[\begin{array}{c|c} M_{i-1} & S_i \\ \hline M_{i-1} & R_i \end{array}\right]$, $I_i = \left[\begin{array}{c} I_{i-1} \\ \hline \lambda_i \\ \rho_i \end{array}\right]$, $B_i = \left[\begin{array}{c|c} & \alpha_i \\ B_{i-1} & \alpha_i \\ & \vdots \\ & \alpha_i \\ \hline & \beta_i \\ B_{i-1} & \beta_i \\ & \vdots \\ & \beta_i \end{array}\right]$

where $S_i = \begin{bmatrix} s_i & 1 \\ s_i & 1 \\ \vdots & \vdots \\ s_i & 1 \end{bmatrix}$ and $R_i = \begin{bmatrix} 1 & r_i \\ 1 & r_i \\ \vdots & \vdots \\ 1 & r_i \end{bmatrix}$ of size $(2^{i-1}, 2)$

- $E_k^{(i)}$ is linked to the row $L_k = (\gamma_1, \gamma_2, \cdots, \gamma_i)$ of B_i with $\gamma \in \{\alpha, \beta\}$ by:

$$E_k^{(i)} = T_1 \cap T_2 \cdots \cap T_i \text{ with } T_j = \begin{cases} Q_j & \text{if } \gamma_j = \alpha_j \\ \overline{Q_j} & \text{if } \gamma_j = \beta_j \end{cases}$$

- For any $i = 1, 2, \cdots, n$, we set:

$$\boxdot_{\min} B_i = [o_k^{(i)}]_{1 \leq k \leq 2^i}$$

$\boxdot_{\min}$: the minimum of the coefficients of each row in a matrix

- For any $k \in \{1, 2, \cdots, 2^i\}$, we deduce:

$$o_k^{(i)} = \begin{cases} \min(o_k^{(i-1)}, \alpha_i) & \text{if } 1 \leq k \leq 2^{i-1} \\ \min(o_{k-2^{i-1}}^{(i-1)}, \beta_i) & \text{if } 2^{i-1} < k \leq 2^i \end{cases}$$

- Finally, we obtain:

Theorem

$$M_i \Box_{\max}^{\min} I_i = \boxdot_{\min} B_i$$

A possibilistic rule-based system composed of $n = 3$ rules

• The sets of the partition $(E_k^{(3)})_{1 \leq k \leq 8}$ are the following: $Q_1 \cap Q_2 \cap Q_3$, $\overline{Q}_1 \cap Q_2 \cap Q_3$, $Q_1 \cap \overline{Q}_2 \cap Q_3$, $\overline{Q}_1 \cap \overline{Q}_2 \cap Q_3$, $Q_1 \cap Q_2 \cap \overline{Q}_3$, $\overline{Q}_1 \cap Q_2 \cap \overline{Q}_3$, $Q_1 \cap \overline{Q}_2 \cap \overline{Q}_3$ and $\overline{Q}_1 \cap \overline{Q}_2 \cap \overline{Q}_3$

• We check the Theorem by direct calculation:

$O_3 = M_3 \Box_{\max}^{\min} I_3$

$$= \begin{bmatrix} s_1 & 1 & s_2 & 1 & s_3 & 1 \\ 1 & r_1 & s_2 & 1 & s_3 & 1 \\ s_1 & 1 & 1 & r_2 & s_3 & 1 \\ 1 & r_1 & 1 & r_2 & s_3 & 1 \\ s_1 & 1 & s_2 & 1 & 1 & r_3 \\ 1 & r_1 & s_2 & 1 & 1 & r_3 \\ s_1 & 1 & 1 & r_2 & 1 & r_3 \\ 1 & r_1 & 1 & r_2 & 1 & r_3 \end{bmatrix} \Box_{\max}^{\min} \begin{bmatrix} \lambda_1 \\ \rho_1 \\ \lambda_2 \\ \rho_2 \\ \lambda_3 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \min(\alpha_1, \alpha_2, \alpha_3) \\ \min(\beta_1, \alpha_2, \alpha_3) \\ \min(\alpha_1, \beta_2, \alpha_3) \\ \min(\beta_1, \beta_2, \alpha_3) \\ \min(\alpha_1, \alpha_2, \beta_3) \\ \min(\beta_1, \alpha_2, \beta_3) \\ \min(\alpha_1, \beta_2, \beta_3) \\ \min(\beta_1, \beta_2, \beta_3) \end{bmatrix} = \boxdot_{\min} B_3$$

- Using the coefficients of $O_i = \boxdot_{\min} B_i$ and the characteristic functions $\mu_{E_1^{(i)}}, \mu_{E_2^{(i)}}, \cdots, \mu_{E_{2^i}^{(i)}}$ of the sets $E_1^{(i)}, E_2^{(i)}, \cdots, E_{2^i}^{(i)}$:

Theorem

The output possibility distribution $\pi^*_{b(x),i}$ associated to the first i rules is:

$$\pi^*_{b(x),i} = \sum_{1 \leq k \leq 2^i} o_k^{(i)} \mu_{E_k^{(i)}}$$

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- Consequence: $\forall u \in D_b$, $\exists k_0$ unique s.t $u \in E_{k_0}^{(i)}$ and $\pi^*_{b(x),i}(u) = o_{k_0}^{(i)}$
- $\pi^*_{b(x),i}$ is normalized iff: $\exists k \in \{1, 2, \cdots, 2^i\}$ s.t $E_k^{(i)} \neq \emptyset$ and $o_k^{(i)} = 1$

Example (continued) : a possibilistic rule-based system composed of $n = 3$ rules

- The characteristic functions of the partition $(E_k^{(3)})_{1 \leq k \leq 8}$ are $\mu_{Q_1 \cap Q_2 \cap Q_3}$, $\mu_{\overline{Q_1} \cap Q_2 \cap Q_3}$, $\mu_{Q_1 \cap \overline{Q_2} \cap Q_3}$, $\mu_{\overline{Q_1} \cap \overline{Q_2} \cap Q_3}$, $\mu_{Q_1 \cap Q_2 \cap \overline{Q_3}}$, $\mu_{\overline{Q_1} \cap Q_2 \cap \overline{Q_3}}$, $\mu_{Q_1 \cap \overline{Q_2} \cap \overline{Q_3}}$ and $\mu_{\overline{Q_1} \cap \overline{Q_2} \cap \overline{Q_3}}$
- The output possibility distribution is:

