10-18-17, Wed

Section 6.12 - Trigonometric interpolation:

Properties of Fourier series:

(There is an enormous literature on this subject. This is just a small sampling.)

① Each Fourier coeffecient A_K, B_K, or C_K is the best possible choice in the L_2 sense.

$$\text{error } E = \int_{-\pi}^{\pi} \left[f(x) - \sum_{k=0}^{n} (A_k \cos kx + B_k \sin kx)\right]^2 dx$$

We want to minimize this error. Check $\frac{\partial E}{\partial B_K}$ for example.

$$\frac{\partial E}{\partial B_k} = 2 \int_{-\pi}^{\pi} \sin kx \left[f(x) - B_k \sin kx\right] dx$$

because

$$\int_{-\pi}^{\pi} \underbrace{\underbrace{\sin kx}_{\text{odd}} \underbrace{\cos jx}_{\text{even}}}_{\text{odd}} \, dx = 0, \text{ and}$$

$$\int_{-\pi}^{\pi} \sin kx \sin jx \, dx = 0 \text{ if } k \neq j. \text{ Then to minimize,}$$

$$0 = 2 \int_{-\pi}^{\pi} \sin kx \left[f(x) - B_k \sin kx\right] dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin kx \, dx = B_k \int_{-\pi}^{\pi} \sin^2 kx \, dx$$

$$B_k = \frac{\int_{-\pi}^{\pi} f(x)\sin kx\,dx}{\int_{-\pi}^{\pi} \sin^2 kx\,dx} = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx}_{\text{Fourier coeffecient}}.$$

Similarly for A_k's.

<u>Interpretation</u>: Sines and cosines are a pendicular set of axis in some function space. So Fourier analysis projects f onto each of these axes.

② Since projections are never larger than the original,

$$\int_{-\pi}^{\pi} [a_0 + a_1\cos x + \dots + b_n \sin nx]^2\,dx \le$$

$$\le \int_{-\pi}^{\pi} (f(x))^2\,dx.$$

This is Bessel's inequality.

As $n \to \infty$, the Fourier series reconstructs f and we get

Fourier coeffecient

$$\int |f(x)|^2 \, dx = (2\pi)^{-n} \int |\hat{f}(\xi)|^2 \, d\xi,$$

Parseval's formula. So the length of f in Fourier space or in spatial domain is "the same".

Section 6.13: Fast Fourier Transform

Typical problem: Find coeffecients c_0, c_1, c_2, c_3 so that

$$\begin{cases} c_0 + c_1 + c_2 + c_3 = 2 \\ c_0 + i c_1 - c_2 + i c_3 = 4 \\ c_0 - c_1 + c_1 - c_3 = 6 \\ c_0 - i c_1 - c_2 + i c_3 = 8 \end{cases} \qquad (1)$$

These equations are asking us to look for a 4-term Fourier series that matches f at 4 equally spaced points. The discrete Fourier transform doesn't do more, in this case, then reproduce f at 4 points.

$$c_0 + c_1 e^{ix} + c_2 e^{i2x} + c_3 e^{i3x} = \begin{cases} 2 & \text{at } x = 0 \\ 4 & \text{at } x = \frac{\pi}{2} \\ 6 & \text{at } x = \pi \\ 8 & \text{at } x = \frac{3\pi}{2} \end{cases}.$$

If $x = \frac{\pi}{2}$,

$$c_0 + c_1 e^{i\frac{\pi}{2}} + c_2 e^{i2\frac{\pi}{2}} + c_3 e^{i3\frac{\pi}{2}} =$$

$$= c_0 + c_1\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] + c_2\left[\cos\pi + i\sin\pi\right] + c_3\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$$

$$= c_0 + ic_1 - c_2 - ic_3 = 4 \text{ by } (1)$$

and so on. If you add all four equations in (1), we get

$$4c_0 = 20$$

$$c_0 = 5.$$

Similarly by multiplying the 2nd equation by $-i$, 3rd equation by -1, and 4th by i and adding gives

$c_1 = -1 + i, \ldots$.

