Optical Flow

Method to estimate apparent motion of scene points from a sequence of images.

Topics:

(1) Motion Field and Optical Flow

(2) Optical Flow Constraint Equation

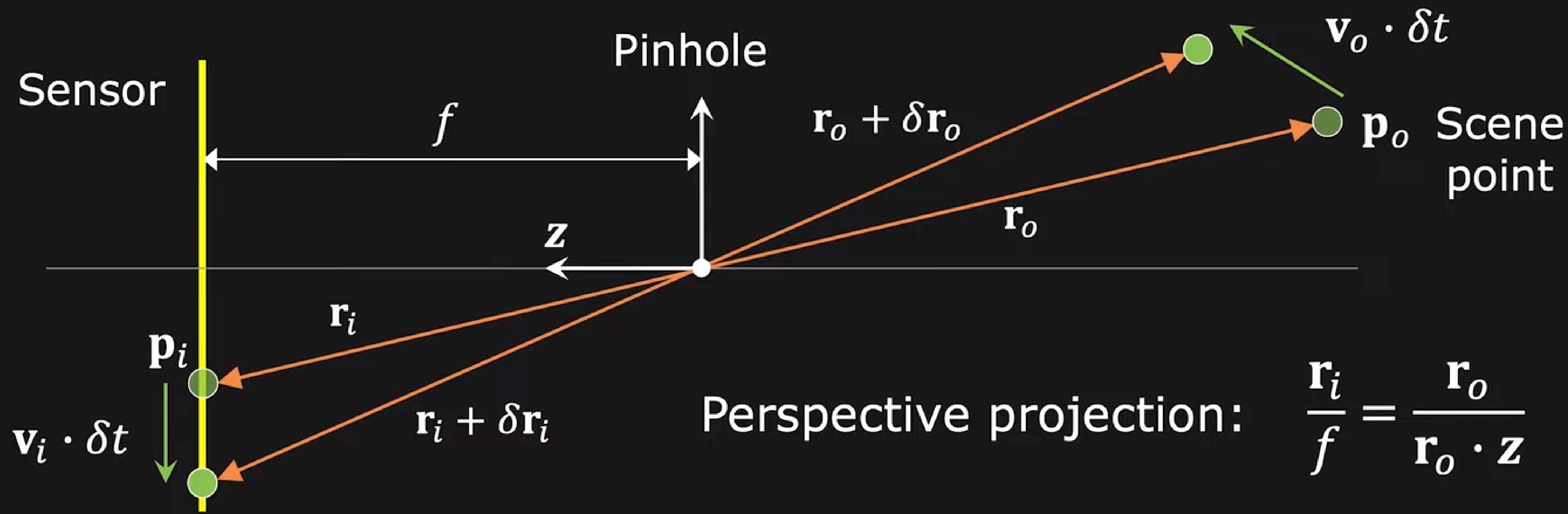
(3) Lucas-Kanade Method

(4) Coarse-to-Fine Flow Estimation

(5) Applications of Optical Flow

Motion Field

Image velocity of a point that is moving in the scene



Perspective projection: $\dfrac{\mathbf{r}_i}{f} = \dfrac{\mathbf{r}_o}{\mathbf{r}_o \cdot \mathbf{z}}$

Image Point Velocity: $\mathbf{v}_i = \dfrac{d\mathbf{r}_i}{dt} = f\dfrac{(\mathbf{r}_o \cdot \mathbf{z})\mathbf{v}_0 - (\mathbf{v}_o \cdot \mathbf{z})\mathbf{r}_0}{(\mathbf{r}_o \cdot \mathbf{z})^2}$

(Motion Field)

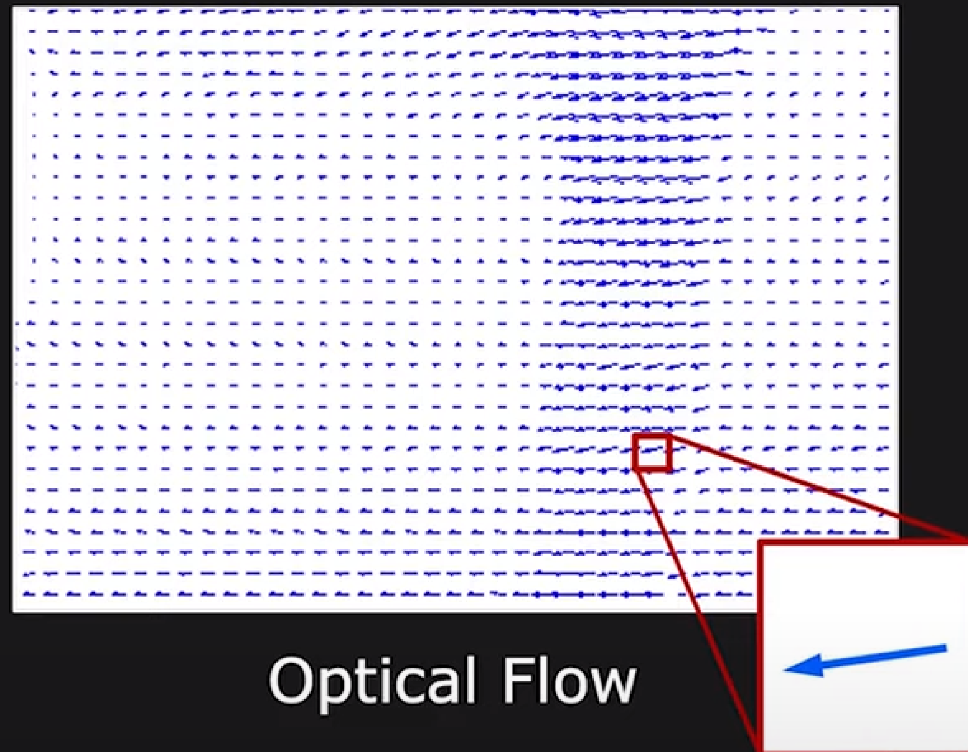
$$\boxed{\mathbf{v}_i = f\frac{(\mathbf{r}_o \times \mathbf{v}_0) \times \mathbf{z}}{(\mathbf{r}_o \cdot \mathbf{z})^2}}$$

[Horn 1981]

Optical Flow

Motion of brightness patterns in the image

Image Sequence
(2 frames)

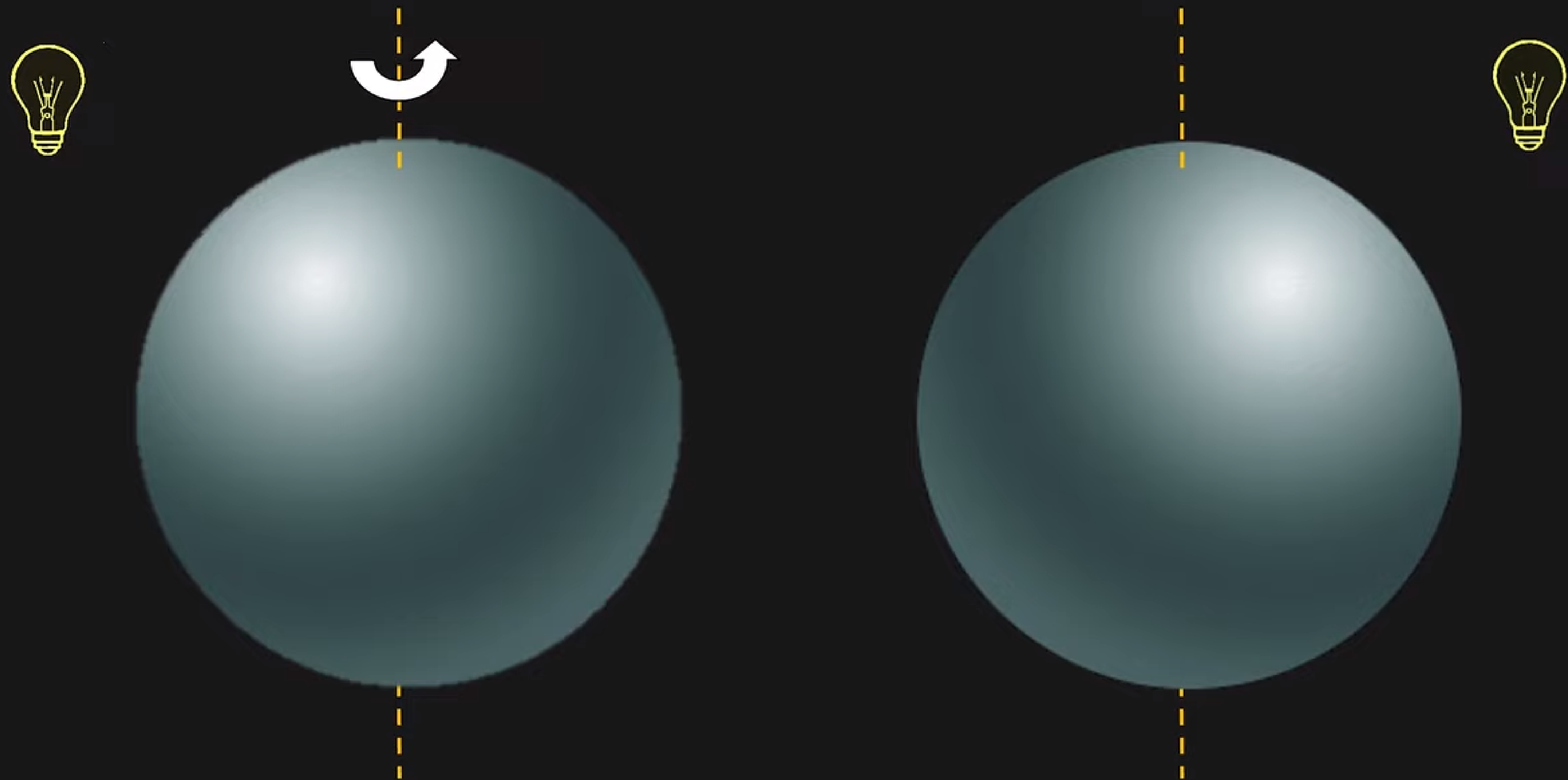


Optical Flow

Velocity of
brightness patt

Ideally, Optical Flow = Motion Field

When is Optical Flow ≠ Motion Field?



