Lab Notes
Measurement Techniques

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Contents

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Lab 1

Generating excitation signals

The goal of this lab is to get acquainted with the use of Matlab to generate multisines and random noise signals, which will be used in later labs as excitation signals for dynamic systems. You will learn

- how the DFT is defined (in Matlab in particular) and how it can be used to analyse periodic signals
- how to generate multisines in the time domain and in the frequency domain
- how you can construct an excitation signal in a given frequency band, and with a given frequency resolution

1.1 Discrete Fourier Transform (DFT)

Consider a discrete-time signal $x(n)$, in the time window $n = 0, \ldots, N-1$. Recall the definition of the DFT of $x(n)$, and of the inverse DFT, in Matlab (the functions `fft` and `ifft` respectively):

DFT	iDFT
$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}},$	$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$ (1.1)
for $k = 0, 1, \ldots, N-1$	for $n = 0, 1, \ldots, N-1$

An interpretation is that the DFT decomposes the time domain signal $x(n)$ into a linear combination of cosines and sines – or complex exponentials – of which the (complex) amplitudes are given by $X(k)$. The frequency axis k is expressed in *bin*. At bin k, the complex exponential $e^{\frac{j2\pi kn}{N}}$ is periodic in n, and has a period which fits exactly k times in the time interval of N points. This is shown graphically in Figure 1.1.

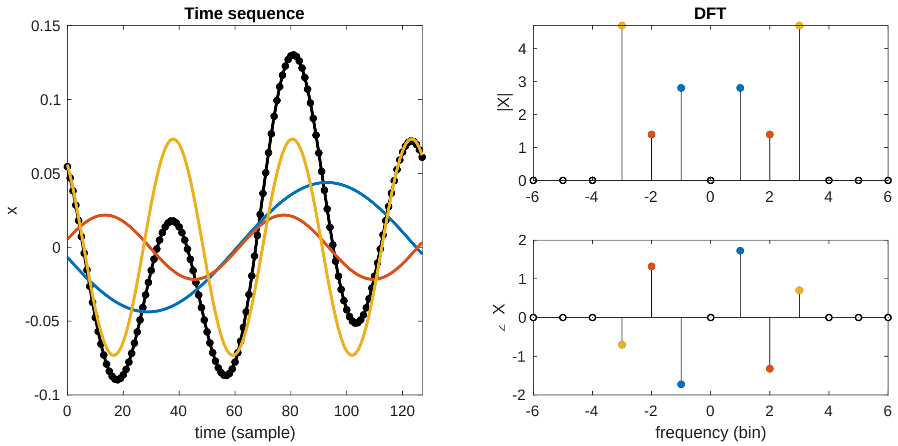

Figure 1.1: Left: time domain signal $x(n)$ (black), and its (co)sine components (coloured). Right: amplitude (top) and phase (bottom) of the individual components of the DFT $X(k)$.

If the discrete-time signal $x(n)$ represents a sampled continuous-time signal, with a sampling time of T_s, then the iDFT can be rewritten as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\omega_k n T_\mathrm{s}} \tag{1.2}$$

where ω_k is the angular frequency at bin k.

TASK 1.1.1. Frequency axis in rad/s. Prove that

$$\omega_k = \frac{2\pi}{T} k \tag{1.3}$$

where $T = NT_\mathrm{s}$ is the window length in seconds. (Hint: compare (1.1) and (1.2)) This will allow you to construct the frequency axis in $\mathrm{rad/s}$: ω_k for $k = 0, 1, ..., N-1$. In fact, the DFT discretises the frequency axis. The resolution of this discretisation increases when the measurement time T increases – i.e. there are more 'bins per Hz' when measuring longer.

TASK 1.1.2. Fundamental frequency. Prove that the fundamental frequency ω_1 of the DFT (which corresponds to one bin) is equal to

$$\omega_1 = \frac{2\pi}{T} = 2\pi \frac{f_\mathrm{s}}{N}. \tag{1.4}$$

where $f_\mathrm{s} = \frac{1}{T_\mathrm{s}}$ is the sampling frequency in Hz.

TASK 1.1.3. Conjugate symmetric DFT. Prove that

$$X(N-k) = X(-k) = X^*(k). \tag{1.5}$$

(Hint: use $e^{j2\pi n} = 1$ for $n \in \mathbb{Z}$, and $x(n) \in \mathbb{R}$.)

This demonstrates that the DFT of a real signal is conjugate symmetric around the origin. Thus, the negative frequencies are obtained from the upper half of the DFT: $X(-k) = X(N-k)$.

1.2 DFT of a (co)sine

TASK 1.2.1. DFT of 3 periods of a cosine. Generate a cosine sequence in Matlab with a randomly selected phase, and with a period that fits exactly 3 times in a data sequence of $N = 1000$ samples. Make a plot of the DFT of this sequence (amplitude and phase).

TASK 1.2.2. Perfect reconstruction. From the DFT plot, check that the condition for perfect reconstruction is satisfied. Is there any leakage visible?

TASK 1.2.3. Interpretation of the frequency axis. At which indices of the DFT do you obtain non-zero values? Explain. (Keep in mind that Matlab indices start at 1.)

TASK 1.2.4. Frequency axis in bins. Construct the frequency axis for the plots, expressed in bins.

TASK 1.2.5. Frequency axis in Hz. Consider that the sample frequency is $f_\mathrm{s} = 100\,\mathrm{Hz}$. Construct the frequency axis for the plots, expressed in Hz. (Hint: use the results from **Task 1.1.1**.)

1.3 Time domain construction of a multisine

A multisine is a sum of cosines, with frequencies that satisfy the condition for perfect reconstruction:

$$x(n) = \sum_{k=1}^{K} A_k \cos(\omega_k n T_\mathrm{s} + \varphi_k) = \sum_{k=1}^{K} A_k \cos\left(\frac{2\pi k n}{N} + \varphi_k\right) \qquad \text{for } n = 0, 1 \ldots, N-1 \tag{1.6}$$

The frequencies ω_k for which the amplitudes A_k are non-zero are called the **excited frequencies**.

TASK 1.3.1. Time domain random phase multisine. Generate a multisine in the time domain, by implementing (1.6), with $N = 1000$ samples and $K = 10$ excited frequencies. Set the amplitudes $A_k = 1$, and choose the phases φ_k randomly between 0 and 2π (i.e. a *random phase* multisine). Check that this multisine satisfies the condition for perfect reconstruction by plotting its DFT. Include the frequency axis, expressed in bin.