We can in fact rewrite the coeffecient matrix as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = A.$$

A is symmetric. Note $\overline{zw} = \bar{z}\bar{w}$. So,

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix}.$$

Then

$$A\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

So

$$A\bar{A} = 4I$$

$$\frac{1}{4}\bar{A} = A^{-1}.$$

function values

Thus, we can solve $Ac = f$ by simple matrix multiplication.

$$c = A^{-1}f \iff c = \frac{1}{4}\bar{A}f. \qquad \begin{pmatrix} c_0 \\ \vdots \\ c_4 \end{pmatrix}$$

Notice if we use GE to find "c", the cost $\Theta(n^3)$ where n is # of rows or columns of matrix. But now with matrix-vector multiplication, the cost is $\Theta(n^2)$.

$$c_0 = \frac{1}{4}(2+4+6+8) = 5.$$

$$c_1 = \frac{1}{4}\left(2(1) + 4(-i) + 6(-i)^2 + 8(-i)^3\right)$$

$$= \frac{1}{4}(2 - 4i + 6 + 8i)$$

$$c_1 = -1 + i.$$

This idea can be generalized to finding the n discrete Fourier coeffecients in the series

$$f(x) \approx \sum_{k=0}^{n} c_k e^{ikx}.$$

Take $n+1$ equally-spaced points in $[0, 2\pi]$ at spacing of $\frac{2\pi}{n}$

Set

$$\omega = e^{2\pi i/n}, \quad \text{i.e.}$$

ω are the n roots of unity.

Example:

1st point at $x=0$:

$$c_0 + c_1 + \cdots + c_{n-1} = f_0$$

$$x = \frac{2\pi}{n} \quad c_0 + c_1\omega + c_2\omega^2 + \cdots + c_{n-1}\omega^{n-1} = f_1$$

$$\vdots$$

$$x = \frac{4\pi}{n} \quad c_0 + c_1\omega^2 + \cdots + c_{n-1}\omega^{2(n-1)} = f_n .$$

Problem becomes solving the system

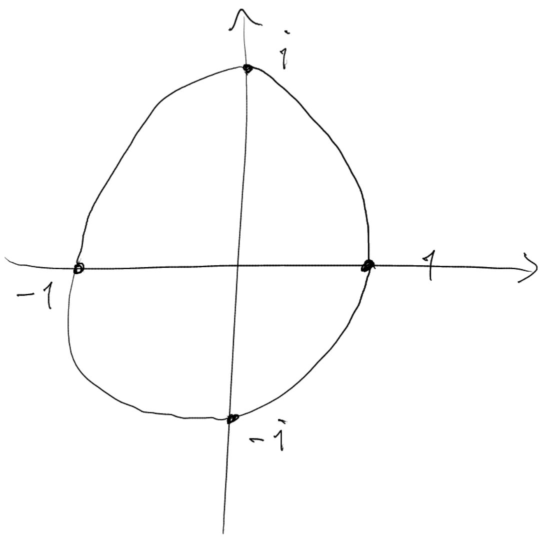
$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)2} \end{bmatrix},$$

Note for $n=4$, ω is the 4th root of 1.
So for example

$$\omega = e^{2\pi i/4} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i.$$

In general,

$$\omega = e^{2\pi i/n} \Rightarrow \omega^n = e^{2\pi i} = 1.$$



So ω lies on the unit circle in $\mathbb{C}$ at an angle $2\pi/n$ from the horizontal. Its square is 2-times as far around unit circle. Its nth power is all the way around.

For $n=2$, we get

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and } F_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Check:

$$F_2 F_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_2.$$

$n=3$:

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i2\pi/3} & e^{i4\pi/3} \\ 1 & e^{i4\pi/3} & e^{i8\pi/3} \end{bmatrix}.$$

$$F_3^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-8\pi i/3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}$$

Idea: Fast-Fourier transform will take this process of finding coeffecients

$$C = \frac{1}{n}\bar{F}f \quad (\mathcal{O}(n^2))$$

and speed it up even more.

In fact for the jk^{th} entry of F_n as

$$(F_n)_{jk} = \omega_n^{jk} = e^{2\pi i jk/n}, \qquad j,k = 0,1,\dots,n.$$

Our interest lies in powers like

$$n = 2^{12} \quad (\text{powers of } 2).$$