Spinning Sphere
Stationary Light Source

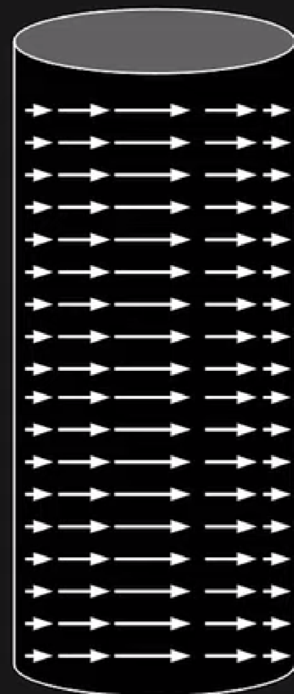
Motion Field exists
But no Optical Flow

Stationary Sphere
Moving Light Source

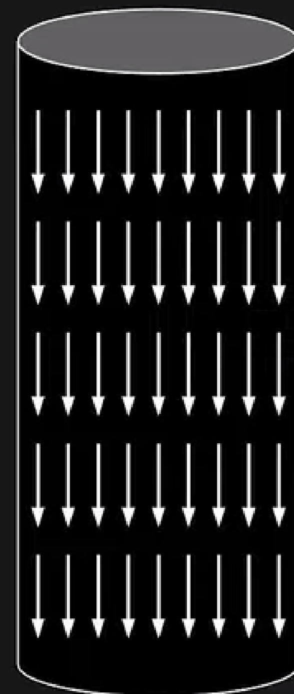
No Motion Field exists
But there is Optical Flow

When is Optical Flow ≠ Motion Field?

