Jonathan notes
Bicycle upstairs

Midterm: (simple calculator)
① Study notes
② Study Hwk problems

10-04-17, Wed

Splines continued:

Question: Does the continuity of S, S', S'' give enough constraints to uniquely define the cubic spline?
Answer: No (endpoint conditions).

Cost: There are $4n$ coeffecients in the cubic spline that define the n cubics.
On each subinterval $[t_i, t_{i+1}]$, there are two interpolation conditions. Namely

$$\left.\begin{aligned} S(t_i) &= y_i \\ S(t_{i+1}) &= y_{i+1} \end{aligned}\right\} \text{ This gives } 2n \text{ constraints}$$

- continuity of S (no new constraints)
- continuity of S' ($n-1$ constraints)
- continuity of S'' ($n-1$ constraints)

So we have now $2n + 2(n-1) = 4n-2$ constraints.
Two degrees of freedom that remain are used up by

$$S''(t_0) = S''(t_n) = 0$$

(linear endpoints).

coeffecients of cubics

Linear system for z_i's is

$$\begin{bmatrix} u_1 & h_1 & & & \\ h_1 & u_2 & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & h_{n-2} \\ & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}.$$

Section 6.8: Best approximation - Least squares

Example: (Classic problem of best approximation)

Can be state as follows:

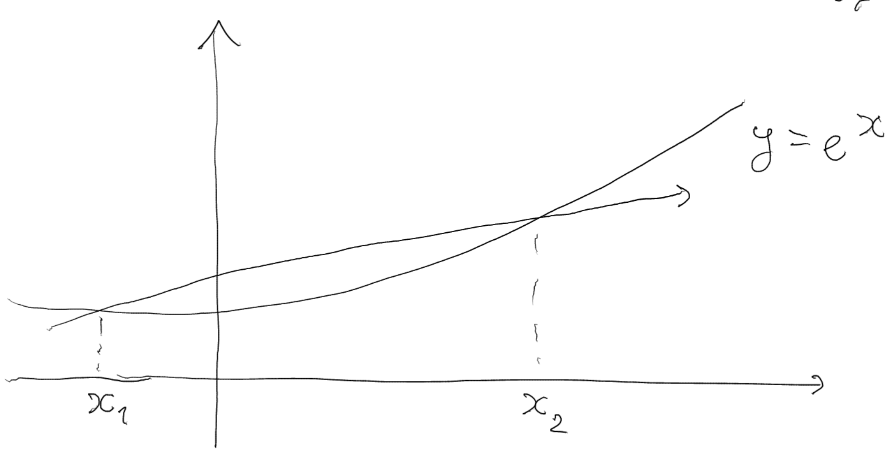
Given a continuos function f on $[a,b]$ and fixed n, we seek polynomial p (of degree $\le n$) so that

$$\max_{a \le x \le b} |f(x) - p(x)| \text{ is as small as possible.}$$

This is a minimax problem.

Example: Compute the minimax polynomial approximation $P_1(x)$ to $f(x) = e^x$ on $-1 \le x \le 1$. Let $P_1(x) = a_0 + a_1 x$.

Goal: We want to find the best a_0, a_1 we can.



Let the error be

$$\varepsilon(x) = e^{x} - [a_0 + a_1 x].$$

Clearly e^x and $P_1(x) = a_0 + a_1 x$ must be equal at two points x_1 and x_2 in $[-1, 1]$.

So,

$$\varepsilon(x_1) = \varepsilon(x_2) = 0.$$

Note also

$$\varepsilon(x) = e^{x} - a_0 - a_1 x$$

$$\varepsilon'(x) = e^{x} - a_0$$

$$\varepsilon'(x) = 0 \Rightarrow$$

$$\Rightarrow e^{x} = a_1$$

① $\boxed{x = \ln a_1}$

We can shift the graph of $P_1(x)$ so that the max error is equal and occurs at 3 points: $x = -1, 1, x_3$

$x_1 < x_3 < x_2$.

So we also have that if ρ = max error then

② $e^{-1} - a_0 - a_1(-1) = \rho$

③ $e^{1} - a_0 - a_1(1) = \rho$

④ $e^{x_3} - a_0 - a_1(x_3) = \rho$.

Thus system ① - ④ give

$a_0 = 1.2643$ $\quad \rho = 0.2788$

$a_1 = 1.1752$

$x_3 = 0.1614$.

$P_1(x) = 1.2643 + 1.1752x$

* The error is evenly distributed over $[-1, 1]$.

<u>Theorem</u>: (Theorem on existence of best approximation)
If G is a finite-dimensional subspace in a normed linear space $\mathbb{E}$, then each point of $\mathbb{E}$ posses at least one best approximation in G.
(Unfortunately these best approximations are often not unique).

