Section 6.13: Fast-Fourier transform

$W = e^{2\pi i/n}$ and $W^n = e^{2\pi i} = 1$.

$(f_n)_{jk} = \omega_n^{jk} = e^{2\pi i jk/n}$, $\quad j,k = 0,1,\ldots,n-1$.

entry of $n \times n$ Fourier matrix

Our interest lies in powers of 2, matrices of size $n \times n$.

Example: $n = 2^{12}$.

F_n has $n^2 = 2^{24}$ entries.

Matrix-vector product is $\Theta(n^2) \approx 2^{24}$ multiplications.

Doing this many times (image processing, time series analysis, etc) is expensive. FFT will do this more cheaply.

$$\underset{\text{matrix}}{F}\,\underset{\text{vector}}{c} = \overset{\text{data}}{f} \iff c = F^{-1} f$$

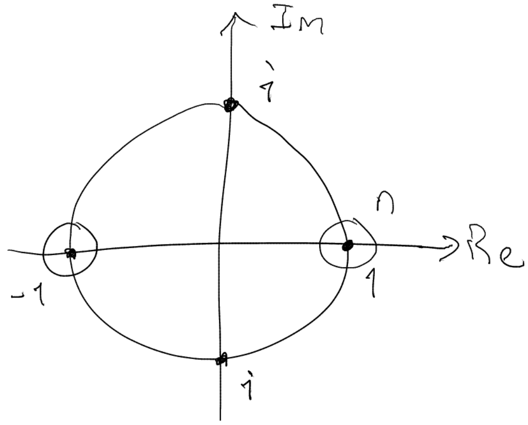
Cost of solving linear systems in general: $\Theta(n^3)$

$$F^{-1} = \frac{1}{n}\overline{F}$$

so cost of solving linear system: $\Theta(n^2)$

$$c = \frac{1}{n}\overline{F}f.$$

Observation: Let $n=2m$.



Example: $n=4$, $m=2$.

$$\boxed{W_n^2 = W_m}$$

If we square a point, its angle is doubled.

Idea:

Want

$$F_n x = y.$$

Start by dividing the vector x into 2 pieces with m components each.

Steps to FFT:

① $x' = \{x_0, x_2, \ldots, x_{n-2}\}$
$x'' = \{x_1, x_3, \ldots, x_{n-1}\}$

② Form

$y' = F_m x'$

$y'' = F_m x''$

③ First m components of

$F_n x = y$

are

ⓐ $y_j = y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1$

and last m components are

$y_{j+m} = y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1.$

"Proof": To verify the formulas, note that

$$y_j = \sum_{k=0}^{n-1} w_n^{kj} x_k = \sum_{k=0}^{m-1} w_n^{2kj} x_{2k} + \sum_{k=0}^{m-1} w_n^{(2k+1)j} x_{2k+1}.$$

Recall that $n = 2m$ and $w_n^2 = w_m$. Then

$$y_j = \sum_{k=0}^{m-1} w_m^{kj} x'_k + w_n^j \sum_{k=0}^{m-1} w_m^{kj} x''_k.$$

So ⓐ is verified.

Note that $j+m$ replaces j. So we get

$$y_{j+m} = \sum_{k=0}^{m-1} w_m^{k(j+m)} x_k' + w_n^{j+m} \sum_{k=0}^{m-1} w_m^{k(j+m)} x_k''.$$

Note

$$w_m^{k(j+m)} = w_m^{kj} w_m^{km}$$

$$w_m^{km} = e^{i \cdot 2\pi km/m} = e^{2i\pi k} = \cos(2\pi k) + i\sin(2\pi k) = 1.$$

Hence,

$$w_m^{k(j+m)} = w_m^{kj}.$$

Look at

$$w_n^{j+m} = w_n^j w_n^m = w_n^j e^{i2\pi m/n} = w_n^j e^{i2\pi}$$

$$= w_n^j \cdot (-1).$$

Therefore,

$$y_{j+m} = \sum_{k=0}^{m-1} w_m^{kj} x' - w_n^j \sum_{k=0}^{m-1} w_n^{kj} x''.$$

So we have shown (b). $\square$

For a matrix of size $n = 2^\ell$ (a power of 2), the cost of matrix vector multiplication is

$$\frac{1}{2} n \ell .$$

<u>Note</u>: $\ell = \log_2(n)$.

So the cost is

$$\boxed{\frac{1}{2} n \log_2(n).}$$

<u>Proof</u>: (by induction)

<u>Base case</u>: $\ell = 0$

Note $n = 2^0 = 1$. So

$$F_1 = [1].$$

In this case there is nothing to do.