Barber Pole Illusion

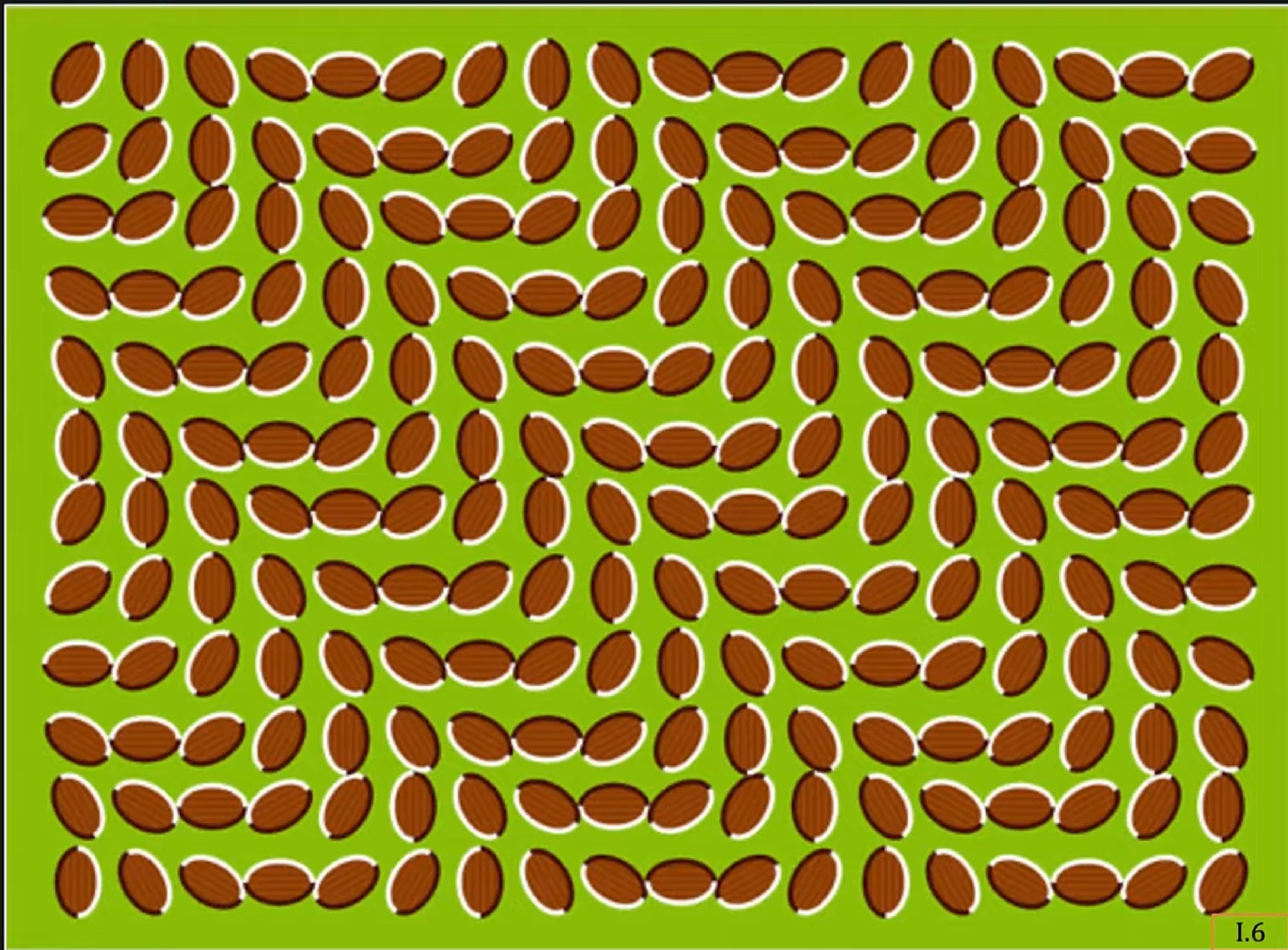


Motion Field



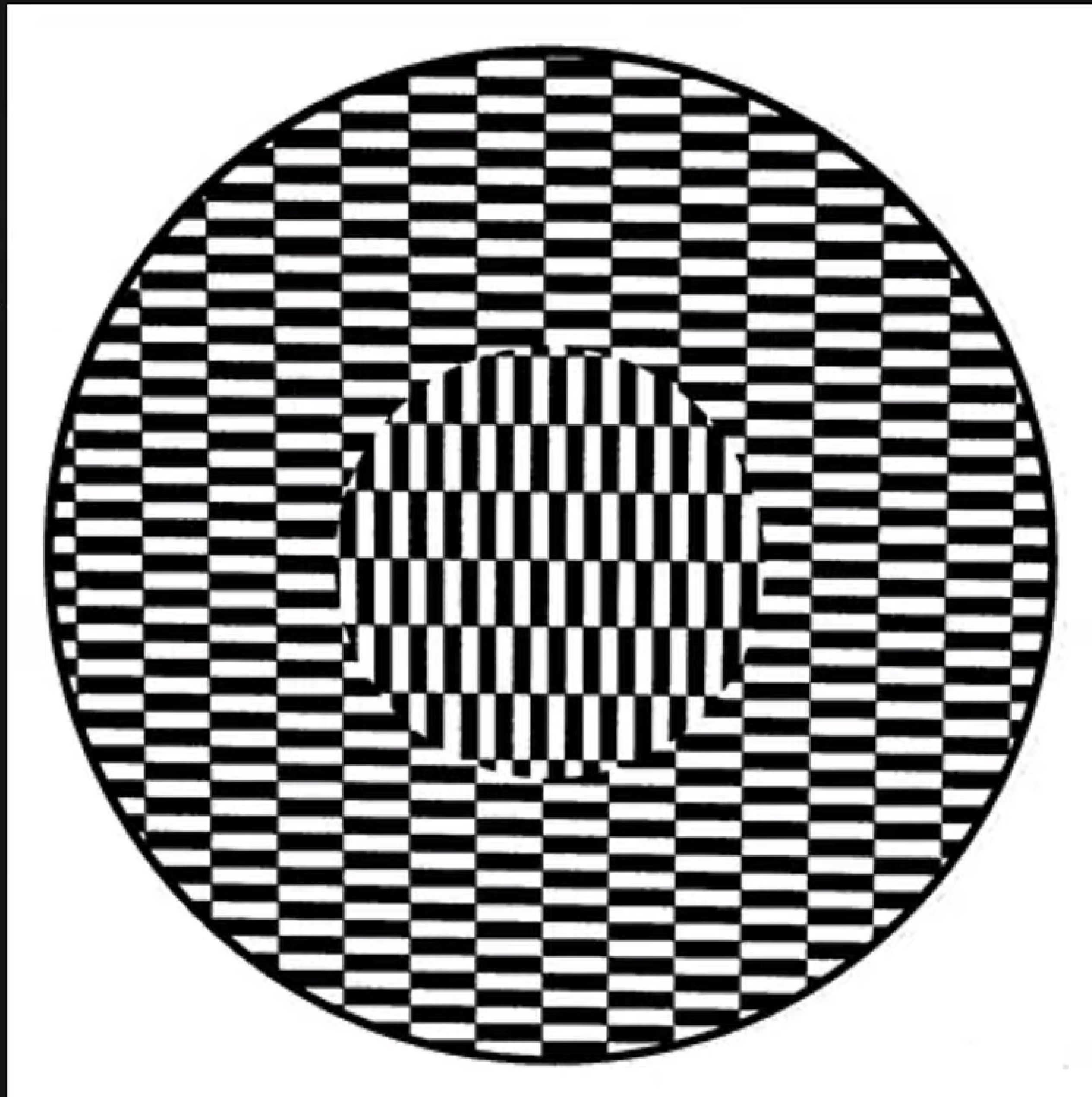
Optical Flow

Motion Illusions



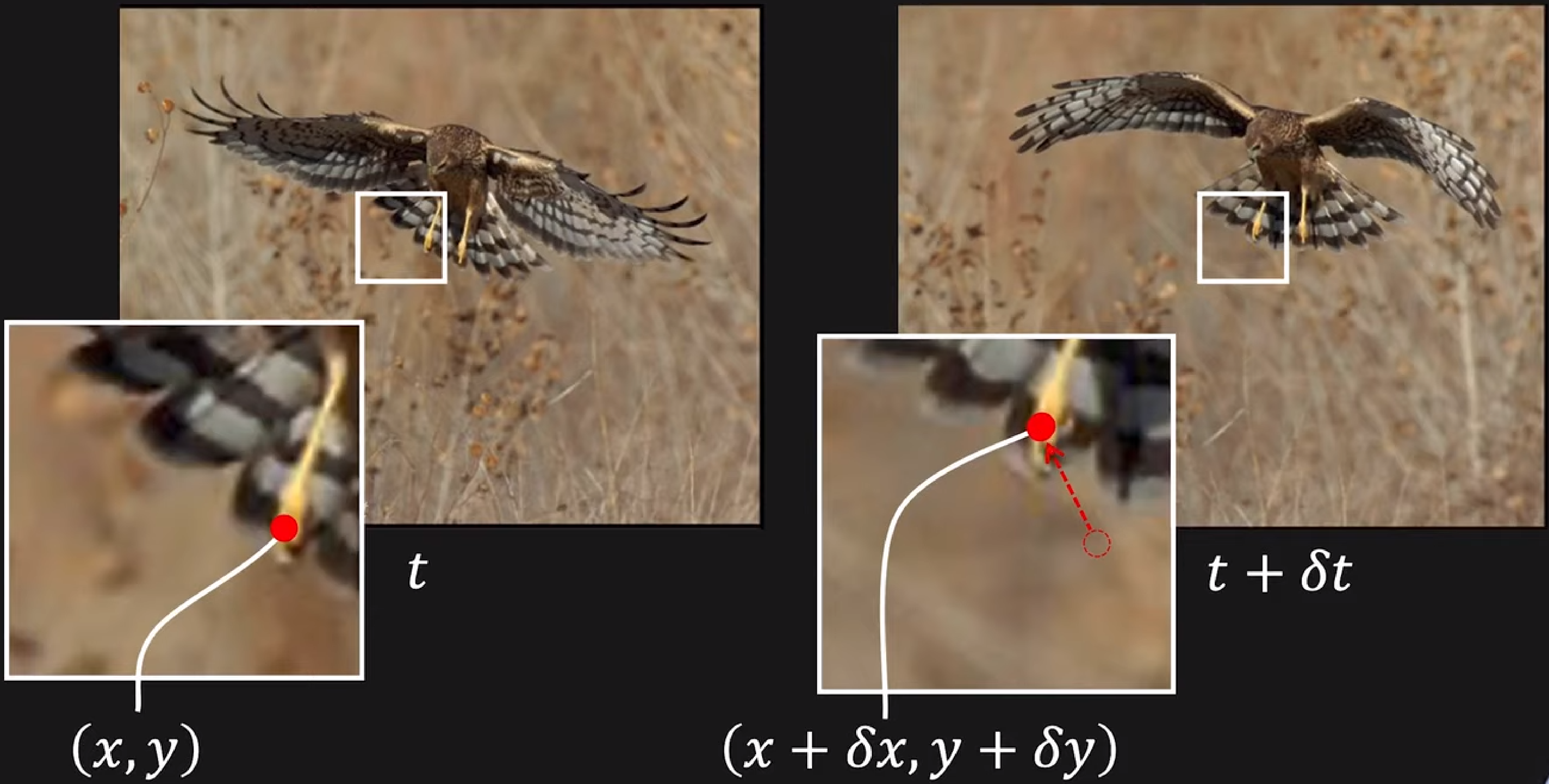
Donguri Wave Illusion

Motion Illusions



Ouchi Pattern

Optical Flow

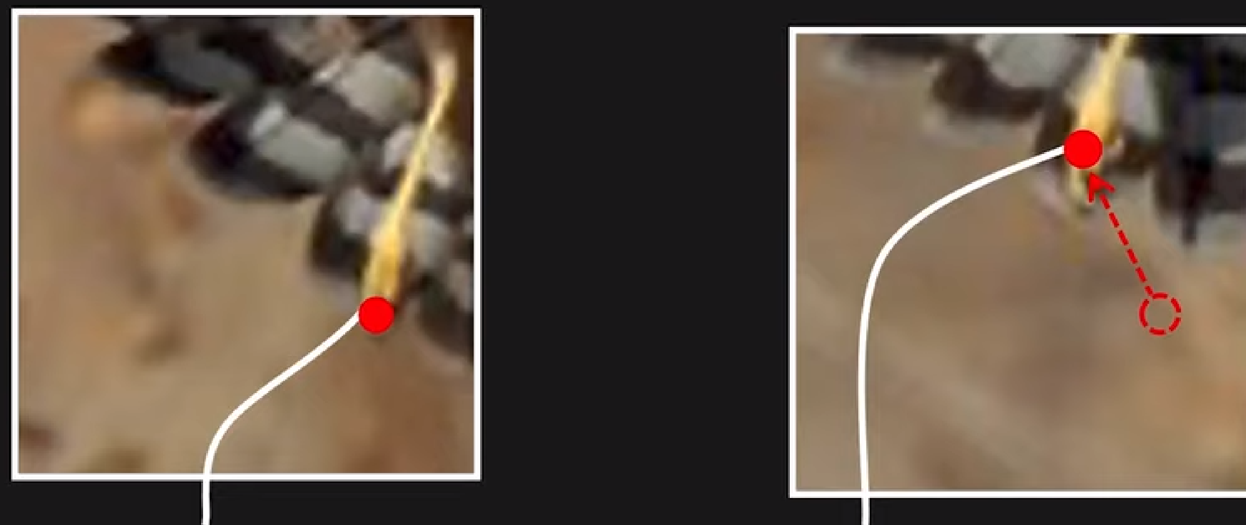


Displacement: $(\delta x, \delta y)$

Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

[Horn 1981]

Optical Flow Constraint Equation



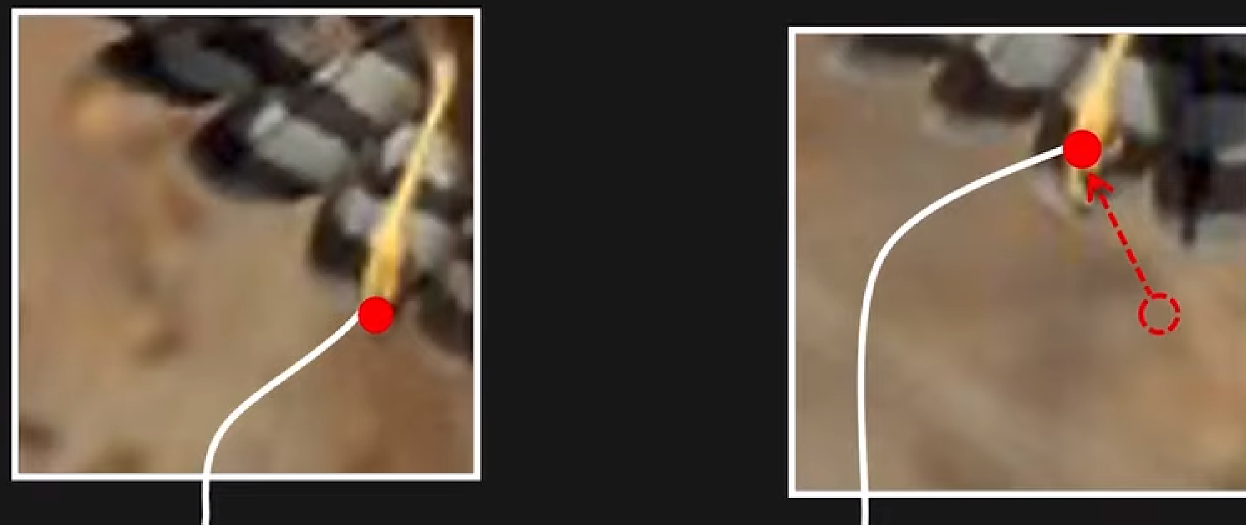
$I(x, y, t)$ $\quad$ $I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #1:

Brightness of image point remains constant over time

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$

Optical Flow Constraint Equation



$I(x, y, t)$ $\quad$ $I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

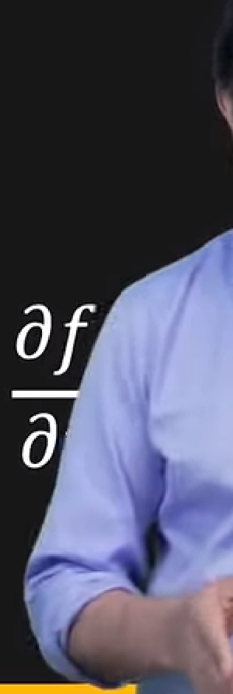
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

If δx is small:

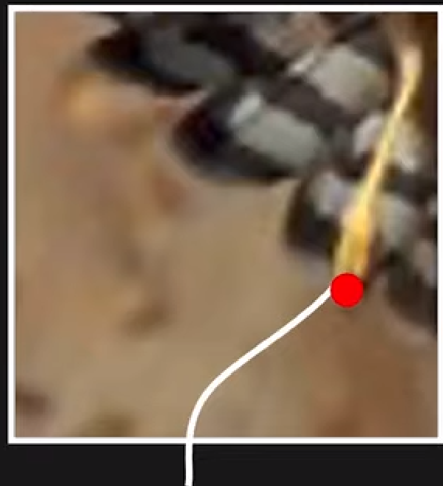
$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial}$$



Optical Flow Constraint Equation



$I(x, y, t)$

$I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t}$$

Optical Flow Constraint Equation

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \quad \text{------------ (1)}$$

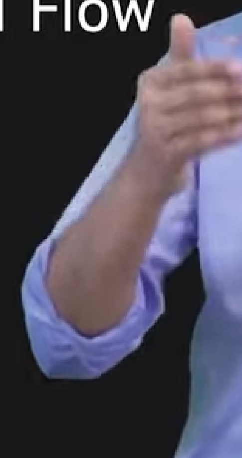
$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t \quad \text{------------ (2)}$$

