National University of Computer & Emerging Sciences, Karachi
Fall-2020 Department of Computer Science
Mid Term-2
24th November 2020, 10:30 AM – 12:00 PM

Course Code: CS302	**Course Name:** Design and Analysis of Algorithm
Instructor Name / Names: Dr. Muhammad Atif Tahir, Dr. Fahad Sherwani, Zeshan Khan, Waqas Sheikh, Sohail Afzal	
Student Roll No:	**Section:**

Instructions:

- Return the question paper.
- Read each question completely before answering it. There are **6 questions** on **3 pages.**
- In case of any ambiguity, you may make assumption. But your assumption should not contradict any statement in the question paper.

Time: 90 minutes. **Max Marks: 17.5**

Question # 1 **[3 marks]**

Consider the following instance of the 0/1 knapsack problem

items	1	2	3	4
values	3	4	5	6
weights	2	3	4	5

The maximum allowable total weight in the knapsack is W = 5.

Find an optimal solution for the above problem with the weights and values given above using Dynamic Programming. Be sure to state both the maximum value as well as the item(s) that will make maximum value in the given capacity W. Show all steps. [see appendix]

Solution

Step-01:

- Draw a table say 'T' with (n+1) = 4 + 1 = 5 number of rows and (w+1) = 5 + 1 = 6 number of columns.
- Fill all the boxes of 0^{th} row and 0^{th} column with 0.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

Step-02:

Start filling the table row wise top to bottom from left to right using the formula-

$$T(i, j) = \max \{ T(i-1, j), value_i + T(i-1, j - weight_i) \}$$

Finding T(1,1)-

We have,

- i = 1
- j = 1
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,1) = max { T(1-1 , 1) , 3 + T(1-1 , 1-2) }

T(1,1) = max { T(0,1) , 3 + T(0,-1) }

T(1,1) = T(0,1) { Ignore T(0,-1) }

T(1,1) = 0

Finding T(1,2)-

We have,

- i = 1
- j = 2
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,2) = max { T(1-1 , 2) , 3 + T(1-1 , 2-2) }

T(1,2) = max { T(0,2) , 3 + T(0,0) }

T(1,2) = max {0 , 3+0}

T(1,2) = 3

Finding T(1,3)-

We have,

- i = 1
- j = 3
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,3) = max { T(1-1 , 3) , 3 + T(1-1 , 3-2) }

T(1,3) = max { T(0,3) , 3 + T(0,1) }

T(1,3) = max {0 , 3+0}

T(1,3) = 3

Finding T(1,4)-

We have,

- i = 1
- j = 4
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,4) = max { T(1-1 , 4) , 3 + T(1-1 , 4-2) }

T(1,4) = max { T(0,4) , 3 + T(0,2) }

T(1,4) = max {0 , 3+0}

T(1,4) = 3

Finding T(1,5)-

We have,

- i = 1
- j = 5
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,5) = max { T(1-1 , 5) , 3 + T(1-1 , 5-2) }

T(1,5) = max { T(0,5) , 3 + T(0,3) }

T(1,5) = max {0 , 3+0}

T(1,5) = 3

Finding T(2,1)-

We have,

- i = 2
- j = 1
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,1) = max { T(2-1 , 1) , 4 + T(2-1 , 1-3) }

T(2,1) = max { T(1,1) , 4 + T(1,-2) }

T(2,1) = T(1,1) { Ignore T(1,-2) }

T(2,1) = 0

Finding T(2,2)-

We have,

- i = 2
- j = 2
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,2) = max { T(2-1 , 2) , 4 + T(2-1 , 2-3) }

T(2,2) = max { T(1,2) , 4 + T(1,-1) }

T(2,2) = T(1,2) { Ignore T(1,-1) }

T(2,2) = 3

Finding T(2,3)-

We have,

- i = 2
- j = 3
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,3) = max { T(2-1 , 3) , 4 + T(2-1 , 3-3) }