TASK 1.3.2. Frequency axis in Hz. For the multisine generated in **Task 1.3.1**, consider that the sampling frequency is $f_\mathrm{s} = 100\,\mathrm{Hz}$. Include the frequency axis expressed in Hz in the DFT plot, and the time axis expressed in seconds in the time domain plot.

TASK 1.3.3. Excite specific frequency lines. Generate a random phase multisine with a sampling frequency of $200\,\mathrm{Hz}$, with excited frequencies

$$[4, 8, 12, 16, 20, 24]\,\mathrm{Hz}. \tag{1.7}$$

Plot the time and frequency domain results, with appropriate axes.

1.4 Frequency domain construction of a multisine

It is also possible to generate a multisine in the frequency domain, so to directly construct $X(k)$, by specifying the amplitudes and phases of the components. One difficulty is that $X(k)$ must be constructed both for the positive and the negative frequencies. However, the following trick can be used such that only the positive frequencies need to be considered.

TASK 1.4.1. Trick for frequency domain multisine. Consider the vector $\tilde{X}(k)$:

$$\begin{aligned} \tilde{X}(k) &= \frac{A_k}{2}e^{j\varphi_k} && \text{for } 1 \leqslant k \leqslant K \\ \tilde{X}(k) &= 0 && \text{otherwise} \end{aligned}$$

Prove that

$$x(n) = 2N\Re\left\{\mathtt{iDFT}\big(\tilde{X}(k)\big)\right\} = \sum_{k=1}^{K} A_k \cos\left(\frac{2\pi kn}{N} + \varphi_k\right) \tag{1.8}$$

where $\Re$ denotes the real part. (Hint: use definition (1.1) of the `iDFT`.)

TASK 1.4.2. Frequency domain multisine. Use the trick from the previous task to construct a random phase multisine in the frequency domain. Let $N = 1000$ and excite the first $K = 30$ bins. Make time and frequency domain plots (time axis in sample number, frequency axis in bin number).

TASK 1.4.3. Specified excited frequency band. Construct a random phase multisine in the frequency domain, which excites the frequency band $[5, 15]\,\mathrm{Hz}$ at 31 equidistantly spaced frequencies. Choose an appropriate sampling frequency. Make time domain and frequency domain plots (time axis in seconds, frequency axis in Hz). How long is one period of this multisine (expressed in seconds)? What is the corersponding frequency resolution (expressed in Hz)?

1.5 Setting the Root-Mean-Square of a signal

The Root-Mean-Square (RMS) of a signal is defined as:

$$\mathrm{RMS}(x) = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x^2(n)}. \tag{1.9}$$

Hence, the RMS value of a signal depends on the number of excited frequencies and on the choice of the amplitudes A_k.

TASK 1.5.1. Set the RMS of the signal from the previous task to $\mathrm{RMS}_{\mathrm{des}} = 3$:

$$x_{\mathrm{des}}(n) = x(n)\frac{\mathrm{RMS}_{\mathrm{des}}}{\mathrm{RMS}(x)}. \tag{1.10}$$

1.6 Influence of the phase of a multisine

Until now, we chose the phases of the multisine components to be random. The choice of the phase has an impact on the time domain properties of the multisine. More specifically, it will influence its crest factor (CF), which is defined as:

$$\mathrm{CF}(x) = \frac{\max(|x|)}{\mathrm{RMS}(x)}. \tag{1.11}$$

In words, the crest factor is the ratio between the peak value of the signal in the time domain and the RMS value of signal.

TASK 1.6.1. Construct a multisine in the frequency domain with $N = 500$ samples, with the first $K = 60$ bins excited and with the following phases:

- Random phase: chosen randomly in $[0, 2\pi]$ (uniform distribution),
- Schroeder phase: $\varphi_k = \frac{k(k+1)\pi}{K}$,
- Linear phase: $\varphi_k = k\pi$.
- Constant phase

Make time and frequency domain plots (in samples and bins), and compute the Crest Factors. Describe, qualitatively, the relationship between the time domain plot and the crest factor. What is the advantage of a low/high crest factor?

1.7 Random noise signals

A popular excitation signal in the literature is random white noise. This is a signal which excites all frequencies, and which is stochastic, both in the time and in the frequency domain.

TASK 1.7.1. White Gaussian random noise. Generate a normally distributed (Gaussian), random, white noise sequence of $N = 1000$ samples, by using the Matlab function `randn`. Make time and frequency domain plots (axes in samples and bins). Observe that all the bins are excited, with random amplitudes and phases.

Note that this approach generates a signal which excites the full available frequency band. This is often not desired, for multiple reasons. For instance, most practical systems are only active in a limited frequency band. Thus, the input energy outside of that frequency band is typically wasted. Also, exciting the full frequency band is prone to cause alias errors, because harmonics created by the system under test will lie beyond the Nyquist frequency. For these reasons, it is better to limit the excitation frequency band, for instance by filtering.

TASK 1.7.2. Filtered random noise. Generate a filtered random noise sequence with $N = 1000$, sampling frequency 100 Hz, from a Gaussian white noise sequence (use `randn`). Do this by using the function `cheby1` to create a lowpass digital Chebyshev filter of order 5, ripple 2 dB, and such that the passband edge lies at

5 Hz. Filter the sequence by using the function `filter`. Make time and frequency domain plots (axes in seconds and Hz), and check that the excited frequency band is as expected. What do you observe in the stop-band of the filter? Is it equal to 0? Explain.

TASK 1.7.3. Periodic band-limited random noise. Generate a Gaussian random noise sequence (use `randn`), with $N = 1000$ and sampling frequency 100 Hz. Compute the DFT, and set the DFT at all frequencies beyond 5 Hz to zero:

$$\tilde{X}(k) = 0 \qquad \text{for } \omega_k > 2\pi 5\,\mathrm{rad/s}, \tag{1.12}$$

$$\tilde{X}(k) = X(k) \qquad \text{otherwise}, \tag{1.13}$$

and use the expression

$$x(n) = 2N\Re\left\{\texttt{iDFT}\left(\tilde{X}(k)\right)\right\} \tag{1.14}$$

to obtain the time domain sequence. Scale the signal such that it has a desired RMS value of 1. Then repeat this sequence by putting multiple copies of the sequence after each other. If you compute the DFT of the result, no leakage should occur. Make time and frequency domain plots (axes in seconds and Hz) to check this, and explain why this is the case. Observe that in the pass-band, some non-excited frequencies appear in between each pair of excited ones. Explain.