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

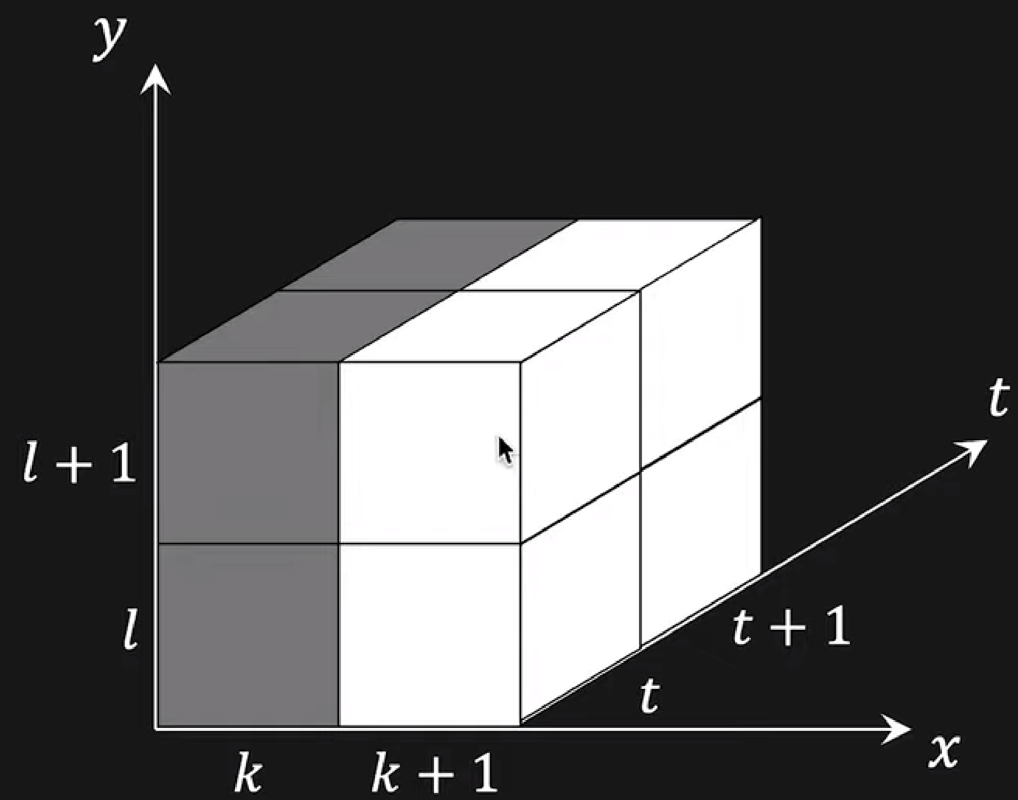
Divide by δt and take limit as $\delta t \to 0$: $I_x \dfrac{\partial x}{\partial t} + I_y \dfrac{\partial y}{\partial t} + I_t = 0$

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u, v): Optical Flow

(I_x, I_y, I_t) can be easily computed from two frames



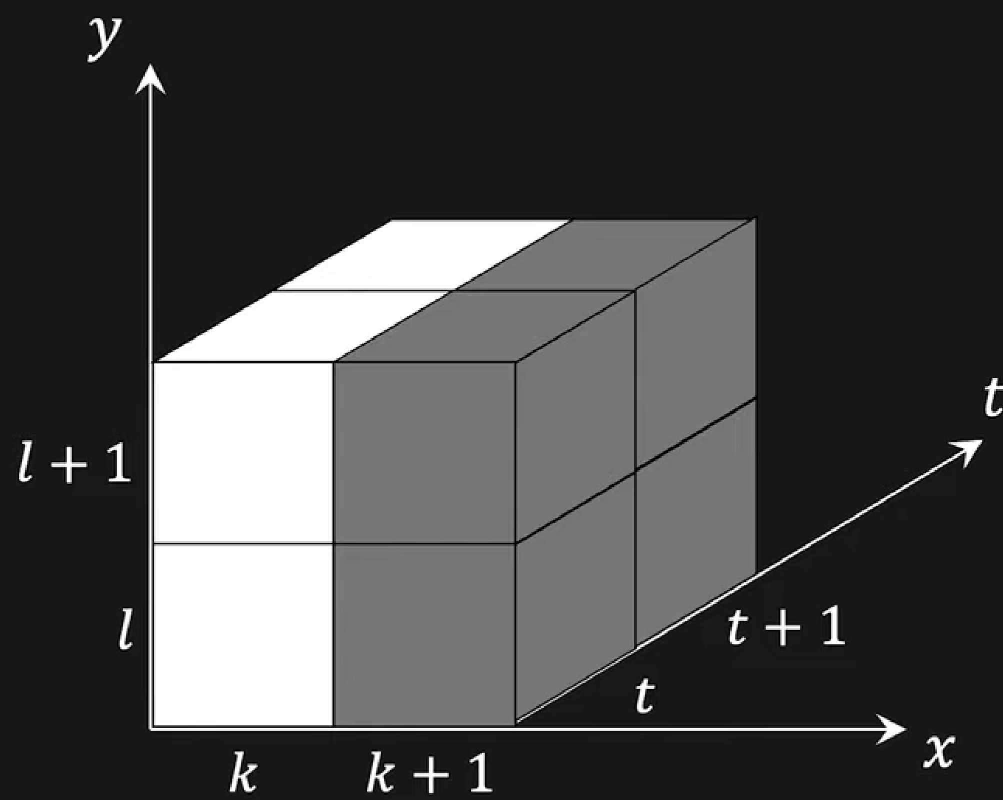
Computing Partial Derivatives I_x, I_y, I_t



$$I_x(k,l,t) =$$

$$\frac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$$

Computing Partial Derivatives I_x, I_y, I_t

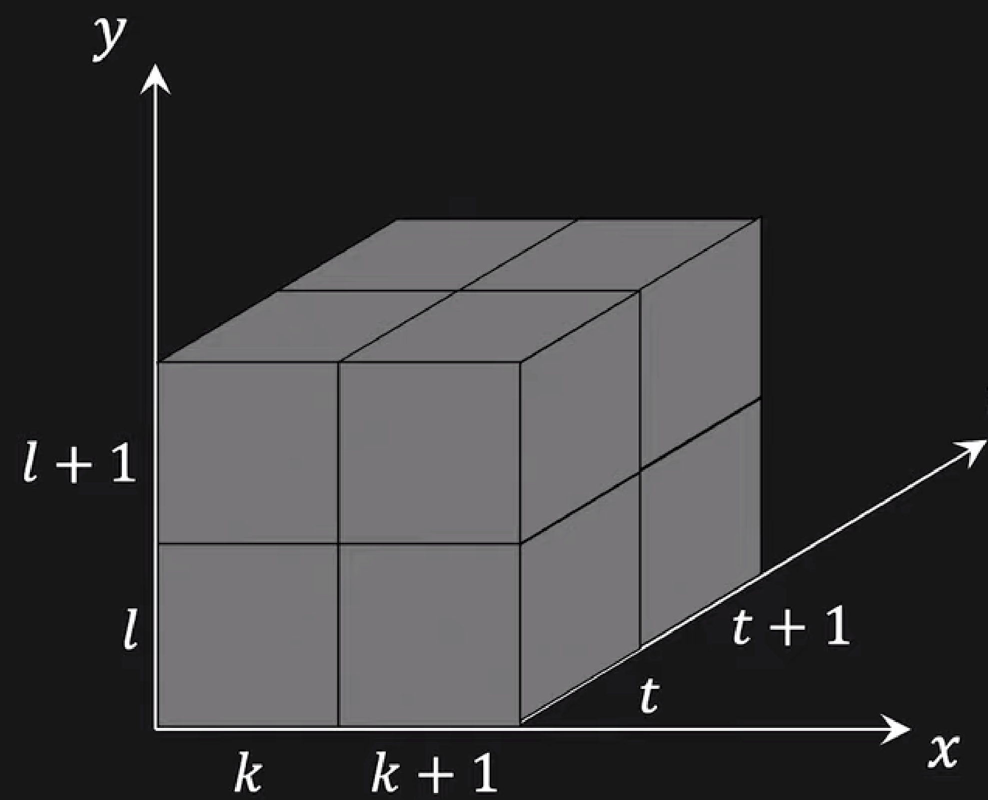


$$I_x(k,l,t) =$$

$$\frac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$$

$$-\frac{1}{4}[I(k,l,t) + I(k,l,t+1) + I(k,l+1,t) + I(k,l+1,t+1)]$$

Computing Partial Derivatives I_x, I_y, I_t



$I_x(k, l, t) =$

$$\frac{1}{4}[I(k+1, l, t) + I(k+1, l, t+1) + I(k+1, l+1, t) + I(k+1, l+1, t+1)]$$