T(2,3) = max { T(1,3) , 4 + T(1,0) }

T(2,3) = max { 3 , 4+0 }

T(2,3) = 4

Finding T(2,4)-

We have,

- i = 2
- j = 4
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,4) = max { T(2-1 , 4) , 4 + T(2-1 , 4-3) }

T(2,4) = max { T(1,4) , 4 + T(1,1) }

T(2,4) = max { 3 , 4+0 }

T(2,4) = 4

Finding T(2,5)-

We have,

- i = 2
- j = 5
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,5) = max { T(2-1 , 5) , 4 + T(2-1 , 5-3) }

T(2,5) = max { T(1,5) , 4 + T(1,2) }

T(2,5) = max { 3 , 4+3 }

T(2,5) = 7

Similarly, compute all the entries.

After all the entries are computed and filled in the table, we get the following table-

	0	1	2	3	4	5
0	0	0	0	0	0	0
✓ 1	0	0	3	3	3	3
✓ 2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

T-Table

- The last entry represents the maximum possible value that can be put into the knapsack.
- So, maximum possible value that can be put into the knapsack = 7.

Identifying Items To Be Put Into Knapsack-

Following Step-04,

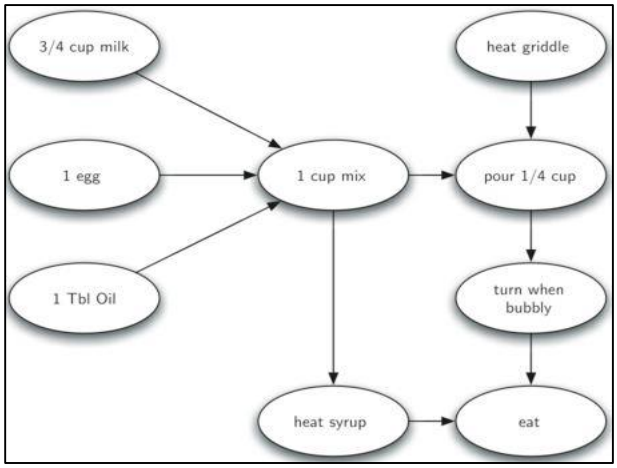
- We mark the rows labelled “1” and “2”.
- Thus, items that must be put into the knapsack to obtain the maximum value 7 are-

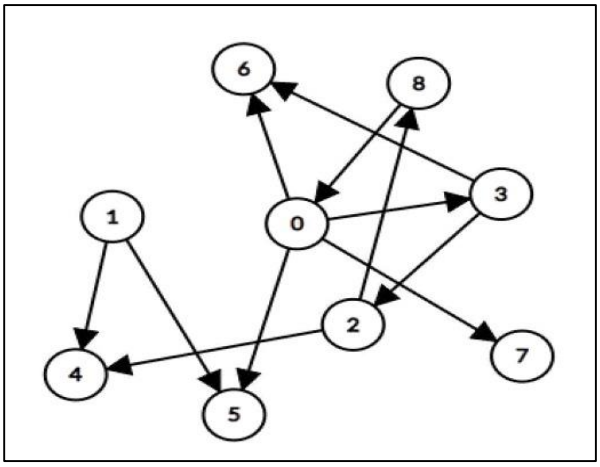
Item-1 and Item-2

Question # 2 **[1+0.5 = 1.5 marks]**

Given two directed graphs in part (a) and part (b), you need to write one possible topological ordering of both graphs.