$$
\begin{aligned}
\pi^*_{b(x),3} &= \min(\pi^{*1}_{b(x)}, \pi^{*2}_{b(x)}, \pi^{*3}_{b(x)}) \\
&= \min(\alpha_1, \alpha_2, \alpha_3)\mu_{Q_1 \cap Q_2 \cap Q_3} + \min(\beta_1, \alpha_2, \alpha_3)\mu_{\overline{Q_1} \cap Q_2 \cap Q_3} \\
&\quad + \min(\alpha_1, \beta_2, \alpha_3)\mu_{Q_1 \cap \overline{Q_2} \cap Q_3} + \min(\beta_1, \beta_2, \alpha_3)\mu_{\overline{Q_1} \cap \overline{Q_2} \cap Q_3} \\
&\quad + \min(\alpha_1, \alpha_2, \beta_3)\mu_{Q_1 \cap Q_2 \cap \overline{Q_3}} + \min(\beta_1, \alpha_2, \beta_3)\mu_{\overline{Q_1} \cap Q_2 \cap \overline{Q_3}} \\
&\quad + \min(\alpha_1, \beta_2, \beta_3)\mu_{Q_1 \cap \overline{Q_2} \cap \overline{Q_3}} + \min(\beta_1, \beta_2, \beta_3)\mu_{\overline{Q_1} \cap \overline{Q_2} \cap \overline{Q_3}}
\end{aligned}
$$

- J contains the indexes of the non-empty sets of the partition:

$$J = \{k \in \{1, 2, \cdots, 2^i\} \mid E_k^{(i)} \neq \emptyset\} \text{ and } \omega = \text{card}(J)$$

Arrange the elements of J as a strictly increasing sequence:
$1 \leq k_1 < k_2 < \cdots < k_\omega \leq 2^i$

We have $\omega \leq \min(d, 2^i)$ where $d = \text{card}(D_b)$

$[\Pi(E_k^{(i)})]_{k \in J} = [o_k^{(i)}]_{k \in J}$

- Let $\mathcal{O}_i$, $\mathcal{M}_i$ and $\mathcal{B}_i$ be the matrices obtained from O_i, M_i and B_i respectively, by deleting each row whose index is <u>not</u> in J

- First set of rules of the cascade of Farreny and Prade ($n = 3$):
- $D_{profession} = \{$businessman, lawyer, doctor, professor, researcher, architect, engineer, others$\}$
- Sets Q_i used in the conclusions of the rules R^i:

$$Q_1 = \{\text{professor, businessman, lawyer, doctor}\},$$
$$Q_2 = \{\text{engineer, researcher, architect}\},$$
$$Q_3 = \{\text{professor, researcher}\}.$$

- Partition of $D_{profession}$: $E^{(3)}_{k_1} = \{\text{researcher}\}$, $E^{(3)}_{k_2} = \{\text{professor}\}$, $E^{(3)}_{k_3} = \{\text{engineer}, \text{architect}\}$, $E^{(3)}_{k_4} = \{\text{businessman}, \text{lawyer}, \text{doctor}\}$ and $E^{(3)}_{k_5} = \{\text{others}\}$
- Equation system:

$$\begin{bmatrix} \Pi(E^{(3)}_{k_1}) \\ \Pi(E^{(3)}_{k_2}) \\ \Pi(E^{(3)}_{k_3}) \\ \Pi(E^{(3)}_{k_4}) \\ \Pi(E^{(3)}_{k_5}) \end{bmatrix} = \begin{bmatrix} 1 & r_1 & s_2 & 1 & s_3 & 1 \\ s_1 & 1 & 1 & r_2 & s_3 & 1 \\ 1 & r_1 & s_2 & 1 & 1 & r_3 \\ s_1 & 1 & 1 & r_2 & 1 & r_3 \\ 1 & r_1 & 1 & r_2 & 1 & r_3 \end{bmatrix} \Box^{\min}_{\max} \begin{bmatrix} \lambda_1 \\ \rho_1 \\ \lambda_2 \\ \rho_2 \\ \lambda_3 \\ \rho_3 \end{bmatrix}$$

The vector $\mathcal{O}_3$ and the matrix $\mathcal{M}_3$ have five rows (while O_3 and M_3 have eight rows).

- Let: $\varepsilon(T) = \begin{cases} 1 \ \text{ si } \ T \neq \emptyset \\ 0 \ \text{ if } \ T = \emptyset \end{cases}$
- To any matrix $A = [a_{ij}]$, we associate $A^\circ = [1 - a_{ij}]$. $(A^\circ)^\circ = A$

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- For $Q \subseteq D_b$, we have $Q = \bigcup_{1 \leq j \leq \omega} E_{k_j}^{(i)} \cap Q$

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- For $Q \subseteq D_b$, we have $Q = \bigcup_{1 \leq j \leq \omega} E_{k_j}^{(i)} \cap Q$
  - The possibility measure is:

$$\Pi^*(Q) = \max_{u \in Q} \pi^*_{b(x)}(u) = \nabla_Q \Box_{\min}^{\max} \mathcal{O}_i$$

where $\nabla_Q = \begin{bmatrix} \varepsilon(E_{k_1}^{(i)} \cap Q) & \varepsilon(E_{k_2}^{(i)} \cap Q) & \cdots & \varepsilon(E_{k_\omega}^{(i)} \cap Q) \end{bmatrix}$

  - The necessity measure is then:

$$N^*(Q) = 1 - \Pi^*(\overline{Q}) = (\Pi^*(\overline{Q}))^\circ = \nabla_{\overline{Q}}{}^\circ \Box_{\max}^{\min} \mathcal{O}_i{}^\circ$$

Equation system properties – *Example (continued)*

First set of possibilistic rules of the cascade of Farreny and Prade:

- Partition of $D_{profession}$: $E^{(3)}_{k_1} = \{\text{researcher}\}$, $E^{(3)}_{k_2} = \{\text{professor}\}$, $E^{(3)}_{k_3} = \{\text{engineer}, \text{architect}\}$, $E^{(3)}_{k_4} = \{\text{business man}, \text{lawyer}, \text{doctor}\}$ and $E^{(3)}_{k_5} = \{\text{others}\}$
- Equation system with $\lambda_1 = 1, \rho_1 = 0.5, \lambda_2 = 0.2, \rho_2 = 1, \lambda_3 = 1, \rho_3 = 0.6$:

$$\begin{bmatrix} \Pi(E^{(3)}_{k_1}) \\ \Pi(E^{(3)}_{k_2}) \\ \Pi(E^{(3)}_{k_3}) \\ \Pi(E^{(3)}_{k_4}) \\ \Pi(E^{(3)}_{k_5}) \end{bmatrix} = \begin{bmatrix} 1 & r_1 & s_2 & 1 & s_3 & 1 \\ s_1 & 1 & 1 & r_2 & s_3 & 1 \\ 1 & r_1 & s_2 & 1 & 1 & r_3 \\ s_1 & 1 & 1 & r_2 & 1 & r_3 \\ 1 & r_1 & 1 & r_2 & 1 & r_3 \end{bmatrix} \Box^{\min}_{\max} \begin{bmatrix} \lambda_1 \\ \rho_1 \\ \lambda_2 \\ \rho_2 \\ \lambda_3 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}$$