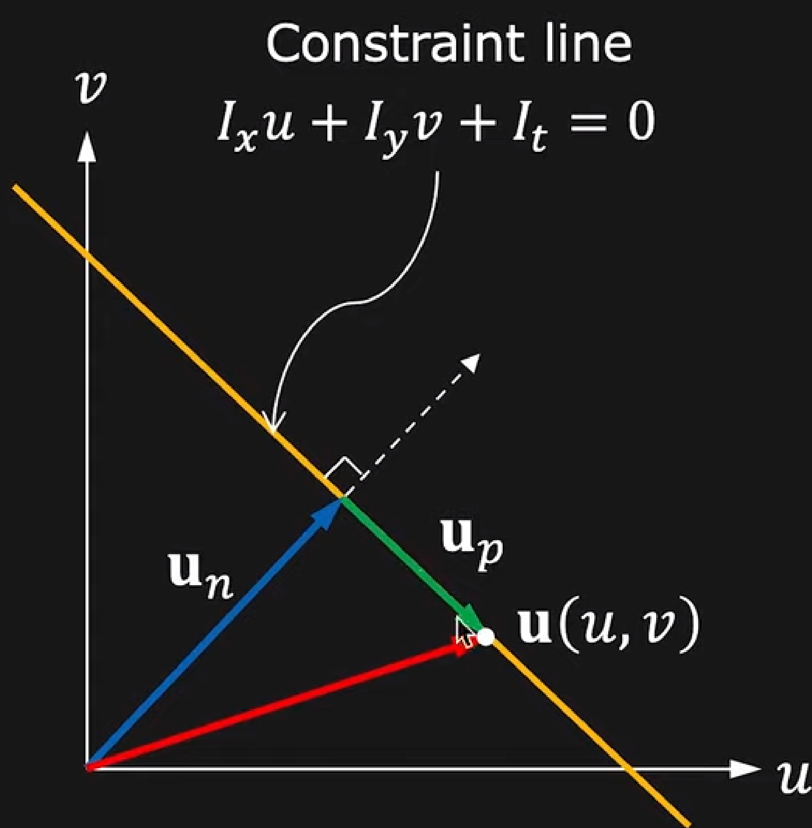
$$-\frac{1}{4}[I(k, l, t) + I(k, l, t+1) + I(k, l+1, t) + I(k, l+1, t+1)]$$

Similarly find $I_y(k, l, t)$ and $I_t(k, l, t)$

Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$



Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

$\mathbf{u}_n$: Normal Flow

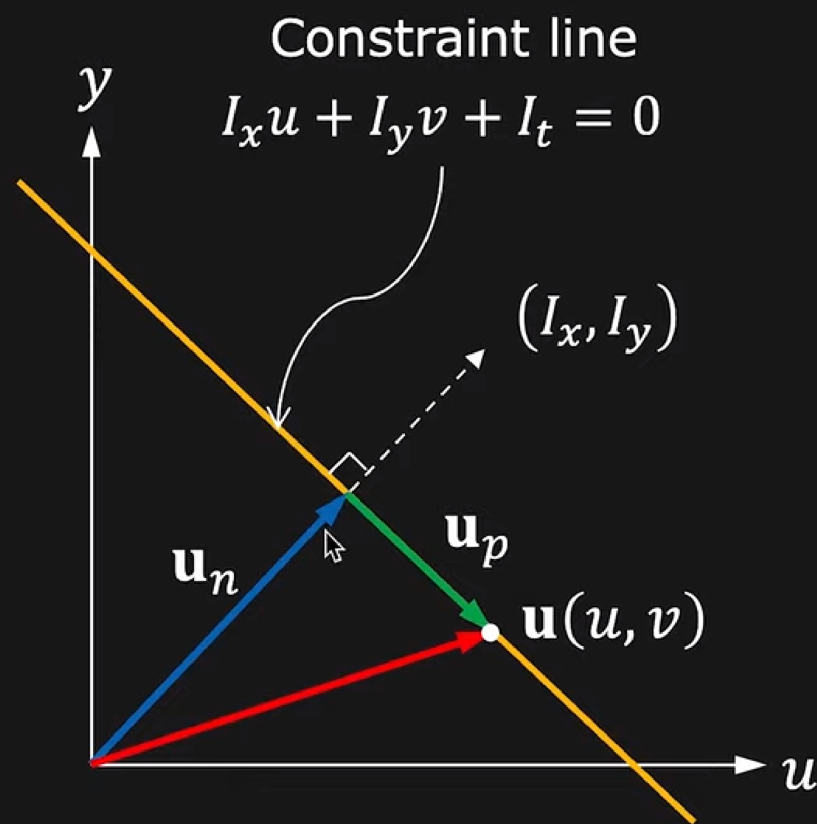
$\mathbf{u}_p$: Parallel Flow

Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$



Magnitude of Normal Flow:

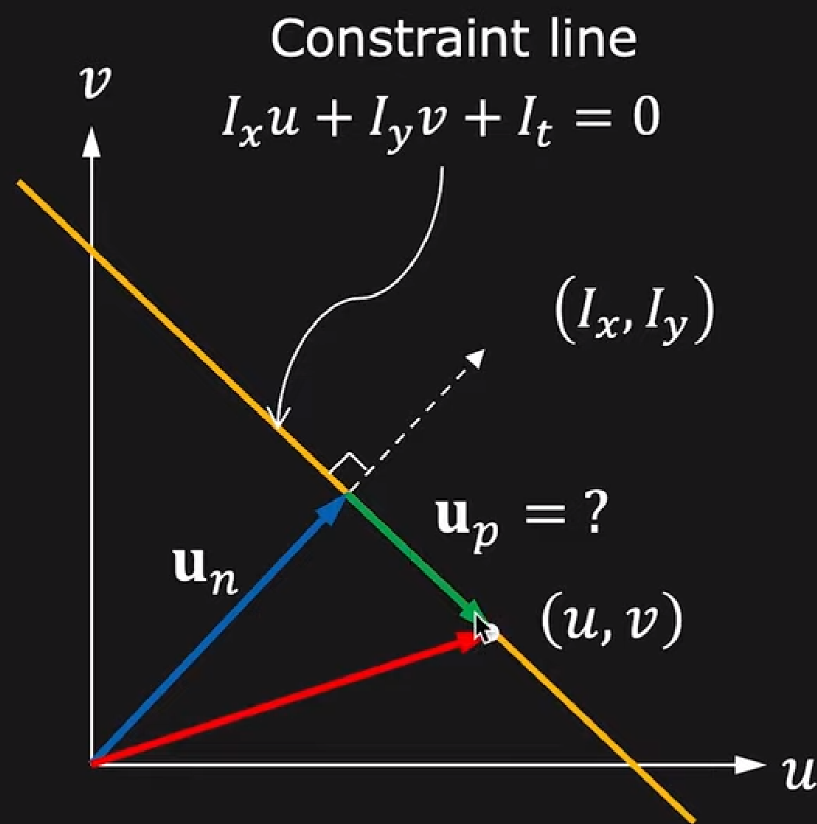
Distance of origin from the constraint line:

$$|\mathbf{u}_n| = \frac{|I_t|}{\sqrt{I_x^2 + I_y^2}}$$

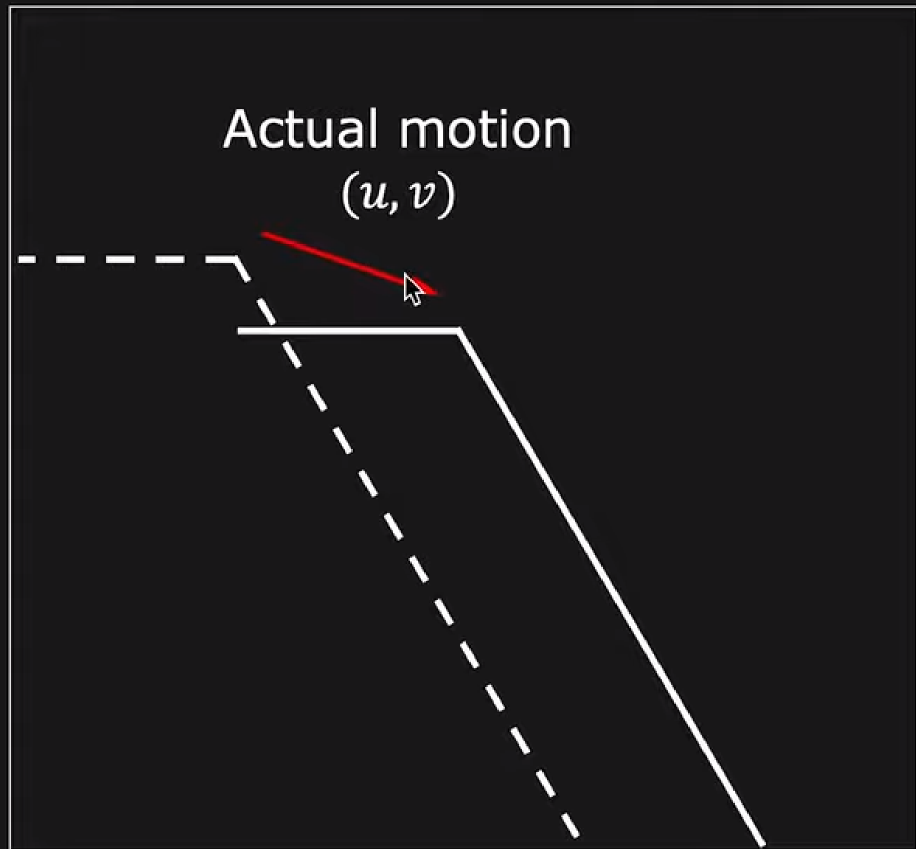
$$\boxed{\mathbf{u}_n = \frac{|I_t|}{(I_x^2 + I_y^2)}(I_x, I_y)}$$

