Midterm: (calcular)

- Through section 6.2 up to Newton divided difference but not Chebyshev polynomials
- Study:
  - notes
    - formulas, e.g. error formula for interpolation
    - proofs
  - homework (up to one due 10-11-17)

Section 6.8: Best approximation, least squares theory

Example: Use the best approximation theorem to determine a cubic polynomial

$$g(x) = c_1 x + c_2 x^2 + c_3 x^3$$

that approximates $f(x) = e^x$ on $[-1, 1]$.

Use the norm

$$\|f\|_2 = \left[\int_{-1}^{1} [f(x)]^2 \, dx \right]^{1/2}$$

Solution: The optimal function g satisfies

$$f - g \perp G$$

where G is the space generated by

$g_1(x) = x$, $g_2(x) = x^2$, and $g_3(x) = x^3$.

Then,

$$\langle g - f, g_1 \rangle = 0 = \langle c_1 x + c_2 x^2 + c_3 x^3 - e^x, x \rangle$$

$$\langle g - f, g_2 \rangle = 0 = \langle c_1 x + c_2 x^2 + c_3 x^3 - e^x, x^2 \rangle$$

$$\langle g - f, g_3 \rangle = 0 = \langle c_1 x + c_2 x^2 + c_3 x^3 - e^x, x^3 \rangle$$

Thus,

$$\begin{cases} c_1 \int_{-1}^{1} x^2\,dx + c_2 \int_{-1}^{1} x^3\,dx + c_3 \int_{-1}^{1} x^4\,dx = \int_{-1}^{1} x\,dx \\ c_1 \int_{-1}^{1} x^3\,dx + c_2 \int_{-1}^{1} x^4\,dx + c_3 \int_{-1}^{1} x^5\,dx = \int_{-1}^{1} x^2\,dx \\ c_1 \int_{-1}^{1} x^4\,dx + c_2 \int_{-1}^{1} x^5\,dx + c_3 \int_{-1}^{1} x^5\,dx = \int_{-1}^{1} x^3\,dx \end{cases}$$

This is a 3×3 linear system to solve for c_1, c_2, c_3.

Often the linear system we get from least squares are poorly conditioned (close to singular).

We get the Hilbert matrix

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & \frac{1}{n+1} \\ \vdots & & & & \\ 1/n & \frac{1}{n+1} & & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

which has a large condition number that grows as n gets big. Badly conditioned even for small n.

So we want to choose the polynomials we use more carefully.

<u>Idea</u>: Use orthogonal polynomials.

<u>Definition</u>: A finite or infinite sequence of vectors (or functions) $f_1, f_2, \ldots$ in an inner product space is orthogonal if

$$\langle f_i, f_j \rangle = 0 \quad (i \neq j).$$

They are orthonormal if for all i, j

$$\langle f_i, f_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

These orthogonal polynomials are good for approximation because of the following theorem.

<u>Theorem</u>: Let the set $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal system in an inner product space E. Then the best approximation of f by an element $\sum_{i=1}^{n} c_i g_i$ is obtained iff

$$c_i = \langle f, g_i \rangle.$$

<u>Proof</u>:

Let G be a subspace generated by $g_1, g_2, \ldots, g_n$.

By the previous theorem, the best approximation

$$\sum_{i=1}^{n} c_i g_i$$

is characterized by the condition

$$f - \sum_{i=1}^{n} c_i g_i \perp G \Leftrightarrow f \text{ is orthogonal to each basis vector } g_j.$$

Thus,

$$0 = \langle f - \sum_{i=1}^{n} c_i g_i, g_j \rangle = \langle f, g_i \rangle - \langle \sum_{i=1}^{n} c_i g_i, g_j \rangle$$

$$= \langle f, g_i \rangle - \langle c_j g_j, g_j \rangle$$

$$c_j = \langle f, g_j \rangle$$

by orthonormality of g_j. $\square$

Recall: Gram-schmidt process.

Strategy: If we want to approximate elements of a function space $\mathbb{E}$ by elements of a subspace G, first orthonomalize the basis $\{g_1, g_2, \ldots, g_n\}$ for G.
The approximate f by $\sum_{i=1}^{n} \langle f, g_i \rangle g_i$.

