Midterm:

- study notes and work homework problems (maybe other problems)

Section 6.8:

Theorem: (On orthogonal polynomials)
The sequence of polynomials defined inductively by

$$P_n(x) = (x - a_n) P_{n-1}(x) - b_n P_{n-2}(x), \quad n \geq 2$$

with $P_0(x) = 1$, $P_1(x) = x - a_1$ and

$$a_n = \frac{\langle x P_{n-1}, P_{n-1}\rangle}{\langle P_{n-1}, P_{n-1}\rangle}, \quad b_n = \frac{\langle x P_{n-1}, P_{n-2}\rangle}{\langle P_{n-2}, P_{n-2}\rangle}$$

is orthogonal.

Proof: (continued)

Induction step:
Assume polynomials are orthogonal for P_{n-1}, $n \geq 1$.
Then, (we want to show $\langle P_n, P_{n+1}\rangle = 0$).

$$\langle P_n, P_{n-1}\rangle = \langle (x - a_n) P_{n-1} - b_n P_{n-2}, P_{n-1}\rangle$$

$$= \langle x P_{n-1}, P_{n-1}\rangle - a_n \langle P_{n-1}, P_{n-1}\rangle$$

$$- b_n \langle P_{n-2}, P_{n-1}\rangle$$

by induction hypothesis

$$= \langle x P_{n-1}, P_{n-1}\rangle - \frac{\langle x P_{n-1}, P_{n-1}\rangle}{\langle P_{n-1}, P_{n-1}\rangle} \langle P_{n-1}, P_{n-1}\rangle$$

$$= \langle x P_{n-1}, P_{n-1}\rangle - \langle x P_{n-1}, P_{n-1}\rangle$$

$$= 0.$$

Next,

$$\langle P_n, P_{n-2}\rangle = \langle (x-a_n)P_{n-1} - b_n P_{n-2}, P_{n-2}\rangle$$

$$= \langle xP_{n-1}, P_{n-2}\rangle - a_n \underbrace{\langle P_{n-1}, P_{n-2}\rangle}_{0 \text{ by induction hypothesis}} -$$

$$- b_n \langle P_{n-2}, P_{n-2}\rangle$$

$$= \langle xP_{n-1}, P_{n-2}\rangle - \frac{\langle xP_{n-1}, P_{n-2}\rangle}{\langle P_{n-2}, P_{n-2}\rangle}\langle P_{n-2}, P_{n-2}\rangle$$

$$= \langle xP_{n-1}, P_{n-2}\rangle - \langle xP_{n-1}, P_{n-2}\rangle$$

$$= 0.$$

In fact, for $i = 0, 1, \ldots, n-3$

$$\langle P_n, P_i\rangle = \langle xP_{n-1}, P_i\rangle - a_n\langle P_{n-1}, P_i\rangle - b_n\langle P_{n-2}, P_i\rangle$$

$$\langle P_n, P_i\rangle = \langle xP_{n-1}, P_i\rangle = \langle P_{n-1}, xP_i\rangle$$

integral inner products (but may not matter)

But

$$P_n(x) = (x-a_n)P_{n-1}(x) - b_n P_{n-2}(x).$$

So,

$$P_{i+1} = (x - a_{i+1})P_i - b_{i+1}P_{i-1}$$

$$\Rightarrow P_{i+1} + a_{i+1}P_i + b_{i+1}P_{i-1} = xP_i$$

or

$$\langle P_n, P_i\rangle = \langle P_{n-1}, xP_i\rangle$$

$$= \langle P_{n-1}, \underset{\text{up to } n-2}{P_{i+1}} + a_{i+1}\underbrace{P_i}_{\text{up to } n-3} + b_{i+1}\underbrace{P_{i-1}}_{\text{up to } n-4}\rangle$$

$= 0. \square$

Section 6.12: Trignometric interpolation

The Fourier transform is a valuable transform. It represents arbitrary functions as combinations of pure harmonic functions (e.g. $\sin kx$, $\cos kx$, e^{ikx}, k-fixed number/frequency).

Example: $f(x) = \cos^2 x$

Note: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

So the function $\cos^2 x$ only needs two frequencies to represent it.

We will assume for simplicity that we are interested in functions which are periodic with period 2π.

Definition: Let f be a function. Its fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Using complex exponentials, the fourier series is

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}.$$

still coeffecients

The fourier series has some nice properties.
The basis functions $\{e^{ik}\}$ are orthogonal in a special "pseudo-inner product space".

<u>Case 1</u>: $j \neq -i$

$$\int_0^{2\pi} e^{ikx} e^{ijx}\,dx = \int_0^{2\pi} e^{i(k+j)x}\,dx =$$

$$= \frac{1}{i(k+j)} e^{i(k+j)x} \Big|_0^{2\pi} = \frac{1}{i(k+j)} \left[e^{i(k+j)2\pi} - 1 \right]$$

Let $j+k=n$ to get

$$= \frac{1}{in}\left[e^{in\cdot 2\pi} - 1 \right] = \frac{1}{in}\left[\cos(2\pi n) + i\cancelto{0}{\sin(2\pi n)} - 1 \right]$$

$$= \frac{1}{in}\left[1-1\right] = 0.$$

Case 2: If $j=-k$, $n=0$.