Let $Q = \{\text{professor}, \text{researcher}\}$. The possibility measure of Q is:

$$\Pi^*(Q) = \nabla_Q \Box^{\max}_{\min} \mathcal{O}_3 = 1$$

where $\nabla_Q = \left[\varepsilon(E^{(i)}_{k_1} \cap Q) \quad \varepsilon(E^{(i)}_{k_2} \cap Q) \quad \cdots \quad \varepsilon(E^{(i)}_{k_\omega} \cap Q)\right] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

The necessity measure of Q is 0.4.

- Two equation systems:

$$\mathcal{O}_n = \mathcal{M}_n \Box^{\min}_{\max} I_n \text{ for } R^1, R^2, \cdots, R^n$$
$$\mathcal{O}'_m = \mathcal{M}'_m \Box^{\min}_{\max} I'_m \text{ for } R'^1, R'^2, \cdots, R'^m$$

- The input vector I'_m is linked to the output vector $\mathcal{O}_n$ by:

$$I'_m = \nabla \Box^{max}_{min} \mathcal{O}_n \quad \text{where} \quad \nabla = \begin{bmatrix} \nabla_{Q'_1} \\ \nabla_{\overline{Q'_1}} \\ \vdots \\ \nabla_{Q'_m} \\ \nabla_{\overline{Q'_m}} \end{bmatrix}$$

- The output vector O'_m is deduced from the first system:

$$\begin{aligned} \mathcal{O}'_m &= \mathcal{M}'_m \Box^{\min}_{\max} I'_m \\ &= \mathcal{M}'_m \Box^{\min}_{\max} (\nabla \Box^{\max}_{\min} \mathcal{O}_n) \\ &= \mathcal{M}'_m \Box^{\min}_{\max} (\nabla \Box^{\max}_{\min} (\mathcal{M}_n \Box^{\min}_{\max} I_n)) \end{aligned}$$

- First set of possibilistic rules:

  R^1: if a person likes meeting people, then recommended professions are professor or businessman or lawyer or doctor

  R^2: if a person is fond of creation/inventions, then recommended professions are engineer or public researcher or architect

  R^3: if a person looks for job security and is fond of intellectual speculation, then recommended professions are professor or public researcher

- Equation system with $\lambda_1 = 1, \rho_1 = 0.5, \lambda_2 = 0.2, \rho_2 = 1, \lambda_3 = 1, \rho_3 = 0.6$:

$$\begin{bmatrix} \Pi(E^{(3)}_{k_1}) \\ \Pi(E^{(3)}_{k_2}) \\ \Pi(E^{(3)}_{k_3}) \\ \Pi(E^{(3)}_{k_4}) \\ \Pi(E^{(3)}_{k_5}) \end{bmatrix} = \begin{bmatrix} 1 & r_1 & s_2 & 1 & s_3 & 1 \\ s_1 & 1 & 1 & r_2 & s_3 & 1 \\ 1 & r_1 & s_2 & 1 & 1 & r_3 \\ s_1 & 1 & 1 & r_2 & 1 & r_3 \\ 1 & r_1 & 1 & r_2 & 1 & r_3 \end{bmatrix} \Box^{\min}_{\max} \begin{bmatrix} \lambda_1 \\ \rho_1 \\ \lambda_2 \\ \rho_2 \\ \lambda_3 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}$$

- Second set of possibilistic rules:

R'^1: if a person is a professor or a researcher, then her salary is rather low
R'^2: if a person is an engineer, a lawyer or an architect, her salary is average or high
R'^3: if a person is a business man or a doctor, then her salary is high

where D_{salary}={low, average, high}, $s'_1 = 1$, $r'_1 = 0.7$, $s'_2 = 0.8$, $r'_2 = 0.2$, $s'_3 = 0.6$ and $r'_3 = 0.4$

- Partition of D_{salary}: $E'^{(3)}_{k_1} = \{\text{high}\}$, $E'^{(3)}_{k_2} = \{\text{average}\}$ and $E'^{(3)}_{k_3} = \{\text{low}\}$

- $$I'_m = \nabla \Box^{max}_{min} \mathcal{O}_3 = \begin{bmatrix} \nabla_{Q'_1} \\ \nabla_{\overline{Q'_1}} \\ \nabla_{Q'_2} \\ \nabla_{\overline{Q'_2}} \\ \nabla_{Q'_3} \\ \nabla_{\overline{Q'_3}} \end{bmatrix} \Box^{max}_{min} \mathcal{O}_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \Box^{max}_{min} \begin{bmatrix} 0.2 \\ 1 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.6 \\ 0.6 \\ 1 \\ 0.6 \\ 1 \end{bmatrix}$$

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- Equation system:

$$\begin{bmatrix} \Pi(E'^{(3)}_{k_1}) \\ \Pi(E'^{(3)}_{k_2}) \\ \Pi(E'^{(3)}_{k_3}) \end{bmatrix} = \begin{bmatrix} 1 & r'_1 & s'_2 & 1 & s'_3 & 1 \\ 1 & r'_1 & s'_2 & 1 & 1 & r'_3 \\ s'_1 & 1 & 1 & r'_2 & 1 & r'_3 \end{bmatrix} \Box^{\min}_{\max} \begin{bmatrix} \lambda'_1 \\ \rho'_1 \\ \lambda'_2 \\ \rho'_2 \\ \lambda'_3 \\ \rho'_3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.7 \\ 1 \end{bmatrix}$$

- (Reminder) The output vector O'_m is deduced from the first system:

$$\mathcal{O}'_m = \mathcal{M}'_m \Box^{\min}_{\max} (\nabla \Box^{\max}_{\min} (\mathcal{M}_n \Box^{\min}_{\max} I_n))$$

- With the matrices $I_n{}^\circ$, $\mathcal{M}_n{}^\circ$ and $\mathcal{M}'_m{}^\circ$, we can express the equations involved in the cascade using only the operator $(A \Box^{\max}_{\min} B)^\circ$:

$$\begin{aligned} \mathcal{O}_n &= (\mathcal{M}_n{}^\circ \Box^{\max}_{\min} I_n{}^\circ)^\circ \\ I'_m{}^\circ &= (\nabla \Box^{\max}_{\min} \mathcal{O}_n)^\circ \\ \mathcal{O}'_m &= (\mathcal{M}'_m{}^\circ \Box^{\max}_{\min} I'_m{}^\circ)^\circ \end{aligned}$$