In fact, the signal generated in **Task** 1.7.3 can be interpreted as a multisine, with random phase *and* random amplitude.

Lab 2

Measuring frequency response functions

In this lab you will

- Design and apply excitation signals to an unknown device under test (DUT) by using measurement equipment
- Compute the FRF of the DUT from the measured input and output signals
- Observe and explain the influence of the choice of the excitation signal (multisine, periodic noise, aperiodic noise) on the measured FRF

2.1 Frequency response function (FRF)

The FRF of a Linear Time Invariant (LTI) system completely describes the system's behavior. It is a system property, that can be measured using:

$$H(j\omega) = \frac{Y(j\omega)}{U(j\omega)} \tag{2.1}$$

with $U(j\omega)$ and $Y(j\omega)$ the Fourier spectra of the input and output signals respectively, as shown in Figure 2.1. In this lab assignment, we will investigate the influence of the choice of $U(j\omega)$ on the quality of the measured FRF. As an example, the FRF of a filter will be measured.

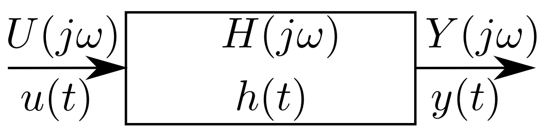

Figure 2.1: Representation of a Linear Time Invariant (LTI) system, where $U(j\omega)$ and $Y(j\omega)$ represent the input and the output spectra respectively.

2.2 Measurement setup

Figure 2.2 shows the block diagram of the measurement setup. The excitation signal is generated by an Arbitrary Waveform Generator (AWG). The voltages that are present at the input and the output of the DUT are both measured by the ELVIS II acquisition channels. To ease the operation, the AWG and the acquisition are integrated in the ELVIS II hardware. They use the same sampling clock. The software drivers that are needed to load and acquire signals are all provided in the LabVIEW environment.

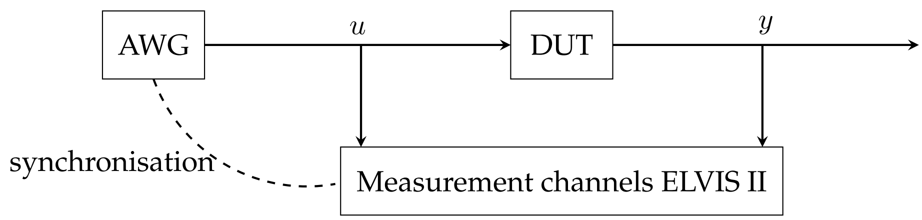

Figure 2.2: Measurement setup.

QUESTION 2.1. Synchronization If the clocks of the generator and the acquisition are not synchronized, which effect can you expect? Is this equally important for low and high frequencies?

2.2.1 Discretization of the measured signals

To generate arbitrary analog signals in continuous time and continuous amplitude, we use an Arbitrary Waveform Generator (AWG). This Digital to Analog Converter (DAC) based device converts a data stream with a predefined spectrum into the analog signal with the same spectrum that is needed to excite the system. To obtain the expected analog excitation spectrum, the data sequence must obey the theoretical rules (Nyquist, leakage avoidance, ...)

To measure the analog signals, it is mandatory to transform them into a data sequence again to obtain a stream of digital numbers. This process discretizes the signals both in time and in amplitude. These discretizations generally result in a loss of information, hereby degrading the measurement significantly. Information can only be preserved if all the theoretical rules are obeyed very carefully.

Quantization of the voltage

The loss of information due to the discretisation of the voltage depends on the resolution of the Analog to Digital Converter (ADC) in the measurement channel. This is determined by the number of bits used to encode the measured samples. ADCs nowadays have a resolution in excess of 10 bit. This means that 1024 different voltage levels can be measured.

The voltage swing associated to the Least Significant Bit (LSB) is the voltage difference (in Volt) between two successive discretization levels of the ADC. The dynamic range of the ADC is then the ratio between the maximum voltage that can be represented and 1 LSB.

QUESTION 2.2. Dynamic range. What is the dynamic range of the ADC used in the ELVIS II board? (refer to the datasheet found in `https://www.ni.com/docs/en-US/bundle/372590b/resource/372590b.pdf`)

Discretisation of the time

To convert the continuous time signals to a data sequence requires to know their value at a discrete set of time instants only. This discretisation in time is bound to hard constraints if the conversion is to happen without loss of information: the sampling theorem or theorem of Nyquist is to be obeyed.

QUESTION 2.3. What is the minimum sampling frequency required to perfectly reconstruct a sinusoidal signal with a frequency of 10 Hz?

2.2.2 The LabVIEW interface

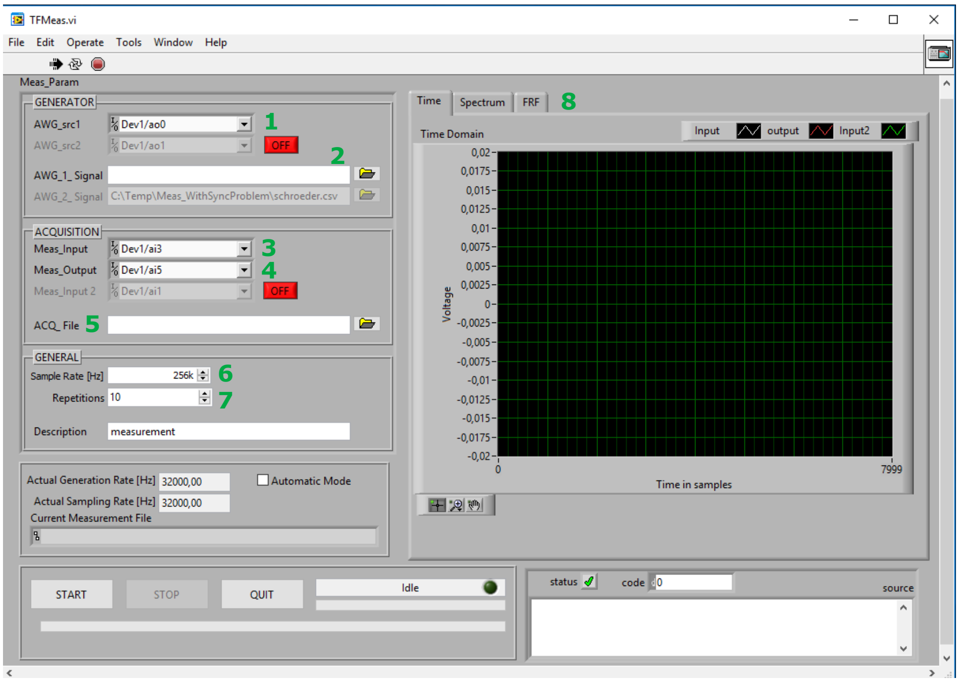

Figure 2.3: LabView interface.

