Example:

The fast fourier transform can solve certain differential equations quickly.

Imagine you want to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} = f(x).$$

If we discretize the PDE to solve it on a computer. We'll use finite differences.

Forward difference: $$u'\left(x+\frac{h}{2}\right) = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$$

$$\approx \frac{u(x+h)-u(x)}{h}.$$

Backward difference: $$u'\left(x-\frac{h}{2}\right) \approx \frac{u(x)-u(x-h)}{h}.$$

So the 2^{nd} derivative is

$$u''(x) = \lim_{h\to 0} \frac{u'\left(x+\frac{h}{2}\right) - u'\left(x-\frac{h}{2}\right)}{h}$$

$$\approx \frac{\dfrac{u(x+h)-u(x)}{h} - \dfrac{u(x)-u(x-h)}{h}}{h}$$

$$= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}.$$

So,

$$u''(x) \approx \frac{u(x+h)-2u(x)+u(x-h)}{h^2}.$$

If we let our x interval be discretized by

Rename

$$u(x) = u_k.$$

So,

$$u(x+h) = u_{k+1}, \text{ and } u(x-h) = u_{k-1}.$$

Finally, our finite difference approximation to

$$\frac{\partial^2 u}{\partial x^2} = f(x)$$

is

$$u_{k+1} - 2u_k + u_{k-1} = h^2 f_k.$$

$$-u_{k+1} + 2u_k - u_{k-1} = -h^2 f_k.$$

So the matrix is tridiagonal.

$$\text{matrix } T \left\{ \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{bmatrix} \right. \begin{bmatrix} \\ \vec{u} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ -h^2 f \\ \\ \end{bmatrix}$$

Periodic boundary conditions

The matrix T is diagonalized by the Fourier-transform.

(Spectral theorem: If A is real symmetric then it is diagonalizable).

Aside: A circulant matrix is a constant diagonal matrix with the form

$$C = \begin{bmatrix} f_0 & f_{n-1} & & \cdots & f_1 \\ f_1 & f_0 & f_{n-1} & \cdots & f_2 \\ \vdots & & & & \\ f_{n-1} & f_{n-2} & & \cdots & f_0 \end{bmatrix}$$

Note: Shift. It is periodic as well.

Example:

$$\begin{bmatrix} 2 & 8 & 6 & 4 \\ 4 & 2 & 8 & 6 \\ 6 & 4 & 2 & 8 \\ 8 & 4 & 6 & 2 \end{bmatrix}$$

In this case, the columns of the fourier matrix are the eigenvectors of C.

Spectral theorem:

$$F^{-1}TF = \Lambda$$

F = $n \times n$ fourier matrix

Λ = diagonable matrix with eigenvalues of T on diagonal

$T\underbrace{u = f}_{\text{rename } -h^2 f}$ can be solved by noting that

$$T = F \Lambda F^{-1}.$$

$$F \Lambda F^{-1} u = f$$

$$u = (F \Lambda F^{-1})^{-1} f$$

$$u = F \Lambda^{-1} F^{-1} f$$

$$u = F \Lambda^{-1} \left(\tfrac{1}{n} \overline{F}\right) f.$$

$$\underbrace{Ax = \lambda x}_{\text{Additional complexity}}$$

This requires 2 FFT multiplications and division by diagonal elements of Λ.

In one dimension, the FFT solve we have just gone through is not so important. But it makes a big difference in 1D or 2D.

$$\underbrace{-u_{xx} - u_{yy} = f(x,y)}_{2D}$$

Aliasing: The discrete fourier transform suffers from the drawback that it can't tell the diffence between

$$u_0 = e^{inx}$$

when $x = \frac{2\pi k}{n}$ and

$$u_0 = 1.$$

So high frequencies are replaced by their "aliases" (low frequency periodic equivalents).

Cool example:

Aliasing makes wagon wheels look like they are going backwards in old westerns.

FFTW - Fastest Fourier Transform in the West.

Package

Section 7.1: Numerical differentiation

We need numerical approximations to derivatives for a few reasons. Two big examples

1. Derive a numerical approximation or method for solving ODE's and PDE's on a computer.
2. We have data that we collect and we need the derivative of the function "behind the data."

Example: In field one collects data points (time t, temp T) on a typical day in August.

$$\{(8AM, 85^\circ), (8{:}30AM, 90^\circ), (9AM, 94^\circ), \ldots\}.$$

$$\underbrace{\frac{\partial T}{\partial t} = \frac{90-85}{8{:}30-8} = \frac{5^\circ}{1/2 \text{ hour}}}_{\text{Rate of change of temp}} = 10^\circ \text{ hour}.$$

Easiest formula for taking a first derivative numerically is

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}.$$

Note: For $f(x) = ax+b$. $f'(x) = a$. (exact derivative)

(approx. derivative) $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$

$$= \frac{a(x+h)+b - ax+b}{h}$$

$$= \frac{a \cdot h}{h} = a.$$

(exact).

Q: Why is the formula exact?

A: The derivative approximation comes from the linear taylor series approximation to the derivative. So linear functions are modeled exactly.

Taylor series gives $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$

or

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(\xi), \qquad x < \xi < x+h.$$

$$f(x+h) - f(x) = hf'(x) + \frac{h^2}{2}f''(\xi)$$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2}f''(\xi).$$

So

$$\frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(\xi) = f'(x).$$

Here we approximate the derivative $f'(x)$ by

$$\frac{f(x+h) - f(x)}{h}$$

and the local truncation error is

$$\frac{h}{2}f''(\xi).$$

(comes from truncating the taylor series, not from round off).

Q: What is the most important in error term?

A:

h is the most important.
We can control h.

The $f''(\xi)$ can generally be bounded.
We have have no ability to change $f''(\xi)$.
h tells us how quickly the approximation
converges to $f'(x)$ as $h \to 0$.
Generally
$0 < h < 1$.

So the higher the power of h the better
the approximation.

If instead we combine two taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(\xi_1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(\xi_2)$$

where

$x < \xi_1 < x+h$, and

$x-h < \xi_2 < x$

then by subtracting we get

$$f(x+h) - f(x-h) = 2hf'(x) + 2\frac{h^3}{3!}f'''(\xi)$$

where ξ is MVT point in $f(x+h) - f(x-h)$.

Then,

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^3}{6} f'''(\xi).$$

$$\underbrace{\frac{f(x+h)-f(x-h)}{2h}}_{\text{approximation formula}} - \underbrace{\frac{h^2}{6} f'''(\xi)}_{\text{error}} = f'(x).$$

Note that this centered difference formula above is better than the previous one-sided difference formula because $O(h^2)$ is better than $O(h)$. Since $h << 1$,

$h^2 \to 0$

than

$h \to 0$.