- We have:

$$\mathcal{O}'_m = (\mathcal{M}'_m{}^\circ \Box^{\max}_{\min} (\nabla \Box^{\max}_{\min} (\mathcal{M}_n{}^\circ \Box^{\max}_{\min} I_n{}^\circ)^\circ)^\circ)^\circ$$

- The cascade construction is represented by a min-max neural network:

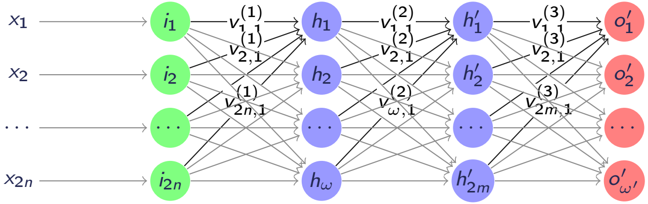

Figure 6: Min-max neural network architecture

- For a neuron x linked by t edges $v_1, v_2, \cdots, v_t$ to t ancestors, whose output values are $y_1, y_2, \cdots, y_t$ we compute:
  - its input value: $1 - \max_{1 \le j \le t} \min(v_j, y_j) = \min_{1 \le j \le t} \max(1 - v_j, 1 - y_j)$
  - its output value: $f(x) = x$

In my thesis:

- We study the *existence of a minimal input vector* for $\pi^*_{b(x)}(u) = 1$
- We define an algorithm to rebuild the equation system *when we remove a rule*. It outputs the equation system associated to the remaining subset of rules.
  Therefore, it enables us to obtain all the equation subsystems of an initial equation system

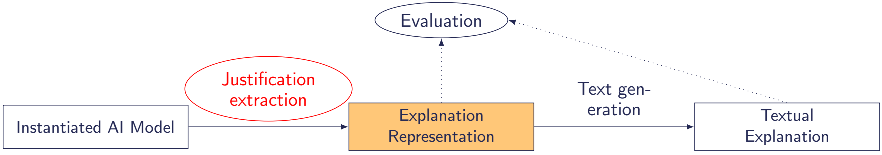

Figure 7: Proposed XAI processing chain (Baaj et al. 2019)

- Representing information given by $\pi(p), n(p)$ of a rule premise p:

Notation

For a premise p, the triplet (p, sem, d) denotes either $(p, P, \pi(p))$ or $(p, C, n(p))$, where sem $\in \{P, C\}$ (P for possible, C for certain) is the semantics attached to the degree $d \in \{\pi(p), n(p)\}$

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- Possibility degree of an output attribute value u computed by:

$$\pi^*_{b(x)}(u) = \min(\gamma_1, \gamma_2, \dots, \gamma_n), \tag{1}$$

$$\text{where } \gamma_i = \pi^{*i}_{b(x)}(u) = \max(t_i, \theta_i) \text{ with } (t_i, \theta_i) = \begin{cases} (s_i, \lambda_i) & \text{if } \gamma_i = \alpha_i \\ (r_i, \rho_i) & \text{if } \gamma_i = \beta_i \end{cases}$$

- Triplets according to the $\gamma_1, \gamma_2, \dots, \gamma_n$ appearing in the relation (1):

$$(p_i, sem_i, d_i) = \begin{cases} (p_i, \mathrm{P}, \lambda_i) & \text{if} \quad \gamma_i = \alpha_i \\ (p_i, \mathrm{C}, 1 - \rho_i) & \text{if} \quad \gamma_i = \beta_i \end{cases} \tag{2}$$

- Blood sugar control system for a patient with type 1 diabetes:

	activity (act)	current-bloodsugar (cbs)	future-bloodsugar (fbs)
R^1	dinner, drink-coffee, lunch	medium, high	high
R^2	long-sleep, sport, walking	low, medium	low
R^3	alcohol-consumption, breakfast	low, medium	low, medium

Table 1: Rule base for the control of the blood sugar level

For each output attribute value:

- $\pi^*_{\mathrm{fbs}(x)}(\mathrm{low}) = \min(\gamma_1^l, \gamma_2^l, \gamma_3^l)$, triplets: $(p_1, \mathrm{C}, 1-\rho_1), (p_2, \mathrm{P}, \lambda_2), (p_3, \mathrm{P}, \lambda_3)$,
  where $\gamma_{\mathbf{1}}^l = \beta_{\mathbf{1}} = \max(r_{\mathbf{1}}, \rho_{\mathbf{1}})$, $\gamma_{\mathbf{2}}^l = \alpha_{\mathbf{2}} = \max(s_{\mathbf{2}}, \lambda_{\mathbf{2}})$, $\gamma_{\mathbf{3}}^l = \alpha_{\mathbf{3}} = \max(s_{\mathbf{3}}, \lambda_{\mathbf{3}})$

- $\pi^*_{\mathrm{fbs}(x)}(\mathrm{medium}) = \min(\gamma_1^m, \gamma_2^m, \gamma_3^m)$ / $(p_1, \mathrm{C}1-\rho_1), (p_2, \mathrm{C}, 1-\rho_2), (p_3, \mathrm{P}, \lambda_3)$,
  where $\gamma_{\mathbf{1}}^m = \beta_{\mathbf{1}}$, $\gamma_{\mathbf{2}}^m = \beta_{\mathbf{2}} = \max(r_{\mathbf{2}}, \rho_{\mathbf{2}})$, $\gamma_{\mathbf{3}}^m = \alpha_{\mathbf{3}}$

- $\pi^*_{\mathrm{fbs}(x)}(\mathrm{high}) = \min(\gamma_1^h, \gamma_2^h, \gamma_3^h)$ / $(p_1, \mathrm{P}, \lambda_1), (p_2, \mathrm{C}, 1-\rho_2), (p_3, \mathrm{C}, 1-\rho_3)$, where
  $\gamma_{\mathbf{1}}^h = \alpha_{\mathbf{1}} = \max(s_{\mathbf{1}}, \lambda_{\mathbf{1}})$, $\gamma_{\mathbf{2}}^h = \beta_{\mathbf{2}}$, $\gamma_{\mathbf{3}}^h = \beta_{\mathbf{3}} = \max(r_{\mathbf{3}}, \rho_{\mathbf{3}})$

Farreny and Prade's approach focuses on two explanatory purposes for an output attribute value $u \in D_b$:

(i) How to get $\pi^*_{b(x)}(u)$ strictly greater or lower than a given $\tau \in [0, 1]$?