Figure 2.3 shows the interface of the LabVIEW Virtual Instrument used in the lab. The main parts are numbered on the figure and are explained below:

1. The connection that is used to connect a signal to the generator.
2. The path to the `.mat` file containing the input signal to be applied.
3. The connection that is used to connect a signal to the measured input.
4. The connection that is used to connect a signal to the measured output.
5. The path to the `.mat` file where the measured signals will be saved.

6. The sampling rate of the generator and the acquisition channels.

7. The integer number of repetitions of the measurement.

8. The first and the second tab show the measured input and output signals in the time domain and the frequency domain respectively. The third tab shows the frequency response function.

Once all the settings are filled in correctly, press the `run` button (arrow at the top of the screen). Next, press the `start` button to fire up the measurement. During the measurement, the progress bar at the bottom of the screen indicates the ratio of the measurements that are already done to the total amount of requested measurements. To stop the measurement, use the `quit` button.

2.3 Choosing an excitation signal

The choice of $u(t)$ can be used to optimize a number of criteria such as the excitation power, the measured frequency band, the minimization of the time needed to perform the experiment, the minimization of the cost of the experiment, the maximization of the accuracy of the results, etc. The outcome of such an optimization is highly dependent on the considered system. In this lab, we will study the effect of the choice of $u(t)$ on the accuracy of the measured FRF.

2.3.1 Influencing measurement accuracy

The random errors on the FRF will depend on the input power that is present in the excitation signal, as more power means a higher Signal-to-Noise Ratio (SNR). To influence the power that is present in an excitation signal of a fixed time domain amplitude, we can tune the crest factor and the power spectrum of the signal.

Power spectrum

The shape of the power spectrum $S_{UU}(j\omega) = U(j\omega)U^*(j\omega)$ can be used to influence the measurement variability. It is clear that it makes no sense to excite the DUT (Device Under Test) in a frequency band where $H(j\omega)$ is of no interest. Since most systems require that the amplitude of the excitation remains bounded, it is an advantage to concentrate the allowable excitation power in the bandwidth that is to be characterized. This will increase the signal to noise ratio in that band, and hence decrease the variability of the measurement. For the characterization of a linear time invariant system, an increase of the excitation power level in the band of interest results in a reduction of the variability of the measured FRF.

Crest factor

If the ratio between the peak value of the signal in the time domain and the RMS value of the signal is very high, the power that can be put in the signal decreases for a fixed maximal signal amplitude. This ratio is called the crest factor (defined in Section 1.6). This quantity somehow quantifies to what extent the signal concentrates its energy in the time.

The crest factor depends on the shape of the signal in the time domain. Consider all the signals with a given power spectrum $S(j\omega)$. The shape of the associated time waveform is determined by the phase spectrum of the signal. The accuracy of the FRF is therefore also affected by the phase spectrum of $U(j\omega)$.

QUESTION 2.4. Explain why the accuracy of the FRF is affected by the phase spectrum.

Measurement errors

Up to now, we have implicitly assumed that measurement errors result from random perturbations of the measured spectra. In general, measurement errors can either be stochastic or systematic. Only stochastic errors can be influenced by a changing signal to noise ratio. Systematic errors or bias errors are more insidious. They often result in a smooth shift of the measured characteristics, that is very hard to detect. To influence them, additional knowledge is mandatory. Systematic errors are removed by calibration. The idea here is to compare the measured and the a priori known FRF of a reference system. In this lab, we will concentrate on the stochastic measurement errors only.

2.3.2 Excitation signals

The idea is to compare FRF measurements of a system that are obtained using different excitation signals. These signals are all normalized to have a constant RMS value in time domain as explained in Section 1.5. The variability of the measurements is obtained by repeating the same measurement and calculating the measurement's sample variance. The signals that will be used in this lab assignment are shortly described below.

Multisine excitation

A multisine is a sum of harmonically related sine waves with a well chosen amplitude and phase spectrum,

$$x(t) = \sum_{k=1}^{K} A_k \sin(2\pi k f_0 t + \varphi_k) \tag{2.2}$$

The excited frequencies kf_0, with $k \in \mathbb{N}$, are always an integer multiple of the fundamental frequency f_0 (the frequencies are therefore called commensurate). The number of excited frequency lines K, the amplitude of the excited lines A_k and their respective phase φ_k can all be chosen freely to match the requirements of the application.

QUESTION 2.5. Generation of multisine signals. Use MATLAB to generate the data sequence of a multisine that consists of 4096 time samples and contains 100 excited spectral lines, located at the low end of the band (from line 1 to 100). Generate the multisine with

(a) a constant phase spectrum,

(b) a random phase spectrum, with phases uniformly distributed between $[0, 2\pi)$

(c) a Schroeder phase spectrum. Choose the phases according to $\varphi_k = \frac{k(k+1)\pi}{K}$, where k is the line number (= the frequency of the line expressed in bins) and K is the number of excited lines in the signal.

QUESTION 2.6. Crest factor. Calculate the crest factor of these signals and explain the differences.

QUESTION 2.7. Plotting and interpretation. Visualize the signals both in the time and the frequency domain then discuss differences and similarities.

Noise excitation

Noise is a very popular choice for an excitation signal. Be aware that there can be a difference between noise and noise in the literature. The noise that we consider here is the purely random (aperiodic) noise. In the literature, many signals are referred to as periodic noise. The difference is quite subtle, but results in a very different behavior.

