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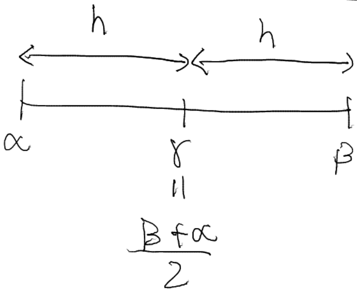
$\boxed{\approx}$

Section 7.5: Adaptive quadrature

$$I_{\alpha,\beta} = \int_\alpha^\beta f(x)\,dx.$$



$$I^1_{\alpha,\beta} = \frac{h}{3}\left[f(\alpha) + 4f\left(\frac{\alpha+\beta}{2}\right) + f(\beta)\right].$$

$$I^2_{\alpha,\beta} = I^1_{\alpha,\gamma} + I^1_{\gamma,\beta}.$$

Suppose we have a tolerance $\varepsilon > 0$. We want our approximation

$$\left| I^{\text{true}} - I^{\text{approx}} \right| \le \varepsilon.$$

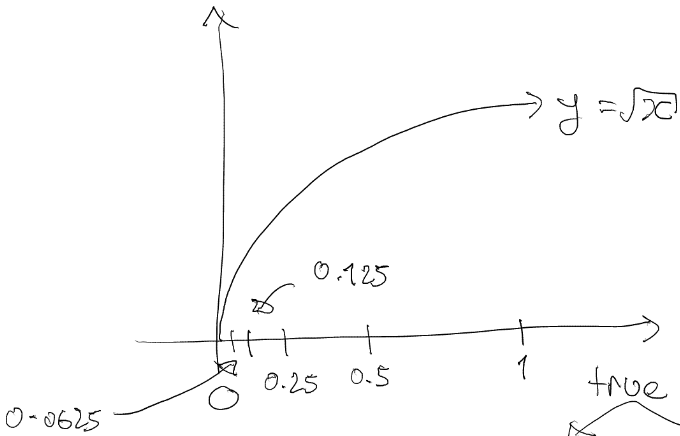
The algorithm starts with

1. Compute $I^1_{\alpha,\beta}$ and $I^2_{\alpha,\beta}$.
2. If $|I^1_{\alpha,\beta} - I^2_{\alpha,\beta}| \le \varepsilon$ then accept $I^2_{\alpha,\beta}$ as integral approx.
3. Otherwise let $\varepsilon' = \frac{\varepsilon}{2}$. Let $I_{\alpha,\beta} = I_{\alpha,\gamma} + I_{\gamma,\beta}$ where each to be computed with error $\le \frac{\varepsilon}{2}$.

Recurse.

The key issue: be careful not to evaluate integrand $f(x)$ at same point twice. Keep track of points you evaluate f at.

Example: $I = \int_0^1 \sqrt{x}\, dx$ with $\varepsilon = 0.005$ on $[0,1]$.



Error estimates

true errors

$[\alpha, \beta]$	I^2	$I - I^2$	$I - I^1$	$\lvert I^2 - I^1 \rvert$	ε
$[0, 0.0625]$	0.010258	$1.6 \cdot 10^{-4}$	$4.5 \cdot 10^{-4}$	$2.9 \cdot 10^{-4}$	0.0003125
$[0.6625, 0.125]$	0.19046	$1.2 \cdot 10^{-7}$	$1.1 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	0.0003125
$[0.125, 0.25]$	0.053871	$4.5 \cdot 10^{-7}$	$3.6 \cdot 10^{-6}$	$4 \cdot 10^{-6}$	0.00625
$[0.25, 0.5]$	0.152368	$9.3 \cdot 10^{-7}$	$1.1 \cdot 10^{-5}$	$1 \cdot 10^{-5}$	0.00125 $\frac{0.005}{4}$
$[0.5, 1]$	0.430962	$2.4 \cdot 10^{-6}$	$3 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$	0.0025

$\frac{0.005}{2}$

add these up to get I^2 on $[0,1]$.

