11-15-17, Wed

Section 8.2: Taylor series methods

IVP: $\begin{cases} x'(t) = f(t, x) \\ x(t_0) = x_0 \end{cases}$

Example:

$\begin{cases} y'(x) = 2x \\ y(0) = 0 \end{cases}$

True solution: $y(x) = x^2$

Euler solution: $y_n(x) = x_n x_{n-1}$.

Local truncation error t_n (or x_n) to t_{n+1} (or x_{n+1}) is $O(h^2)$. Global error is $O(h)$.

	x	approx. sol. from Euler $y_h(x)$	True sol. $y(x)$	Error $y(x) - y_h(x)$
$h=0.2$	0.0	0.0	0.0	0.0
	0.4	0.37631	0.34483	−0.03148
	0.8	0.54228	0.487870	−0.051448
	1.2	0.52709	0.49180	
	1.6	0.46623	0.44944	
$h=0.1$	0.0		" "	
	0.4	0.36085		−0.1603
	0.8	0.51377		−0.0250
	1.2	0.50961		−0.01781
	1.6	0.45871		−0.00928
$h=0.05$	0.0		" "	
	0.4	0.35287		−0.00804
	0.8	0.50049		−0.01268
	1.2	0.50073		−0.00892
	1.6	0.45425		−0.00489

(Different ODE)

In fact, we can generalize the global error result as general.

<u>Theorem</u>: Let y_n be the approximate solution to the IVP

$$\begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

gotten from Euler's method. If the exact solution $y(x)$ has a continuos 2nd derivative on an interval around the initial point and if

$$\left| \frac{\partial f}{\partial y} \right| \leq L$$

and

$$|y''(x)| < C$$

are satisfied for L and C, then the error

$$e_n = y(x_n) - y_n$$

of Euler's method at point x_n is

$$|e_n| \leq \frac{hC}{2L} \left(e^{(x_n - x_0)L} - 1 \right).$$ (upper bound on global error)

In other words, the error is $O(h)$.

The proof uses difference equation theory.

<u>Example</u>: Look at IVP

$$\begin{cases} y'(x) = y \\ y(0) = 1 \end{cases}, \quad x \in (0, 1].$$

$f(x, y) = y$, $\frac{\partial f}{\partial y} = 1$, so $L = 1$.

$$\frac{dy}{y} = dx$$

$$\ln y = x + c$$

$$y = Ce^{x}.$$

$$y(0) = 1 \Rightarrow 1 = Ce^{0} \Rightarrow C = 1.$$

<u>True solution</u>: $y(x) = e^{x}$.

$y''(x) = e^{x}$ and so

$|y''(x)| \leq e$ on $[0, 1]$.

$C = e$.

So

$$|e_n| \leq \frac{hC}{2L}\left(e^{(x_n - x_0)L} - 1\right)$$

$$|e_n| \leq \frac{he}{2 \cdot 1}\left(e^{(1-0)(1)} - 1\right)$$

$$|e_n| \leq \frac{he}{2}(e - 1) < 2.4h.$$

If $h = 0.1$ then

$|\text{error}| \leq 0.24$.

Question: How good is this error bound?

Answer: Euler gives

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + hy_n$$

$$y_{n+1} = y_n(1+h).$$

(difference equation)

Difference equation:

Differential equations move forward via an in infinite number of infinitesimal steps. Difference equations move forward in a finite number of finite steps.

Example: Suppose you invest \$1000 for 5 years at 6% interest.

If it is computed once a year, then

$$P_{k+1} = 1.06P_k.$$

This is a difference equation with a time step of 1 year. After 5 years,

$$P_5 = (1.06P_4) = (1.06)(1.06P_3) = \dots = (1.06)^5 P_0 =$$

$$= (1.06)^5 P_0 \simeq \$1338.$$

If we reduce the time step to one month, the new difference equation is

$$P_{k+1} = \left(1 + \frac{0.06}{12}\right) P_k .$$

Thus after 5 years,

$$P_{60} = \left(1 + \frac{0.06}{12}\right)^{60} P_0 = \left(1 + \frac{0.06}{12}\right)^{60} (1000) =$$

$$= \$1349.$$

Next step is compounding interest daily ... and then continuosly compounding.

When you compound the interest more and more often, we switch to a differential equation

$$P_{k+1} = (1 + 0.06\,\Delta t) P_k \Rightarrow$$

$$\frac{P_{k+1} - P_k}{\Delta t} = 0.06 P_k \Rightarrow$$

$$\Rightarrow \frac{dP}{dt} = 0.06 P.$$

Solution: $P(t) = P_0 e^{0.06t}$.