For periodic noise, the signal generator is loaded with one noise record that is repeated periodically and is used for all the experiments. In fact, this boils down to the creation of a periodic excitation. Periodic noise is a kind of multisine whose amplitude and phase spectra are both selected in a random way. Clearly, this is not what we call a random excitation in the context of this lab.

A purely random (aperiodic) noise excitation requires that the signal generator is loaded with a fresh realization of the random signal for each experiment that is performed. Therefore, there is no periodicity at all and the signal will behave as predicted by the theoretical analysis for a noise excitation.

As mentioned in the theory, two complications arise when working with arbitrary excitations. First, the system never reaches a steady state, hence a transient term is present. To address this problem, the signals can be multiplied by a window (e.g. a Hann window), prior to computing the DFTs. This will reduce the influence of the transient error. Secondly, the input spectra can be very small or even zero at certain frequencies, leading to problems in the calculation of the resulting FRF. This can be solved by using the spectral averaging technique, as will be done in Lab 3.

2.4 Now, it's up to you!

2.4.1 Get to know your DUT

The DUT that is to be characterized is unknown a priori. To obtain maximal information, the FRF is to be measured in a well chosen frequency band. We will select this frequency band first using a 'get-to-know-your-system' measurement that covers a wide frequency band. In the lab, it is known that the DUT has a bandwidth which is smaller than 500 Hz. The first experiment is executed as follows:

- Configure the instrument to use a sampling frequency $f_s = 8\,\mathrm{kHz}$.
- Construct a multisine with a Schroeder phase such that

  - The excited lines are present at frequencies between 1 Hz and 500 Hz
  - A frequency resolution of 1 Hz is obtained
  - The RMS value of the input signal is $V_{\mathrm{RMS}} = 100\,\mathrm{mV}$.

- Perform the measurement. Measure a few (5 to 10) periods of the input and output signals.

QUESTION 2.8. **Differences between periods** Compare the measurements of the different periods of the signal in the time domain and calculate the FRF for each period of the signals separately. Explain what you see and decide which periods to select or leave out and why.

QUESTION 2.9. **Frequency resolution** Visualize the spectra of input and output signals. What does this measurement show? Focus on the bandwidth where all the important features of the system are included and modify the frequency resolution to improve the representation of these features. Fix the frequency resolution for the remainder of this lab.

2.4.2 Compare the FRFs obtained with different excitation signals

To quantify the influence of the choice of the excitation signal on the quality of the measured FRF, the experiment is repeated for the different signals given below. A distinction is made between periodic and aperiodic signals. With the procedure explained in the previous section, perform the measurements for the following signals:

Periodic

(a) The Schroeder multisine designed in the previous section.

(b) A multisine with constant phase and amplitude in the specified analysis band only.

(c) A multisine with an arbitrary (random) phase and constant amplitude in the specified analysis band only.

(d) A periodic noise signal.

Aperiodic

(e) An aperiodic noise signal. Remember that here measuring P 'periods' requires to load P different signals in the AWG.

(f) Same as (e), but multiply the measured aperiodic input and output signals by a Hann window and observe the results. How are the results in comparison to the results with a rectangular window?

  - **Hint:** You can generate a Hann window using the Matlab command as follows: `[Hann] = hanning(N,'periodic')`

QUESTION 2.10. **FRF computation** For each measurement, compute the FRF and provide a plot in the report. Which differences do you observe? Explain.

Lab 3

Measuring frequency response functions in the presence of noise

Unlike simulations, real life experiments will always be influenced by noise. If not taken into account, this noise can lead to a bias and/or high variability in your model. A solution to this problem is averaging over multiple experiments to obtain the frequency response function (FRF). In this lab we will study different averaging methods to calculate the FRF.

3.1 Averaging methods to calculate the FRF

Consider that the measured input and output records $u(n)$ and $y(n)$ have been obtained for $n = 0, \ldots, N-1$ with N the number of measured samples. Let $U(k)$ and $Y(k)$ be the discrete Fourier transforms of the measured time records:

$$U(k) = \mathrm{DFT}\{u(n)\} \tag{3.1}$$

$$Y(k) = \mathrm{DFT}\{y(n)\} \tag{3.2}$$

for $k = 0, 1, \ldots, N-1$. In the ideal noiseless case, the FRF $H(j\omega_k)$ is simply the division of the DFT spectra:

$$H(j\omega_k) = \frac{Y(k)}{U(k)} \tag{3.3}$$

where the angular frequency $\omega_k = \frac{2\pi k f_s}{N} = \frac{2\pi k}{T}$, with f_s the sampling frequency and T the measurement time, corresponds to the kth bin.

Under practical conditions, the measurements will always be distorted by measurement noise. As a result, the calculated FRF will only approximate the ideal one with a finite accuracy. To improve the accuracy, one can measure the data records more than once (spending more time and money) and then average these repetitions $u_i(n)$ and $y_i(n)$, for $i = 1, 2, \ldots, M$. This averaging can be performed in different ways, giving different results, even when fed by the same data. We will now look at the different averaging methods and determine their advantages and disadvantages.

1. Averaging the time records:

$$u(n) = \frac{1}{M}\sum_{i=1}^{M} u_i(n) \Rightarrow U(k) = \mathrm{DFT}\left\{u(n)\right\} \tag{3.4}$$

$$y(n) = \frac{1}{M}\sum_{i=1}^{M} y_i(n) \Rightarrow Y(k) = \mathrm{DFT}\left\{y(n)\right\} \tag{3.5}$$

$$H(j\omega_k) = \frac{Y(k)}{U(k)} \tag{3.6}$$

2. Averaging the DFT spectra:

$$U_i(k) = \mathrm{DFT}\left\{u_i(n)\right\} \Rightarrow U(k) = \frac{1}{M}\sum_{i=1}^{M} U_i(k) \tag{3.7}$$

$$Y_i(k) = \mathrm{DFT}\left\{y_i(n)\right\} \Rightarrow Y(k) = \frac{1}{M}\sum_{i=1}^{M} Y_i(k) \tag{3.8}$$