Looking back at the previous example, we approximated $f(x) = e^x$ in least squares sense by

$$g(x) = c_1 x + c_2 x^2 + c_3 x^3, \quad x \in [-1, 1].$$

$$e^x \simeq c_1 x + c_2 x^2 + c_3 x^3.$$

We need an orthonormal basis.

One example is Chebyshev polynomials.

$$T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0$$

and an inner product weighted by

$$w(x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Aside: We define the inner product of two continuos functions f and g by

$$\langle f, g\rangle = \int_a^b w(x) f(x) g(x), \quad f, g \in C([a,b])$$

where w is a non-negative weight function on (a,b).

This is an orthogonal family of polynomials with degree $(T_n) = n$.

Recall:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$\vdots$$

Example: $\langle T_0, T_1\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\, dx.$

Let

$$\begin{cases} u = 1 - x^2 \\ du = -2x\,dx \end{cases} \Rightarrow x\,dx = -\frac{1}{2} du$$

Then,

$$\langle T_0, T_1\rangle = -\frac{1}{2}\int_0^0 \frac{du}{\sqrt{u}} = 0$$

Since

$x = -1 \Rightarrow u = 0$, and

$x = 1 \Rightarrow u = 0$.

Example: (Legendre polynomials)

Let $w(x) = 1$ on $[-1,1]$.

The Legendre polynomials are given by

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x^2)^n\right], \quad n \geq 1.$$

Note, $P_0(x) = 1$.

Recall we had a theorem that said that the best approximating polynomial has the form

$$\sum_{i=1}^{n} c_i g_i$$

where $c_i = \langle f_i, g_i \rangle$.

Continuing Chebyshev example

$$c_1 = \int_{-1}^{1} w(x) T_1(x) e^x dx \qquad c_2 = \int_{-1}^{1} w(x) T_2(x) e^x dx =$$

$$c_1 = \int_{-1}^{1} \frac{x e^x}{\sqrt{1-x^2}}, \qquad = \int_{-1}^{1} \frac{(2x^2-1)e^x}{\sqrt{1-x^2}} dx,$$

$$c_3 = \int_{-1}^{1} \frac{(4x^3-3x)e^x}{\sqrt{1-x^2}} dx.$$

Then polynomial is

$$c_1 T_1 + c_2 T_2 + c_3 T_3.$$

Theorem: (on orthogonal polynomials)
A sequence of polynomial defined inductively by

$$P_n(x) = (x - a_n) P_{n-1}(x) - b_n P_{n-2}(x), \quad n \geq 2$$

with $P_0(x) = 1$, $P_1(x) = x - a_1$ and

$$a_n = \frac{\langle x P_{n-1}, P_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle}, \quad \text{and } b_n = \frac{\langle x P_{n-1}, P_{n-2} \rangle}{\langle P_{n-2}, P_{n-2} \rangle}.$$

Proof: Each P_n is monic of degree n.
Note: $P_0 = 1$. $P_1 = x - a_1$.

$$P_2 = (x - a_2) P_1(x) - b_2 P_0(x)$$

$$P_2 = (x - a_2)(x - a_1) - b_2$$

$$P_2 = x^2 - a_1 x - a_2 x + a_1 a_2.$$

$\vdots$

So one could show coeff. of highest order is 1.

Thus, $P_n \neq 0$. So inner product

$$\langle P_{n-1}, P_{n-1} \rangle \neq 0 \text{ and } \langle P_{n-2}, P_{n-2} \rangle \neq 0.$$

Need to show

$$\langle P_n, P_i \rangle = 0, \quad i = 0, 1, \ldots, n.$$

Base case: $n=1$

For $n=1$, we have

$$\langle P_1, P_0 \rangle = \langle x - a_1, 1 \rangle =$$

$$= \langle (x - a_1) P_0, P_0 \rangle$$

$$= \langle x P_0, P_0 \rangle - \langle a_1 P_0, P_0 \rangle$$

$$= \langle x P_0, P_0 \rangle - \frac{\langle x P_0, P_0 \rangle}{\langle P_0, P_0 \rangle} \langle P_0, P_0 \rangle$$

$$= 0.$$

Induction step: Assume the polynomials are orthogonal for P_{n-1}, $n \geq 2$. Then,

$$\langle P_n, P_{n-1} \rangle$$