$$\int_0^{2\pi} e^{ikx} e^{ijx}\,dx = \int_0^{2\pi} e^{ikx} e^{-ikx}\,dx = \int_0^{2\pi} dx =$$

$$= 2\pi.$$

So,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikx} e^{i(-j)x}\,dx = \begin{cases} 1, & \text{if } j=-k \\ 0, & \text{if } j\neq -k \end{cases}.$$

Complex ℓ_2 norm

The functions e^{ikx} $(k=0,\pm1,\pm2,\dots)$ form an orthonormal system of functions with the L_2 inner product on $[-\pi,\pi]$ provided that we define the inner product as

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

Recall from last class that when we have an orthogonal set of polynomials $\{g_i\}$, the best approximation of f is

$$f = \sum_{i=1}^{n} c_i g_i$$

where $c_i = \langle f, g_i \rangle$.

So

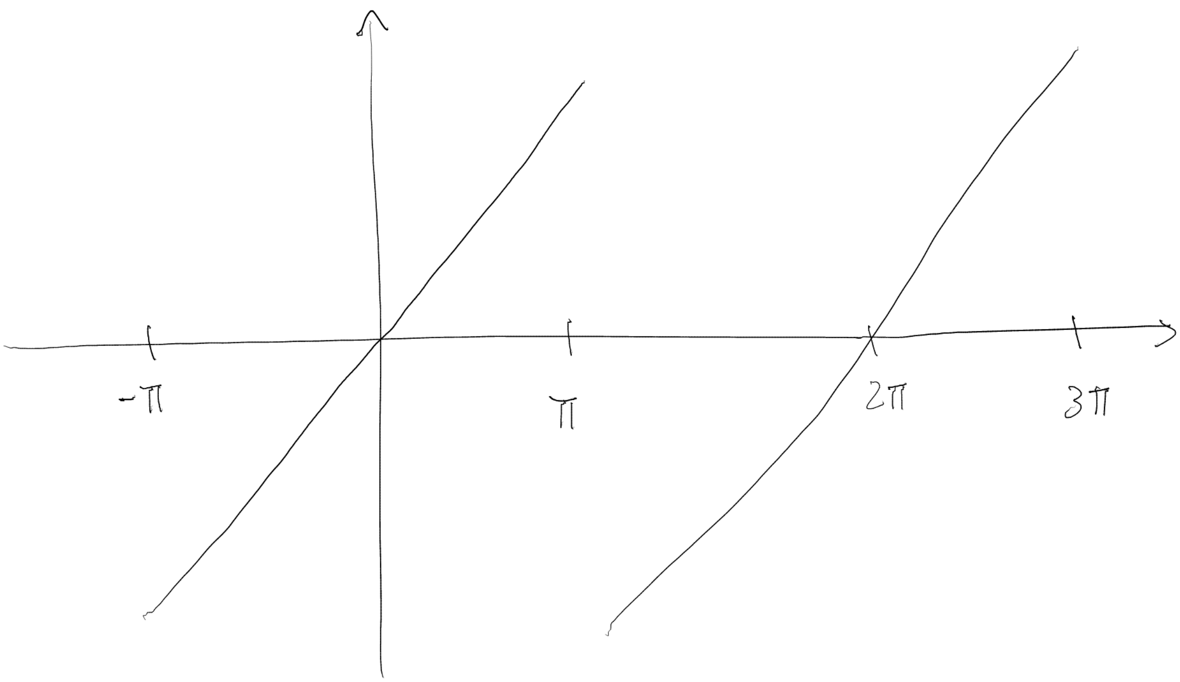
$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt = \langle f, g_i \rangle.$$

The 2nd property of the orthogonal fourier functions is that they form a basis for the space.
And they are easy to work with.

- Fourier series/transforms are used to solve differential equations.
- You can use fourier series/transforms to investigate stability of differential equations, signal processing, sampling theory, etc.

Example: $f(x) = x$

Since f has to be 2π periodic, we have



$$f(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \quad k \geq 1,$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad k \geq 1, \text{ and}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

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So,

$$a_k = \int_{-\pi}^{\pi} x \cos kx \, dx = 0.$$

since x odd and $\cos kx$ even implies $x \cos kx$ is odd.

Next,

$$b_k = \int_{-\pi}^{\pi} x \sin(kx) \, dx.$$

Apply integration by parts with

$$\begin{cases} u = x & du = dx \\ dv = \sin(kx)dx & v = -\frac{1}{k}\cos(kx) \end{cases}$$

Hence,

$$b_k = \frac{1}{\pi}\left[-\frac{x\cos(kx)}{k}\right]\Bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{k}\cos(kx)dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos kx}{k}\right]\Bigg|_{-\pi}^{\pi} + \frac{1}{k^2}\sin(kx)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos k\pi}{k} - \frac{-\pi\cos k(-\pi)}{k} + \frac{1}{\pi^2}\sin(k\pi) + \right.$$
$$\left. + \frac{1}{\pi^2}\sin(-k\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi \cos k\pi}{k}\right] = \underbrace{\pm \frac{2}{k}}_{\text{depends on } k.}.$$

So we get the series

$$x = 2\left\{\sin x - \frac{\sin 2x}{3} + \frac{\sin 3x}{3} - \cdots\right\}.$$