<u>Problem</u>: Given a function f on the interval $[a,b]$ find a polynomial of degree $\leq n$ that deviates as little as possible from f over the whole interval.

$$\min \max_{a \leq x \leq b} |f(x) - p(x)|$$

We need a way to measure distance between functions.
An inner product space is a linear space $\mathbb{E}$ with an associated inner product and norm that satisfy these properties:

<u>Axioms for inner products</u>:

(a) $\|f\| = \sqrt{\langle f,f \rangle}$

symmetry (b) $\langle f,g \rangle = \langle g,f \rangle$

linearity (c) $\langle f, \alpha g + \beta h \rangle = \alpha \langle f,g \rangle + \beta \langle f,h \rangle$

positivity (d) $\langle f,f \rangle > 0$ if $f \neq 0$

Example: Inner product on $\mathbb{R}^n$ gotten from Euclidean distance metric

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Note:

$$\sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2} = \|x\|_2 .$$

Note: $f \perp g$ if $\langle f, g \rangle = 0$.

$$\underset{\text{function}}{f} \perp \underset{\text{space}}{G} \quad \text{if} \quad \langle f, g \rangle = 0, \; g \in G.$$

Theorem: (Theorem on characterizing best approximation)
Let G be a subspace in an inner product E.
For a function $f \in E$ and $g \in G$, these properties are equivalent.

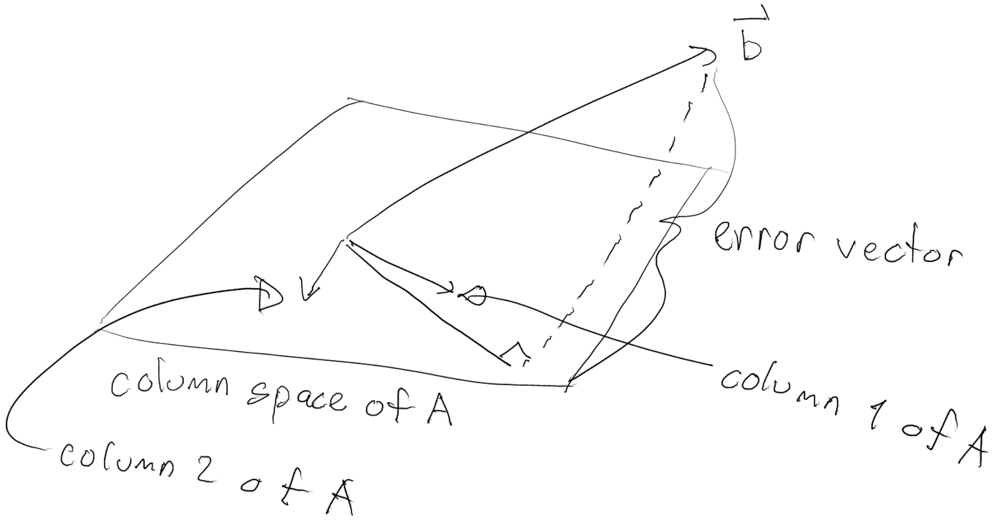
1. g is the best approximation to f in G
2. $f - g \perp G$.

Example: (Least squares)

We want to solve $Ax=b$. $A \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^m$, $b \in \mathbb{R}^n$.

Take $A \in \mathbb{R}^{3\times 2}$, $x \in \mathbb{R}^2$, $b \in \mathbb{R}^3$.

Clearly A does not have an inverse in the typical sense.



$\bar{x}$ = least squares solution of $Ax=b$.

Normal equation:

$$A^T A x = A^T b \Rightarrow$$

$$\Rightarrow A^T(\underbrace{Ax-b}_{\text{error vector}}) = 0$$

So error vector is $\perp$ to columns of A.

Proof: (Theorem on characterizing best approximation)

($\Leftarrow$):

If $f-g \perp G$ then for any $h \in G$

$$\|f-h\|^2 = \|f-g+g-h\|^2 =$$

$$= \|(f-g)\|^2 + \|g-h\|^2 + 2\langle f-g, g-h\rangle$$

$$\Downarrow$$

$$\langle f-g, g\rangle + \langle f-g, -h\rangle$$

$$= \|(f-g)\|^2 + \|g-h\|^2 \quad (f-g \perp G)$$

$$\geq \|(f-g)\|^2. \quad (\|g-h\|^2 \geq 0)$$

Proof: ($\Rightarrow$)

Suppose g is the best approximation to f.

Let $h \in G$ and $\lambda > 0$. Then

$$0 \leq \|f - \underbrace{g + \lambda h}_{\in G}\|^2 - \|f-g\|^2$$

$$= \langle f-g+\lambda h, f-g+\lambda h\rangle - \|f-g\|^2$$

$$= \langle f-g, f-g\rangle + 2\lambda\langle h, f-g\rangle + \lambda^2\langle h,h\rangle - \|f-g\|^2$$

$$= \|f-g\|^2 + 2\lambda\langle h, f-g\rangle + \lambda^2\|h\|^2 - \|f-g\|^2$$

$$= \lambda\{2\langle f-g, h\rangle + \lambda\|h\|^2\}.$$

Take the limit as $\lambda \to \infty$ in

$$0 \leq \lambda \{ 2\langle f-g, h\rangle + \lambda \|h\|^2 \}.$$

So 2nd term goes to 0 faster than the first.
Hence,

$$0 \leq \langle f-g, h\rangle.$$

Similarly using $-h \in G$,

$$0 \geq \langle f-g, -h\rangle.$$

Thus,

$$\langle f-g, h\rangle = 0. \quad \square$$