Parallel Flow

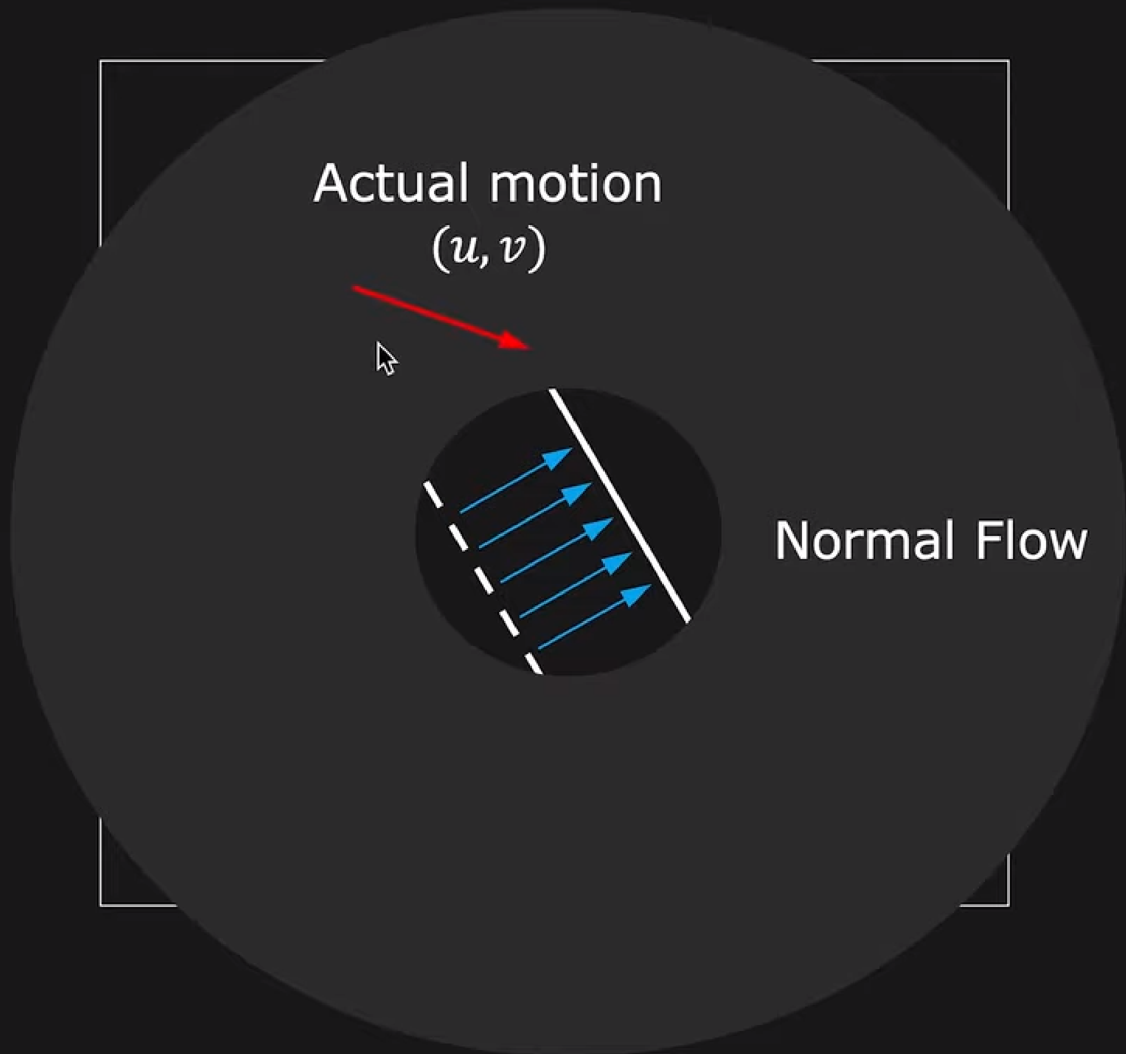
We cannot determine $\mathbf{u}_p$, the optical flow component parallel to the constraint line.



Aperture Problem



Aperture Problem



Locally, we can only determine normal flow!

Optical Flow is Under Constrained

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$

2 unknowns, 1 equation.

We need additional constraints.

Lucas-Kanade Solution

Assumption: For each pixel, assume Motion Field, and hence Optical Flow (u, v), is constant within a small neighborhood W.

W

That is for all points $(k, l) \in W$:

$$I_x(k,l)u + I_y(k,l)v + I_t(k,l) = 0$$

[Lucas 1981]

Lucas-Kanade Solution

For all points $(k, l) \in W$: $I_x(k,l)u + I_y(k,l)v + I_t(k,l) = 0$

Let the size of window W be $n \times n$

In matrix form:

$$\underbrace{\begin{bmatrix} I_x(1,1) & I_y(1,1) \\ I_x(k,l) & I_y(k,l) \\ \vdots & \vdots \\ I_x(n,n) & I_y(n,n) \end{bmatrix}}_{\substack{A \\ \text{(Known)} \\ n^2 \times 2}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\substack{\mathbf{u} \\ \text{(Unknown)} \\ 2\times 1}} = \underbrace{\begin{bmatrix} I_t(1,1) \\ I_t(k,l) \\ \vdots \\ I_t(n,n) \end{bmatrix}}_{\substack{B \\ \text{(Known)} \\ n^2 \times 1}}$$

n^2 Equations, 2 Unknowns: Find Least Squares Solution

Least Squares Solution

Solve linear system: $A\mathbf{u} = B$

$$A^T A\mathbf{u} = A^T B$$ (Least-Squares using Pseudo-Inverse)

In matrix form:

$$\underbrace{\boxed{\begin{bmatrix} \sum_W I_x I_x & \sum_W I_x I_y \\ \sum_W I_x I_y & \sum_W I_y I_y \end{bmatrix}}}_{\substack{A^T A \\ \text{(Known)} \\ 2\times2}} \underbrace{\boxed{\begin{bmatrix} u \\ v \end{bmatrix}}}_{\substack{\mathbf{u} \\ \text{(Unknown)} \\ 2\times1}} = \underbrace{\boxed{\begin{bmatrix} -\sum_W I_x I_t \\ -\sum_W I_y I_t \end{bmatrix}}}_{\substack{A^T B \\ \text{(Known)} \\ 2\times1}}$$

Indices (k, l) not written for simplicity

$$\boxed{\mathbf{u} = (A^T A)^{-1} A^T B}$$

Fast and Easy to Solve

When Does Optical Flow Estimation Work?

$\boxed{A\mathbf{u} = B}$ $\boxed{A^T A\mathbf{u} = A^T B}$

- $A^T A$ must be invertible. That is $\det(A^T A) \neq 0$
- $A^T A$ must be well-conditioned.

  If λ_1 and λ_2 are eigen values of $A^T A$, then

  $\lambda_1 > \epsilon$ and $\lambda_2 > \epsilon$

  $\lambda_1 \geq \lambda_2$ but not $\lambda_1 \ggg \lambda_2$

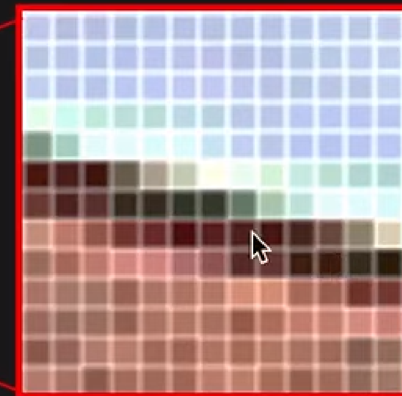
Smooth Regions (Bad)

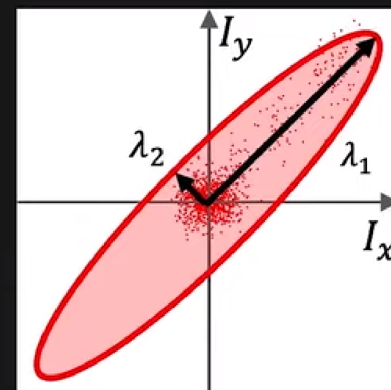
$\lambda_1 \sim \lambda_2$
Both are Small

Equations for all pixels in window are more or less the sam

Cannot reliably compute flow!

Edges (Bad)



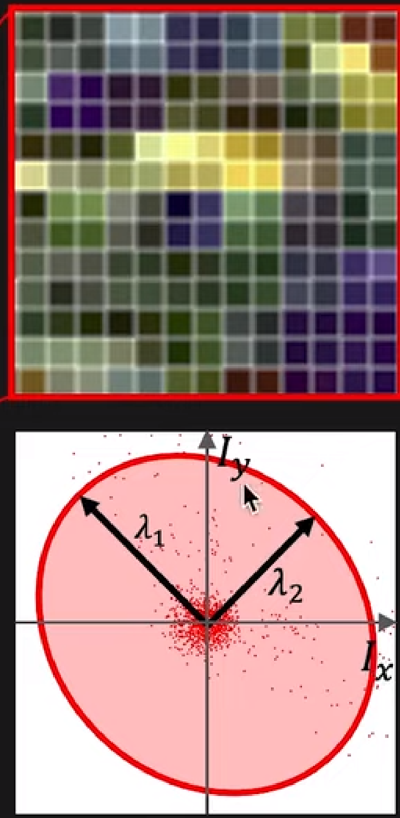


$\lambda_1 \ggg \lambda_2$

Badly conditioned. Prominent gradient in one direction.

Cannot reliably compute flow!
Same as Aperture Problem.

Textured Regions (Good)



$\lambda_1 \sim \lambda_2$
Both are Large

Well conditioned. Large and diverse gradient magnitudes.

Can reliably compute optical flow.

What if we have Large Motion?

