Section 7.2: Numerical integration using polynomial interpolation

$$I(f) = \int_a^b f(x)\,dx$$

Trapezoid rule:

$$I(f) \approx \frac{b-a}{2}\left[f(b)+f(a)\right].$$

$$\text{Error} = f(x) - \sum_{j=0}^{n} f(x_j)\,l_j(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x-x_i).$$

$n=1$ in our case.

$$\text{Error} = \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(\xi) = \frac{(x-a)(x-b)}{2} f^{(2)}(\xi)$$

$\uparrow$

in approximating f

$$I_{\text{Error}} = \frac{1}{2}\int_a^b (x-b)(x-a) f''(\xi)\,dx = \frac{f''(\xi)}{2}\int_a^b (x-b)(x-a)\,dx =$$

$$= \frac{f''(\xi)}{2}\left[\frac{x^3}{3} - a\frac{x^2}{2} - b\frac{x^2}{2} + abx\right]\Bigg|_a^b$$

$$= \frac{f''(\xi)}{2}\left[\left(\frac{a^3}{3} - \frac{a^3}{2} - \frac{a^2 b}{2} + a^2 b\right) - \left(\frac{b^3}{3} - a\frac{b^2}{2} - \frac{b^3}{2} + ab^2\right)\right]$$

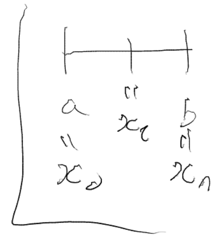
$$= \frac{f''(\xi)}{2}\left[-\frac{1}{6}(b-a)^3\right]$$

$$= -\frac{f''(\xi)}{12}(b-a)^3.$$

If $b-a$ is big then the error will be large.

In that case, break $b-a$ into smaller subintervals.

Apply the trapezoid rule to each subinterval.



$\frac{b-a}{2} = h$

Let $h = \frac{b-a}{n}$ (n = # subintervals).

Let $x_j = a + jh$, $j = 0, \ldots, n$.

Then

$$I(f) = \int_a^b f(x)\,dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx$$

$$= \sum_{j=1}^{n} \left[\frac{h}{2} [f(x_j) + f(x_{j-1})] - \frac{h^3}{12} f''(\xi_j) \right].$$

This is the composite trapezoid rule. So

$$I(f) \approx h\left[\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n\right], \quad n \geq 1$$

where $f_i = f(x_i)$. The error is

$$\sum_{j=1}^{n} \left(-\frac{h^3}{12} f''(\xi_j)\right) = -\frac{h^3}{12} n \cdot \frac{1}{n} \sum_{j=1}^{n} f''(\xi_j).$$

Now assume that $f''(x)$ is continuos on $[a,b]$. Then the mean value theorem says that the average value of $f''(x)$ must be be attained at some $\xi \in [a,b]$.

IVT instead?

$$\min_{[a,b]} f''(x) \leq M = \frac{1}{n} \sum_{j=1}^{n} f''(\xi_j) \leq \max_{a \leq x \leq b} f''(x).$$

So

$$-\frac{h^3}{12} n \cdot \frac{1}{n} \sum_{j=1}^{n} f''(\xi_j) = -\frac{h^3}{12} n \cdot f''(\xi).$$

Then

$$-\frac{h^3}{12} n f''(\xi) = -\frac{h^3}{12}\left(\frac{b-a}{h}\right) f''(\xi)$$

$$= -\frac{h^2}{12}(b-a) f''(\xi), \quad \xi \in [a,b].$$

So

$$I_{error} = -\frac{h^2}{12}(b-a) f''(\xi), \quad \xi \in [a,b].$$

The trapezoid rule idea can be generalized (Newton-Cotes formulas nodes are equally spaced).

Use Lagrange polynomials with node points $\{x_0, x_1, \ldots, x_n\}$ in $[a,b]$.

$$l_i(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j}, \quad 0 \leq i \leq n.$$

So,

$$P(x) = \sum_{i=0}^{n} f(x_i)\, l(x_i).$$

Then,

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx = \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x_i)dx}_{\text{weights } A_i} =$$

$$= \sum_{i=0}^{n} A_i f(x_i).$$

Trapezoid is a Newton-Cotes formula for $n=1$.