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Reminder: the parameters of the rules are set:
$\pi^*_{b(x)}(u)$ ranges between $\omega = \min(t_1, t_2, \dots, t_n)$ and 1

- for $\pi^*_{b(x)}(u) > \tau$: $\forall i \in \{j \in \{1, 2, \dots, n\} \mid t_j \leq \tau\}$ we have $\theta_i > \tau$
- for $\pi^*_{b(x)}(u) < \tau$ with $\omega < \tau \leq 1$: $\exists i \in \{j \in \{1, 2, \dots, n\} \mid t_j < \tau\}$ s.t. $\theta_i < \tau$

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Example

For our blood sugar control system with $s_1 = 1, r_2 = r_3 = 0$, we have $\pi^*_{fbs(x)}(high) > 0.5$ iff $\rho_2 > 0.5$ and $\rho_3 > 0.5$

Justifying inference results

Farreny and Prade's approach focuses on two explanatory purposes for an output attribute value $u \in D_b$:

(i) How to get $\pi^*_{b(x)}(u)$ strictly greater or lower than a given $\tau \in [0, 1]$?
Reminder: the parameters of the rules are set:
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- for $\pi^*_{b(x)}(u) > \tau$: $\forall i \in \{j \in \{1, 2, \ldots, n\} \mid t_j \leq \tau\}$ we have $\theta_i > \tau$
- for $\pi^*_{b(x)}(u) < \tau$ with $\omega < \tau \leq 1$: $\exists i \in \{j \in \{1, 2, \ldots, n\} \mid t_j < \tau\}$ s.t. $\theta_i < \tau$

(ii) What are the degrees of the premises justifying $\pi^*_{b(x)}(u) = \tau$?

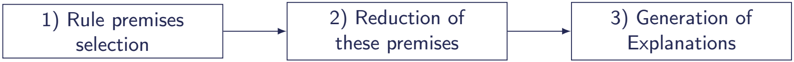

Figure 8: Process for extracting a justification of $\pi^*_{b(x)}(u) = \tau$

- Let two sets compare the parameters $t_1, t_2, \cdots t_n$ of the rules to the degrees $\theta_1, \theta_2, \cdots, \theta_n$:

$$J^P = \{i \in \{1, 2, \cdots, n\} \mid t_i \leq \theta_i\} \text{ and } J^R = \{i \in \{1, 2, \cdots, n\} \mid t_i \geq \theta_i\}$$

We have $\{1, 2, \cdots, n\} = J^P \cup J^R$ but J^P or J^R may be <u>empty</u>

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We have $\{1, 2, \cdots, n\} = J^P \cup J^R$ but J^P or J^R may be empty

- With the convention $\min_\emptyset = 1$, we take:

$c_\theta = \min_{i \in J^P} \theta_i$: the *lowest possibility degree justifiable by premises* (if $J^P \neq \emptyset$)

$c_t = \min_{i \in J^R} t_i$: the *lowest possibility degree justifiable by rule parameters* (if $J^R \neq \emptyset$)

- Let two sets compare the parameters $t_1, t_2, \cdots t_n$ of the rules to the degrees $\theta_1, \theta_2, \cdots, \theta_n$:

$$J^P = \{i \in \{1,2,\cdots,n\} \mid t_i \leq \theta_i\} \text{ and } J^R = \{i \in \{1,2,\cdots,n\} \mid t_i \geq \theta_i\}$$

We have $\{1,2,\cdots,n\} = J^P \cup J^R$ but J^P or J^R may be empty

- With the convention $\min_\emptyset = 1$, we take:

$c_\theta = \min_{i \in J^P} \theta_i$: the *lowest possibility degree justifiable by premises* (if $J^P \neq \emptyset$)

$c_t = \min_{i \in J^R} t_i$: the *lowest possibility degree justifiable by rule parameters* (if $J^R \neq \emptyset$)

- By using the properties of the min function, we establish:

Proposition

$$\tau = \min(c_\theta, c_t)$$

- *When we <u>can't explain by degrees of premises</u>*: <u>if $J^P = \emptyset$</u>

In that case, $c_\theta = 1$, $J^R = \{1, 2, \cdots, n\}$ and:

$$\pi^*_{b(x)}(u) = c_t = \min(t_1, t_2, \cdots, t_n)$$

- Blood sugar control system for a patient with type 1 diabetes:

	activity (act)	current-bloodsugar (cbs)	future-bloodsugar (fbs)
R^1	dinner, drink-coffee, lunch	medium, high	high
R^2	long-sleep, sport, walking	low, medium	low
R^3	alcohol-consumption, breakfast	low, medium	low, medium

Table 2: Rule base for the control of the blood sugar level

Inputs: the patient wants to drink a coffee and his current blood sugar level is medium:

$$\pi_{act(x)}(\textit{drink-coffee}) = 1, \quad \pi_{cbs(x)}(\textit{medium}) = 1 \quad \text{and} \quad \pi_{cbs(x)}(\textit{low}) = 0.3$$

$s_1 = s_3 = 1$, $s_2 = 0.7$, $r_1 = r_2 = r_3 = 0$. and $\lambda_1 = 1$, $\rho_1 = 0.3$, $\lambda_2 = \lambda_3 = 0$, $\rho_2 = \rho_3 = 1$.

- Blood sugar control system for a patient with type 1 diabetes:

	activity (act)	current-bloodsugar (cbs)	future-bloodsugar (fbs)
R^1	dinner, drink-coffee, lunch	medium, high	high
R^2	long-sleep, sport, walking	low, medium	low
R^3	alcohol-consumption, breakfast	low, medium	low, medium

Table 2: Rule base for the control of the blood sugar level

Inputs: the patient wants to drink a coffee and his current blood sugar level is medium:

$$\pi_{act(x)}(drink\text{-}coffee) = 1, \quad \pi_{cbs(x)}(medium) = 1 \quad \text{and} \quad \pi_{cbs(x)}(low) = 0.3$$

$s_1 = s_3 = 1, s_2 = 0.7, r_1 = r_2 = r_3 = 0.$ and $\lambda_1 = 1,\ \rho_1 = 0.3,\ \lambda_2 = \lambda_3 = 0,\ \rho_2 = \rho_3 = 1.$

We form the following sets for each output attribute value and deduce their respective c_θ, c_t:

- for *low*: $J_l^P = \{1\}$ and $J_l^R = \{2, 3\}$: $c_\theta^l = 0.3 = \pi^*_{fbs(x)}(\text{low})$ and $c_t^l = 0.7$
- for *medium*: $J_m^P = \{1, 2\}$ and $J_m^R = \{3\}$: $c_\theta^m = 0.3 = \pi^*_{fbs(x)}(\text{medium})$ and $c_t^m = 1$
- for *high*: $J_h^P = \{1, 2, 3\}$ and $J_h^R = \{1\}$: $c_\theta^h = c_t^h = 1 = \pi^*_{fbs(x)}(\text{high})$

To explain the inference results of our possibilistic rule-based system, we introduce a threshold $\eta > 0$:

Definition

If a possibility (resp. necessity) degree is higher than the threshold η, it intuitively means that the information it models is relevantly possible (resp. certain)

Extraction of premises justifying $\pi^*_{b(x)}(u) = \tau$

To explain the inference results of our possibilistic rule-based system, we introduce a threshold $\eta > 0$:

Definition

If a possibility (resp. necessity) degree is higher than the threshold η, it intuitively means that the information it models is relevantly possible (resp. certain)

For a given output value $u \in D_b$, we extract the rule premises justifying the possibility degree $\pi^*_{b(x)}(u) = \tau$ by the following formula:

$$J_{b(x)}(u) = \begin{cases} \{(p_i, \text{sem}_i, d_i) \mid i \in J^P \text{ and } \theta_i = \tau\} & \text{if } \tau \geq \eta \\ \{(p_i, \text{sem}_i, d_i) \mid i \in \{1, 2, \ldots, n\} \text{ and } \gamma_i < \eta\} & \text{if } \tau < \eta \end{cases}$$

Notes: if $\tau \geq \eta$, we rely on J^P (which may be empty) and <u>the condition $\tau = c_\theta$</u>
Otherwise, if $\tau < \eta$ it <u>always exists</u> at least a premise justifying $\pi^*_{b(x)}(u)$

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- Blood sugar control system for a patient with type 1 diabetes:

	activity (act)	current-bloodsugar (cbs)	future-bloodsugar (fbs)
R^1	dinner, drink-coffee, lunch	medium, high	high
R^2	long-sleep, sport, walking	low, medium	low
R^3	alcohol-consumption, breakfast	low, medium	low, medium

Table 3: Rule base for the control of the blood sugar level

Let us take $\eta = 0.1$. We obtain for each output attribute value:

- $J^*_{\mathrm{fbs(x)}}(\mathrm{low}) = J^*_{\mathrm{fbs(x)}}(\mathrm{medium}) = \{(p_1, \mathrm{C}, 0.7)\}$
- $J^*_{\mathrm{fbs(x)}}(\mathrm{high}) = \{(p_1, \mathrm{P}, 1), (p_2, \mathrm{C}, 0), (p_3, \mathrm{C}, 0)\}$

If instead of $r_1 = 0$, we take $r_1 > 0.3$, then for $u = low$, the corresponding set J^P is empty: no justification in terms of premises could be given in that case

Purpose : for an output attribute value $u \in D_b$, apply reduction functions $(\mathscr{R}_\pi, \mathscr{R}_n, \mathscr{C}_\pi, \mathscr{C}_n)$ to the selected premises in $J_{b(x)}(u)$ in order to **form explanations of** $\pi^*_{b(x)}(u)$:

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- Using $\mathscr{R}_\pi, \mathscr{R}_n$: **the justification of** $\pi^*_{b(x)}(u)$: *A set of possibilistic expressions that are sufficient to justify "b(x) is u at* $\pi^*_{b(x)}(u)$*"*

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- Using $\mathscr{C}_\pi, \mathscr{C}_n$: **the unexpectedness of** $\pi^*_{b(x)}(u)$: *A set of possible or certain possibilistic expressions, which may appear to be incompatible with* $\pi^*_{b(x)}(u)$ *while not being involved in its determination*
  - In Simplicity Theory (Dessalles 2008), the unexpectedness aims at capturing exactly what people consider *surprising* in a given situation
  - An unexpectedness X let us formulate statements such as:

    "*even if X, b(x) is u at* $\pi^*_{b(x)}(u)$"
  - It is in the same vein as the "even-if-because" of (Darwiche 2020)

Proposition reduction functions

Preliminaries : definition of two proposition reduction functions $\mathscr{P}_\pi$, $\mathscr{P}_n$

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- Let a be an attribute with a normalized possibility distribution $\pi_{a(x)}$ on D_a and a proposition p of the form "$a(x) \in P$", where $P \subseteq D_a$.
- We introduce the following two subsets of D_a:

$$(P)_\pi = \{v \in P \mid \pi(v) = \Pi(P)\} \text{ related to a proposition } p_\pi,$$
$$(P)_n = P \cup \{v \in \overline{P} \mid 1 - \pi(v) > N(P)\} \text{ related to } p_n$$

We have $\overline{(P)_n} = \{v \in \overline{P} \mid 1 - \pi(v) = N(P)\}$, $\overline{(P)_n} = (\overline{P})_\pi$, and $\overline{(P)_\pi} = (\overline{P})_n$. Therefore:

$$(P)_n = \overline{(\overline{P})_\pi} \text{ and } (P)_\pi = \overline{(\overline{P})_n}$$

Example

Let us take the possibility distribution π on $D_{cbs} = \{low, medium, high\}$ defined by:

$$\pi(low) = 0.3, \quad \pi(medium) = 1, \quad \pi(high) = 0.$$

Given $P = \{medium, high\}$, we have $(P)_\pi = \{medium\}$ and $(P)_n = P$.
For $P' = \{low\}$, we have $(P')_\pi = P'$ and $(P')_n = \{low, high\}$

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Preliminaries : definition of two proposition reduction functions $\mathscr{P}_\pi$, $\mathscr{P}_n$

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$$(P)_n = \overline{(\overline{P})_\pi} \text{ and } (P)_\pi = \overline{(\overline{P})_n}$$

- Definitions: $\mathscr{P}_\pi$ reduces P if $\pi(p) \geq \eta$ and $\mathscr{P}_n$ reduces $\overline{P}$ if $n(p) \geq \eta$:

$$\mathscr{P}_\pi(p) = \begin{cases} p_\pi & \text{if } \pi(p) \geq \eta \\ p & \text{if } \pi(p) < \eta \end{cases} \text{ and } \mathscr{P}_n(p) = \begin{cases} p_n & \text{if } n(p) \geq \eta \\ p & \text{if } n(p) < \eta \end{cases}$$