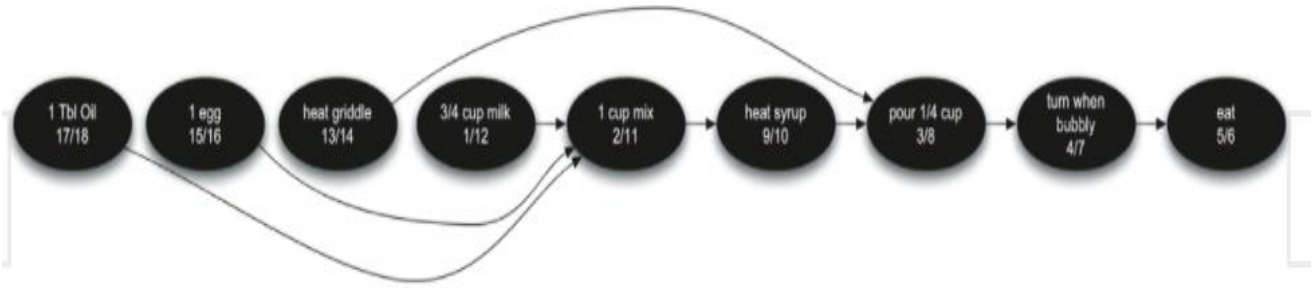
Part (a) **Part (b)**





Solution

Part (a):



Part (b):
Not possible as there is cycle among nodes (8,0,3,2)

Question # 3 **[1 + 1 = 2 marks]**

Compute the time complexity for both below mentioned algorithms. Show all steps.

```
void Algorithm1(A, int n){
    for(int i = 0; i ≤ √n; i++){
        for (int j = 0; j ≤ √n; j++){
            int temp = A[j][i]/A[i][i]
            for (int k = i + 1; k ≤ n + i; k++){
                A[j][k − i] −= temp ∗ A[i][k − i]
            }
        }
    }
}
```

```
void Algorithm2(A, int n){
    for(int i = 0; i ≤ n/2; i++){
        for (int j = 1; j ≤ n/2; j++){
            for (int k = i + 1; k ≤ i + 5; k++){
                for (int l = 1; l ≤ n; l++){
                    sum+= A[i + j][l]
                }
            }
        }
    }
}
```

Solution

Algorithm 1:

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + \left(\sum_{j=0}^{\sqrt{n}} 2 + \left(\sum_{k=i+1}^{n+i} (3) \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + \left(\sum_{j=0}^{\sqrt{n}} 2 + 3n \right) \right)$$

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + 2\sqrt{n} + 3n\sqrt{n} + 2 + 3n \right)$$

$$T(n) = \left(4 + 2\sqrt{n} + 3n\sqrt{n} + 3n \right)(\sqrt{n} + 1)$$

$$T(n) = \left(4\sqrt{n} + 2n + 3n^2 + 3n\sqrt{n} + 4 + 2\sqrt{n} + 3n\sqrt{n} + 3n \right)$$

$$T(n) = 3n^2 + 6n\sqrt{n} + 5n + 6\sqrt{n} + 4$$

$$T(n) = \theta(n^2)$$

Algorithms 2:

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + \left(\sum_{k=i+1}^{i+5} 2 + \left(\sum_{l=1}^{n} 3 \right) \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + \left(\sum_{k=i+1}^{i+5} 2 + 3n \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + 10 + 6n \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 12 + 6n \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} (2 + 6n + 3n^2)$$

$$T(n) = (n + 3n^2 + \frac{3}{2}n^3 + 2 + 6n + 3n^2)$$

$$T(n) = (\frac{3}{2}n^3 + 6n^2 + 7n + 2)$$

$$T(n) = \theta(n^3)$$

Question # 4 **[3 marks]**

Let A and B be two sequences of *"n"* integers each, in the range $[1, n^2]$, from square series. Given an integer *x*, design an O(n)-time algorithm for determining if there is an integer "*a"* in A and an integer "*b"* in B such that *x = a + b*. (You may write algorithm in plain text)

Solution

Sort both a and b by count sort altered version

```
countingSort(array):

  size = len(array)

  output = [0] * size

  count = [0] * 10

  for i in range(0, size):

          count[sqrt(array[i])] += 1//Modification

  for i in range(1, 10):

          count[i] += count[i - 1]

  i = size – 1

  while i >= 0:

          output[count[array[i]] - 1] = array[i]

          count[array[i]] -= 1

          i -= 1

  for i in range(0, size):

          array[i] = (output[i]* output[i]) //Modification
```