$$H(j\omega_k) = \frac{Y(k)}{U(k)} \tag{3.9}$$

3. Averaging the FRF:

$$H_i(j\omega_k) = \frac{Y_i(k)}{U_i(k)} \tag{3.10}$$

$$H(j\omega_k) = \frac{1}{M}\sum_{i=1}^{M} H_i(j\omega_k) \tag{3.11}$$

4. Averaging the auto-power of the input signal and the cross-power:

$$S_{YU}(k) = \frac{1}{M}\sum_{i=1}^{M} Y_i(k)U_i^*(k) \tag{3.12}$$

$$S_{UU}(k) = \frac{1}{M}\sum_{i=1}^{M} U_i(k)U_i^*(k) \tag{3.13}$$

$$H_1(j\omega_k) = \frac{S_{YU}(k)}{S_{UU}(k)} \tag{3.14}$$

5. Averaging the auto-power of the output signal and the cross-power:

$$S_{YY}(k) = \frac{1}{M}\sum_{i=1}^{M} Y_i(k)Y_i^*(k) \tag{3.15}$$

$$S_{UY}(k) = \frac{1}{M}\sum_{i=1}^{M} U_i(k)Y_i^*(k) \tag{3.16}$$

$$H_2(j\omega_k) = \frac{S_{YY}(k)}{S_{UY}(k)} \tag{3.17}$$

The first two methods will only work properly if the data repetitions are identical up to the noise contribution. This calls for a periodic excitation signal and a triggering signal that ensures that each measurement starts at a fixed point in the period of the excitation.

The last two methods are also applicable to the measurements obtained with noise as an excitation signal (i.e. with *arbitrary* excitation). They do not at all require a triggering of the records but come at the cost of a bias. Fortunately, this bias is proportional to the signal-to-noise-ratio of the signal. Therefore, it can often safely be neglected. However, when the signal-to-noise-ratio is small or the required accuracy is high, it can be important. Remember that it can be removed completely without an increase in the measurement time if a different (periodic) excitation would be selected! The bias can be proven to be such that the true value of the FRF is bound by the H_1 and the H_2 estimate,

$$|H_1| \leqslant |H| \leqslant |H_2| \tag{3.18}$$

For a fast conversion of the signals between the time and the frequency domain the use of the FFT is preferred.

QUESTION 3.1. Length data records. Note that this algorithm works faster when the length of the data records N is a power of 2. Do you know why?

QUESTION 3.2. Trigger signal What is the purpose of a trigger signal? A number of processing methods were discussed in the theory to improve the SNR of the FRF by averaging of measurements. Which methods require a trigger signal to operate properly? Explain what happens if the trigger is absent while needed.

3.2 Measurements

3.2.1 Measurement setup

The measurement setup is the same as in the previous lab, with the addition of a noise generator interposed between the Elvis and the DUT. This noise generator adds noise to the measured input and output signals. The noise level that is added to the measurements can be tuned manually. Perform the measurements both noiseless and with an appropiate noise level.

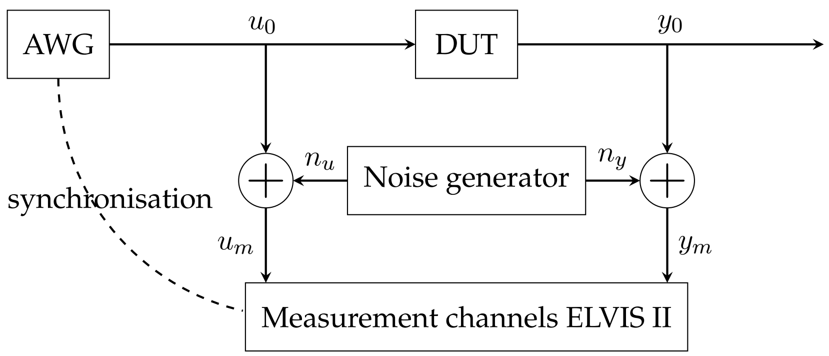

Figure 3.1: Measurement setup.

3.2.2 Excitation signals

Periodic multisine signal

- Construct a multisine with random or Schroeder phase in the frequency domain such that
  - The sampling frequency is 16 kHz
  - The excited lines are present at frequencies between 4 Hz and 1000 Hz
  - The RMS value of the input signal is $V_{\text{RMS}} = 500\,\text{mV}$.
  - The signal contains $N_1 = 4000$ samples, and this corresponds to one period of the multisine.
- Perform the measurement on the DUT. Measure 40 repetitions of the input and output signals.

QUESTION 3.3. Frequency domain multisine. In your report, show the Matlab code you used to construct the multisine signal (with random or Schroeder phase) in the frequency domain. Make sure that the code is sufficiently commented to improve its readability.

Aperiodic noise signal

- Construct an aperiodic noise signal such that
  - The sampling frequency is 16 kHz.
  - The RMS value of the input signal is $V_{\text{RMS}} = 500\,\text{mV}$.
  - The signal contains $N_2 = 40 \cdot N_1 = 160\,000$ samples.
- Perform the measurement on the DUT. Measure a single repetition of the input and output signals.

3.2.3 Preprocessing the data

Before applying the different averaging techniques, we need to preprocess the data to make sure that they can easily be analyzed. For each measurement, we are going to discard the first 8 repetitions. Then, you need to create an input matrix and and output matrix of size $N_1 \times 32$. For the measurement with the multisine, you can use the provided function `ReadData`, while for the aperiodic noise, you can use the Matlab function `reshape`.

3.2.4 Averaging the measurements

Apply the averaging techniques described in Section 3.1 to the preprocessed data. Calculate both the FRF and its standard deviation with all the applicable averaging methods for both the multisine and the periodic noise excitation.

QUESTION 3.4. Which averaging techniques are applicable to which exciation signals? Explain.

QUESTION 3.5. Provide relevant plots of the estimated FRF, obtained with the different methods, with the different exciation signals and with a different number of averaged records.

QUESTION 3.6. Determine the effect of the number of averaged records on the variability of the averaged result. Compare the standard deviation.

QUESTION 3.7. Discuss the differences and explain according to you, which will deliver the best result. Discuss the pros and cons of each excitation signal.

Good to know: the Standard Deviation (STD) is obtained as the sample variance of the FRFs that were calculated for each repetition of the measurement. For example, if 32 repetitions are available and one wants to obtain the variance based on the processing of 4 averaged data records per FRF calculation, only 8 repeated FRF measurements can be obtained. One of these FRFs is considered to be the result, while the standard deviation is evaluated using the 8 obtained FRFs. Note that as only 32 repetitions of the measurements are present, there is only 1 FRF that can be calculated by taking the 32 repeated experiments into account and therefore the provided processing routine can not calculate the sample standard deviation of this FRF.