$\pi(\mathscr{P}_\pi(p)) = \pi(p)$ and $n(\mathscr{P}_n(p)) = n(p)$

- Let $p = p_1 \wedge p_2 \wedge \cdots \wedge p_k$ be a compounded premise, where p_j for $j = 1, 2, \cdots, k$, is a proposition of the form "$a_j(x) \in P_j$" with $P_j \subseteq D_{a_j}$

- $\mathscr{R}_\pi$ returns the structure responsible for $\pi(p)$:

$$\mathscr{R}_\pi(p) = \begin{cases} \bigwedge_{j=1}^{k} \mathscr{P}_\pi(p_j) & \text{if } \pi(p) \geq \eta \\ \bigwedge_{p_j \in \{p_s \mid \pi(p_s) < \eta \text{ for } s=1,\cdots,k\}} p_j & \text{if } \pi(p) < \eta \end{cases}$$

- $\mathscr{R}_n$ returns the structure responsible for $n(p)$:

$$\mathscr{R}_n(p) = \begin{cases} \bigwedge_{j=1}^{k} \mathscr{P}_n(p_j) & \text{if } n(p) \geq \eta \\ \bigwedge_{p_j \in \{p_s \mid n(p_s) < \eta \text{ for } s=1,\cdots,k\}} p_j & \text{if } n(p) < \eta \end{cases}$$

$\pi(\mathscr{R}_\pi(p)) = \pi(p)$ and $n(\mathscr{R}_n(p)) = n(p)$

- Let $p = p_1 \wedge p_2 \wedge \cdots \wedge p_k$ be a compounded premise, where p_j for $j = 1, 2, \cdots, k$, is a proposition of the form "$a_j(x) \in P_j$" with $P_j \subseteq D_{a_j}$

- When $\pi(p) < \eta$ and $A_p^\pi = \{p_j \mid \pi(p_j) \geq \eta \text{ for } j = 1, \cdots, k\} \neq \emptyset$, $\mathscr{C}_\pi$ returns a conjunction of propositions, which is *not* involved in the determination of $\pi(p)$:

$$\mathscr{C}_\pi(p) = \bigwedge_{p_j \in A_p^\pi} \mathscr{P}_\pi(p_j)$$

- When $n(p) < \eta$ and $A_p^n = \{p_j \mid n(p_j) \geq \eta \text{ for } j = 1, \cdots, k\} \neq \emptyset$, $\mathscr{C}_n$ returns a conjunction of propositions, which is *not* involved in the determination of $n(p)$:

$$\mathscr{C}_n(p) = \bigwedge_{p_j \in A_p^n} \mathscr{P}_n(p_j)$$

- If $\pi(p) < \eta$, (resp. $n(p) < \eta$), each proposition p_j composing p, is either used in $\mathscr{R}_\pi(p)$ or in $\mathscr{C}_\pi(p)$ (resp. $\mathscr{R}_n(p)$ or in $\mathscr{C}_n(p)$), according to $\pi(p_j)$ (resp. $n(p_j)$)

- Blood sugar control system for a patient with type 1 diabetes:

	activity (act)	current-bloodsugar (cbs)	future-bloodsugar (fbs)
R^1	dinner, drink-coffee, lunch	medium, high	high
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R^3	alcohol-consumption, breakfast	low, medium	low, medium

Table 4: Rule base for the control of the blood sugar level

Let us take $\eta = 0.1$

- For the proposition "act(x) $\in$ {dinner, drink-coffee, lunch}" of p_1 of R^1:

  $\mathscr{P}_\pi$ reduces it to "act(x) $\in$ {drink-coffee}"

  $\mathscr{P}_n$ keeps it as is

- For the premise p_2:"act(x) $\in$ {long-sleep, sport, walking} and cbs(x) $\in$ {low, medium}" of R^2:

  $\mathscr{R}_\pi$ returns the proposition "act(x) $\in$ {long-sleep, sport, walking}"

  $\mathscr{C}_\pi$ returns "cbs(x) $\in$ {medium}"

Reminder: $J_{b(x)}(u)$ is composed of triplets (p, sem, d)

- To apply the reduction functions $\mathscr{R}_\pi$ and $\mathscr{R}_n$ to the premise p of a triplet (p, sem, d), we introduce the function $\mathscr{S}_\mathscr{R}$:

$$\mathscr{S}_\mathscr{R}(p, sem, d) = \begin{cases} (\mathscr{R}_\pi(p), sem, d) \text{ if } sem = \mathrm{P} \\ (\mathscr{R}_n(p), sem, d) \text{ if } sem = \mathrm{C} \end{cases}$$

- Similarly, to apply $\mathscr{C}_\pi$ and $\mathscr{C}_n$, we introduce the function $\mathscr{S}_\mathscr{C}$:

$$\mathscr{S}_\mathscr{C}(p, sem, d) = \begin{cases} (\mathscr{C}_\pi(p), sem, \pi(\mathscr{C}_\pi(p))) \text{ if } sem = \mathrm{P}, d < \eta \text{ and } A_p^\pi \neq \emptyset \\ (\mathscr{C}_n(p), sem, n(\mathscr{C}_n(p))) \text{ if } sem = \mathrm{C}, d < \eta \text{ and } A_p^n \neq \emptyset \end{cases}$$

- **The justification of** $\pi^*_{b(x)}(u)$: A set of possibilistic expressions that are sufficient to justify "b(x) is u at $\pi^*_{b(x)}(u)$":

$$\text{Justification}_{b(x)}(u) = \{\mathscr{S}_{\mathscr{R}}(p, sem, d) \mid (p, sem, d) \in J_{b(x)}(u)\}$$

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- **The justification of** $\pi^*_{b(x)}(u)$: A set of possibilistic expressions that are sufficient to justify "b(x) is u at $\pi^*_{b(x)}(u)$":

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- **The unexpectedness of** $\pi^*_{b(x)}(u)$: a set of possible or certain possibilistic expressions, which may appear to be incompatible with $\pi^*_{b(x)}(u)$ while not being involved in its determination:

$$\text{Unexpectedness}_{b(x)}(u) = \{\mathscr{S}_{\mathscr{C}}(p, sem, d) \mid (p, sem, d) \in J_{b(x)}(u)\}$$

- Purpose: represent graphically explanations of possibilistic inference decisions by **conceptual graphs** (Chein & Mugnier 2008)

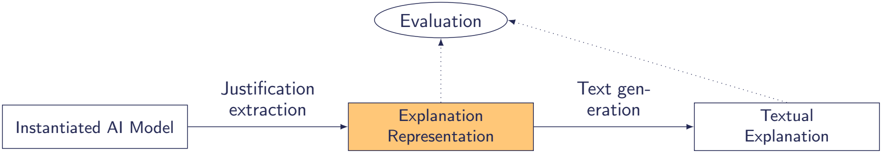

Figure 9: Proposed XAI processing chain (Baaj et al. 2019)