After 5 years, we get

$$P(5) = 1000 \cdot e^{0.06(5)} \approx \$1349.87.$$

Some difference equations do not correspond to differential equations at all.

Example: Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13

Difference equation: $F_{k+2} = F_{k+1} + F_k$

Question: How can we find the 1000th Fibonacci number other than starting with $F_0 = 0$ and $F_1 = 1$.

$$F_{k+2} = F_{k+1} + F_k .$$

Lets write this as

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}, \quad \text{so}$$

$$F_{k+2} = F_{k+1} + F_k$$
$$F_{k+1} = F_{k+1}$$
$$\Updownarrow$$

$$u_{k+1} = A u_k \quad \Rightarrow \quad \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

Then,

$$u_{k+1} = A u_k$$

is easy to solve.

$$u_1 = A_0$$
$$u_2 = A u_1 = A(A u_0) = A^2 u_0$$
$$\vdots$$
$$u_{k+1} = A^{k+1} u_0 .$$

The real issue is finding an effecient way to calculate powers of A. In fact, we don't compute powers of A.

<u>Theory</u>: If A can be diagonalized (e.g. A is real symmetric) then

$$A = S\Lambda S^{-1}$$

with column of A eigenvectors of A and Λ a diagonal matrix whose diagonal entries are the diagonals of A.

Then

$$u_k = A^k u_0 = \underbrace{S\Lambda^k S^{-1}}_{= (S\Lambda S^{-1})(S\Lambda S^{-1}) \ldots (S\Lambda S^{-1})} u_0 .$$

So,

$$u_k = [x_1 | \ldots | x_n] \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{S^{-1}u_0}$$

$$u_k = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \ldots + c_n\lambda_n^k x_n .$$

<u>Back to Fibonacci differential equation example:</u>

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} .$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1 \pm \sqrt{5}}{2} .$$

Finding eigenvectors we get solution F_k.

Difference equations show up in Markov processes, finance, etc.

Back to Euler method example:

$$y_{n+1} = y_n(1+h)$$

$$\left(u_{k+1} = A u_k = A^{k+1} u_0\right)$$

$$y_n = (1+h)^n y_0 .$$

For $h=0.1$, $n=10$ we get

$$y_{10} = (1+0.1)^{10}(1) = (1.1)^{10} \approx 2.5937.$$

Then,

$$y_{exact} - y_{10} \approx e - 2.5937 = 2.71828 - 2.5937 = 0.12458.$$

Predicted error was 0.24.

(True error is smaller than upper bound).

We can try to improve the accuracy of the solution by using higher-order Taylor series methods.

Example:

$$\begin{cases} xy' = x - y \\ y(2) = 2 \end{cases} \iff \begin{cases} y' = \dfrac{x-y}{x} \\ y(2) = 2 \end{cases} \iff \begin{cases} y' = 1 - \dfrac{y}{x} \\ y(2) = 2 \end{cases}.$$

Taylor series expansion about $x_0 = 2$ is

$$y(x) = y(2) + (x-2)y'(2) + \frac{(x-2)^2}{2} y''(2) + \frac{(x-2)^3}{3!} y'''(2) + \dots$$

We need derivatives of the RHS of DE.
We have the 1st derivative

$$y' = 1 - \frac{y}{x}.$$

$$y'(x) = 1 - yx^{-1}$$

$$y''(x) = -y'x^{-1} + yx^{-2}y' = -\frac{y'}{x} + \frac{y}{x^2} =$$

$$= -\frac{1 - yx^{-1}}{x} + \frac{y}{x^2}.$$

$$y'''(x) = -\frac{y''}{x} + \frac{y'}{x^2} + \frac{y'}{x^2} - \frac{2y}{x^3}.$$

So TS method of order 2 here is

$$y(x) = y_0 + (x-2)y_0' + \tfrac{1}{2}(x-2)^2 y_0'', \text{ or}$$

$$y(x) = 2 + (x-2)\left[1 - \frac{2}{2}\right] + (x-2)^2\left[0 + \frac{2}{2^2}\right]$$

$$y(x) = 2 + \tfrac{1}{4}(x-2)^2.$$

This scheme stops before the h^3 term, so the local truncation error is $O(h^3)$.
Global error $O(h^2)$.