Lab 4

Measuring nonlinear distortions

The goal of this lab is to characterize the nonlinearities in the response of a nonlinear system using a single sine input signal.

4.1 Static nonlinear system

Per definition, any system that obeys the superposition principle is a linear system. Consequently, any system that does not obey the superposition principle is a nonlinear system. This negative definition has the disadvantage that it is too general. Therefore, we will restrict the class to systems that have a close to linear behaviour.

A *static nonlinear system* is a nonlinear system where the relation between the input signal u and the output signal y can be described by a function. We will assume that this function can be written as a polynomial series expansion of the response with respect to the input signal in a given interval:

$$y\left(t\right)=\sum_{k=0}^{\infty}K_k u^k\left(t\right)=K_0+K_1 u\left(t\right)+K_2 u^2\left(t\right)+K_3 u^3\left(t\right)+\dots \tag{4.1}$$

4.1.1 Harmonic and intermodulation distortions

It has been proven before that when a linear time-invariant (LTI) system is excited by a single sine input signal $u\left(t\right)=\cos\left(\omega_1 t\right)$, the response is again a sine at the same frequency ω_1, but with a different amplitude and a different phase. When a nonlinear system is considered, this is no longer the case. Consider a static nonlinear system whose response $y\left(t\right)$ to an excitation $u\left(t\right)$ is given by:

$$y\left(t\right)=u\left(t\right)+u^2\left(t\right)+u^3\left(t\right). \tag{4.2}$$

For a single sine input $u\left(t\right)=\cos\left(\omega_1 t\right)$, the output $y\left(t\right)$ can then be rewritten as:

$$y\left(t\right)=\left[\cos\left(\omega_1 t\right)\right]+\left[\frac{1}{2}+\frac{1}{2}\cos\left(2\omega_1 t\right)\right]+\left[\frac{3}{4}\cos\left(\omega_1 t\right)+\frac{1}{4}\cos\left(3\omega_1 t\right)\right] \tag{4.3}$$

The linear term results in an output component at the fundamental frequency ω_1. The quadratic term creates a DC component and a component at the second harmonic $2\omega_1$. The cubic term leads to a component at the fundamental frequency ω_1 and a component at the third harmonic $3\omega_1$. Hence, the response contains energy at frequencies that are integer multiples, also called harmonics, of the excited fundamental frequency ω_1. This type of behaviour is called *harmonic distortion*.

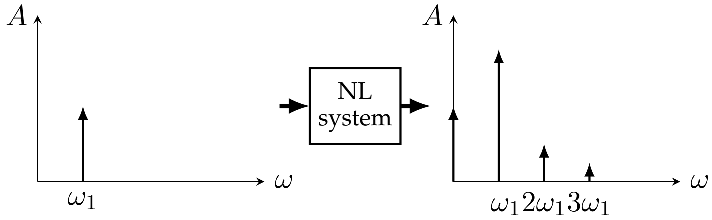

Figure 4.1: Input and output spectrum of a nonlinear system excited by a single sine.

If the input signal is a sum of two sinewaves, $u(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$, the output signal becomes more complex:

$$y(t) = [\cos(\omega_1 t) + \cos(\omega_2 t)] \tag{4.4}$$
$$+ \left[1 + \frac{1}{2}\Big(\cos(2\omega_1 t) + \cos(2\omega_2 t)\Big) + \Big(\cos((\omega_1 + \omega_2)t) + \cos((\omega_1 - \omega_2)t)\Big)\right]$$
$$+ \left[\frac{9}{4}\Big(\cos(\omega_1 t) + \cos(\omega_2 t)\Big) + \frac{1}{4}\Big(\cos(3\omega_1 t) + \cos(3\omega_2 t)\Big)\right.$$
$$\left. + \frac{3}{4}\Big(\cos((\omega_1 + 2\omega_2)t) + \cos((\omega_1 - 2\omega_2)t) + \cos((\omega_2 + 2\omega_1)t) + \cos((\omega_2 - 2\omega_1)t)\Big)\right]$$

The linear term now results in components at the frequencies ω_1 and ω_2. The quadratic term not only produces contributions at DC and at the frequencies $2\omega_1$ and $2\omega_2$, but also at the sum and difference frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$. The cubic term creates components at the frequencies ω_1, ω_2, $3\omega_1$ and $3\omega_2$, and in addition to that, contributions also appear at $\omega_1 + 2\omega_2$, $\omega_1 - 2\omega_2$, $\omega_2 + 2\omega_1$ en $\omega_2 - 2\omega_1$. Hence, besides the expected harmonic distortion at multiples of ω_1 and ω_2, a new type of distortion, called *intermodulation distortion*, also appears at frequencies that are a combination of ω_1 and ω_2.

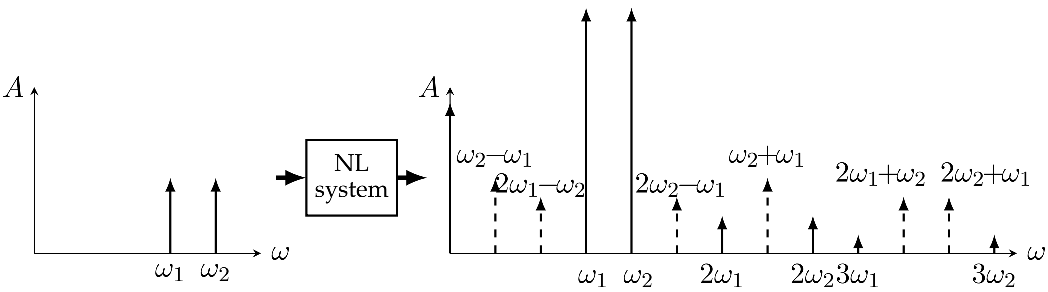

Figure 4.2: Input and output spectrum of a nonlinear system (full line = harmonic distortion, dashed line = intermodulation distortion).

Determining the frequencies where energy is expected to pop up at the output fortunately does not always require to fully work out the equations (4.3) and (4.4). Instead, a simple rule of thumb can be derived based on the example given above. It is important to be remember that a sinewave does not only have a contribution at the positive frequency ω_1, but also at the corresponding negative frequency.