- Representation of a *possibilistic expression* of an explanation (justification, unexpectedness) by a possibilistic conceptual graph :

Definition

A possibilistic conceptual graph (PCG) is a basic conceptual graph (BG) $G = (C, R, E, l)$, where C is the concept nodes set, R the relation nodes set, E is the multi-edges set and the label function l is extended by allowing a degree and a semantics in the label of any concept node $c \in C$:

$$l(c) = (\text{type}(c) : \text{marker}(c) | sem_c, d_c),$$

where $sem_c \in \{\text{P}, \text{C}\}$

- The definition of a *star BG* i.e., a BG restricted to a relation node and its neighbors, is naturally extended as a star PCG

From an explanation (justification, unexpectedness), we form an ontology (called: a *vocabulary* in the conceptual graph framework) to build $m+1$ *possibilistic conceptual graphs* (m : number of possibilistic expressions in an explanation) and a *basic conceptual graph*:

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- D : PCG representing the observed phenomenon : $\pi^*_{b(x)}(u)$
- $N_1, \cdots, N_m$: star PCGs representing the m extracted possibilistic expressions of an explanation
- R (*root*) : star BG structuring the explanation by representing the link (*isJustifiedBy* or *evenIf*) between D and $N_1, \cdots, N_m$

Representation of Explanations

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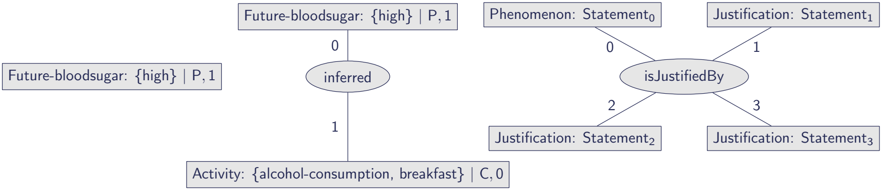

Figure 10: examples of D, N_i and R with $m = 3$

The representation is a **nested conceptual graph** G by its associated tree $Tree(G)$ where the graphs $D, N_1, \cdots, N_m$ are nested in R:

Definition

$Tree(G) = (V_T, U_T, l_T)$ is given by:
- $V_T = \{R, D, N_1, N_2, \cdots, N_m\}$ is the set of nodes,
- $U_T = \{(R, D), (R, N_1), (R, N_2), \cdots, (R, N_m)\}$ is the set of edges and the node R is the root of $Tree(G)$,
- the labels of the edges are given by $l_T(R, D) = (R, c_0, D)$ and $l_T(R, N_i) = (R, c_i, N_i)$ for $i = 1, 2, \cdots, m$.

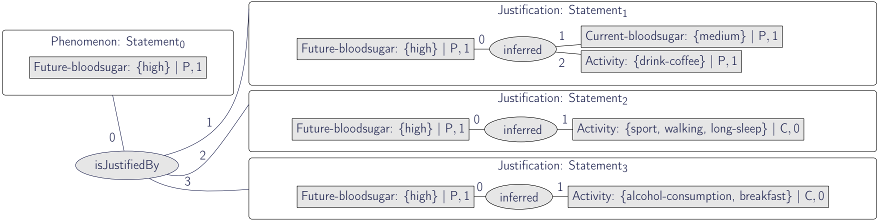

Figure 11: Representation of an explanation

A natural language explanation could be: "*It is possible that the patient's blood sugar level will become high. In fact, his activity is drinking coffee and his current blood sugar level is medium. In addition, it is assessed as not certain that he chose sport, walking, sleeping, eating breakfast or drinking alcohol as an activity.*"

In my thesis, the framework is extended to represent an explanation that is a combination of a justification and an unexpectedness:

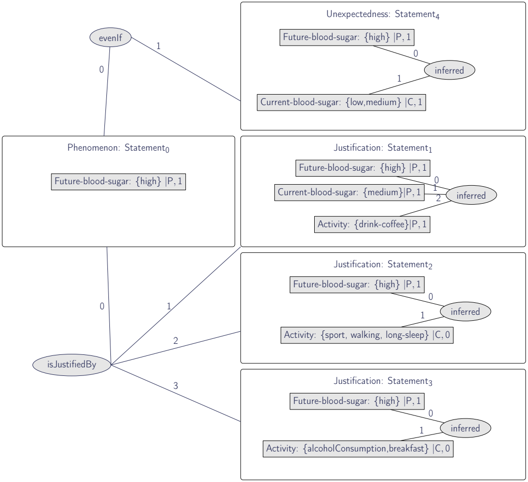

Figure 12: Representation: combination of the justification of $\pi^*_{fbs(x)}(high)$ and its unexpectedness

In my thesis, the same explanatory paradigms are developed for a Mamdani inference system, where fuzzy rules (possibility rules) encode positive information.

We show the total inferred conclusion of a Mamdani inference system satisfies the semantics α^*-possible in the sense of (Dubois-Prade 1998), where α^*: the maximum of the activation degrees of the rules

- Canonical construction for the matrices governing the min-max equation system of Farreny and Prade (1989)
- Formulas for $\pi^*_{b(x)}$ and its possibility and necessity measures
- Representation of the cascade by a min-max neural network

- Canonical construction for the matrices governing the min-max equation system of Farreny and Prade (1989)
- Formulas for $\pi^*_{b(x)}$ and its possibility and necessity measures
- Representation of the cascade by a min-max neural network
- Necessary and sufficient condition for justifying by rule premises the possibility degree $\pi^*_{b(x)}(u)$
- Two explanations of $\pi^*_{b(x)}(u)$: the justification and the unexpectedness (also for Mamdani inference system)
- Representation of explanations of possibilistic/fuzzy inference decisions by nested conceptual graphs, which may be used by natural language generation systems

- Min-max equation system:
  - Sensitivity analysis and verifying the coherence of the rule base (conditions on the parameters of the rules and the input vector)
  - Possibilistic learning: the rule parameters s_i, r_i may be learned with algorithms for solving systems of fuzzy relational equations (Sanchez 1977, Peeva 2013) or by adapting a min-max gradient descent method

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  - Evaluation: Are the explanations suitable for users?

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- Explainability of possibilistic/fuzzy rule-based systems:
  - How to determine the threshold?
  - Evaluation: Are the explanations suitable for users?
- Representation of explanations:
  - Query mechanism and logical semantics of possibilistic/fuzzy conceptual graphs
  - Representation of new explanations (e.g. for a cascade)
  - Text Generation (NLG) of the explanations

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