△ Get notes from Jonathan!

Classic example:

Let $f(x) = \frac{1}{x^2+1}$, for $x \in [-5, 5]$.

This is called the Runga function.

With eveningly spaced node points on $[-5, 5]$, we can show that

$$\lim_{n\to\infty} \|f - P_n\|_\infty = \infty$$

where

$$\|f\|_\infty = \max_x |f(x)|.$$

The idea behind Chebyshev Polynomials is to optimize placement of the nodes so that if possible

$$\|f - P_n\|_\infty \to 0 \text{ as } n \to \infty.$$

The idea is we want to do uniformly well (hence ∞ norm) at approximating f on whole interval $a \le x \le b$.

Chebyshev polynomials of degree n (on $-1 \le x \le 1$) are defined by the rule

① $T_n(\cos\theta) = \cos n\theta$

The book uses

② $T_n(x) = \cos(n \cos^{-1} x)$, $n \ge 0$

Let $x = \cos\theta$

$\cos^{-1} x = \theta$

$T_0(x) = \cos(0 \cdot \cos^{-1} x) = 1$

$T_1(x) = \cos(1 \cdot \cos^{-1} x) = x$.

In fact, to find $T_2, T_3, \ldots$ we can use previous values.

Recall: $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

$T_{n+1}(x) = \cos[(n+1)\cos^{-1} x] = \cos((n+1)\theta)$

since $\cos^{-1} x = \theta$.

(3) $\cos((n+1)\theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$

$\cos((n-1)\theta) = \cos n\theta \cos(-\theta) - \sin n\theta \sin(-\theta)$

$= \cos n\theta \cos\theta + \sin n\theta \sin\theta$

$\left(\begin{array}{l} \cos \text{ is even on symmetric interval} \\ \sin \text{ is odd} \quad - \quad - \quad - \end{array} \right)$

Adding (3) and (4)

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cos\theta$$

$$\underbrace{\cos(n+1)\theta}_{T_{n+1}(x)} = \underbrace{2\cos n\theta \cos\theta}_{2T_n(x)\cdot x} - \underbrace{\cos(n-1)\theta}_{T_{n-1}(x)}$$

$$\boxed{T_{n+1}(x) = 2T_n(x)\cdot x - T_{n-1}(x)}, \quad n \geq 1.$$

$T_0(x) = \cos(0 \cdot \cos^{-1} x) = 1$

$T_1(x) = \cos(1 \cdot \cos^{-1} x) = x.$

$T_2(x) = 2T_1(x) \cdot x - T_0 = 2x^2 - 1.$

$T_3(x) = 2T_2(x) \cdot x - T_1 = 2(2x^2-1)x - x$

$\qquad = 4x^3 - 3x.$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

and so on...

Properties of Chebyshev polynomials:

① $|T_n(x)| = |\cos(n \cos^{-1} x)| \le 1, \quad -1 \le x \le 1$

② $T_n(\cos \frac{j\pi}{n}) = \cos(n \cos^{-1}(\cos \frac{j\pi}{n}))$

$$= \cos(j\pi)$$

$$= (-1)^j, \quad 0 \le j \le n.$$

③ $T_n(\cos \frac{2j-1}{2n}\pi) = 0, \quad 1 \le j \le n.$

Pf: $T_n(\cos \frac{2j-1}{2n}\pi) = \cos(n \cos^{-1}(\cos \frac{2j-1}{2n}\pi))$

$$= \cos(\frac{2j-1}{2}\pi).$$

Terminology:

Definition: A monic polynomial is a polynomial where the term of highest degree has coeffecient 1.

In fact, the Chebyshev polynomials $T_n(x)$ have a 1st term of the form

$$2^{n-1} x^n \quad \text{for } n > 0.$$

So,

$$\frac{1}{2^{n-1}} T_n(x) = 2^{n-1} T_n(x) \quad \text{is monic.}$$

Theorem: If P is a monic polynomial of degree n then

$$\|p\|_\infty = \max_{-1 \le x \le 1} |p(x)| \ge 2^{1-n}.$$

Proof:

Suppose $|p(x)| < 2^{1-n}$. $(|x| \le 1)$

Let $q = 2^{1-n} T_n$ and $x_i = \cos\left(\frac{i\pi}{n}\right)$.

As noted, q is monic.

Note: $(-1)^i p_i(x_i) \le |p(x_i)|$.

By assumption $|p(x_i)| < 2^{1-n}$.

By the properties of Chebyshev polynomials,

$$q(x_i) = 2^{1-n} T_n(x_i) = 2^{1-n} T_n\left(\cos\left(\frac{i\pi}{n}\right)\right) = 2^{1-n} \cdot (-1)^i$$

$$\Rightarrow (-1)^i q(x_i) = 2^{1-n}.$$

So $(-1)^i p(x_i) \le |p(x_i)| < 2^{1-n} = (-1)^i q(x_i)$.
Then,

$$0 < (-1)^i (q(x_i) - p(x_i)), \quad 0 \le i \le n$$

Thus $q-p$ oscillates in sign $n+1$ times on $[-1,1]$.
So there must be at least n roots of the polynomial $q-p$ in $[-1,1]$.
But $q-p$ were both monic polynomials of degree n, so the leading term of both q and p was x^n.
So $q-p$ is a polynomial of degree $n-1$.

This is a contradiction. Hence,

$$|p(x)| \ge 2^{1-n}, \quad |x| \le 1 \quad \square.$$

Recall the theorem on interpolation error:

Theorem: Let f be a function in $C^{n+1}[a,b]$ and let p be the polynomial of degree $\le n$ that interpolates f at $n+1$ distinct points $x_0, x_1, \dots, x_{n+1}$ on $[a,b]$.
For each $x \in [a,b]$, there is a point $\xi_x \in (a,b)$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i).$$

If we assume our nodes for interpolation and x are in $[-1,1]$, then

$$\max_{|x|\leq 1} |f(x) - P(x)| \leq \frac{1}{(n+1)!} \max_{|x|\leq 1} |f^{(n+1)}(x)| \cdot \max_{|x|\leq 1} \left| \prod_{i=0}^{n} (x - x_i) \right| .$$

From the theorem on monic polynomials, we have

$$\max_{|x|\leq 1} \underbrace{\left| \prod_{i=0}^{n} (x - x_i) \right|}_{\text{Monic polynomial of degree } n+1} \geq 2^{1-(n+1)} = 2^{-n} .$$

It turns out that the best you can do

(i.e. $\max_{|x|\leq 1} \left| \prod_{i=0}^{n} (x - x_i) \right| = 2^{-n}$)

is when $\prod_{i=0}^{n} (x - x_i) = 2^{-n} T_{n+1}(x)$

and the nodes $x_0, x_1, \ldots, x_n$ will be the roots of T_{n+1}, i.e. roots are

$$\cos\left[\frac{2j-1}{2n}\pi\right] . \quad 1 \leq j \leq n$$

So the roots of T_{n+1} are

$$\cos\left[\frac{2j-1}{2n}\pi\right], \quad 0 \leq j \leq n.$$

<u>Theorem</u>: (on interpolation error)

If the nodes x_i are the roots of Chebyshev polynomials T_{n+1}, then for $x \in [-1, 1]$

$$|f(x) - p(x)| \leq \frac{1}{2^n (n+1)!} \max_{|x| \leq 1} f^{(n+1)}(x) .$$

This means that

$$\|f(x) - p_n(x)\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

for Chebyshev polynomials.