Error analysis:

For Simpson's rule,

$$\int_a^b f(x)dx = \underbrace{\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]}_{S(a,b)} - $$

$$\underbrace{-\frac{1}{90}\left[\frac{b-a}{2}\right]^5 f^{(4)}(\xi)}_{\text{Error}(a,b)}, \quad \xi \in (a,b).$$

If Simpson's rule isn't accurate enough, divide our interval in half and apply the rule in each half. Repeat as needed. So

(3) $I = S^1 + E^1, \qquad S^1 = S(a,b),\ E^1 = -\frac{1}{90}\left(\frac{h}{2}\right)^5 f^{(4)}(\xi)$

where $h = b-a$. Apply Simpson twice to $[a,b]$ to get

(4) $I = S^2 + E^2.$

Here,

$$S^2 = S(a,c) + S(c,b), \quad c = \text{midpoint}\left(\frac{a+b}{2}\right)$$

$$E^2 = -\frac{1}{90}\left(\frac{h/2}{2}\right)^5 f^{(4)} - \frac{1}{90}\left(\frac{h/2}{2}\right)^5 f^{(4)} =$$

$$= \frac{1}{16}E^1.$$

Now subtract (3) from (4) to get

$$S^2 - S^1 = E^1 - E^2 = 16E^2 - E^2 = 15E^2, \text{ or}$$

$$I = S^2 + E^1 = S^2 + \underbrace{\frac{1}{15}(S^2 - S^1)}$$

Error approximation.
(we have this!)
Require this to be $< \varepsilon$.

If this test $\frac{1}{15}(S^2 - S^1)$ is not $\leq \varepsilon$ then subdivide intervals $[a, c]$ and $[c, d]$ and require error $\leq \frac{\varepsilon}{2}$ on each subinterval.

Section 8.2: Taylor series methods

In this chapter we devise methods for solving ODE's.
We consider only 1^{st}-order ODE's.

1. $$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases} \quad \text{(IVP)}$$

We will assume that (at a minimum) $f(t,x)$ is continuos for all (t,x) in some subset of $\mathbb{R}^2$.
Recall if f and $\frac{\partial f}{\partial x}$ are continuos in the rectangle
$R = \{(t,x): |t-t_0| \le \alpha, \ |x-x_0| \le \beta\}$
(rectangle around initial point (t_0, x_0))
then the initial value problem (1) has a unique solution in an interval around t_0.

Example: $x'(t) = -x(t)$.

$f(x) = -x(t)$ is continuos for all (t,x).

$\frac{\partial f}{\partial x} = -1$ " "

So we have a unique solution.

$$\frac{dx}{dt} = -x \qquad\qquad \int -\frac{dx}{x} = \int dt$$

$$-\frac{dx}{x} = dt \qquad\qquad -\ln x = t + c$$

$$\ln x = -t + c$$

$$x(t) = e^{-t+c} = e^c e^{-t} = ce^{-t}.$$

LHS:

$x(t) = ce^{-t}$

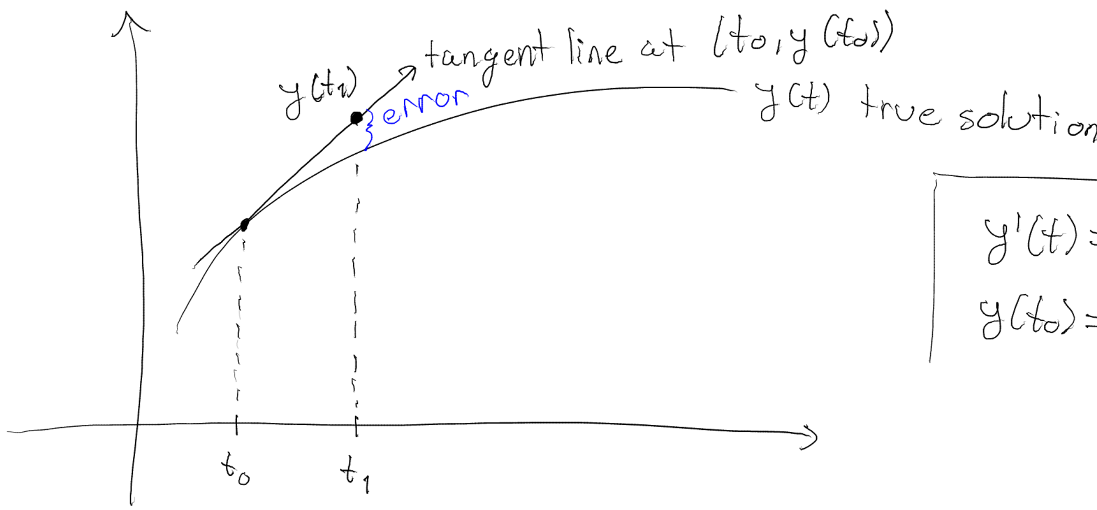
$x'(t) = -ce^{-t}$

RHS:

$-x(t) = -ce^{-t}$.