Taylor Series approximation of

$I(x + \delta x, y + \delta y, t + \delta t)$ is not valid

Our simple linear constraint equation not valid

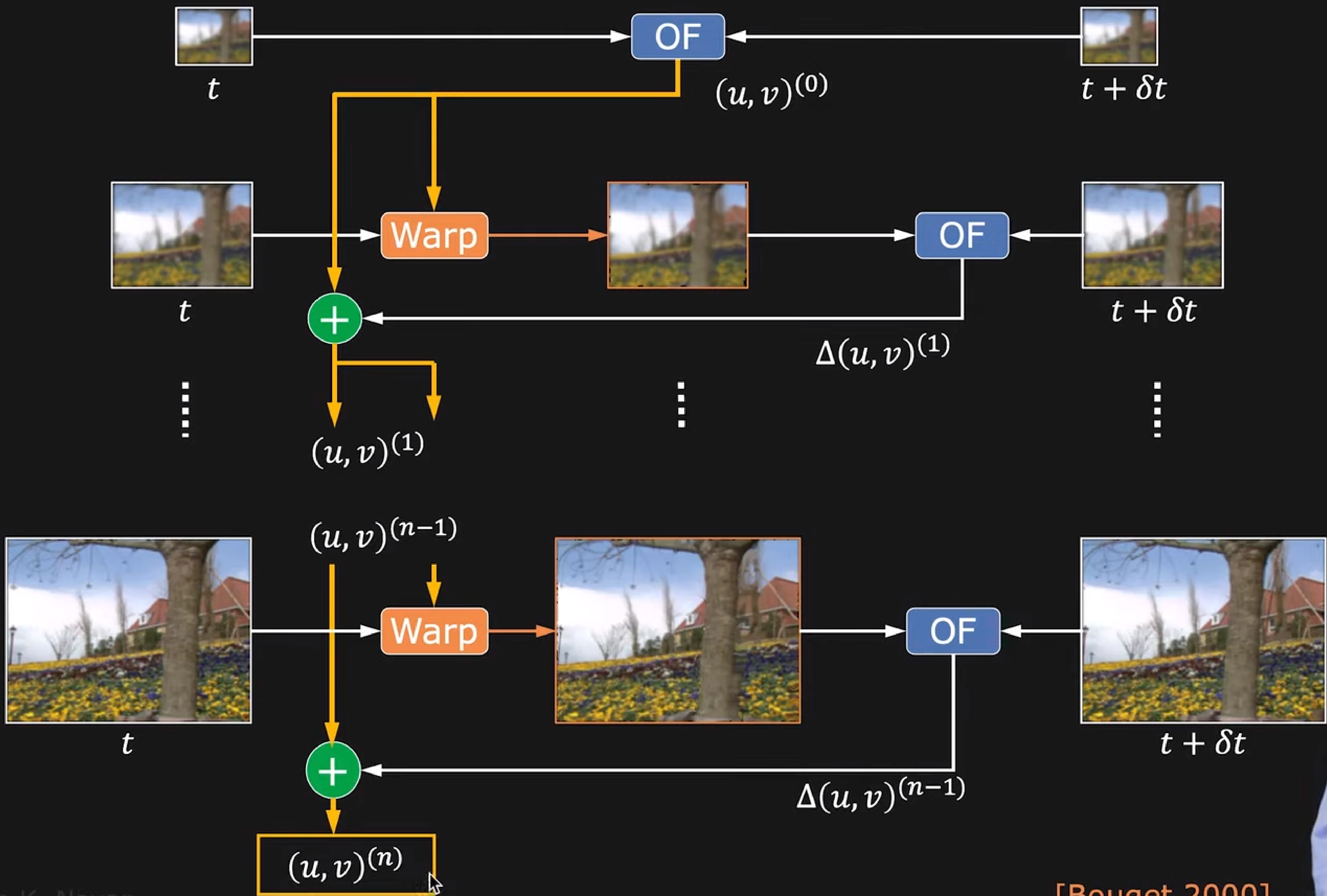
$$I_x u + I_y v + I_t \neq 0$$

Large Motion: Coarse-to-Fine Estimation

Resolution $N \times N$	Resolution $\frac{N}{2} \times \frac{N}{2}$	Resolution $\frac{N}{4} \times \frac{N}{4}$	Resolution $\frac{N}{8} \times \frac{N}{8}$

At lowest resolution, motion $\leq$ 1 pixel

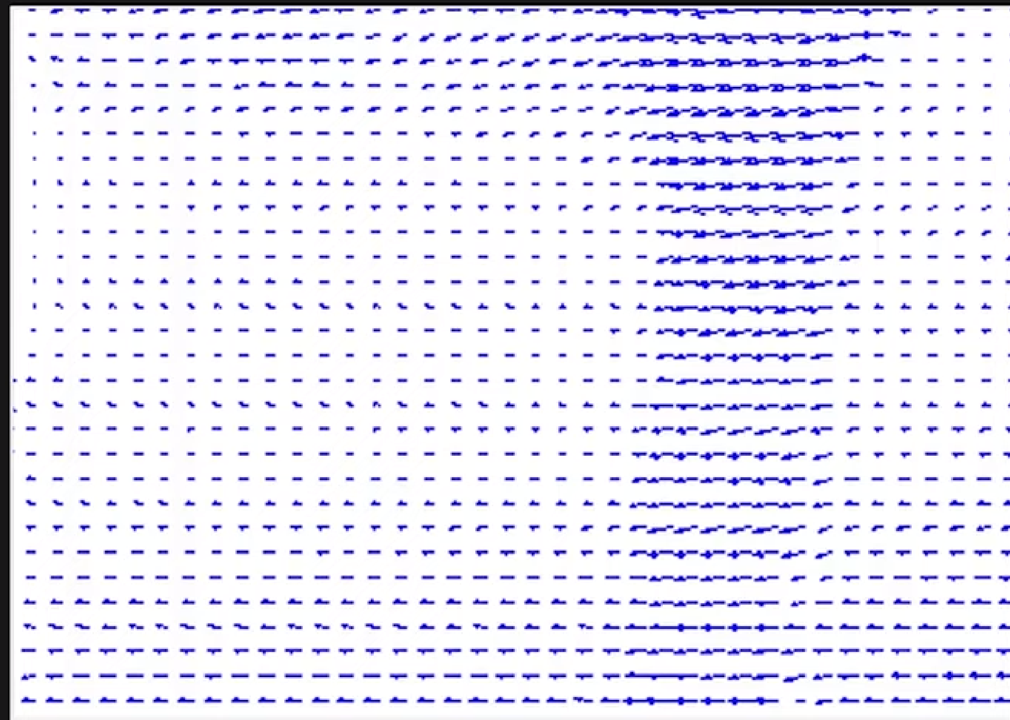
Coarse-to-Fine Estimation Algorithm



[Bouget 2000]

Results: Tree Sequence

Image Sequence



Optical Flow

Results: Rotating Ball

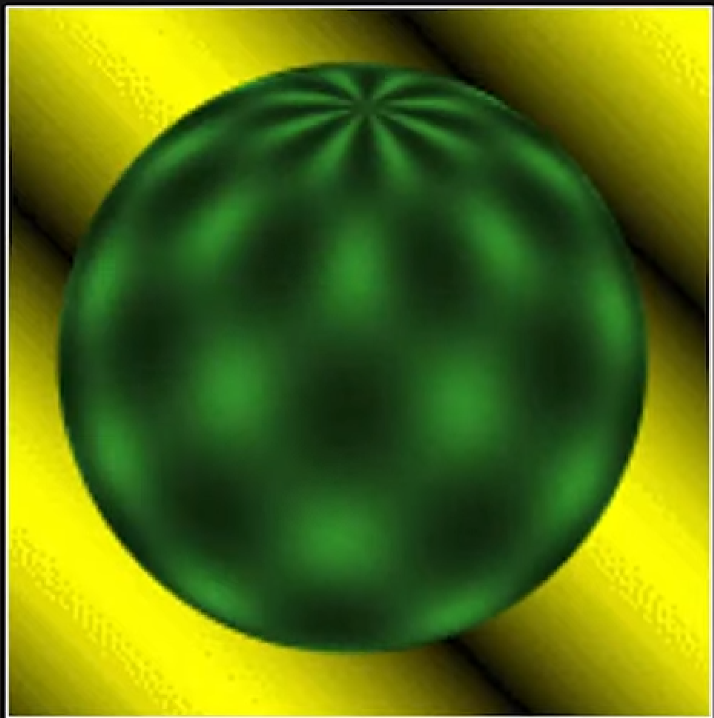
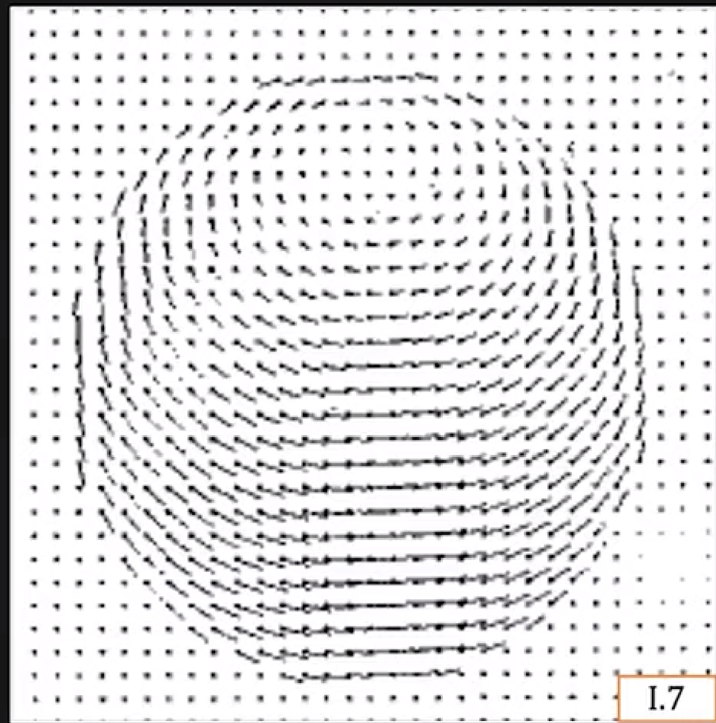


