Statistical Computing HW 1

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```
## Set seed
set.seed(1234)
```

Problem 3.2)

Question

If $x_0 = 3$ and

$$x_n = (5x_{n-1} + 7) \mod 200$$

find $x_1, ...x_{10}$.

Answer

The code below finds $x_1, ...x_{10}$:

```
# Create a dataframe to store the values
df2 <- data.frame(x_i=c(0:10), equals=rep(0,11))

# Initialize x0 = 3
df2[1,2] = 3

# Loop for x1 to x10
for (i in 1:10){
  df2[i+1, 2] = (5*df2[i, 2]+7) %% 200
}
```

```
# Print results
kable(df2)
```

x_i	equals
0	3
1	22
2	117
3	192
4	167
5	42
6	17
7	92
8	67
9	142
10	117

Problem 3.5)

Question

Use simulation to approximate

$$\int_{-2}^{2} e^{x+x^2} dx$$

Compare answer with the exact answer if known.

Answer

1) Derivation

A change of variables is needed. Let:

$$y = \frac{x+2}{4}$$

This is a one to one (increasing) function that maps the interval $(-2, 2)$ to $(0, 1)$. Then:

$$\begin{aligned} x &= 4y - 2 \\ dx &= 4\ dy \end{aligned}$$

Substituting these values in the integral we get:

$$\int_{-2}^{2} e^{x+x^2} dx = \int_{0}^{1} e^{4y-2+(4y-2)^2} (4) dy$$

Let:

$$h(y) = e^{4y-2+(4y-2)^2} (4) dy$$

Then:

$$\begin{aligned} \int_{0}^{1} e^{4y-2+(4y-2)^2} (4) dy &= \int_{0}^{1} h(y) dy \\ &= \int_{0}^{1} h(y)(1) dy \\ &= E(h(y)), \quad Y \sim U(0, 1) \end{aligned}$$

The final integral is then the expected value of $h(y)$, where Y is a standard uniform random variable.

2) Algorithm

1) Generate y from $U(0, 1)$
2) Evaluate $h(y)$, store value
3) Repeat (1) and (2) 100000 times
4) Find the average of the 100000 values

3) Simulation

```
# initialize a vector of length 10000
vector <- rep(0, 100000)

# loop to generate y values and evalute h(y)
for(i in 1:100000){
  # generate random number from U(0,1)
  y <- runif(1)
  # evalute and store h(y)
  vector[i] <- exp(4*y-2+(4*y-2)^2)*4
}

# compute the average
average <- mean(vector)
average
```

```
## [1] 93.49606
```

The average value of the 100000 $h(y)$ values is 93.4960624.

4) Analytical Result

To my knowledge, there is no simple way to integrate this function. WolframAlpha provides an estimate of 93.1628 however.

Problem 3.6)

Question

Use simulation to approximate

$$\int_0^\infty x(1+x^2)^{-2}dx$$

Compare answer with the exact answer if known.

Answer

1) Derivation

First, it worthwhile to check that this integral converges. This will be done in Part 4 of the answer, and it will be shown that it does in fact converge. To use simulation, a change of variables is needed. Let:

$$y = \frac{x}{x+1} = 1 - \frac{1}{x+1}$$

This is a one to one (increasing) function that maps the interval $(0, \infty)$ to $(0, 1)$. Then:

$$x = \frac{y}{1-y}$$
$$dx = \frac{1}{(1-y)^2} dy$$

Then making the change of variables, we have:

$$\int_0^\infty x(1+x^2)^{-2} dx = \int_0^1 \frac{y}{1-y} \left(1 + \left(\frac{y}{1-y}\right)^2\right)^{-2} \frac{1}{(1-y)^2} dy$$
$$= \int_0^1 \frac{y}{(1-y)^3(1+(\frac{y}{1-y})^2)^2} dy$$

Then let:

$$h(y) = \frac{y}{(1-y)^3(1+(\frac{y}{1-y})^2)^2}$$

Then:

$$\int_0^1 \frac{y}{(1-y)^3(1+(\frac{y}{1-y})^2)^2} = \int_0^1 h(y)dy$$
$$= \int_0^1 h(y)(1)dy$$
$$= E(h(y)), \;\; Y \sim U(0,1)$$

The final integral is then the expected value of $h(y)$, where Y is a standard uniform random variable.

2) Algorithm

1) Generate y from $U(0, 1)$
2) Evaluate $h(y)$, store value
3) Repeat (1) and (2) 100000 times
4) Find the average of the 100000 values

3) Simulation

```
# initialize a vector of length 10000
vector <- rep(0, 100000)

# loop to generate y values and evalute h(y)
for(i in 1:100000){
  # generate random number from U(0,1)
  y <- runif(1)
  # evalute and store h(y)
  vector[i] <- y / ((1-y)^3 * (1+(y/(1-y))^2)^2)
}
```

```r
# compute the average
average <- mean(vector)
average
```

```
## [1] 0.4998166
```

The average value of the 100000 $h(y)$ values is 0.4998166.

4) Analytical Result

The exact value of the integral can be found (using a change of variable: $y = x^2$, $dy = 2x\ dx$):

$$\begin{aligned}
\int_0^\infty x(1+x^2)^{-2}dx &= \int_0^\infty \frac{x}{(1+x^2)^2}dx \\
&= \frac{1}{2}\int_0^\infty \frac{2x}{(1+x^2)^2}dx \\
&= \frac{1}{2}\int_0^\infty \frac{1}{(1+y)^2}dy \\
&= -\frac{1}{2}\left[\frac{1}{1+y}\right]_0^\infty \\
&= -\frac{1}{2}(0-1) \\
&= \frac{1}{2}
\end{aligned}$$

The exact value of the integral is $\frac{1}{2}$.