<u>Example:</u> NC formula with $n=3$

$$A_0 = \int_a^b l_0(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx.$$

For $x = x_0 + ih$ where $i \in \mathbb{R}$. So then

$$A_0 = \int_{x_0}^{x_3} \frac{[x_0+ih-(x_0+h)][x_0+ih-(x_0+2h)][x_0+ih-(x_0-3h)]}{\underbrace{(h)(2h)(3h)}_{(-1)(-1)(-1)}}\,dx$$

$$A_0 = -\int_0^3 \frac{(i-1)h\,(i-2)h\,(i-3)h}{(h)(2h)(3h)}\,h\,di$$

$$= -\frac{h}{6}\int_0^3 (i-1)(i-2)(i-3)\,di = -\frac{h}{6}\int_0^3 (i^2-3i+2)(i-3)\,di$$

$$= -\frac{h}{6}\int_0^3 (i^3-3i^2+2i-3i^2+9i-6)\,di$$

$$= -\frac{h}{6}\int_0^3 (i^3-6i^2+11i-6)\,di = \frac{3h}{8}.$$

So for $n=3$ the complete formula is

$$I_3(f) = \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right].$$

Error for a general NC formula comes from error in polynomial interpolation.

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) .$$

$$\int_a^b f(x) - \sum_{i=0}^{n} A_i f(x_i) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i) dx.$$

Since $f^{(n+1)}$ is continuos, it is bounded on the finite interval $[a,b]$. So,

$$|f^{(n+1)}(x)| \leq M, \quad x \in [a,b].$$

Thus

$$\left| \int_a^b f(x) - \sum_{i=0}^{n} A_i f(x_i) \right| \leq \frac{M}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) dx .$$

There are special cases where we can simplify this. For example, consider when the integrand

$\prod_{j=0}^{n} (x - x_j) = \varphi_k(x)$ is such that

$$\int_a^b \varphi_k(x) dx = 0 .$$

Example 2:

Prove that Simpson's rule correctly integrates all cubic polynomials.

Proof:

Simpson is $\int_a^b f(x)\,dx \simeq \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

Try: $f(x)=1$.

LHS:

$$\int_a^b f(x)\,dx = \int_a^b 1\,dx = b-a$$

RHS:

$$\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$= \frac{b-a}{6}\left[1+4+1\right]$$

$$= b-a$$

$f(x)=x$:

$$\int_a^b f(x)\,dx = \int_a^b x\,dx =$$

$$= \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2}-\frac{a^2}{2}$$

$$\frac{b-a}{6}\left[\;\ldots\;\right] =$$

$$\frac{b-a}{6}\left[a+4\,\frac{b+a}{2}+b\right] =$$

$$= \frac{b-a}{6} + a + 2b + 2a + b$$

$$= \frac{b-a}{6}\left[3a+3b\right]$$

$$= \frac{b-a}{2}(a+b) = \frac{b^2}{2}-\frac{a^2}{2}.$$

$f(x) = x^2$:

$$\int_a^b f(x)\,dx = \int_a^b x^2\,dx =$$

$$= \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

$$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] =$$

$$= \frac{b-a}{6}\left[a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right]$$

$$= \frac{b-a}{6}\left[a^2 + b^2 + (a+b)^2\right]$$

$$= \frac{b-a}{6}\left[2a^2 + 2ab + 2b^2\right]$$

$$= \frac{b-a}{3}\left[a^2 + ab + b^2\right]$$

$$= \frac{b^3 - a^3}{3}.$$

Example:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right].$$

$f(x) = x^2$

LHS:

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{x^3}{3} - \frac{x^3}{3}$$

RHS:

$$\frac{b-a}{2}\left[a^2 + b^2\right] = \frac{ba^2 - a^3 + b^3 - ab^2}{2}$$

LHS $\neq$ RHS

Section 7.3: Gaussian quadrature

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

weights nodes

Recall: For polynomial quadrature formulas, the nodes were equally spaced.

Idea: Choose nodes and weights so that we can exactly integrate polynomials of as high a degree as possible.

The hope is that this will allow us to accurately approximate integrals of functions "similar" to these polynomials.

The N.C. formulas for integration used evenly spaced nodes $\{x_i\}$ and we know there are functions for which these quadrature formulas won't converge even as $n \to \infty$.

Recall: $f(x) = \frac{1}{1+x^2}$, $x \in [-5, 5] \Rightarrow$

$$\Rightarrow \int_a^b f(x)\,dx \neq \lim_{n\to\infty} \int_a^b p_n(x)\,dx$$

had big oscillations.

Imagine you want to approximate

$$\int_{-1}^{1} f(x)dx$$

by $\sum_{i=1}^{n} A_i f(x_i)$, the error is

$$E_n(f) = \int_{-1}^{1} f(x)dx - \sum_{i=1}^{n} A_i f(x_i).$$

We want $E_n(f) = 0$ as high degree polynomial as possible.

Case $n=1$:

$$\int_{-1}^{1} f(x)dx \simeq A_1 f(x_1)$$

Try $f(x) = 1$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} dx = x\Big|_{-1}^{1} = 2. \qquad A_1 = 2.$$