So the cost is

$$0 = \frac{1}{2} 1 \cdot \log_2(1).$$

<u>Induction step</u>: Assume the cost for $m = 2^\ell$ FFT is

$$\frac{1}{2} m \log m.$$

Then,

$$n = 2^{\ell+1} = 2(2)^\ell.$$

So (count multiplications)

$$\text{cost} = \underset{(\text{step } 2)}{2 \cdot \left(\tfrac{1}{2} 2^{\ell} \ell\right)} + \underset{(\text{step } 3)}{2^{\ell}}$$

$$= \tfrac{1}{2} \cdot 2 \cdot 2^{\ell} (\ell + 1)$$

$$= \tfrac{1}{2} 2^{\ell+1} (\ell + 1). \quad \square$$

Example: To go from $n=4$ to $m=2$, we first split the x vector.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} F_2 x' \\ F_2 x'' \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$$

Step ①:

$$x = (2, 4, 6, 8)^t$$

splits into

$$x' = (2, 6) \text{ and } x'' = (4, 8).$$

Step ②: Multiplying by F_2 gives

$$y' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}.$$

$$y'' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x'' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix}.$$

Step 3:

$$y_j = y_j' + W_n^j y_j'', \quad j=0,\dots,m-1$$

$$y_{j+m} = y_j' - W_n^j y_j'' \quad j=0,\dots,m-1$$

$$y_0 = y_0' + W_4^0 y_0'' = 8 + e^{2\pi i(0)/4} \cdot (12) = 8+12=20.$$

$$y_1 = y_1' + W_4^1 y_1'' = -4 + e^{2\pi i/4} \cdot (-4) = -4-4i.$$

$$y_2 = y_0' - W_4^0 y_0'' = 8-12=-4.$$

$$y_3 = y_1' - W_4^1 y_1'' = -4 - i(-4) = -4+4i.$$

Originally we discussed finding coeffecients c in $Fc=f$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}.$$

$$c = \frac{1}{4}\overline{F}f = \begin{pmatrix} 5 \\ i-1 \\ 1 \\ -i-1 \end{pmatrix}.$$

Our example is $F_n f$. This previous example is $\frac{1}{4}\overline{F}f$.

In fact, the FFT is really a giant factorization of the Fourier matrix.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & & 1 & i \\ & 1 & & \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}} .$$

$y_j = y_j' + w_n^j y_j''$

$y_{j+m} = y_j' - w_n^j y_j''$

(Step 3)

F_2 appearing twice (Step 2)

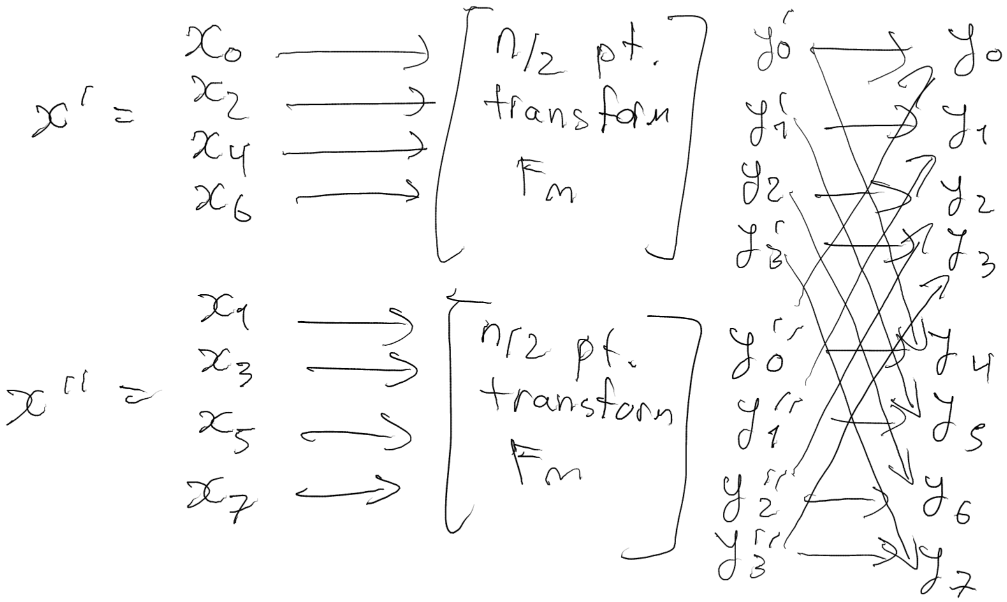
separates x into x' and x'' (Step 1)

Single matrix F with n^2 non-zeros becomes a $\log n$ product of matrices with $n \log n$ non-zeros.

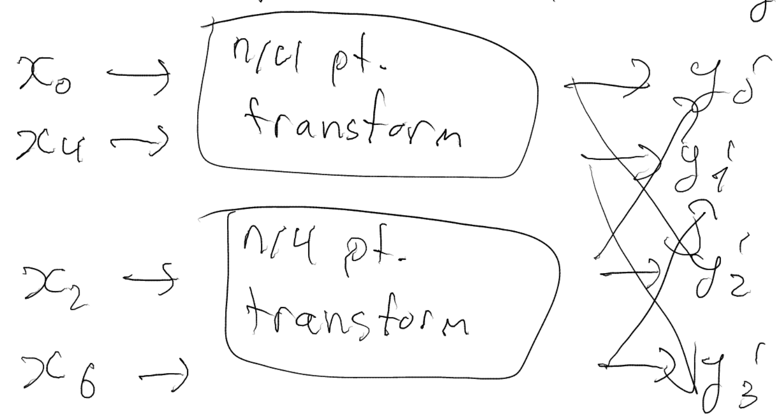
$n = 4$

$\log_2 4 = 2$.

The first step of the FFT changes multiplication by F_n into 2 multiplications by $F_m = F_{n/2}$.



Key idea: Replace each F_4 box by 2 F_2 boxes.



Then, each of these F_2 boxes is a single butterfly.

$\left(F_2 = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right)$.