Then apply the following linear algorithm

While(a<A.size and b>0):

```
if(A[a] + B[b] < x) -> update index a to be a + 1
if(A[a] + B[b] > x) -> update index b to be b - 1
if(A[a] + B[b] = x) -> success
```

Question # 5 **[4 marks]**

In Fairyland, a network is to be constructed to connect N cities. During K weeks, you keep receiving tenders for constructing a direct connection between two cities. For every week–if possible– give a plan, based on the tenders received so far, regarding which cities are to be connected directly so that every city will be accessible from every other city directly or via other cities but such that the total cost of construction will be minimum. The plan contains the pairs of cities to be connected and the total cost of constructing the network. Design algorithm for this problem. (You may write algorithm in plain text)

Solution

The problem is the online version of creating a minimum spanning tree. The difficulty the competitors are to face is that they rarely, if ever, meet such algorithms at school or at the preparatory courses for the competitions [1]. The problem, however, can be deduced to the following case: if there is a minimum spanning forest and one single new graph edge, how can you make a minimum spanning forest? For example, a modification to Kruskal's algorithm could be the following:

```
Procedure online(G: graph; var A: set of edges)
  A := {};
  for all v∈ G.V do Make-a-set(v);
  for all (u,v) ∈ G.E
    do if Find-a-set(u) ≠ Find-a-set(v)
         then begin A := A ∪ {(u, v)}; Join(u,v); end
         else begin
              (p, q):=the maximum weight edge of path (u, v);
              if weight(p, q) > weight(u, v)
                then A := A ∪ {(u, v)} − {(p, q)};
            end;
  end;
```

Question # 6 **[0.5 + 0.5 + 3 = 4 marks]**

Answer the following.

a) How do we decide to split the matrix-chain and parenthesize for the optimization of multiplications operations? (0.5 marks)

Solution (a)

The matrix-chain problem can be solved recursively to determine the best value of k, we will consider all possible values of k, and pick the best of them to split the matrix-chain accordingly, providing optimal multiplication operations for each subproblem. Parenthesizing can be done in the form of (A[1…k])+(A[k+1…..n]).

b) Which of the following is the recurrence relation for the matrix-chain multiplication problem where mat[i-1] * mat[i] gives the dimension of the i^{th} matrix? (0.5 marks)

1) M[i, j] = 1 if i=j

   M[i, j] = min{ M[i, k] + M[k+1, j]}

2) M [i, j] = 0 if i=j

   M [i, j] = min{M[i, k] + M[k+1, j]}

3) M [i, j] = 0 if i=j

   M [i, j] = max{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

4) M [i, j] = 0 if i=j

   M [i, j] = min{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

Solution (b)

4) M[i,j] = 0 if i=j

   M[i,j] = min{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

c) Design a **recursive algorithm** to compute the minimum number of scalar multiplications for the chain matrix product $A_{i...j}$ in a **top-down manner**. You can also write the algorithm in English. (3 marks)

Solution (c)

```
Rec-Matrix-Chain(array p, int i, int j) {
    if (i = = j) m[i, i] = 0;                          // basic case
    else {
            m[i, j] = infinity;                        // initialize
            for k = i to j − 1 do {                    // try all possible splits
                cost=Rec-Matrix-Chain(p, i, k) + Rec-Matrix-Chain(p, k + 1, j) + p[i − 1]*p[k]*p[j];
                if (cost<m[i, j]) then
                    m[i, j]= cost;
                }
             }                                         // update if better
     return m[i,j];                                    // return final cost
}
```

Appendix

$$V[i,j] = \begin{cases} V[i-1,j] & \text{if } w_i > j \\ \max\{V[i-1,j],\ \ v_i + V[i-1,j-w_i]\} & \text{if } w_i \le j \end{cases}$$

BEST OF LUCK