Consider again a single sine input as in (4.3). The linear term only creates components in the output signal at the frequencies of the input signal.

input	output
$-f,f$	$-f,f$

The quadratic term creates contributions at all possible frequencies that can be obtained by summing 2 frequencies present in the input signal, also taking into account the negative frequencies:

input	output	
$-f, f$	$-f-f$	$-2f$
	$-f+f$	DC
	$f+f$	$2f$

For the cubic term, we have to combine 3 frequencies:

input	output	
$-f, f$	$-f-f-f$	$-3f$
	$-f-f+f$	$-f$
	$-f+f+f$	f
	$f+f+f$	$3f$

To obtain the output contributions for a two-tone input as in (4.4), the reasoning is completely similar. The linear term only creates components at the excited frequencies, the quadratic term creates components at all frequencies that can be obtained by the sum of 2 input frequencies, and the cubic term creates components at all frequencies that can be written as the sum of 3 input frequencies.

input = linear output	quadratic output		cubic output	
$-f_2,-f_1,f_1,f_2$	$-f_2-f_2$	$-2f_2$	$-f_2-f_2-f_2$	$-3f_2$
	$-f_2-f_1$	$-f_2-f_1$	$-f_2-f_2-f_1$	$-2f_2-f_1$
	$-f_1-f_1$	$-2f_1$	$-f_2-f_1-f_1$	$-f_2-2f_1$
	$-f_2+f_1$	$-f_2+f_1$	$-f_1-f_1-f_1$	$-3f_1$
	f_1-f_1	DC	$-f_2-f_2+f_1$	$-2f_2+f_1$
	f_2-f_2	DC	$-f_2-f_1+f_1$	$-f_2$
	f_2-f_1	f_2-f_1	$-f_2-f_2+f_2$	$-f_2$
	f_1+f_1	$2f_1$	$-f_2-f_1+f_2$	$-f_1$
	f_2+f_1	f_2+f_1	$-f_1-f_1+f_1$	$-f_1$
	f_2+f_2	$2f_2$	$-f_1-f_1+f_2$	$-2f_1+f_2$
			$\vdots$	$\vdots$

QUESTION 4.1. Consider a static nonlinear system whose response is given by:

$$y(t) = u(t) - \frac{1}{2}u^3(t) - \frac{1}{4}u^4(t). \tag{4.5}$$

At which frequencies will energy appear at the output when the input is excited by the sum of 2 sinewaves, one at frequency 4Hz and one at frequency 11Hz?

4.1.2 1 dB compression and expansion points

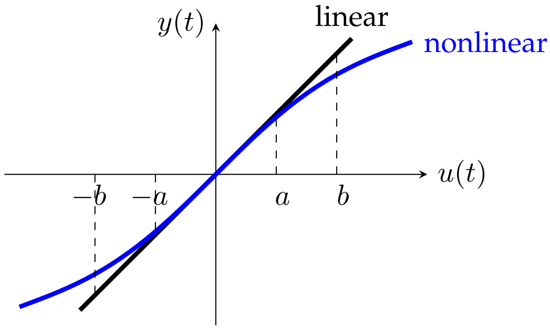

Figure 4.3: A static nonlinear response.

Figure 4.3 clearly illustrates that for an input amplitude between $-a$ and a, the nonlinear response is very close to the linear one, while for an amplitude between $-b$ and b, the difference with the linear response is much larger. Stated differently, one sees that the nonlinearity is much less excited by the first range of excitations, and hence the harmonic contributions will be much smaller.

In general, when a nonlinear system is excited with a low amplitude input signal, the influence of the nonlinearity will be small. The key issue is to know which amplitudes are small for a given system. To quantify the amplitude level at which a system becomes significantly nonlinear, a number of figures of merit have been defined. Here we will consider the 1 dB compression and expansion points.

For a linear system, the input and output power are linearly related considering equation (4.1):

$$P_y = K_1^2 P_u \tag{4.6}$$

For a nonlinear system, when the input power is increased, the output power will gradually start to deviate from this linear law. When the output power increases slower than the linear law, the nonlinear behaviour is called *compression*. When the output power increases faster than the linear law, we speak of *expansion*.

The input powers where the difference between the nonlinear and the linear characteristic reaches 1dB are called the 1 dB compression point for a system that compresses and the 1 dB expansion point for a system that expands the amplitude.

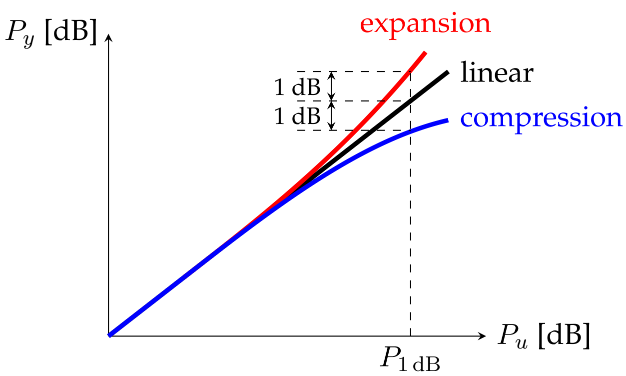

Figure 4.4: 1 dB compression and expansion points.

4.2 Measurements

In this lab, we will measure records that contain 4096 samples with a sampling frequency of 10 kHz. Remember that real systems are dynamic and thus bound to exhibit transient behaviour. Hence, the measurements are repeated several times to allow for the transients to damp. Make sure that you remove the transients before you start the data processing!

QUESTION 4.2. Design an excitation sinewave with a frequency of 100 Hz and an amplitude of 1 V. Plot the input signal and the output signal of the DUT in the frequency domain. What do you observe? Give a list of all possible solutions to get rid of this behaviour. Design experiments to show that the proposed solutions do indeed work.

QUESTION 4.3. Change the excitation frequency to a frequency in the neighborhood of 100 Hz in order to solve the problem in the previous step. Use 10 different amplitudes that are spaced logarithmicaly from 100 mV to 1.1 V [`logspace` command in MATLAB]). Plot the DFT of the output for each amplitude. What are the frequencies at which you expect distortion to appear? Are they all present in reality? Explain.

QUESTION 4.4. Plot the measurements from the previous question on an input power – output power plot. Do you see compression or expansion? From this plot, estimate the 1 dB compression or expansion point. Generate a single sine wave with the corresponding input amplitude to see how accurate your estimate is.