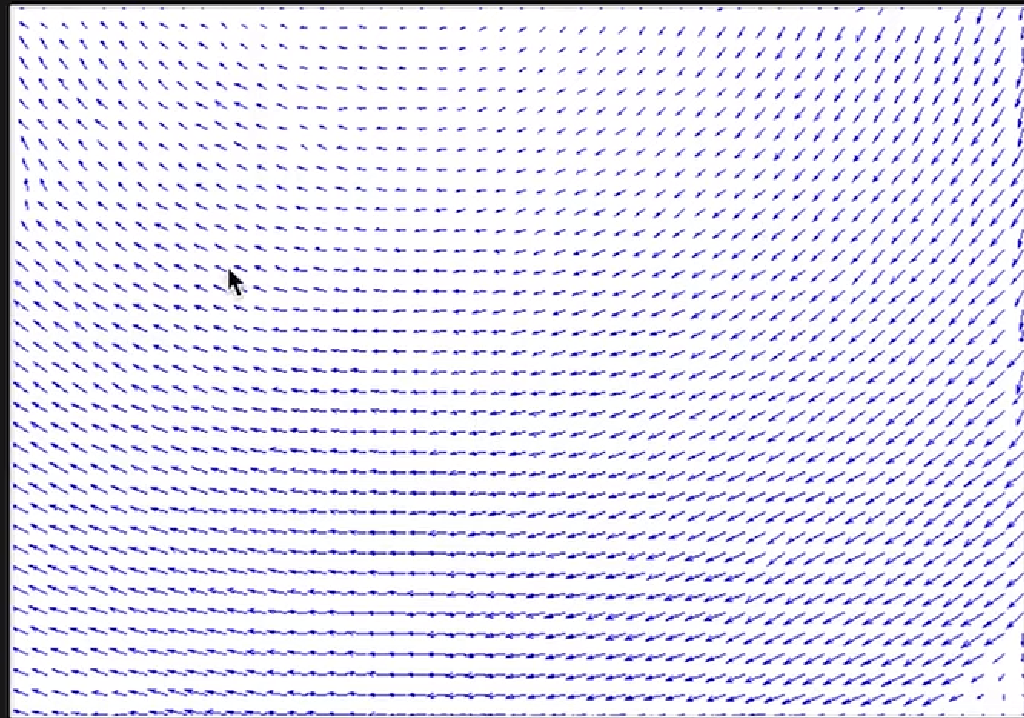
Image Sequence



Optical Flow

Results: Rotating Camera

Image Sequence



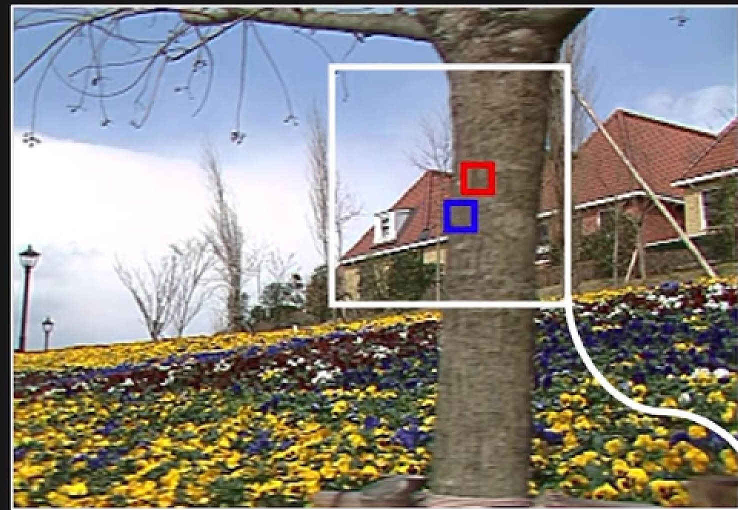
Optical Flow

Alternative Approach: Template Matching

Determine Flow using Template Matching

Template Window T

Image I_1 at time t



Search Window S

Image I_2 at time $t + \delta t$

For each template window T in image I_1,
find the corresponding match in image I_2.

Large Motion: Template matching

Determine Flow using Template Matching

Template Window T

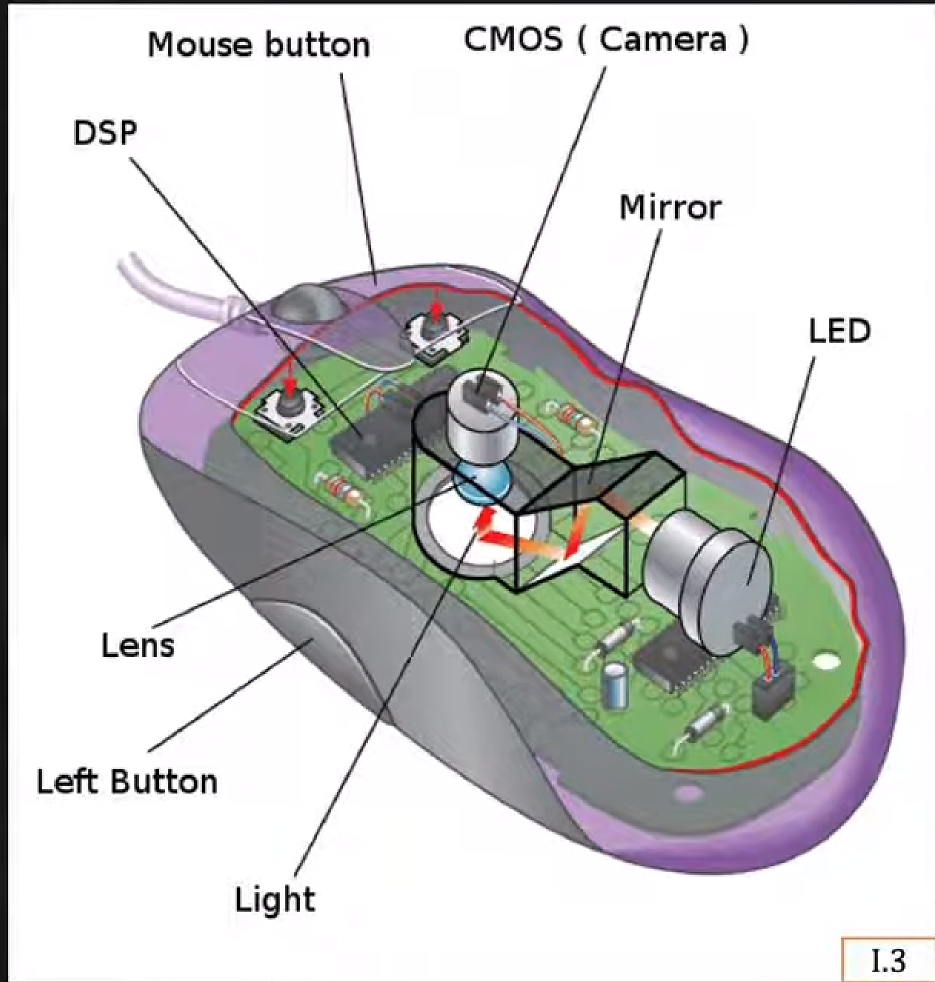
Image I_1 at time t

Search Window S

Image I_2 at time $t + \delta t$

Template matching is slow
when search window S is large.
Also, mismatches are possible.

Optical Mouse



Estimating Mouse Movements

Traffic Monitoring

Finding Velocities of Vehicles

Video Retiming

Optical Flow is used to determine the intermediate frames to produce slow-motion effect.

Video Retiming

Optical Flow is used to determine the intermediate frames to produce slow-motion effect.

Image Stabilization

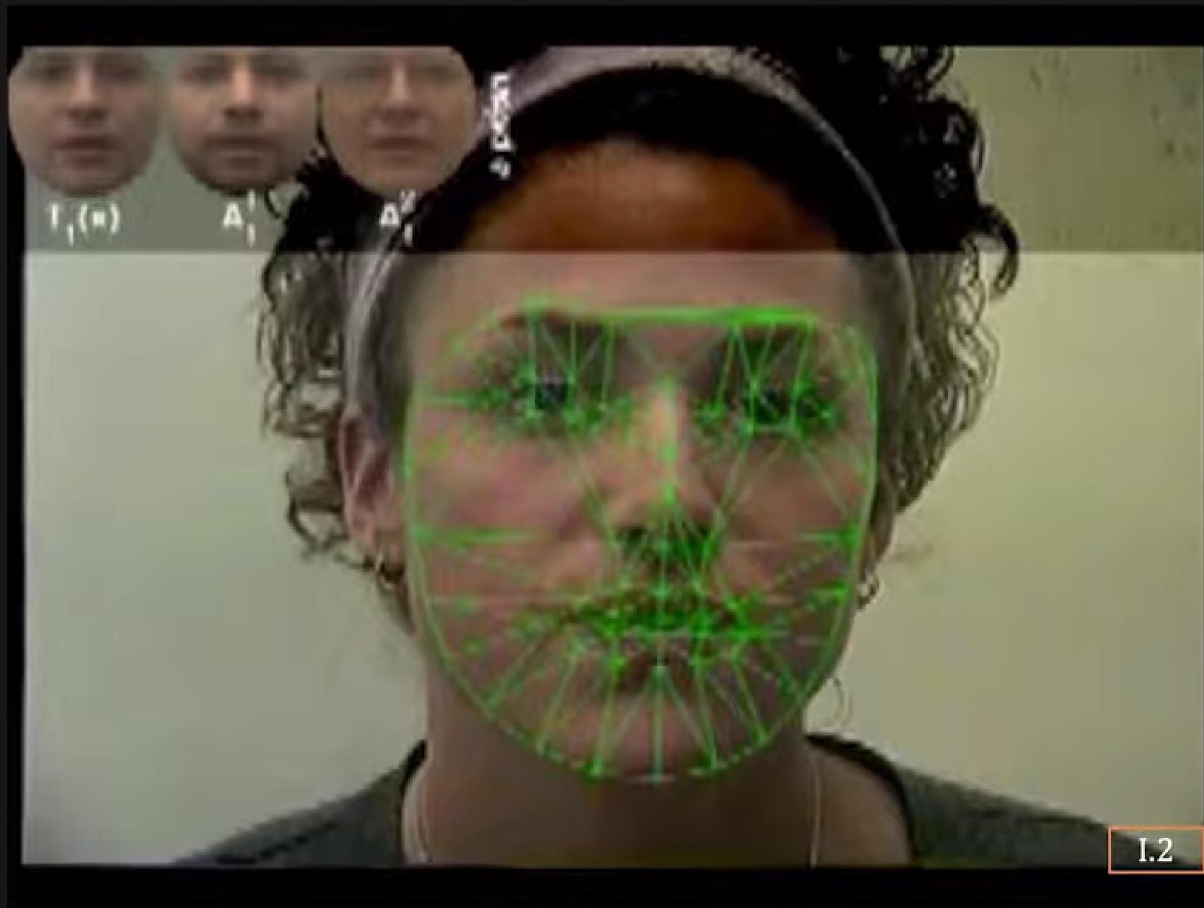
Captured Video

Stabilized Video

Optical Flow is used to remove camera shake.

[Liu 2014]

Face Tracking



Tracking of Facial Features