The simplest method for solving 1^{st} order ODEs numerically is Euler's method.

Geometric derivation:



$$y'(t) = f(t,y)$$
$$y(t_0) = y_0.$$

We will solve the DE at a discrete set of points (t_j, y_j) and

$t_j = t_0 + jh, \quad j = 0, 1, \dots,$

$h \Rightarrow$ fixed (equal) grid spacing between node points.

2 equations for slope of tangent line:

$$\frac{y(t_1) - y(t_0)}{t_1 - t_0} = \frac{y(t_1) - y(t_0)}{h} = y'(t_0) = f(t_0, y_0).$$

$$\underbrace{y(t_1)}_{\text{New value of solution to DE.}} = y_0 + hf(t_0, y_0)$$

Let $y(t_n)=y_n$. In general,

$$y_{n+1} = y_n + hf(t_n, y_n), \quad n=0,1,\ldots.$$

Second derivation: Use Taylor series to expand $y(t_{n+1})$.

local truncation error

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(\xi_n), \quad t_n < \xi_n < t_{n+1}.$$

Euler's method is a linear TS approximation to solution y.

$$y(t_{n+1}) = y(t_n) + hf(t_n, y_n).$$

Remainder term is

$$\frac{h^2}{2}y''(\xi_n).$$

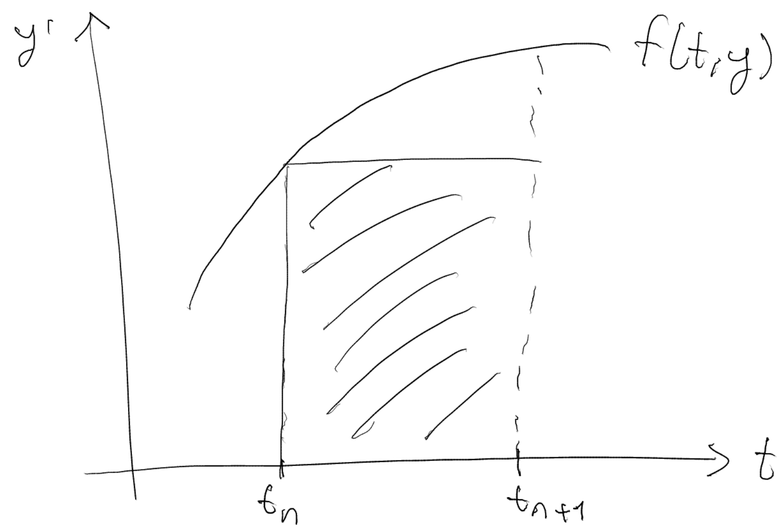
Third derivative: Integrate $y'(t)=f(t,y)$ over $[t_n, t_{n+1}]$.

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

FTC gives: | Using the left-handed rectangle rule

$$y(t_{n+1}) - y(t_n) = f(t_n, y_n)(t_{n+1} - t_n)$$

$$y(t_{n+1}) = y(t_n) + hf(t_n, y_n).$$



This idea leads to a class of methods called multi-step methods which we will see later.

Example:

IVP: $\begin{cases} y'(x) = 2x \\ y(0) = 0 \end{cases}$

True solution:

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

$$\int dy = \int 2x\,dx$$

$$y(x) = x^2 + c$$

$$0 = 0^2 + c \Rightarrow c = 0.$$

$$y(x) = x^2.$$

Euler's method for this problem:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + 2h x_n$$

$$y_0 = 0.$$

So

$$y_1 = y_0 + h x_0 = 0 + 2h(0) = 0.$$

Assume that the solution from Euler's method is of the form

$$y_n = x_n x_{n-1}, \quad n \geq 1.$$

Proof: (Induction)

Base case:

$$y_1 = x_1 x_0 = x_1 (0) = 0.$$

Induction step:

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \text{Euler's method.}$$

$$y_{n+1} = x_n x_{n-1} + 2h x_n = x_n (x_{n-1} + 2h) = x_n x_{n+1}. \quad \square$$

So the error is

$$y(x_n) - y_n = x_n^2 - x_n x_{n-1} = x_n(x_n - x_{n-1}) = x_n h.$$

$y(x_n)$: true

y_n: approx from Euler

So even though Euler had a local truncation error of $O(h^2)$ from T.S., the global error is $O(h)$.